fractal and fractional

Article

Functional Integro-Differential Equations with State-Dependent Delay and Non-Instantaneous Impulsions: Existence and Qualitative Results

Abdelhamid Bensalem [1], Abdelkrim Salim [1,2], Mouffak Benchohra [1] and Gaston M. N'Guérékata [3,*]

[1] Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbès, Sidi Bel-Abbès 22000, Algeria
[2] Faculty of Technology, Hassiba Benbouali University of Chlef, Chlef 02000, Algeria
[3] NEERLab, Department of Mathematics, Morgan State University, 1700 E. Cold Spring Lane, Baltimore, MD 21252, USA
* Correspondence: gaston.nguerekata@morgan.edu

Abstract: This paper addresses some existence, attractivity and controllability results for semilinear integrodifferential equations having non-instantaneous impulsions on an infinite interval via resolvent operators in case of neutral and state-dependent delay problems. Our criteria were obtained by applying a Darbo's fixed-point theorem combined with measures of noncompactness. The obtained result is illustrated by an example at the end.

Keywords: attractivity; controllability; fixed point; infinite delay; integrodifferential equation; measures of noncompactness; mild solution; resolvent operator; state-dependent delay

MSC: 34D23; 93B05; 45J05; 34K45

check for updates

Citation: Bensalem, A.; Salim, A.; Benchohra, M.; N'Guérékata, G.M. Functional Integro-Differential Equations with State-Dependent Delay and Non-Instantaneous Impulsions: Existence and Qualitative Results. *Fractal Fract.* **2022**, *6*, 615. https://doi.org/10.3390/fractalfract6100615

Academic Editor: Natália Martins

Received: 20 September 2022
Accepted: 11 October 2022
Published: 21 October 2022

1. Introduction

Models using instantaneous impulses do not appear to be able to explain the specific dynamics of the evolution process in pharmacotherapy. For example, when one analyzes a person's hemodynamic equilibrium, the entry of drugs into the bloodstream and the subsequent absorption for the body are gradual and ongoing processes. Hernéndez and O'Regan [1] and Pierri et al. [2] began by investigating Cauchy problems for first-order evolution equations with instantaneous and non-instantaneous impulses. The works in [3–11] and their references include current results for evolution equations with non-instantaneous impulses. Many authors have examined qualitative properties such as existence, uniqueness, and stability for many integral, differential, and integrodifferential equations, see [12] for more details.

Whenever the system's behavior relies not just on its present condition, but also on its history, the past history is important in the analysis of a system represented as functional and partial functional differential equations. We assume that the histories y_ϑ belong to some abstract phase space $\mathcal{B}$, to be specified later. When the delay is infinite, we introduce the phase space concept $\mathcal{B}$. It is crucial in the study of both qualitative and quantitative theory, see [13]. A usual choice is a seminormed space satisfying suitable axioms, which was introduced by Hale and Kato in [14].

Functional evolution equations with state-dependent delay appear frequently in mathematical modeling of a variety of real-world problems, and as a result, the study of these equations has received considerable attention in recent years, see, for instance, [15–17].

In fact, the resolvent operator, which takes the place of the C_0-semigroup in evolution equations, is critical in solving (1), in both the weak and strict senses. Based on these important works, many authors have done extensive work in recent years on various topics such as existence, regularity of solutions and control problems for semilinear integro-differential evolution equations using the theory of resolvent operator, see [18–21], and the references therein.

Motivated by the works [4,19,22–26], we will investigate the existence and attractivity of mild solutions for non-instantaneous integrodifferential equations via resolvent operators with infinite delay:

$$\begin{cases} \phi'(\vartheta) = A\phi(\vartheta) + f(\vartheta, \phi_\vartheta, (H\phi)(\vartheta)) + \int_0^\vartheta B(\vartheta-\delta)\phi(\delta)d\delta, \text{ if } \vartheta \in I_k, k \in \mathbb{N}_0, \\ \phi(\vartheta) = \Xi_k\left(\vartheta, \phi(\vartheta_k^-)\right), \text{if } \vartheta \in J_k, k \in \mathbb{N}, \\ \phi(\vartheta) = \Phi(\vartheta), \text{ if } \vartheta \in \mathbb{R}_-, \end{cases} \tag{1}$$

and with state-dependent delay:

$$\begin{cases} \phi'(\vartheta) = A\phi(\vartheta) + f\left(\vartheta, \phi_{\rho(\vartheta,\phi_\vartheta)}, (H\phi)(\vartheta)\right) + \int_0^\vartheta B(\vartheta-\delta)\phi(\delta)d\delta, \text{ if } \vartheta \in I_k, k \in \mathbb{N}_0, \\ \phi(\vartheta) = \Xi_k\left(\vartheta, \phi(\vartheta_k^-)\right), \text{if } \vartheta \in J_k, k \in \mathbb{N}, \\ \phi(\vartheta) = \Phi(\vartheta), \text{ if } \vartheta \in \mathbb{R}_-, \end{cases} \tag{2}$$

where $I_0 = [0, \vartheta_1], I_k = (\delta_k, \vartheta_{k+1}]$ et $J_k = (\vartheta_k, \delta_k]$, $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ with $0 = \delta_0 < \vartheta_1 \leq \delta_1 \leq \vartheta_2 < \ldots < \delta_{m-1} \leq \vartheta_m \leq \delta_m \leq \vartheta_{m+1} \to +\infty, (k \to +\infty)$, $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $\{T(\vartheta)\}_{\vartheta \geq 0}$, $B(\vartheta)$ is a closed linear operator with domain $D(A) \subset D(B(\vartheta))$, the operator H is defined by

$$(H\phi)(\vartheta) = \int_0^a h(\vartheta, \delta, \phi(\delta))d\delta, \ \ a > 0,$$

the nonlinear term $f : J \times \mathcal{B} \times E \to E$; $k \in \mathbb{N}_0$, $\Xi_k : J_k \times E \to E$, $k \in \mathbb{N}, \ldots,$ $\Phi : \mathbb{R}_- \to E$, $\rho : J \times \mathcal{B} \to (-\infty, \infty)$ are a given functions, and $(E, \|\cdot\|)$ is a Banach space.

The following is how this manuscript is structured. Section 2 is reserved for some preliminary results and definitions which will be utilized throughout this manuscript. After we present and prove the existence and attractivity of solutions for problems (1) and (2), we study as well the controllability of solutions. Finally, we provide a relevant illustration.

2. Preliminaries

We introduce in this section some of the notations, definitions, fixed-point theorems and preliminary facts that will be used in the remainder of this paper.

Let $BC(J, E)$ be the Banach space of all bounded and continuous functions y mapping $J := [0, +\infty)$ into E, with the usual supremum norm

$$\|y\|_\infty = \sup_{\vartheta \in J} \|y(\vartheta)\|.$$

A measurable function $u : [0, +\infty) \to E$ is Bochner integrable if and only if $\|u\|$ is Lebesgue integrable. (For the Bochner integral properties, see [27], for instance). Let us denote by $L^1([0, +\infty), E)$ the Banach space of measurable functions $u : [0, +\infty) \to E$ which are Bochner integrable, with the norm

$$\|u\|_{L^1} = \int_0^{+\infty} \|u(\vartheta)\|dt.$$

We consider the following Cauchy problem

$$\begin{cases} \phi'(\vartheta) = A\phi(\vartheta) + \int_0^{\vartheta} B(\vartheta - \delta)\phi(\delta)d\delta; \ \text{for } \vartheta \geq 0, \\ \phi(0) = \phi_0 \in E. \end{cases} \tag{3}$$

The existence and properties of a resolvent operator is discussed in [28–30]. In what follows, we suppose the following assumptions:

(H_1) A is the infinitesimal generator of a uniformly continuous semigroup $\{T(\vartheta)\}_{\vartheta>0}$;

(H_2) For all $\vartheta \geq 0, B(\vartheta)$ is closed linear operator from $D(A)$ to E and $B(\vartheta) \in B(D(A), E)$. For any $y \in D(A)$, the map $\vartheta \to B(\vartheta)y$ is bounded, differentiable, and the derivative $\vartheta \to B'(\vartheta)y$ is bounded uniformly continuous on $\mathbb{R}^+$.

Theorem 1 ([29])**.** *Assume that (H_1)–(H_2) hold, then there exists a unique resolvent operator for the Cauchy problem (3).*

Let

$$PC(\mathbb{R}, E) = \Big\{ y : \mathbb{R} \to E \ : \ y|_{\mathbb{R}^-} \in \mathcal{B}, y|_{J_k} = \Xi_k; \ k \in \mathbb{N}, y|_{I_k}; \ k \in \mathbb{N}_0, \text{is continuous,}$$
$$y(\delta_k^+), y(\vartheta_k^-) and y(\vartheta_k^+) \text{ exists with } y(\delta_k^+) = g_k(\delta_k, y(\delta_k^-)) \text{ and } y(\vartheta_k^-) = y(\vartheta_k) \Big\}.$$

In this paper, we assume that the state space $(\mathcal{B}, \|.\|_{\mathcal{B}})$ is a seminormed linear space of functions mapping $(-\infty, 0]$ into $\mathbb{R}$, and satisfying the following fundamental axioms which were introduced by Hale and Kato in [14],

(A_1) If $y \in PC$ and $y_0 \in \mathcal{B}$, then for every $\vartheta \in J$, the following conditions hold:
- (i) $y_\vartheta \in \mathcal{B}$;
- (ii) There exists a positive constant H such that $|y(\vartheta)| \leq H\|y_\vartheta\|_{\mathcal{B}}$;
- (iii) There exist two functions $L(\cdot)$ and $M(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of y with L continuous and bounded and M locally bounded such that :

$$\|y_\vartheta\|_{\mathcal{B}} \leq L(\vartheta) \sup\{|y(\delta)| : 0 \leq \delta \leq \vartheta\} + M(\vartheta)\|y_0\|_{\mathcal{B}}.$$

(A_2) For the function y in (A_1), y_ϑ is a $\mathcal{B}$-valued continuous function on $\mathbb{R}^+ \setminus J_k$.

(A_3) The space $\mathcal{B}$ is complete. Denote $L_* = \sup\{L(\vartheta) : \vartheta \in J\}$, $M_* = \sup\{M(\vartheta) : \vartheta \in J\}$, $\aleph = \max\{L_*, M_*\}$.

Now, let $(\theta_k)_{k\in\mathbb{N}}$ be a sequence defined by

$$\theta_k = \vartheta_k - \vartheta, \ k \in \mathbb{N}, \ \vartheta \in \mathbb{R}^-.$$

Then, for $I_\theta = \mathbb{R}^- \setminus \{\theta_k \ : \ k \in \mathbb{N}\}$, we define the space

$$PC_\theta(\mathbb{R}^-, E) = \{y : \mathbb{R}^- \to E : \ y|_{I_\theta} \text{ is continuous and}$$
$$y(\theta_k^-), \ y(\theta_k^+) \text{ exist with } y(\theta_k^-) = y(\theta_k)\},$$

and the space

$$C_\theta := \{\phi \in PC_\theta(\mathbb{R}^-, E) : \lim_{\tau\to-\infty} \phi(\tau) \ \text{ exist in } \ E\},$$

endowed with the norm

$$\|\phi\|_\theta = \sup\{|\phi(\tau)| : \tau \leq 0\}.$$

Then, the axioms (A_1)–(A_3) are satisfied in the space C_θ. Thus, in all that follows, we consider the phase space $\mathcal{B} = C_\theta$, and let

$$BPC(\mathbb{R}, E) = \{y \in PC(\mathbb{R}, E) : y \ \text{ is bounded on } \mathbb{R}^+ \text{ with the norm } \|\cdot\|_{BPC}\},$$

such that

$$\|y\|_{BPC} = \sup_{\vartheta\in\mathbb{R}}\{\|y(\vartheta)\|\}.$$

Definition 1 ([31]). *Let X be a Banach space and Ω_X—the bounded subsets of X. The Kuratowski measure of noncompactness is the map $\mu : \Omega_X \to [0,\infty]$ defined by*

$$\mu(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^{n} B_i \text{ and } diam(B_i) \leq \epsilon\};\ here\ B \in \Omega_X,$$

where

$$diam(B_i) = sup\{\|u - v\|_E : u, v \in B_i\}.$$

Lemma 1 ([32]). *If Y is a bounded subset of a Banach space X, then for each $\epsilon > 0$, there is a sequence $\{y_k\}_{k=1}^{\infty} \subset Y$ such that*

$$\mu(Y) \leq 2\mu(\{y_k\}_{k=1}^{\infty}) + \epsilon.$$

Lemma 2 ([33]). *If $\{y_k\}_{k=0}^{\infty} \subset L^1$ is uniformly integrable, then the function $\vartheta \to \alpha(\{y_k(\vartheta)\}_{k=0}^{\infty})$ is measurable and*

$$\mu\left(\left\{\int_0^{\vartheta} y_k(\delta)d\delta\right\}_{k=0}^{\infty}\right) \leq 2\int_0^{\vartheta} \mu(\{y_k(\delta)\}_{k=0}^{\infty})d\delta.$$

More properties of the Kuratowski measure of noncompactness can be found in [31,34,35].

3. Global Existence and Attractivity for Functional Integro-Differential Equations

In this section, we will demonstrate the existence and attractivity of mild solutions of the problem (1). We will begin with the existence result, which is based on Mönch's fixed point theorem with the noncompactness. We move next to the attractivity of solutions [36].

3.1. Existence of Mild Solutions

In order to define a measure of noncompactness in the space $\mathcal{X} = BPC(\mathbb{R}, E)$, let us recall the following special measure of noncompactness which originates from [37], and will be used in our main results.

Let us fix a nonempty bounded subset S of the space $\mathcal{X}$. For $v \in H$, $T > 0$, $\epsilon > 0$, $\kappa_1, \kappa_2 \in [-T, T]$, such that $|\kappa_1 - \kappa_2| \leq \epsilon$. We denote $\omega^T(v,\epsilon)$ the modulus of continuity of the function v on the interval $[-T, T]$, namely,

$$\begin{aligned}
\omega^T(v,\epsilon) &= \sup\{\|e^{-\kappa_1}u(\kappa_1) - e^{-\kappa_2}u(\kappa_2)\|\ ;\ \kappa_1, \kappa_2 \in [-T, T]\},\\
\omega^T(H,\epsilon) &= \sup\{\omega^T(v,\epsilon)\ ;\ v \in H\},\\
\omega_0^T(S) &= \lim_{\epsilon\to 0}\{\omega^T(S,\epsilon)\},\\
\omega_0(S) &= \lim_{T\to+\infty}\omega_0^T(H).
\end{aligned}$$

If ϑ is fixed from $\mathbb{R}$, let us denote $S(\vartheta) = \{v(\vartheta) \in E\ ;\ v \in S\}$ and

$$d^{\Delta}(S(\vartheta)) = diam\ (S(\vartheta)) = \sup\{\|e^{-\vartheta}u(\vartheta) - e^{-\vartheta}v(\vartheta)\|\ ;\ u,\ v \in S\}.$$

Finally, consider the function χ defined on the family of subset of $\mathcal{X}$ by the formula

$$\chi_{BPC}(S) = \omega_0(S) + \lim_{|\vartheta|\to\infty} \sup d^{\Delta}(S(\vartheta)).$$

It can be shown similar to [38], that the function χ_{BPC} is a sublinear measure of noncompactness on the space $\mathcal{X}$.

Definition 2. *A function $\phi \in \mathcal{X}$ is called a mild solution of problem (1), if it satisfies*

$$\phi(\vartheta) = \begin{cases} R(\vartheta)\Phi(0) + \int_0^\vartheta R(\vartheta-\delta) f(\delta, \phi_\delta, (H\phi)(\delta)) d\delta; \text{ if } \vartheta \in I_0, \\ R(\vartheta-\delta_k)\left[\Xi_k(\delta_k, \phi(\delta_k^-))\right] + \int_{\delta_k}^\vartheta R(\vartheta-\delta) f(\delta, \phi_\delta, (H\phi)(\delta)) d\delta, \vartheta \in I_k,\ k \in \mathbb{N}, \\ \Xi_k(\vartheta, \phi(\vartheta_k^-)), \vartheta \in J_k,\ k \in \mathbb{N}, \\ \Phi(\vartheta); \text{ if } \vartheta \in \mathbb{R}_-. \end{cases}$$

The following assumption will be needed throughout the paper:

$(C1)$ $f : J \times \mathcal{B} \times E \to E$ is a Carathéodory function and there exist two functions $p_f^1, p_f^2 \in L^1(J, \mathbb{R}^+)$ and a continuous nondecreasing functions $\psi_f^1, \psi_f^2 : J \to (0, +\infty)$ such that:

$$||f(\vartheta, \phi_1, \phi_2)|| \le p_f^1(\vartheta)\psi_f^1(\|\phi_1\|_\mathcal{B}) + p_f^2(\vartheta)\psi_f^2(\|\phi_2\|), \quad \text{for } \phi_1 \in \mathcal{B},\ \phi_2 \in E,$$

and for every $M_1, M_2 \ge 0$,

$$\lim_{\vartheta \to +\infty} \sup_{\vartheta \in J} \int_0^\vartheta e^{-\mu(\vartheta-\delta)} \Big(M_1 p_f^1(\delta) + M_2 p_f^2(\delta) \Big) d\delta = 0.$$

$(C2)$ The function $h : D_h \times E \to E$ is continuous and there exists a continuous function $h_{c_1} : D_h \to (0, +\infty)$ such that,

$$\|h(\vartheta, \delta, \phi_1) - h(\vartheta, \delta, \phi_2)\| \le h_{c_1}(\vartheta, \delta)\|\phi_1 - \phi_2\|, \text{ for each } (\vartheta, \delta) \in D_h \text{ and } \phi_1,\ \phi_2 \in E.$$

$$\max\left\{ \sup_{D_h}\{h_{c_1}(\vartheta, \delta)\}, \sup_{D_h}\{\|h(\vartheta, \delta, 0)\|\} \right\} = \max\{h_{c_1}^*, h^*\} < \infty.$$

$(C3)$ $\Xi_k : J_k \times E \to E$ are continuous and there exist functions $L_{\Xi_k} : J \to (0, +\infty)$, $k \in \mathbb{N}$, such that

$$\|\Xi_k(\vartheta, \phi_1) - \Xi_k(\vartheta, \phi_2)\| \le L_{\Xi_k}(\vartheta)\|\phi_1 - \phi_2\|, \quad \text{for all } \phi_1, \phi_2 \in E,\ k \in \mathbb{N},$$

and

$$\max_{k \in \mathbb{N}} \sup_{\vartheta \in J}\{L_{\Xi_k}(\vartheta),\ k \in \mathbb{N}\} = L_{\Xi_k}^* < +\infty.$$

$(C4)$ Assume that (H_1)–(H_2) hold, and there exist $M_R \ge 1$ and $\mu \ge 0$, such that

$$\|R(\vartheta)\|_{B(E)} \le M_R e^{-\mu\vartheta}.$$

Theorem 2. *Assume that the conditions $(C1)$–$(C4)$ are satisfied. If*

$$M_R L_{\Xi_k}^* < 1,$$

then, the system (1) has at least one mild solution.

Proof. Transform the problem (1) into a fixed-point problem, consider the operator $\Lambda : \mathcal{X} \to \mathcal{X}$ define by :

$$\Lambda\phi(\vartheta) = \begin{cases} R(\vartheta)\Phi(0) + \int_0^\vartheta R(\vartheta-\delta) f(\delta, \phi_\delta, (H\phi)(\delta)) d\delta; \text{ if } \vartheta \in I_0, \\ \\ R(\vartheta-\delta_k)\left[\Xi_k(\delta_k, \phi(\delta_k^-))\right] \\ + \int_{\delta_k}^\vartheta R(\vartheta-\delta) f(\delta, \phi_\delta, (H\phi)(\delta)) d\delta; \text{ if } \vartheta \in I_k,\ k \in \mathbb{N}, \\ \\ \Xi_k(\vartheta, \phi(\vartheta_k^-)); \text{ if } \vartheta \in J_k,\ k \in \mathbb{N}, \\ \\ \Phi(\vartheta); \text{ if } \vartheta \in \mathbb{R}_-. \end{cases} \tag{4}$$

The transformation that we are going to use now is to simplify the calculations and the conditions and not to have a norm as soon as our space is already a Banach space. Let $x(\cdot) : (-\infty, +\infty) \to E$ be the function defined by:

$$x(\vartheta) = \begin{cases} R(\vartheta)\Phi(0), & \text{if } \vartheta \in I_0, \\ \\ 0, & \text{if } \vartheta \in (\vartheta_1, +\infty), \\ \\ \Phi(\vartheta), & \text{if } \vartheta \in \mathbb{R}_-. \end{cases}$$

Then, $x_0 = \Phi$, and for each $z \in \mathcal{X}$, with $z(0) = 0$, we denote by $\overline{z}$ the function

$$\overline{z}(\vartheta) = \begin{cases} z(\vartheta), & \text{if } \vartheta \in \mathbb{R}^+, \\ \\ 0, & \text{if } \vartheta \in \mathbb{R}_-. \end{cases}$$

If ϕ satisfies Definition 2, then we can decompose it as $\phi(\vartheta) = z(\vartheta) + x(\vartheta)$, which implies $\phi_\vartheta = z_\vartheta + x_\vartheta$, and the function $z(.)$ satisfies

$$z(\vartheta) = \begin{cases} \int_0^\vartheta R(\vartheta-\delta) f(\delta, z_\delta + x_\delta, H(z+x)(\delta)) d\delta; \text{ if } \vartheta \in I_0, \\ \\ R(\vartheta-\delta_k)\left[\Xi_k(\delta, (z)(\delta_k^-))\right] \\ + \int_{\delta_k}^\vartheta R(\vartheta-\delta) f(\delta, z_\delta + x_\delta, H(z)(\delta)) d\delta; \text{ if } \vartheta \in I_k,\ k \in \mathbb{N}, \\ \\ \Xi_k(\vartheta, (z)(\vartheta_k^-)); \text{ if } \vartheta \in J_k,\ k \in \mathbb{N}. \end{cases}$$

Set

$$\Omega = \{z \in \mathcal{X} \ : \ z(0) = 0\}.$$

Let the operator $\widehat{\Lambda} : \Omega \to \Omega$ defined by

$$\widehat{\Lambda}z(\vartheta) = \begin{cases} \int_0^\vartheta R(\vartheta-\delta) f(\delta, z_\delta + x_\delta, H(z+x)(\delta)) d\delta; \text{ if } \vartheta \in I_0, \\ \\ R(\vartheta-\delta_k)\left[\Xi_k(\delta_k, (z)(\delta_k^-))\right] \\ + \int_{\delta_k}^\vartheta R(\vartheta-\delta) f(\delta, z_\delta + x_\delta, H(z)(\delta)) d\delta; \text{ if } \vartheta \in I_k,\ \ k \in \mathbb{N}, \\ \\ \Xi_k(\vartheta, (z)(\vartheta_k^-)), \text{ if } \vartheta \in J_k,\ \ k \in \mathbb{N}. \end{cases}$$

Obviously, the operator Λ has a fixed point is equivalent to $\widehat{\Lambda}$ having a fixed point, and so we turn to proving that $\widehat{\Lambda}$ has a fixed point. We shall use Mönch's fixed-point theorem [33] to prove that $\widehat{\Lambda}$ has a fixed point.

Let $\Delta_\theta = \{z \in \Omega : \|y\|_{\mathcal{X}} \leq \theta\}$, with

$$0 < \max\left\{\Delta_1^\theta, \Delta_2^\theta, \Delta_3^\theta\right\} \leq \theta,$$

such that

$$\begin{aligned}
\Delta_1^{\theta} &= M_R\Big(\|p_f^1\|_{L^1}\psi_f^1(\aleph^*)+\|p_f^2\|_{L^1}\psi_f^2(H^*)\Big),\\
\Delta_2^{\theta} &= \frac{M_R(\Xi_k^0+\|p_f^1\|_{L^1}\psi_f^1(\widetilde{\aleph^*})+\|p_f^2\|_{L^1}\psi_f^2(\widetilde{H^*}))}{1-M_R L^*_{\Xi_k}},\\
\Delta_3^{\theta} &= L^*_{\Xi_k}\theta+\Xi_k^0,
\end{aligned}$$

and $\aleph^*, H^*,\ \widetilde{H^*},\ \widetilde{\aleph^*}$ are constants, they will be specific later.

The set Δ_θ is bounded, closed, and convex. We have divided the proof into four steps:

Step 1 : $\widehat{\Lambda}(\Delta_\theta)\subset\Delta_\theta$.

- *Case* 1: $\vartheta\in I_0$.
  For each $z\in\Delta_\theta$ and from $(C1)$–$(C3)$, it follows that

$$\begin{aligned}
\|z_\vartheta+x_\vartheta\|_{\mathcal{B}} &\le \|z_\vartheta\|_{\mathcal{B}}+\|x_\vartheta\|_{\mathcal{B}}\\
&\le L(\vartheta)|z(\vartheta)|+L(\vartheta)(M_R(\|\Phi(0)\|))+M(\vartheta)(\|\Phi\|_{\mathcal{B}})\\
&\le \aleph(\theta+(M_R+1)\|\Phi\|_{\mathcal{B}})=\aleph^*.
\end{aligned}$$

  And
$$\|H(z+x)(\delta)\|\le a(h^*_{c_1}(\theta+M_R\|\Phi\|_{\mathcal{B}})+h^*)=H^*.$$

  Then, we have

$$\begin{aligned}
\|\widehat{\Lambda}z(\vartheta)\| &\le M_R\int_0^{\vartheta}\Big(p_f^1(\delta)\psi_f^1(\aleph^*)+p_f^2(\delta)\psi_f^2(H^*)\Big)d\delta\\
&\le M_R\Big(\|p_f^1\|_{L^1}\psi_f^1(\aleph^*)+\|p_f^2\|_{L^1}\psi_f^2(H^*)\Big)\\
&\le \theta.
\end{aligned}$$

- *Case* 2: $\vartheta\in I_k$.
  For each $z\in\Delta_\theta$, by $(C1)$, $(C2)$ and $(C3)$, we obtain

$$\|\Xi_k(\vartheta,u(\cdot))\|\le L_{\Xi_k}(\vartheta)\|u(\vartheta)\|+\Xi_k^0.$$

  Hence, for
$$\widetilde{H^*}=a(h^*_{c_1}\theta+h^*)\text{ and }\widetilde{\aleph^*}=\aleph(\theta+\|\Phi\|_{\mathcal{B}}),$$
  we obtain

$$\begin{aligned}
\|\widehat{\Lambda}z(\vartheta)\| &\le M_R\Big[L^*_{\Xi_k}\theta+\Xi_k^0+\|p_f^1\|_{L^1}\psi_f^1(\widetilde{\aleph^*})+\|p_f^2\|_{L^1}\psi_f^2(\widetilde{H^*})\Big]\\
&\le \theta.
\end{aligned}$$

- *Case* 3: $\vartheta\in J_k$.
  For each $z\in\Delta_\theta$ and from $(C3)$, we obtain

$$\begin{aligned}
\|\widehat{\Lambda}z(\vartheta)\| &\le L^*_{\Xi_k}\theta+\Xi_k^0\\
&\le \theta.
\end{aligned}$$

  Thus,

$$\|\widehat{\Lambda}z\|_{\mathcal{X}} \le \theta.$$

  Consequently, $\widehat{\Lambda}(\Delta_\theta)\subset\Delta_\theta$ and $\widehat{\Lambda}(\Delta_\theta)$ is bounded.

Step 2: $\widehat{\Lambda}$ is continuous.

Let $\{z^n\}_{n\in\mathbb{N}}$ be a sequence, such that $z_n\to z^*$,

- *Case* 1: $\vartheta \in I_0$. We have

$$\|(\widehat{\Lambda}z^n)(\vartheta) - (\widehat{\Lambda}z^*)(\vartheta)\| \leq M_R \int_0^{\vartheta} \|f(\delta, z_\delta^n + x_\delta, H(z^n + x)(\delta)) \\ - f(\delta, (z_\delta^* + x_\delta), H(z^* + x)(\delta))\| d\delta.$$

By the continuity of h and f, we get

$$h(\vartheta, \delta, (z_\delta^n + x)(\delta)) \to h(\vartheta, \delta, (z^* + x)(\delta)) \quad as \;\; n \to +\infty,$$

and

$$\|h(\vartheta, \delta, (z^n + x)(\delta)) - h(\vartheta, \delta, (z^* + x)(\delta))\| \leq h_{c_1}^* \|z^n - z^*\|.$$

By the Lebesgue dominated convergence theorem, we obtain

$$\int_0^{\vartheta} h(\vartheta, \delta, (z^n + x)(\delta)) d\delta \to \int_0^{\vartheta} h(\vartheta, \delta, (z^* + x)(\delta)) d\delta, \quad as \;\; n \to +\infty.$$

Hence, from the continuity of the function f, and also by the Lebesgue dominated convergence theorem, we obtain

$$\|(\widehat{\Lambda}z^n) - (\widehat{\Lambda}z^*)\|_{\mathcal{X}} \to 0, \quad as \;\; n \to +\infty.$$

- *Case* 2: $\vartheta \in I_k$. We have

$$\|\widehat{\Lambda}(z^n)(\vartheta) - \widehat{\Lambda}(z^*)(\vartheta)\| \leq M_R \|\Xi_k(\delta_k, (z^n)(\delta_k^-)) - \Xi_k((\delta_k, (z^*)(\delta_k^-)))\| \\ + M_R \int_{\delta_k}^{\vartheta} \|f(\delta, (z_\delta^n + x_\delta)(\delta), H(z^n)(\delta)) \\ - f(\delta, (z_\delta^* + x_\delta), H(z^*)(\delta))\| d\delta.$$

Similar to Case 1, by the continuity of h, f *and* Ξ_k, we obtain

$$\|(\widehat{\Lambda}z^n) - (\widehat{\Lambda}z^*)\|_{\mathcal{X}} \to 0, \quad as \;\; n \to +\infty.$$

- *Case* 2: $\vartheta \in J_k$. We have

$$\|(\widehat{\Lambda}(z^n))(\vartheta) - \widehat{\Lambda}(z^*)(\vartheta)\| \quad \leq \quad \|\Xi_k(\vartheta, (z^n)(\vartheta_k^-)) - \Xi_k(\vartheta, (z^*)(\vartheta_k^-))\|.$$

By the continuity of Ξ_k, we obtain

$$\|(\widehat{\Lambda}z^n) - (\widehat{\Lambda}z^*)\|_{\mathcal{X}} \to 0, \quad as \;\; n \to +\infty.$$

Thus, $\widehat{\Lambda}$ is continuous.

Step 3: the set $\widehat{\Lambda}(\Delta_\theta)$ is equicontinuous.

For $\Pi \subset \Delta_\theta$, $T > 0$, *and* $k_0 \in \mathbb{N}$ with $T \geq \vartheta_{k_0}$ and $z \in \Pi$, we have

- *Case* 1: $\kappa_1, \kappa_2 \in I_0$.

$$\begin{aligned}
&\|\widehat{\Lambda}z(\kappa_1) - \widehat{\Lambda}z(\kappa_2)\| \\
&\quad \leq \int_0^{\kappa_1} \|R(\kappa_1-\delta) - R(\kappa_2-\delta)\| \Big(p_f^1(\delta)\psi_f^1(\aleph^*) + p_f^2(\delta)\psi_f^2(H^*) \Big) d\delta \\
&\qquad + \int_{\kappa_1}^{\kappa_2} \|R(\kappa_2-\delta)\| \Big(p_f^1(\delta)\psi_f^1(\aleph^*) + p_f^2(\delta)\psi_f^2(H^*) \Big) d\delta \\
&\quad \leq \psi_f^1(\aleph^*) \int_0^{\kappa_1} \|R(\kappa_1-\delta) - R(\kappa_2-\delta)\| p_f^1(\delta) d\delta \\
&\qquad + \psi_f^2(H^*) \int_0^{\kappa_1} \|R(\kappa_1-\delta) - R(\kappa_2-\delta)\| p_f^2(\delta) d\delta \\
&\qquad + M_R \int_{\kappa_1}^{\kappa_2} \Big(\psi_f^1(\aleph^*) p_f^1(\delta) + \psi_f^2(H^*) p_f^2(\delta) \Big) d\delta.
\end{aligned}$$

By the strong continuity of $R(\cdot)$ and $(C1)$, we have

$$\|\widehat{\Lambda}v(\kappa_1) - \widehat{\Lambda}v(\kappa_2)\| \to 0, \ as\ \kappa_1 \to \kappa_2.$$

- *Case* 2: $\kappa_1, \kappa_2 \in I_k$.

$$\begin{aligned}
&\|\widehat{\Lambda}z(\kappa_1) - \widehat{\Lambda}z(\kappa_2)\| \\
&\quad \leq \|R(\kappa_1-\delta_k) - R(\kappa_2-\delta_k)\| \|\Xi_k(\delta_k, (z)(\delta_k^-))\| \\
&\qquad + \int_{\delta_k}^{\kappa_1} \|R(\kappa_1-\delta) - R(\kappa_2-\delta)\| \Big(p_f^1(\delta)\psi_f^1(\widetilde{\aleph^*}) + p_f^2(\delta)\psi_f^2(\widetilde{H^*}) \Big) d\delta \\
&\qquad + \int_{\kappa_1}^{\kappa_2} \|R(\kappa_2-\delta)\| \Big(p_f^1(\delta)\psi_f^1(\widetilde{\aleph^*}) + p_f^2(\delta)\psi_f^2(\widetilde{H^*}) \Big) d\delta \\
&\quad \leq \|R(\kappa_1-\delta_k) - R(\kappa_2-\delta_k)\| (L_{\Xi_k}^* \theta + \Xi_k^0) \\
&\qquad + \psi_f^1(\widetilde{\aleph^*}) \int_0^{\kappa_1} \|R(\kappa_1-\delta) - R(\kappa_2-\delta)\| p_f^1(\delta) d\delta \\
&\qquad + \psi_f^2(\widetilde{H^*}) \int_0^{\kappa_1} \|R(\kappa_1-\delta) - R(\kappa_2-\delta)\| p_f^2(\delta) d\delta \\
&\qquad + M_R \int_{\kappa_1}^{\kappa_2} \Big(\psi_f^1(\widetilde{\aleph^*}) p_f^1(\delta) + \psi_f^2(\widetilde{H^*}) p_f^2(\delta) \Big) d\delta.
\end{aligned}$$

By the strong continuity of $R(\cdot)$ and assumption $(C1)$, we obtain

$$\|\widehat{\Lambda}z(\kappa_1) - \widehat{\Lambda}z(\kappa_2)\| \to 0, \ as\ \kappa_1 \to \kappa_2.$$

- *Case* 3: $\kappa_1, \kappa_2 \in J_k$.

$$\|\widehat{\Lambda}z(\kappa_1) - \widehat{\Lambda}z(\kappa_2)\| \quad = \quad \|\Xi_k(\kappa_1, z(({\kappa_1})_k^{\ -})) - \Xi_k(\kappa_2, z(({\kappa_2})_k^{\ -}))\|.$$

From $(C3)$, the set $\{\Xi_k(\vartheta, y_\vartheta^-)\}_{k=1}^{k_0}$ is equicontinuous, then

$$\|\widehat{\Lambda}z(\kappa_1) - \widehat{\Lambda}z(\kappa_2)\| \to 0, \ as\ \kappa_1 \to \kappa_2.$$

Hence, the set $\widehat{\Lambda}(\Pi)$ as equicontinuous, then $\omega_0(\widehat{\Lambda}(\Pi)) = 0$.

Step 4: the set $\widehat{\Lambda}(\Delta_\theta)$ is equiconvergent.

- *Case* 1: $\vartheta \in I_0$.
  For each $z \in \Delta_\theta$ and by $(C1)$, $(C3)$, we have

$$\|\widehat{\Lambda}z(\vartheta)\| \quad \leq \quad M_R \int_0^{\vartheta} e^{-\mu(\vartheta-\delta)} \Big(\psi_f^1(\aleph^*) p_f^1(\delta) + \psi_f^2(H^*) p_f^2(\delta) \Big) d\delta \xrightarrow[\vartheta \to +\infty]{} 0.$$

Then,

$$\|\widehat{\Lambda}z(\vartheta) - \widehat{\Lambda}z(+\infty)\| \xrightarrow[\vartheta \to +\infty]{} 0.$$

- *Case* 2: $\vartheta \in I_k$.
  For each $z \in \Delta_\theta$ by $(C1)$, $(C2)$ and $(C3)$, we obtain

$$\begin{aligned} \|\widehat{\Lambda}z(\vartheta)\| \quad &\leq \quad M_R\left(L^*_{\Xi_k}\theta + \Xi^0_k + \int_{\delta_k}^{\vartheta} e^{-\mu(\vartheta-\delta_k)}\left(\psi^1_f(\widetilde{\aleph^*})p^1_f(\delta) + \psi^2_f(\widetilde{H^*})p^2_f(\delta)\right)d\delta\right) \\ &\xrightarrow[\vartheta \to +\infty]{} M_R(L^*_{\Xi_k}\theta + \Xi^0_k). \end{aligned}$$

  Therefore,

$$\begin{aligned} \|\widehat{\Lambda}z(\vartheta) - \widehat{\Lambda}z(+\infty)\| \quad &\leq \quad M_R\left(\int_{\delta_k}^{\vartheta} e^{-\mu(\vartheta-\delta_k)}\left(\psi^1_f(\widetilde{\aleph^*})p^1_f(\delta) + \psi^2_f(\widetilde{H^*})p^2_f(\delta)\right)d\delta\right) \\ &\xrightarrow[\vartheta \to +\infty]{} 0. \end{aligned}$$

- *Case* 3: $\vartheta \in J_k$.
  For each $z \in \Delta_\theta$, by $(C3)$

$$\|\widehat{\Lambda}z(\vartheta)\| \quad \leq \quad L^*_{\Xi_k}\theta + \Xi^0_k \xrightarrow[\vartheta \to +\infty]{} L^*_{\Xi_k}\theta + \Xi^0_k.$$

  Then, we obtain

$$\|\widehat{\Lambda}z(\vartheta) - \widehat{\Lambda}z(+\infty)\| \xrightarrow[\vartheta \to +\infty]{} 0.$$

Now, let Π be a subset of Δ_θ, such that $\Pi \subset \overline{\widehat{\Lambda}(\Pi)} \cup \{0\}$. Π is bounded and equicontinuous, therefore, the function $\vartheta \to \varphi(\vartheta) = \chi(\Pi(\vartheta))$ is continuous. By $(C3)$ and the properties of the measure χ_{BPC}, we have

$$\begin{aligned} \varphi(\vartheta) \quad &\leq \quad \chi\Big((\widehat{\Lambda}(\Pi))(\vartheta) \cup \{0\}\Big), \\ &\leq \quad \chi\Big((\widehat{\Lambda}(\Pi))(\vartheta)\Big). \end{aligned}$$

As the set $\widehat{\Lambda}(\Delta_\theta)$ is equicontinuous, we get

$$\varphi(\vartheta) \quad \leq \quad \lim_{\vartheta \to \infty} \sup_{\vartheta \in J} d^\Delta\Big((\widehat{\Lambda}(\Pi))(\vartheta)\Big).$$

Now for $z, \bar{z} \in \Pi$, we have three cases:

- *Case* 1: $\vartheta \in I_0$. We have

$$\begin{aligned} &\|(\widehat{\Lambda}z)(\vartheta) - (\widehat{\Lambda}\bar{z})(\vartheta)\| \\ &\leq M_R \int_0^{\vartheta} \|f(\delta, z_\delta + x_\delta, H(z+x)(\delta)) - f(\delta, \bar{z}_\delta + x_\delta, H(\bar{z}+x)(\delta))\| d\delta \\ &\leq 2M_R \int_0^{\vartheta} e^{-\mu(\vartheta-\delta)}\left(\psi^1_f(\aleph^*)p^1_f(\delta) + \psi^2_f(H^*)p^2_f(\delta)\right) \xrightarrow[\vartheta \to +\infty]{} 0. \end{aligned}$$

- *Case* 2: $\vartheta \in I_k$. We have

$$
\begin{aligned}
&\|(\widehat{\Lambda} z)(\vartheta) - (\widehat{\Lambda}\overline{z})(\vartheta)\| \\
&\quad \leq M_R\|\Xi_k(\delta_k,(z)(\delta_k^-)) - \Xi_k((\delta_k,(\overline{z})(\delta_k^-)))\| + M_R\int_{\delta_k}^{\vartheta}\|f(\delta,z_\delta+x_\delta,H(z)(\delta)) \\
&\quad\quad - f(\delta,\overline{z_\delta}+x_\delta,H(\overline{z})(\delta))\|d\delta \\
&\quad \leq M_R L^*_{\Xi_k}\|z(\vartheta)-\overline{z}(\vartheta)\| \\
&\quad\quad + 2M_R\int_{\delta_k}^{\vartheta} e^{-\mu(\vartheta-\delta_k)}\Big(\psi_f^1(\widetilde{\aleph^*})p_f^1(\delta) + \psi_f^2(\widetilde{H^*})p_f^2(\delta)\Big) \xrightarrow[\vartheta\to+\infty]{} 0.
\end{aligned}
$$

When $\vartheta \to +\infty$ and by $(C1)$, we obtain

$$\varphi(\vartheta) \leq (M_R L^*_{\Xi_k})\chi_{PBC}(\Pi).$$

Then,

$$(1 - M_R L^*_{\Xi_k})\|\varphi\|_{\mathcal{X}} \leq 0.$$

- *Case* 2: $\vartheta \in J_k$.
We have

$$
\begin{aligned}
\|(\widehat{\Lambda} z)(\vartheta) - (\widehat{\Lambda}\overline{z})(\vartheta)\| &\leq \|\Xi_k(\vartheta,(z)(\vartheta_k^-)) - \Xi_k((\vartheta,(\overline{z})(\vartheta_k^-)))\| \\
&\leq L^*_{\Xi_k}\|z(\vartheta)-\overline{z}(\vartheta)\|.
\end{aligned}
$$

Hence,

$$(1 - L^*_{\Xi_k})\|\varphi\|_{\mathcal{X}} \leq 0.$$

Consequently, $\|\varphi\|_{BPC} = 0$, implies that $\varphi(\vartheta) = \chi(\Pi(\vartheta)) = 0$, then $\Pi(\vartheta)$ is relatively compact in E. In view of the Corduneanu theorem, Π is relatively compact in Δ_θ. Applying now Mönch's fixed-point theorem [33], we conclude that $\widehat{\Lambda}$ has at least one fixed point z^*. Then, $\phi^* = z^* + x$ is a fixed point of the operator Λ, which is a mild solution of problem (1). □

Remark 1. *The transformation we used allows us to find a mild solution without imposing conditions on the function $x(\cdot)$ and with simple calculations, but it imposes a strong condition on the space $\mathcal{X}$ ($z(0) = 0$ is necessary for decomposition), then, to avoid this constraint, we can directly show the existence of the fixed point for the operator Λ without imposing this condition. Indeed, if we assume that*

$$\max\left\{\overline{\Delta_1^\theta}, \overline{\Delta_2^\theta}\right\} \leq \theta,$$

with

$$
\begin{aligned}
\overline{\Delta_1^\theta} &= \frac{M_R\Big(\max\{\|\Phi\|_{\mathcal{B}}, \Xi_k^0\} + \|p_f^1\|_{L^1}\psi_f^1(\aleph^*) + \|p_f^2\|_{L^1}\psi_f^2(H^*)\Big)}{1 - L^*_{\Xi_k}M_R}, \\
\overline{\Delta_2^\theta} &= L^*_{\Xi_k}\theta + \Xi_k^0.
\end{aligned}
$$

then $\Lambda(\Delta_\theta) \subset \Delta_\theta$ and $\Lambda(\Delta_\theta)$ is bounded.

In addition to the estimates that we have obtained in the proof of Theorem 2, we can see that the map $\vartheta \to R(\vartheta)\Phi(0)$ and $\vartheta \to \Phi(\vartheta)$, are continuous on I_0, $\mathbb{R}^-$, respectively. We have also the set $\{S_i\}_{i=0,1} = \big\{R((1-i)\vartheta)(\Phi(it))\big\}_{i=0,1}$ which is equicontinuous and equiconvergent.

From similar analysis as in the proof of Theorem 2 and from Mönch's fixed point theorem [33], we can conclude that Λ has at least one fixed point which is a mild solution of problem (1).

3.2. Attractivity of Solutions

Firstly, we introduce the following concept of attractivity of solutions.

Definition 3 ([36])**.** *We say that solutions of Equation (1) are locally attractive if there exists a closed ball $B(\phi^*,\rho)$ in the space $\mathcal{X}$ for some $\phi^* \in \mathcal{X}$ such that for arbitrary solutions ϕ and $\widetilde{\phi}$ of Equation (1) belonging to $B(\phi^*,\rho)$ we have that*

$$\lim_{\vartheta\to+\infty}\left(\phi(\vartheta)-\widetilde{\phi}(\vartheta)\right)=0.$$

When the last limit is uniform with respect to $B(\phi^*,\rho)$, solutions of problem (1) are said to be uniformly locally attractive (or equivalently that solutions of Equation (1) are locally asymptotically stable).

Let ϕ^* be a solution of problem (1), such that $\phi^* = z^* + x$, such that z^* is a fixed point of operator $\widehat{\Lambda}$, then for $\phi = z + x$ and $\widetilde{\phi} = \widetilde{z} + x$, we have

$$\lim_{\vartheta\to+\infty}\left(\phi(\vartheta)-\widetilde{\phi}(\vartheta)\right)=0 \Leftrightarrow \lim_{\vartheta\to+\infty}(z(\vartheta)-\widetilde{z}(\vartheta))=0.$$

Theorem 3. *Suppose that the hypotheses $(C1)-(C4)$ hold, and for $\gamma>0$,*

$$\max\left\{T_1,T_2,T_3\right\}\le\gamma,\ \text{and}\ \gamma+\|x\|_{\mathcal{X}}\le\rho,$$

such that

$$\begin{aligned}
T_1 &= 2M_R\Big(\|p_f^1\|_{L^1}\psi_f^1(\aleph_\gamma^*)+\|p_f^2\|_{L^1}\psi_f^2(H_\gamma^*)\Big),\\
T_2 &= 2M_R\left(L^*_{\Xi_k}\gamma+\Xi_k^0+\left(\|p_f^1\|_{L^1}\psi_f^1(\widetilde{\aleph_\gamma^*})+\|p_f^2\|_{L^1}\psi_f^2(\widetilde{H_\gamma^*})\right)\right),\\
T_3 &= 2L^*_{\Xi_k}\gamma,
\end{aligned}$$

with

$$\begin{aligned}
\aleph_\gamma^* &= \aleph(\gamma+(M_R+1)\|\Phi\|_{\mathcal{B}}),\\
\widetilde{\aleph_\gamma^*} &= \aleph(\gamma+\|\Phi\|_{\mathcal{B}}),\\
H_\gamma^* &= a\big(h_{c_1}^*(\gamma+M_R\|\Phi\|_{\mathcal{B}})+h^*\big),\\
\widetilde{H_\gamma^*} &= a(h_{c_1}^*\gamma+h^*).
\end{aligned}$$

Then, the problem (1) is attractive.

Proof. For $z\in B(z^*;\gamma)$ by $(C1)$ and $(C3)$, we get

- *Case* 1: $\vartheta\in I_0$. We have

$$\begin{aligned}
\|(\widehat{\Lambda}z)(\vartheta)-z^*(\vartheta)\| &= \|\widehat{\Lambda}(z)(\vartheta)-\widehat{\Lambda}(z^*)(\vartheta)\|\\
&\le 2M_R\Big(\|p_f^1\|_{L^1}\psi_f^1(\aleph_\gamma^*)+\|p_f^2\|_{L^1}\psi_f^2(H_\gamma^*)\Big)\\
&\le \gamma.
\end{aligned}$$

Therefore, we obtain $\widehat{\Lambda}(B_\gamma)\subset B_\gamma$.
Now, for each $z,\widetilde{z}\in B(z^*;\gamma)$ solutions of problem (1) and $\vartheta\in I_0$, we have

$$\begin{aligned}
\|z(\vartheta)-\widetilde{z}(\vartheta)\| &= \|(\widehat{\Lambda}z)(\vartheta)-(\widehat{\Lambda}\widetilde{z})(\vartheta)\|\\
&\le 2M_R\sup_{\vartheta\in I_0}\int_0^\vartheta e^{-\mu(\vartheta-\delta)}\Big(\psi_f^1(\aleph_\gamma^*)p_f^1(\delta)+\psi_f^2(H_\gamma^*)p_f^2(\delta)\Big)d\delta.
\end{aligned}$$

Then, from $(C1)$, we conclude that

$$\|z(\vartheta)-\widetilde{z}(\vartheta)\|\to0,\ \ \text{as}\ \vartheta\to+\infty.$$

- *Case* 2: $\vartheta \in I_k$. We have

$$\begin{aligned}
\|(\widehat{\Lambda}z)(\vartheta) - z^*(\vartheta)\| &= \|(\widehat{\Lambda}z)(\vartheta) - (\widehat{\Lambda}z^*)(\vartheta)\| \\
&\leq M_R\|\Xi_k(\delta_k, (z)(\delta_k^-)) - \Xi_k(\delta_k, (z^*)(\delta_k^-))\| \\
&\quad + M_R \int_{\delta_k}^{\vartheta} \|f(\delta, (z_\delta + x_\delta), H(z)(\delta)) \\
&\quad - f(\delta, (z^*_\delta + x_\delta)(\delta), H(z^*)(\delta))\| d\delta \\
&\leq 2M_R \Big(L^*_{\Xi_k}\gamma + \Xi^0_k + \Big(\|p^1_f\|_{L^1}\psi^1_f(\widetilde{\aleph^*_\gamma}) + \|p^2_f\|_{L^1}\psi^2_f(\widetilde{H^*_\gamma}) \Big) \Big) \\
&\leq \gamma.
\end{aligned}$$

Therefore, we obtain $\widehat{\Lambda}(B_\gamma) \subset B_\gamma$.
So, for each $z, \widetilde{z} \in B(z^*; \gamma)$ solutions of problem (1) and $\vartheta \in I_k$, we have

$$\begin{aligned}
\|z(\vartheta) - \widetilde{z}(\vartheta)\| &= \|(\widehat{\Lambda}z)(\vartheta) - (\widehat{\Lambda}\widetilde{z})(\vartheta)\| \\
&\leq M_R e^{-\mu(\vartheta-\delta_k)}\|\Xi_k(\delta_k, z(\delta_k^-))) - \Xi_k(\delta_k, \widetilde{z}(\delta_k^-))\| \\
&\quad + 2M_R \left(\sup_{\vartheta \in I_k} \int_{\delta_k}^{\vartheta} e^{-\mu(\vartheta-\delta)} \Big(\psi^1_f(\widetilde{\aleph^*_\gamma}) p^1_f(\delta) + \psi^2_f(\widetilde{H^*_\gamma}) p^2_f(\delta) \Big) d\delta. \right) \\
&\leq M_R e^{-\mu(\vartheta-\delta_k)} L^*_{\Xi_k}\|z(\delta_k) - \widetilde{z}(\delta_k)\| \\
&\quad + 2M_R \sup_{\vartheta \in J} \int_0^{\vartheta} e^{-\mu(\vartheta-\delta)} \Big(\psi^1_f(\widetilde{\aleph^*_\gamma}) p^1_f(\delta) + \psi^2_f(\widetilde{H^*_\gamma}) p^2_f(\delta) \Big) d\delta.
\end{aligned}$$

Then, from $(C1)$, we conclude that

$$\|z(\vartheta) - \widetilde{z}(\vartheta)\| \to 0, \ \ as\ \vartheta \to +\infty.$$

- *Case* 3: $\vartheta \in J_k$. We have

$$\begin{aligned}
\|(\widehat{\Lambda}z)(\vartheta) - z^*(\vartheta)\| &= \|(\widehat{\Lambda}z)(\vartheta) - (\widehat{\Lambda}z^*)(\vartheta)\| \\
&\leq \|\Xi_k(\vartheta, (z)(\vartheta_k^-)) - \Xi_k((\vartheta, (z^*)(\vartheta_k^-)))\| \\
&\leq 2L^*_{\Xi_k}\gamma \\
&\leq \gamma.
\end{aligned}$$

Therefore, we obtain $\widehat{\Lambda}(B_\gamma) \subset B_\gamma$.
Thus, for each $z, \widetilde{z} \in B(z^*; \gamma)$ solutions of problem (1) and $\vartheta \in J_k$, we have

$$\begin{aligned}
\|z(\vartheta) - \widetilde{z}(\vartheta)\| &= \|(\widehat{\Lambda}z)(\vartheta) - (\widehat{\Lambda}\widetilde{z})(\vartheta)\| \\
&\leq \|\Xi_k(\vartheta, (z)(\vartheta_k^-)) - \Xi_k(\vartheta, (\widetilde{z})(\vartheta_k^-))\| \\
&\leq L^*_{\Xi_k}\|z(\vartheta) - \widetilde{z}(\vartheta)\|,
\end{aligned}$$

then

$$(1 - L^*_{\Xi_k})\|z(\vartheta) - \widetilde{z}(\vartheta)\| \quad \leq \quad 0,$$

hence,

$$\|z(\vartheta) - \widetilde{z}(\vartheta)\| = 0.$$

Consequently, the solutions of the problem (1) are uniformly locally attractive.

□

4. Functional Integro-Differential Equations with State-Dependent Delay

4.1. Existence Results

To prove our results on the existence, we introduce the following conditions.

(C5) (i) There exists a function $l_f \in L^1(J,\mathbb{R}^+)$, such that for any bounded set $B \subset E$, and $B_\vartheta \in \mathcal{B}$ and each $\vartheta \in \mathbb{R}$, we have

$$\mu(f(\vartheta, B_\vartheta, H(B(\vartheta)))) \le l_f(\vartheta)\mu(B(\vartheta)).$$

(ii) There exists $\tau > 2$, such that $M_R L^*_{\Xi_k} \le \frac{1}{\tau}$.

(C_H) Set $\mathcal{R}(\rho^-) = \{\rho(\delta,\varphi) : (\delta,\varphi) \in J \times \mathcal{B}, \rho(\delta,\varphi) \le 0\}$. We assume that $\rho : J \times \mathcal{B} \to \mathbb{R}$ is continuous. Moreover, we assume the following assumption and hypothesis:
• (H_Φ) The function $\vartheta \to \Phi_\vartheta$ is continuous from $\mathcal{R}(\rho^-)$into $\mathcal{B}$ and there exists a continuous and bounded function $L^\Phi : \mathcal{R}(\rho^-) \to (0,\infty)$ such that

$$\|\Phi_\vartheta\|_{\mathcal{B}} \le L^\Phi(\vartheta)\|\Phi\|_{\mathcal{B}}, \quad \text{for every } \vartheta \in \mathcal{R}(\rho^-).$$

Remark 2. *The condition (H_Φ) is frequently verified by functions continuous and bounded. For more details, see, for instance, [39].*

Lemma 3 ([40])**.** *If $y : (-\infty,+\infty) \to E$ is a function such that $y_0 = \Phi$, then*

$$\|y_\delta\|_{\mathcal{B}} \le \left(M + \mathcal{L}^\Phi\right)\|\Phi\|_{\mathcal{B}} + l\sup\{|y(\theta)|\ ;\ \theta \in [0,\max\{0,\delta\}]\},\ \ \delta \in \mathcal{R}(\rho^-) \cup J,$$

where $\mathcal{L}^\Phi = \sup_{\vartheta\in\mathcal{R}(\rho^-)} \mathcal{L}^\Phi(\vartheta)$.

We define on $\mathcal{X}$ measures of non-compactness by

$$\mu_{BPC}(\Pi) = \omega_0(\Pi) + \lim_{|\vartheta|\to\infty} \sup\left\{e^{-\tau\widetilde{\Sigma}(\vartheta)}\mu(\Pi(\vartheta))\right\},$$

with $\widetilde{\Sigma}(\vartheta) = \int_0^\vartheta \Sigma(\delta)d\delta$, $\quad \Sigma(\vartheta) = 4M_R l_f(\vartheta)$ and $\quad \Pi(\vartheta) = \{\delta(\vartheta) \in E\ ;\ \delta \in \Pi\}$.
Notice that if the set Π is equicontinuous, then $\omega_0(\Pi) = 0$.

Theorem 4. *Assume that the conditions (C1)–(C5) and (C_H) are satisfied. Then, the system (2) has at least one mild solution.*

Proof. Define the operator, $\widehat{\Upsilon}_1 : \mathcal{X} \to \mathcal{X}$, by :

$$\widehat{\Upsilon}_1\phi(\vartheta) = \begin{cases} R(\vartheta)\Phi(0) + \int_0^\vartheta R(\vartheta-\delta)f(\delta,\phi_{\rho(\delta,\phi_\delta)},(H\phi)(\delta))d\delta; \text{ if } \vartheta \in I_0, \\[2mm] R(\vartheta-\delta_k)\left[\Xi_k(\delta_k,\phi(\delta_k^-))\right] \\ +\int_{\delta_k}^\vartheta R(\vartheta-\delta)f(\delta,\phi_{\rho(\delta,\phi_\delta)},(H\phi)(\delta))d\delta; \text{ if } \vartheta \in I_k,\ k \in \mathbb{N}, \\[2mm] \Xi_k(\vartheta,\phi(\vartheta_k^-)); \text{ if } \vartheta \in J_k,\ k \in \mathbb{N}, \\[2mm] \Phi(\vartheta), \text{ if } \vartheta \in \mathbb{R}_-. \end{cases}$$

If ϕ is a fixed point of $\widehat{\Upsilon}_1$, then similar transformation to that in the proof of Theorem 2, give the following decomposition $\phi(\vartheta) = w(\vartheta) + x(\vartheta)$, which implies $\phi_\vartheta = w_\vartheta + x_\vartheta$. Thus, consider the operator $\widehat{\Upsilon}_2 : \Omega \to \Omega$ defined by,

$$\widehat{\Upsilon}_2 w(\vartheta) = \begin{cases} \int_0^\vartheta R(\vartheta-\delta)f(\delta,w_{\rho(\delta,w_\delta+x_\delta)} + x_{\rho(\delta,w_\delta+x_\delta)},H(w+x)(\delta))d\delta, \text{ if } \vartheta \in I_0, \\[2mm] R(\vartheta-\delta_k)\left[\Xi_k(\delta,w(\delta_k^-))\right] \\ +\int_{\delta_k}^\vartheta R(\vartheta-\delta)f(\delta,w_{\rho(\delta,w_\delta+x_\delta)} + x_{\rho(\delta,w_\delta+x_\delta)},H(w)(\delta))d\delta, \text{ if } \vartheta \in I_k,\ k \in \mathbb{N}, \\[2mm] \Xi_k(\vartheta,(w)(\vartheta_k^-)), \text{ if } \vartheta \in J_k,\ k \in \mathbb{N}. \end{cases}$$

The operator $\widehat{\Upsilon}_1$ having a fixed point is equivalent to saying that $\widehat{\Upsilon}_2$ has one, so it turns to prove that $\widehat{\Upsilon}_2$ has a fixed point. We shall check that operator $\widehat{\Upsilon}_2$ satisfies all conditions of Darbo's theorem [41].

Let $\Delta_{\theta'} = \{w \in \Omega : \|w\|_{\mathcal{X}} \leq \theta'\}$, with

$$0 < \max\left\{\Delta_1^{\theta'}, \Delta_2^{\theta'}, \Delta_3^{\theta'}\right\} \leq \theta',$$

such that

$$\begin{aligned}
\Delta_1^{\theta'} &= M_R\Big(\|p_f^1\|_{L^1}\psi_f^1(H_{\theta'}^*) + \|p_f^2\|_{L^1}\psi_f^2(\overline{H}^*)\Big), \\
\Delta_2^{\theta'} &= \frac{M_R(\Xi_k^0 + \|p_f^1\|_{L^1}\psi_f^1(H_{\theta'}^+) + \|p_f^2\|_{L^1}\psi_f^2(\overline{H}^+))}{1 - M_R L_{\Xi_k}^*}, \\
\Delta_3^{\theta'} &= L_{\Xi_k}^*\theta' + \Xi_k^0,
\end{aligned}$$

where $H_{\theta'}^*$, $\overline{H}^*$, $\overline{H}^+$, $H_{\theta'}^+$ are constants, they will be specific later.

The set $\Delta_{\theta'}$ is bounded, closed, and convex. We have divided the proof into four steps:

Step 1: $\Theta(\Delta_{\theta'}) \subset \Delta_{\theta'}$.

This step is similar to Step 1 in the proof of Theorem 2, we need only to change constants $\aleph^*, H^*, \widetilde{H^*}, \widetilde{\aleph^*}$ with $H_{\theta'}^*$, $\overline{H}^*$, $\overline{H}^+$, $H_{\theta'}^+$, which we are going to define now:

- *Case* 1: $\vartheta \in I_0$.
  For $w \in \Delta_{\theta'}$, $\vartheta \in I_0$ and by (C1)–(C3), we have

$$\begin{aligned}
\left\|w_{\rho(\delta,w_\delta+x_\delta)} + x_{\rho(\delta,w_\delta+x_\delta)}\right\|_{\mathcal{B}} &\leq \left\|w_{\rho(\delta,w_\delta+x_\delta)}\right\|_{\mathcal{B}} + \left\|x_{\rho(\delta,w_\delta+x_\delta)}\right\|_{\mathcal{B}} \\
&\leq L(\vartheta)\sup_{[0,\delta]}|w(\vartheta)| + \Big(M(\vartheta) + \mathcal{L}^{\Phi}\Big)\|\Phi\|_{\mathcal{B}} \\
&\quad + L(\vartheta)\sup_{[0,\delta]}\|x(\theta)\| \\
&\leq L_*\theta' + \Big(M_* + \mathcal{L}^{\Phi}\Big)\|\Phi\|_{\mathcal{B}} + L_* M_R\|\Phi\|_{\mathcal{B}} \\
&\leq \aleph\theta' + \Big(\mathcal{L}^{\Phi} + \aleph(M_R + 1)\Big)\|\Phi\|_{\mathcal{B}} = H_{\theta'}^*.
\end{aligned}$$

  Furthermore,

$$\|H(w + x)(\delta)\| \leq ah_{c_1}^*(\theta' + M_R\|\Phi\|_{\mathcal{B}}) + ah^* = \overline{H}^*.$$

  Then,

$$\|\widehat{\Upsilon}_2 w(\vartheta)\| \leq M_R\Big(\psi_f^1(H_{\theta'}^*)\|p_f^1\|_{L^1_{loc}} + \psi_f^2(\overline{H}^*)\|p_f^2\|_{L^1_{loc}}\Big) = \Delta_1^{\theta'}.$$

- *Case* 2: $\vartheta \in I_k$.
  For each $w \in \Delta_{\theta'}$, from (C1), (C2) *and* (C3), we have

$$H_{\theta'}^+ = \aleph\theta' + (\aleph + \mathcal{L}^{\Phi})\|\Phi\|_{\mathcal{B}} \text{ and } \overline{H}^+ = ah_{c_1}^*\theta' + ah^*.$$

  Then

$$\|\widehat{\Upsilon}_2 w(\vartheta)\| \leq M_R\Big(L_{\Xi_k}^*\theta' + \Xi_k^0 + \psi_f^1(H_{\theta'}^+)\|p_f^1\|_{L^1_{loc}} + \psi_f^2(\overline{H}^+)\|p_f^2\|_{L^1_{loc}}\Big) = \Delta_2^{\theta'}.$$

- *Case* 3: $\vartheta \in J_k$.
  For each $w \in \Delta_{\theta'}$ and by (C3), we obtain

$$\|\widehat{\Upsilon}_2 w(\vartheta)\| \leq L_{\Xi_k}^*\theta' + \Xi_k^0 = \Delta_3^{\theta'}.$$

Thus,

$$\|\widehat{\Upsilon}_2 w\|_{\mathcal{X}} \quad \leq \quad \theta',$$

Step 2: $\widehat{\Upsilon}_2$ is continuous.

Let $\{w_m\}_{m\in\mathbb{N}}$ be a sequence such that $w_m \to w^*$ in $\Delta_{\theta'}$. First, we study the convergence of the sequences $\left(w^m_{\rho(\delta,w^m_\delta)}\right)_{m\in\mathbb{N}}$, $\delta \in J$. If $\delta \in J$ is such that $\rho(\delta, w_\delta) > 0$, then we have

$$\begin{aligned}
\left\|w^m_{\rho(\delta,w^m_\delta)} - w^*_{\rho(\delta,w^*_\delta)}\right\|_{\mathcal{B}} &\leq \left\|w^m_{\rho(\delta,w^n_\delta)} - w^*_{\rho(\delta,w^m_\delta)}\right\|_{\mathcal{B}} + \left\|w^*_{\rho(\delta,w^m_\delta)} - w^*_{\rho(\delta,w^*_\delta)}\right\|_{\mathcal{B}} \\
&\leq L\|w_m - w^*\| + \left\|w^*_{\rho(\delta,w^n_\delta)} - w^*_{\rho(\delta,w^*_\delta)}\right\|_{\mathcal{B}},
\end{aligned}$$

which proves that $w^m_{\rho(\delta,w^m_\delta)} \to w^*_{\rho(\delta,w_\delta)}$ in $\mathcal{B}$, as $m \to \infty$, for every $\delta \in J$ such that $\rho(\delta, w_\delta) > 0$. Similarly, if $\rho(\delta, w_\delta) < 0$, we get

$$\left\|w^n_{\rho(\delta,w^m_\delta)} - w^*_{\rho(\delta,w_\delta)}\right\|_{\mathcal{B}} = \left\|\Phi^m_{\rho(\delta,w^m_\delta)} - \Phi_{\rho(\delta,w^*_\delta)}\right\|_{\mathcal{B}} = 0,$$

which also shows that $w^m_{\rho(\delta,w^m_\delta)} \to w^*_{\rho(\delta,w_\delta)}$ in $\mathcal{B}$, as $m \to \infty$, for every $\delta \in J$ such that $\rho(\delta, w_\delta) < 0$.

- *Case* 1: $\vartheta \in I_0$. We have

$$\begin{aligned}
&\|(\widehat{\Upsilon}_2 w^m)(\vartheta) - (\widehat{\Upsilon}_2 w^*)(\vartheta)\| \\
&\quad \leq M_R \int_{\delta_k}^{\vartheta} \|f(\delta, w^m_{\rho(\delta,w^m_\delta)} + x_{\rho(\delta,w^m_\delta+x_\delta)}, H(w^m + x)(\delta)) \\
&\qquad - f(\delta, (w^*_{\rho(\delta,w^*_\delta)} + x_{\rho(\delta,w^*_\delta+x_\delta)}), H(w^* + x)(\delta))\| d\delta.
\end{aligned}$$

  Since h and f are continuous, we obtain

$$h(\vartheta, \delta, (w^m + x)(\delta)) \to h(\vartheta, \delta, (w^* + x)(\delta)), \quad as \;\; m \to +\infty.$$

  Additionally,

$$\|h(\vartheta, \delta, (w^m + x)(\delta)) - h(\vartheta, \delta, (w^* + x)(\delta))\| \leq h^*_{c_1}\|w^m - w^*\|.$$

  We have by the Lebesgue dominated convergence theorem

$$\int_0^{\vartheta} h(\vartheta, \delta, (w^m + x)(\delta)) d\delta \xrightarrow[m\to+\infty]{} \int_0^{\vartheta} h(\vartheta, \delta, (w^* + x)(\delta)) d\delta,$$

  Then, by $(H1)$, we obtain

$$\begin{aligned}
&f(\delta, w^m_{\rho(\delta,w^m_\delta)} + x_{\rho(\delta,w^m_\delta+x_\delta)}, H(w^m + x)(\delta)) \\
&\quad \xrightarrow[m\to+\infty]{} f(\delta, (w^*_{\rho(\delta,w^*_\delta)} + x_{\rho(\delta,w^*_\delta+x_\delta)}), H(w^* + x)(\delta)).
\end{aligned}$$

  By the Lebesgue dominated convergence theorem, we obtain

$$\|(\widehat{\Upsilon}_2 w^m) - (\widehat{\Upsilon}_2 w^*)\| \to 0, \quad as \;\; m \to +\infty.$$

- *Case* 2: $\vartheta \in I_k$. We have

$$\begin{aligned}\|\widehat{\Upsilon}_2(w^m)(\vartheta) - \widehat{\Upsilon}_2(w^*)(\vartheta)\| \leq M_R \|\Xi_k(\delta_k, (w^m)(\delta_k^-)) - \Xi_k((\delta_k, (w^*)(\delta_k^-)))\| \\ + M_R \int_0^\vartheta \|f(\delta, (w_\delta^n + x_\delta), H(w^m)(\delta)) \\ - f(\delta, (w_\delta^* + x_\delta), H(w^*)(\delta))\| d\delta.\end{aligned}$$

Similar to Case 1, by the continuity of h, f and Ξ_k, we obtain

$$\|(\widehat{\Upsilon}_2 w^m) - (\widehat{\Upsilon}_2 w^*)\| \to 0, \quad as \ \ m \to +\infty.$$

- *Case* 3: $\vartheta \in J_k$. we have

$$\|(\widehat{\Upsilon}_2(w^m))(\vartheta) - \widehat{\Upsilon}_2((w^*))(\vartheta)\| \quad \leq \quad M_R \|\Xi_k(\vartheta_k, (w^m)(\vartheta_k^-)) - \Xi_k(\vartheta_k, (w^*)(\vartheta_k^-))\|.$$

Since Ξ_k are continuous, we obtain

$$\|\widehat{\Upsilon}_2(w^m) - \widehat{\Upsilon}_2(w^*)\| \to 0, \quad as \ \ m \to +\infty.$$

Thus, $\widehat{\Upsilon}_2$ is continuous.

Step 3: We have $\widehat{\Upsilon}_2(\Delta_{\theta'}) \subset \Delta_{\theta'}$, which implies that $\widehat{\Upsilon}_2(\Delta_{\theta'})$ is bounded.

Step 4:

Let Π be a bounded equicontinuous subset of $\Delta_{\theta'}$, for $\Pi \subset (\Delta_{\theta'})$ and $w \in \Pi$, similar to Step 3 in the proof of Theorem 2, we obtain

- *Case* 1: $\kappa_1, \kappa_2 \in I_0$.

$$\begin{aligned}\|\widehat{\Upsilon}_2 w(\kappa_1) - \widehat{\Upsilon}_2 w(\kappa_2)\| \quad \leq \quad & \psi_f^1(H_{\theta'}^*) \int_0^{\kappa_1} \|R(\kappa_1 - \delta) - R(\kappa_2 - \delta)\| p_f^1(\delta) d\delta \\ & + \psi_f^2(\overline{H}^*) \int_0^{\kappa_1} \|R(\kappa_1 - \delta) - R(\kappa_2 - \delta)\| p_f^2(\delta) d\delta \\ & + M_R \int_{\kappa_1}^{\kappa_2} \Big(\psi_f^1(H_{\theta'}^*) p_f^1(\delta) + \psi_f^2(\overline{H}^*) p_f^2(\delta)\Big) d\delta, \\ & \xrightarrow[\kappa_1 \to \kappa_2]{} 0.\end{aligned}$$

- *Case* 2: $\kappa_1, \kappa_2 \in I_k$.

$$\begin{aligned}\|\widehat{\Upsilon}_2 w(\kappa_1) - \widehat{\Upsilon}_2 w(\kappa_2)\| \quad \leq \quad & \|R(\kappa_1 - \delta_k) - R(\kappa_2 - \delta_k)\| (L_{\Xi_k}^* \theta' + \Xi_k^0) \\ & + \psi_f^1(H_{\theta'}^+) \int_0^{\kappa_1} \|R(\kappa_1 - \delta) - R(\kappa_2 - \delta)\| p_f^1(\delta) d\delta \\ & + \psi_f^2(\overline{H}^+) \int_0^{\kappa_1} \|R(\kappa_1 - \delta) - R(\kappa_2 - \delta)\| p_f^2(\delta) d\delta \\ & + M_R \int_{\kappa_1}^{\kappa_2} \Big(\psi_f^1(H_{\theta'}^+) p_f^1(\delta) + \psi_f^2(\overline{H}^+) p_f^2(\delta)\Big) d\delta, \\ & \xrightarrow[\kappa_1 \to \kappa_2]{} 0.\end{aligned}$$

- *Case* 3: $\vartheta \in J_k$.

$$\|\widehat{\Upsilon}_2 v(\kappa_1) - \widehat{\Upsilon}_2 v(\kappa_2)\| \quad = \quad \|\Xi_k(\kappa_1, v(\kappa_{1k}{}^-)) - \Xi_k(\kappa_2, v(\kappa_{2k}{}^-))\| \to 0 \; as \; \kappa_1 \to \kappa_2.$$

Hence, the set $\widehat{\Upsilon}_2(\Pi)$ is equicontinuous, then $\omega_0(\widehat{\Upsilon}_2(\Pi)) = 0$. $\square$

Now for any $\varrho > 0$, there exists a sequence $\{w^k\}_{k=0}^{\infty} \subset \Pi$ such that

- *Case* 1: $\vartheta \in I_0$. We have

$$\begin{aligned}
&\mu(\widehat{\Upsilon}_2(\Pi)(\vartheta)) \\
&\quad \leq \mu\left(\left\{\int_0^{\vartheta} R(\vartheta-\delta)f(\delta, w_{\rho(\delta,w_\delta)} + x_{\rho(\delta,w_\delta+x_\delta)}, H(w+x)(\delta))d\delta \;;\; w \in \Pi\right\}\right) \\
&\quad \leq 2\mu\left(\left\{\int_0^{\vartheta} R(\vartheta-\delta)f(\delta, w^k_{\rho(\delta,w^k_\delta)} + x_{\rho(\delta,w^k_\delta+x_\delta)}, H(w^k+x)(\delta))d\delta \;;\; v \in \Pi\right\}\right) \\
&\qquad + \varrho \\
&\quad \leq 4\int_0^{\vartheta} M_R l_f(\delta)\mu(\{\Pi(\delta)\})d\delta + \varrho \\
&\quad \leq \int_0^{\vartheta} e^{\tau\widetilde{\Sigma}(\delta)}e^{-\tau\widetilde{\Sigma}(\delta)}\Xi(\delta)\mu(\Pi(\delta))d\delta + \varrho \\
&\quad \leq \int_0^{\vartheta} \Sigma(\delta)e^{\tau\widetilde{\Sigma}(\delta)} \sup_{\delta\in[0,\vartheta]} e^{-\tau\widetilde{\Sigma}(\delta)}\mu(\Pi(\delta))d\delta + \varrho \\
&\quad \leq \mu_{BPC}(\Pi)\int_0^{\vartheta}\left(\frac{e^{\tau\widetilde{\Sigma}(\delta)}}{\tau}\right)' d\delta + \varrho \\
&\quad \leq \frac{e^{\tau\widetilde{\Sigma}(\vartheta)}}{\tau}\mu_{BPC}(\Pi) + \varrho.
\end{aligned}$$

Since ϱ is arbitrary, we obtain

$$\mu(\widehat{\Upsilon}_2(\Pi)(\vartheta)) \quad \leq \quad \frac{e^{\tau\widetilde{\Xi}(\vartheta)}}{\tau}\mu_{BPC}(\Pi),$$

thus

$$\mu_{BPC}(\widehat{\Upsilon}_2(\Pi)) \quad \leq \quad \frac{1}{\tau}\mu_{BPC}(\Pi).$$

- *Case* 2: $\vartheta \in I_k$.
We have

$$\begin{aligned}
&\mu(\widehat{\Upsilon}_2(\Pi)(\vartheta)) \\
&\quad \leq M_R\ \mu(\{\Xi_k(\delta, w(\delta_k^-)); w \in \Pi\}) \\
&\qquad + \mu\left(\left\{\int_0^{\vartheta} R(\vartheta-\delta)f(\delta, w_{\rho(\delta,w_\delta)} + x_{\rho(\delta,w_\delta+x_\delta)}, H(w)(\delta))d\delta \;;\; w \in \Pi\right\}\right) \\
&\quad \leq \frac{1}{\tau}\mu(\Pi(\vartheta)) + 4\int_0^{\vartheta} M_R l_f(\delta)\mu(\{\Pi(\delta)\})d\delta + \varrho \\
&\quad \leq \frac{2e^{\tau\widetilde{\Sigma}(\vartheta)}}{\tau}\mu_{BPC}(\Pi) + \varrho.
\end{aligned}$$

Therefore,

$$\mu_{BPC}\left(\widehat{\Upsilon}_2(\Pi)\right) \quad \leq \quad \frac{2}{\tau}\mu_{BPC}(\Pi).$$

- *Case* 3: $\vartheta \in J_k$. By $(C3)$, we obtain

$$\begin{aligned}
\mu\Big(\widehat{\Upsilon}_2(\Pi)(\vartheta)\Big) &= \mu\big(\{\Xi_k(\delta, w(\delta_k^-)); w \in \Pi\}\big) \\
&\leq \frac{1}{M_R \tau}\mu(\Pi(\vartheta)) \\
&\leq \frac{e^{\tau\widetilde{\Sigma}(\vartheta)}}{\tau\, M_R}\mu_{BPC}(\Pi),
\end{aligned}$$

then

$$\mu_{BPC}\Big(\widehat{\Upsilon}_2(\Pi)\Big) \leq \frac{1}{\tau\, M_R}\mu_{BPC}(\Pi).$$

As a consequence of Darbo's theorem [41], we deduce that $\widehat{\Upsilon}_2$ has at least one fixed point w^*. Then $\phi^* = w^* + x$ is a fixed point of the operator $\widehat{\Upsilon}_1$, which is a mild solution of the problem (2).

4.2. Attractivity Results

Theorem 5. *Suppose that the hypotheses* $(C1)$–$(C5)$ *and* (C_H) *hold, and for* $\widetilde{\gamma} > 0$,

$$\max\{S_1, S_2, S_3\} \leq \widetilde{\gamma}, \text{ and } \widetilde{\gamma} + \|x\|_{\mathcal{X}} \leq \widetilde{\rho},$$

such that

$$\begin{aligned}
S_1 &= 2M_R\Bigg(\|p_f^1\|_{L^1}\psi_f^1\Big(\aleph\widetilde{\gamma} + \Big(\mathcal{L}^{\Phi} + \aleph(M_R+1)\Big)\|\Phi\|_{\mathcal{B}}\Big) \\
&\quad + \|p_f^2\|_{L^1}\psi_f^2\big(ah_{c_1}^*(\widetilde{\gamma} + M_R\|\Phi\|_{\mathcal{B}}) + ah^*\big)\Bigg), \\
S_2 &= 2M_R\Bigg(L_{\Xi_k}^*\widetilde{\gamma} + \Xi_k^0 + \|p_f^1\|_{L^1}\psi_f^1\Big(\aleph\widetilde{\gamma} + (\aleph + \mathcal{L}^{\Phi})\|\Phi\|_{\mathcal{B}}\Big) \\
&\quad + \|p_f^2\|_{L^1}\psi_f^2\big(ah_{c_1}^*\widetilde{\gamma} + ah^*\big)\Bigg), \\
S_3 &= 2L_{\Xi_k}^*\widetilde{\gamma}.
\end{aligned}$$

Then, the problem (2) is attractive.

Proof. The proof is similar to this of Theorem 3, then, by parallel steps, we can prove that the solutions of problem (2) are locally attractive. □

4.3. Controllability Results

Now, we present a controllability result for the system:

$$\begin{cases}
\phi'(\vartheta) = A\phi(\vartheta) + f\Big(\vartheta, \phi_{\rho(\vartheta,\phi_\vartheta)}, (H\phi)(\vartheta)\Big) \\
\qquad\qquad + \int_0^{\vartheta} B(\vartheta-\delta)\phi(\delta)d\delta + Cu(\vartheta); \text{ if } \vartheta \in I_k,\ k \in \mathbb{N}_0, \\
\phi(\vartheta) = \Xi_k\big(\vartheta, \phi(\vartheta_k^-)\big); \text{if } \vartheta \in J_k,\ k \in \mathbb{N}, \\
\phi(\vartheta) = \Phi(\vartheta); \text{ if } \vartheta \in \mathbb{R}_-,
\end{cases} \tag{5}$$

where the control function u is a given function in $L^2(J, U)$ Banach space of admissible control with U as a Banach space. C is a bounded linear operator from U into E. Before this, we introduce the the following type of solutions for the problem (5).

Definition 4. *The system (5) is said to be controllable on the interval J, if for every initial function $\Phi \in \mathcal{B}$ and $\hat{v} \in E$, there is for some $\widehat{n} > 0$, some control $u \in L^2([0, \widehat{n}], E)$ such that the mild solution $v(\cdot)$ of this problem satisfies the terminal condition $v(\widehat{n}) = \hat{v}$.*

We will need to introduce the following hypotheses:

$(C6)$ (i) For each $\widehat{n}$, the linear operator $W : L^2([0, \widehat{n}], U) \to X$, defined by

$$Wu = \int_0^{\widehat{n}} R(\widehat{n} - \delta)Cu(\delta)d\delta,$$

has a pseudo inverse operator W^{-1}, which takes values in

$$L^2([0, \widehat{n}], U) \diagdown Ker(W),$$

(ii) There exist positive constants m_1, m_2, such that

$$\|C\| \leq m_1 \ \ and \ \ \|W^{-1}\| \leq m_2.$$

(iii) There exists $q_w \in L^1(J, \mathbb{R}^+),\ m_C \geq 0$, such that for any bounded sets $\widetilde{M}_1 \subset E,\ \widetilde{M}_2 \subset U$,

$$\mu((W^{-1}\widetilde{M}_1)(\vartheta)) \leq q_w(\vartheta)\mu(\widetilde{M}_1), \quad \mu((C\widetilde{M}_2)(\vartheta)) \leq m_C\mu(\widetilde{M}_2(\vartheta)).$$

$(C7)$ There exists a positive constant ϖ, such that $\max\{\varphi_1^{\varpi}, \varphi_2^{\varpi}, L^*_{\Xi_k}\varpi + \Xi_k^0\} \leq \varpi$, with

$$\begin{aligned}
\varphi_1^{\varpi} &= M_R\Bigg(\psi_f^1(H^*_{\varpi})\|p_f^1\|_{L^1} + \psi_f^2(\widetilde{H}^*)\|p_f^2\|_{L^1} \\
&\quad + m_1m_2\Big(\frac{\varpi + \|\Phi\|_{\mathcal{B}}}{M_R} + \psi_f^1(H^*_{\varpi})\|p_f^1\|_{L^1} + \psi_f^2(\widetilde{H}^*)\|p_f^2\|_{L^1}\Big)\Bigg), \\
\varphi_2^{\varpi} &= M_R\Bigg(L^*_{\Xi_k}\varpi + \Xi_k^0 + \psi_f^1(R_{\varpi})\|p_f^1\|_{L^1} + \psi_f^2(R^*_{\varpi})\|p_f^2\|_{L^1} \\
&\quad + m_1m_2\Big(\frac{\varpi}{M_R} + L^*_{\Xi_k}\varpi + \Xi_k^0 + \psi_f^1(R_{\varpi})\|p_f^1\|_{L^1} + \psi_f^2(R^*_{\varpi})\|p_f^2\|_{L^1}\Big)\Bigg),
\end{aligned}$$

and

$$\begin{aligned}
H^*_{\varpi} &= \aleph\varpi + \big(\mathcal{L}^{\Phi} + \aleph(M_R + 1)\big)\|\Phi\|_{\mathcal{B}}, \\
\widetilde{H}^* &= ah^*_{c_1}(\varpi + M_R\|\Phi\|_{\mathcal{B}}) + ah^*, \\
R_{\varpi} &= \aleph\varpi + (\aleph + \mathcal{L}^{\Phi})\|\Phi\|_{\mathcal{B}}, \\
R^*_{\varpi} &= ah^*_{c_1}\varpi + ah^*.
\end{aligned}$$

Theorem 6. *Suppose that the hypotheses $(C1)$–$(C7)$ and (C_H) are valid. Then, the problem (5) is controllable.*

Proof. The steps of the proof will not be presented in detail, since the calculation methods have been discussed in detail in the previous proofs. □

We define in $\mathcal{X}$ measures of noncompactness as in Section 4, but we change $\widetilde{\Sigma}$ by $\widetilde{\varkappa}$, such that for $\widetilde{\varkappa}(\vartheta) = \int_0^{\vartheta} \varkappa(\delta)d\delta$, $\varkappa(\vartheta) = 4M_R\Big(l_f(\vartheta) + m_C(M_R\|l_f\|_{L^1})q_w(\vartheta)\Big)$.

Now, using $(C6)$ and defining the control:

$$u_v(\vartheta) = \begin{cases} W^{-1}\Big(v(\widehat{n}) - R(\widehat{n})\Phi(0) - \int_0^{\widehat{n}} R(\widehat{n}-\delta) f(\delta, v_{\rho(\delta,v_\delta)}, H(v)(\delta))d\delta\Big); \text{ if } \vartheta \in I_0, \\ \\ W^{-1}\Big(v(\widehat{n}) - R(\widehat{n}-\delta_k)\big(\Xi_k(\delta, v(\delta_k^-))\big) \\ - \int_{\delta_k}^{\widehat{n}} R(\vartheta-\delta) f(\delta, v_{\rho(\delta,v_\delta)}, H(v)(\delta))d\delta\Big); \text{ if } \vartheta \in I_k,\ k \in \mathbb{N}. \end{cases}$$

We shall note that when using the control $u(\cdot)$, the operator $\Upsilon_3' : \mathcal{X} \to \mathcal{X}$ defined by:

$$\Upsilon_3' v(\vartheta) = \begin{cases} R(\vartheta)\Phi(0) + \int_0^{\vartheta} R(\vartheta-\delta) f(\delta, v_{\rho(\delta,v_\delta)}, H(v)(\delta))d\delta \int_0^{\vartheta} R(\vartheta-\delta) C u_v(\delta) d\delta, \\ \qquad \text{if } \vartheta \in I_0, \\ \\ R(\vartheta-\delta_k)\big(\Xi_k(\delta, v(\delta_k^-))\big) + \int_{\delta_k}^{\vartheta} R(\vartheta-\delta) f(\delta, v_{\rho(\delta,v_\delta)}, H(v)(\delta))d\delta \\ + \int_{\delta_k}^{\vartheta} R(\vartheta-\delta) C u_v(\delta) d\delta; \text{ if } \vartheta \in I_k,\ k \in \mathbb{N}, \\ \\ \Xi_k(\vartheta, v(\vartheta_k^-)); \text{ if } \vartheta \in J_k,\ k \in \mathbb{N}, \\ \\ \Phi(\vartheta), \text{ if } \vartheta \in \mathbb{R}_-, \end{cases} \tag{6}$$

has a fixed point, this fixed point is a mild solution of system (5), and this implies that the system is controllable.

If ϕ is a fixed point of Υ_3', then there is similar transformation to that in the proof of Theorem 2, given the following decomposition $\phi(\vartheta) = y(\vartheta) + x(\vartheta)$, which implies $\phi_\vartheta = y_\vartheta + x_\vartheta$.

Let the operator $\Upsilon_3 : \Omega \to \Omega$ defined by

$$\Upsilon_3 y(\vartheta) = \begin{cases} \int_0^{\vartheta} R(\vartheta-\delta) f(\delta, y_{\rho(\delta,y_\delta+x_\delta)} + x_{\rho(\delta,y_\delta+x_\delta)}, H(y+x)(\delta))d\delta \\ + \int_0^{\vartheta} R(\vartheta-\delta) C u_{y+x}(\delta) d\delta, \text{ if } \vartheta \in I_0, \\ \\ R(\vartheta-\delta_k)\big[\Xi_k(\delta, y(\delta_k^-))\big] \\ + \int_{\delta_k}^{\vartheta} R(\vartheta-\delta) f(\delta, y_{\rho(\delta,y_\delta+x_\delta)} + x_{\rho(\delta,y_\delta+x_\delta)}, H(y)(\delta))d\delta \\ + \int_{\delta_k}^{\vartheta} R(\vartheta-\delta) C u_y(\delta) d\delta; \text{ if } \vartheta \in I_k,\ k \in \mathbb{N}, \\ \\ \Xi_k(\vartheta, (y)(\vartheta_k^-)), \text{ if } \vartheta \in J_k,\ k \in \mathbb{N}. \end{cases}$$

The operator Υ_3' having a fixed point is equivalent to saying that Υ_3 has one, so it turns to prove that Υ_3 has a fixed point. We shall check that operator Υ_3' satisfies all conditions of Darbo's theorem [41].

Let $B_\varpi = B(0,\varpi) = \{y \in \Omega \ :\ \|y\|_{\mathcal{X}} \leq \varpi\}$, then the set B_ϖ is closed, bounded, and convex.

Step 1: $\Upsilon_3(B_\varpi) \subset B_\varpi$.

For $\vartheta \in I_0$ and $y \in B_\varpi$, we have

$$\begin{aligned} \|\Upsilon_3 y(\vartheta)\| \ \leq\ & \Bigg(\psi_f^1(H_\varpi^*)\|p_f^1\|_{L^1} + \psi_f^2(\widetilde{H}^*)\|p_f^2\|_{L^1} \\ & + m_1 m_2 \bigg(\frac{\varpi + \|\Phi\|_{\mathcal{B}}}{M_R} + \psi_f^1(H_\varpi^*)\|p_f^1\|_{L^1} + \psi_f^2(\widetilde{H}^*)\|p_f^2\|_{L^1}\bigg)\Bigg). \end{aligned}$$

If $\vartheta \in I_k$ and $y \in B_\varpi$, we obtain

$$\begin{aligned}\|\Upsilon_3 y(\vartheta)\| \quad &\leq \quad M_R\bigg(L^*_{\Xi_k}\varpi + \Xi^0_k + \psi^1_f(R_\varpi)\|p^1_f\|_{L^1} + \psi^2_f(R^*_\varpi)\|p^2_f\|_{L^1} \\ &\quad + m_1 m_2\bigg(\frac{\varpi}{M_R} + L^*_{\Xi_k}\varpi + \Xi^0_k + \psi^1_f(R_\varpi)\|p^1_f\|_{L^1} + \psi^2_f(R^*_\varpi)\|p^2_f\|_{L^1}\bigg)\bigg).\end{aligned}$$

Additionally, for $\vartheta \in J_k$ and $y \in B_\varpi$, we obtain

$$\|\Upsilon_3 y(\vartheta)\| \quad \leq \quad L^*_{\Xi_k}\varpi + \Xi^0_k.$$

Thus, we deduce from $(C7)$ that $\Upsilon_3(B_\varpi) \subset B_\varpi$ and $\Upsilon_3(B_\varpi)$ is bounded.

Step 2: Υ_3 is continuous.

Let $\{y_n\}_{n\in\mathbb{N}}$ be a sequence such that $y_n \to y_*$ in B_ϖ.

Since f, h, Ξ_k, C are continuous, and by the Lebegue dominated convergence theorem, we have

$$\int_0^\vartheta R(\vartheta-\delta)Cu_{y_n+x}(\delta)d\delta \xrightarrow[n\to+\infty]{} \int_0^\vartheta R(\vartheta-\delta)Cu_{y_*+x}(\delta)d\delta.$$

Then similar to Step 2 in proof of Theorem 4, we obtain

$$\|(\Upsilon_3 y_n)(\vartheta) - (\Upsilon_3 y_*)(\vartheta)\| \to 0, \quad as \;\; n \to +\infty.$$

Consequently, Υ_3 is continuous.

Step 3:

Let Π a bounded equicontinuous subset of B_ϖ, we have $\{\Upsilon_3(\Pi)\}$ is equicontinuous, implies $\omega_0(\Upsilon_3(\Pi)) = 0$, and for any $\varrho > 0$ there exists a sequence $\{y_k\}_{k=0}^\infty \subset \Pi$, such that for $\vartheta \in I_0$, we have

$$\begin{aligned}\mu(\Upsilon_3(\Pi)(\vartheta)) \quad &\leq \quad 4\int_0^\vartheta M_R(l_f(\delta) + m_C(M_R\|l_f\|_{L^1})q_y(\delta))\mu(\{\Pi(\delta)\})d\delta + \varrho \\ &\leq \quad \frac{e^{\tau\widetilde{\varkappa}(\vartheta)}}{\tau}\mu_{BPC}(\Pi) + \varrho,\end{aligned}$$

therefore

$$\mu_{BPC}(\Upsilon_3(\Pi)) \leq \frac{1}{\tau}\mu_{BPC}(\Pi).$$

Now, for $\vartheta \in I_k$, similar to Case 01, we obtain

$$\begin{aligned}\mu(\Upsilon_3(\Pi)(\vartheta)) \quad &\leq \quad \frac{1}{\tau}\mu(\Pi(\vartheta)) + 2\mu\bigg(\bigg\{\int_0^\vartheta R(\vartheta-\delta) \\ &\quad \Big(f(\delta, y^k_{\rho(\delta,y^k_\delta)} + x_{\rho(\delta,y^k_\delta+x_\delta)}, H(y^k)(\delta)) + u_{y_k}(\delta)\Big)d\delta\,;\, y_k \in \Pi\bigg\}\bigg) + \varrho \\ &\leq \quad \frac{2e^{\tau\widetilde{\varkappa}(\vartheta)}}{\tau}\mu_{BPC}(\Pi) + \varrho,\end{aligned}$$

thus

$$\mu_{BPC}(\Upsilon_3(\Pi)) \leq \tfrac{2}{\tau}\mu_{BPC}(\Pi).$$

Additionally, for $\vartheta \in J_k$, we obtain

$$\mu_{BPC}(\Upsilon_3(\Pi)) \quad \leq \quad \frac{1}{\tau\, M_R}\mu_{BPC}(\Pi).$$

By Darbo's fixed-point theorem [41], we conclude that Υ_3 has at least one fixed point y^*. Consequently, $\phi^* = y^* + x$ is a fixed point of the operator Υ_3', which implies that the system is controllable.

5. An Example

Consider the following class of partial integrodifferential system:

$$
\begin{cases}
\frac{\partial}{\partial\vartheta}\zeta(\vartheta,x) - \frac{\partial}{\partial x}\Big(\frac{\partial\zeta(\vartheta,x)}{\partial x} - \theta_1\zeta(\vartheta,x)\Big) - \theta_2\zeta(\vartheta,x) \\
\qquad - \int_0^\vartheta \Gamma(\vartheta-\delta)\Big(\frac{\partial}{\partial x}\Big(\frac{\partial\zeta(\delta,x)}{\partial x} + \theta_1\zeta(\delta,x)\Big) + \theta_2\zeta(\delta,x)\Big)d\delta \\
\qquad = \int_{-\infty}^{-\vartheta} \frac{\eta\sin(\tau)e^{-\Delta(\vartheta,\tau)-\gamma(\vartheta-\tau)}}{((\vartheta+\tau)^2+1)}d\tau - \frac{\eta\cos\left(e^{-\gamma\pi}-(\vartheta+\pi)^{-\gamma}\right)\zeta(\vartheta,x)}{(\vartheta^2+1)(1+|\zeta(\vartheta,x)|)} + \widehat{\lambda}Cu \\
\qquad + \frac{\eta\sin\left(e^{-\gamma\vartheta}\right)}{(\vartheta^2+1)}\int_0^a \frac{\ln(1+e^{-\vartheta^2})(1+\zeta(\delta,x))e^{-\gamma(\vartheta-\delta)}}{1+2\vartheta^2+\delta^2}d\delta,\ \vartheta\in I_k,\ x\in(0,1), \\
\zeta(\vartheta,0)=\zeta(\vartheta,1)=0, \quad \text{for } \vartheta\geq 0, \\
\zeta(\vartheta,x) = \frac{1}{5+e^\vartheta}\sin(\zeta(k^-,x)),\ \text{if } \vartheta\in J_k,\ \ x\in(0,1), \\
\zeta(\vartheta,x) = \Phi(\vartheta,x),\ \text{if } \vartheta\in\mathbb{R}_-\ \text{ and }\ x\in(0,1),
\end{cases}
\tag{7}
$$

where $I_k = (2k, 2k+1];\ k\in\mathbb{N}_0,\ J_k = (2k-1, 2k];\ k\in\mathbb{N},\ \Gamma:\mathbb{R}^+\mapsto\mathbb{R}$ is continuous, $\theta_1,\theta_2\in\mathbb{R},\ \eta\in\left(0,\pi^{-1}\right),\ \widehat{\lambda}\in\{0,1\}$. u is given in $L^2(J;U)$ Banach space of admissible control functions with U as a Banach space. C is a bounded linear operator.

Let

$$H := L^2(0,1) = \left\{u:(0,1)\longrightarrow\mathbb{R} : \int_0^1 |u(x)|^2dx < \infty\right\},$$

be the Hilbert space with the scalar product $\langle u,v\rangle = \int_0^1 u\ (x)v(x)dx$, and the norm

$$\|u\|_2 = \left(\int_0^1 |u(x)|^2dx\right)^{1/2},$$

and the phase space $\mathcal{B}$ be $BUC(\mathbb{R}^-,H)$, the space of bounded uniformly continuous functions endowed with the following norm: $\|\psi\|_{\mathcal{B}} = \sup_{-\infty<\tau\leq 0}\|\psi(\tau)\|_{L^2}, \psi\in\mathcal{B}$. It is well known that $\mathcal{B}$ satisfies the axioms (A_1) and (A_2) with $K=1$ and $L(\vartheta)=M(\vartheta)=1$, (see [39]), and put $Y = BPC(\mathbb{R}^+,X)$.

We define the operator A induced on H as follows:

$$\mathrm{A}z = z'' + \theta_1 z' + \theta_2 z,\ \ \theta_1,\theta_2\in\mathbb{R} \text{ and } D(\mathrm{A}) = H^2(0,1)\cap H_0^1(0,1),$$

which is the infinitesimal generator of an analytic semigroup $(G(\vartheta))_{\vartheta\geq 0}$ on H. Since the semigroup generated by A is analytic, then it is norm continuous for $\vartheta>0$. THis implies that the resolvent operator is operator-norm continuous for $\vartheta>0$ (see [42]).

As in [29,43], for some $\widehat{r}>r>1$, we assume that $\|\Gamma(\vartheta)\|\leq\frac{e^{-\widehat{r}\vartheta}}{r}$, and $\|\Gamma'(\vartheta)\|\leq\frac{e^{-\widehat{r}\vartheta}}{r^2}$, we get that $\|R(\vartheta)\|\leq e^{-\widehat{\sigma}\vartheta}$, where $\widehat{\sigma} = 1-r^{-1}$.

We define also the operators $B(\vartheta): H\mapsto H$ as follows:

$$B(\vartheta)z = \Gamma(\vartheta)Az,\ for\ \vartheta\geq 0,\ z\in D(\mathrm{A}).$$

More appropriate conditions on operator B, (C4) hold with $M_R = 1$ and $\mu = 1-r^{-1}$.
Case 1 : $\Delta(\vartheta,\tau) = \zeta(\vartheta+\tau,x),\ \widehat{\lambda}=0$.

We assign $\zeta(\vartheta)(x) = \zeta(\vartheta,x)$, for $\vartheta\in[0,+\infty)$, and define

$$
\begin{aligned}
\mathrm{f}(\vartheta,\phi_1,\phi_2)(x) = & \int_{-\infty}^{-\vartheta}\frac{\eta\sin(\tau)e^{-\phi_1(\vartheta+\tau,x)-\gamma(\vartheta-\tau)}}{((\vartheta+\tau)^2+1)}d\tau \\
& - \frac{\eta\cos(e^{-\gamma\pi}-(\vartheta+\pi)^{-\gamma})}{(\vartheta^2+1)}\frac{\phi_1(\vartheta,x)}{(1+|\phi_1(\vartheta,x)|)} \\
& + \frac{\eta\sin\left(e^{\gamma\vartheta}\right)}{(\vartheta^2+1)}e^{-\gamma\vartheta}\phi_2(\vartheta)(x),
\end{aligned}
$$

$$\begin{aligned}\phi_2(\vartheta)(x) &= H(\phi_1)(x)\\ &= \frac{\sin\left(e^{-\gamma\vartheta}\right)}{(\vartheta^2+1)}\int_0^a \frac{\ln(1+e^{-\vartheta^2})(1+\zeta(\delta,x))e^{-\gamma(\vartheta-\delta)}}{1+2\vartheta^2+\delta^2}d\delta,\end{aligned}$$

$$\Xi_k(\vartheta,\phi(\vartheta,x)) = \frac{1}{5+e^{\vartheta}}\sin(\phi(k^-,x)).$$

Using these definitions, we can represent the system (7) in the following abstract form

$$\begin{cases} \phi'(\vartheta) = A\phi(\vartheta) + f(\vartheta,\phi_\vartheta,(H\phi)(\vartheta)) + \int_0^\vartheta B(\vartheta-\delta)\phi(\delta)d\delta, \text{ if } \vartheta\in I_k,\ k\in\mathbb{N}_0,\\ \phi(\vartheta) = \Xi_k\left(\vartheta,\phi\left(\vartheta_k^-\right)\right), \text{if } \vartheta\in J_k,\ k\in\mathbb{N},\\ \phi(\vartheta) = \Phi(\vartheta), \text{ if } \vartheta\in\mathbb{R}_-, \end{cases} \tag{8}$$

For $\vartheta\in I_k$, we have

$$\|\mathrm{f}(\vartheta,\varkappa_{1(\vartheta)},\varkappa_2(\vartheta))\| \le \frac{\eta e^{-\gamma\vartheta}}{(\vartheta^2+1)}(1+\|\varkappa_1\|_{\mathcal{B}}) + \frac{\eta|\cos(\pi^{-\gamma}-(\vartheta+\pi)^{-\gamma})|}{(\vartheta^2+1)}(\|\varkappa_2(\vartheta)\|).$$

So, $\psi_{i+1}(\vartheta) = \vartheta+i$, are continuous nondecreasing functions from $\mathbb{R}^+$ to $[i,+\infty), i=0,1$. And, we have

$$p_f^{i+1}(\vartheta) = \frac{\eta|\cos(e^{-\gamma\pi}-(\vartheta(1+i)+\pi)^{-\gamma})|}{(\vartheta^2+1)e^{\gamma i\vartheta}},\ \ i=0,1,$$

this clearly forces $(p_f^{i+1})_{i=0,1}\in L^1(J,\mathbb{R}^+)$, and

$$\lim_{\vartheta\to+\infty}\sup_{\vartheta\in J}\int_0^\vartheta e^{-\mu(\vartheta-\delta)}p_f^{i+1}(\delta)d\delta = \eta\lim_{\vartheta\to+\infty}\sup_{\vartheta\in J}e^{-\mu\vartheta}\int_0^\vartheta \frac{e^{-(\mu+\gamma i)\delta}}{1+\delta^2}d\delta = 0,\ \ i=0,1,$$

and

$$M_R L^*_{\Xi_k} = \frac{1}{5+e^1} < 1.$$

Now, for h and Ξ_k, we have

$$\begin{aligned}\|h(\vartheta,\delta,\varkappa_1)-h(\vartheta,\delta,\varkappa_2)\| &\le \frac{\ln(1+e^{-\vartheta^2})e^{-\gamma(\vartheta-\delta)}}{(1+2\vartheta^2+\delta^2)(\vartheta^2+1)}\|\varkappa_1(\vartheta)-\varkappa_2(\vartheta)\|\\ &\le \ln(2)\|\varkappa_1(\vartheta)-\varkappa_2(\vartheta)\|,\end{aligned}$$

$$\begin{aligned}\|\Xi_k(\varkappa_1)(\vartheta)-\Xi_k(\varkappa_2)(\vartheta)\| &\le \frac{1}{5+e^\vartheta}\|\varkappa_1(\vartheta)-\varkappa_2(\vartheta)\|\\ &\le \frac{1}{5+e^1}\|\varkappa_1(\vartheta)-\varkappa_2(\vartheta)\|.\end{aligned}$$

Additionally, for some positive constant c_θ, we have

$$\begin{aligned}M_R\left(\|p_f^1\|_{L^1}\psi_f^1(\aleph^*)+\|p_f^2\|_{L^1}\psi_f^2(H^*)\right) &= (1+2\|\Phi\|_{\mathcal{B}}+c_\theta)\|p_f^1\|_{L^1}\\ &\quad +(1+\|\Phi\|_{\mathcal{B}}+c_\theta)a\ln(2)\|p_f^2\|_{L^1}\\ &\le \frac{\eta\pi}{2}(1+a\ln(2)+(2+\ln(2))\|\Phi\|_{\mathcal{B}})\\ &\quad +\left(\frac{\pi}{2}+a\ln(\sqrt{2})\pi\right)\eta c_\theta.\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{M_R(\Xi_k^0+\|p_f^1\|_{L^1}\psi_f^1(\widetilde{\aleph^*})+\|p_f^2\|_{L^1}\psi_f^2(\widetilde{H^*}))}{1-M_RL^*_{\Xi_k}} &= \frac{(1+c_\theta+\|\Phi\|_{\mathcal{B}})\|p_f^1\|_{L^1}}{1-\frac{1}{5+e^1}}\\ &\quad+\frac{a\ln(2)(c_\theta+1)\|p_f^2\|_{L^1}}{1-\frac{1}{5+e^1}}\\ &\leq \frac{1}{5}\Big(3\pi\eta(1+\|\Phi\|_{\mathcal{B}})+6\pi a\ln(\sqrt{2})\eta\Big)\\ &\quad+\frac{1}{5}\Big(3\pi\eta+6\pi\ln(\sqrt{2})a\eta\Big)c_\theta.\end{aligned}$$

Hence, from the previous estimate, we assign

$$\begin{aligned}\aleph_1 &= \frac{\pi\eta(1+a\ln(\sqrt{2}))(1+\|\Phi\|_{\mathcal{B}})}{1-a\ln(\sqrt{2})\pi\eta-\pi\eta},\\ \aleph_2 &= \frac{3\pi\eta(1+a\ln(2)+\|\Phi\|_{\mathcal{B}})}{5-3\pi\eta(1+a\ln(2))}.\end{aligned}$$

Therefore, we can choose θ,γ as the following:

$$\max(\aleph_1,\aleph_2)<\theta<2\theta<\gamma.$$

Thus, all conditions of Theorems 2 and 3 are verified. Then, the problem (7) has at least one mild solution, which is locally attractive.

Case 2: $\Delta(\vartheta,\tau)=\zeta(\vartheta+\sigma(\vartheta,\zeta(\vartheta+\tau,x)),x)$, $\sigma: J\times\mathbb{R}\to\mathbb{R}$ is given function, $\widehat{\lambda}=0$.

In addition to the estimates that we have obtained in Case 01, we have for any bounded set $\Pi\subset X$, and $\Pi_\vartheta\in\mathcal{B}$,

$$\mu(f(\vartheta,\Pi_\vartheta,H(\Pi)))\leq\eta\left(\vartheta^2+1\right)^{-1}\mu(\Pi),\ \text{and}\ \left(\vartheta^2+1\right)^{-1}\in L^1(J,\mathbb{R}^+).$$

For $\Phi\in BUC(\mathbb{R}^-,H)$, we assign $\rho(\vartheta,\Phi)(\zeta)=\sigma(\vartheta,\zeta(\vartheta+\tau,x))$, such that (C_Φ) hold, and let $\vartheta\to\Phi_\vartheta$ be continuous on $\mathcal{R}(\rho^-)$.

Consequently, the assumptions of Theorems 4 and 5 are satisfied, which guarantees the existence and attractivity of solutions for the problem (7).

Case 3: $\Delta(\vartheta,\tau)=\zeta(\vartheta+\sigma(\vartheta,\zeta(\vartheta+\tau,x)),x)$, $\sigma: J\times\mathbb{R}\to\mathbb{R}$ is given function, $\widehat{\lambda}=1$.

In addition to the estimations obtained in Case 1 and Case 2, we assume that the operator W given by $Wu=\int_0^{\widehat{n}}R(\widehat{n}-\delta)Cu(\delta)d\delta$, satisfies $(C6)$. Then, all the assumptions given in Theorem (6) are verified. Therefore, the problem (7) is controllable.

6. Conclusions

Under certain conditions and by employing Darbo's fixed-point theorem with the measure of noncompactness, we demonstrated the existence, attractivity, and controllability results for semilinear integro-differential equations with non-instantaneous impulses on an infinite interval via resolvent operators in the case of neutral and state-dependent delay problems. We believe that the provided results will have an influence on the relevant literature and have various potential applications. The results may be extended to a variety of fields, notably in fractional calculus.

Author Contributions: Conceptualization: A.B., A.S. and M.B.; Writing original draft preparation: A.B. and A.S.; Supervision: M.B. and G.M.N.; Validation: M.B. and G.M.N. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Hernàndez, E.; O'Regan, D. On a new class of abstract impulsive differential equations. *Proc. Amer. Math. Soc.* **2013**, *141*, 1641–1649. [CrossRef]
2. Pierri, M.; O'Regan, D.; Rolnik, V.; Existence of solutions for semi-linear abstract differential equations with non instantaneous impulses. *Appl. Math. Comput.* **2013**, *219*, 6743–6749.
3. Abbas, S.; Benchohra, M. Advanced Functional Evolution Equations and Inclusions. In *Developments in Mathematics*; Springer: Cham, Switzerland, 2015; Volume 39.
4. Abbas, S.; Benchohra, M.; N'Guérékata, G.M. Instantaneous and noninstantaneous impulsive integrodifferential equations in Banach spaces. *J. Anal. Appl.* **2020**, *38*, 143–156. [CrossRef]
5. Bai, L.; Nieto, J.J. Variational approach to differential equations with not instantaneous impulses. *Appl. Math. Lett.* **2017**, *73*, 44–48. [CrossRef]
6. Benchohra, M.; Henderson, J.; Ntouyas, S.K. *Impulsive Differential Equations and Inclusions*; Hindawi Publishing Corporation: New York, NY, USA, 2006.
7. Benkhettou, N.; Aissani, K.; Salim, A.; Benchohra, M.; Tunc, C. Controllability of fractional integro-differential equations with infinite delay and non-instantaneous impulses. *Appl. Anal. Optim.* **2022**, *6*, 79–94.
8. Chen, P.; Zhang, X.; Li, Y. Existence of mild solutions to partial differential equations with noninstantaneous impulses. *Electron. J. Differ. Equ.* **2016**, *241*, 1–11.
9. Hao, X.; Liu, L. Mild solution of semilinear impulsive integro-differential evolution equation in Banach spaces. *Math. Meth. Appl. Sci.* **2017**, *40*, 4832–4841. [CrossRef]
10. Yang, D.; Wang, J. Integral boundary value problems fornonlinear non-instataneous impulsive differential equations. *J. Appl. Math. Comput.* **2017**, *55*, 59–78. [CrossRef]
11. Yu, X.; Wang, J. Periodic boundary value problems for nonlinear impulsive evolution equations on Banach spaces. *Commun. Nonlinear Sci. Numer. Simulat.* **2015**, *22*, 980–989. [CrossRef]
12. Li, Z.; Chen, Q.; Wang, Y.; Li, X. Solving two-sided fractional super-diffusive partial differential equations with variable coefficients in a class of new reproducing kernel spaces. *Fractal Fract.* **2022**, *6*, 492. [CrossRef]
13. Baliki, A.; Benchohra, M.; Graef, J. Global existence and stability for second order functional evolution equations with infinite delay. *Electron. J. Qual. Theory Differ. Equ.* **2016**, *2016*, 1–10.
14. Hale, J.; Kato, J. Phase space for retarded equations with infinite delay. *Funk. Ekvac.* **1978**, *21*, 11–41.
15. Santos, J.P.C.d. On state-dependent delay partial neutral functional integrodifferential equations. *Appl. Math. Comput.* **2010**, *100*, 1637–1644.
16. Hernandez, E.; McKibben, M.A. On state-dependentdelay partial neutral functional-differential equations. *Appl. Math. Comput.* **2007**, *186*, 294–301.
17. Hernandez, E.; Prokopczyk, A.; Ladeira, L. A note on partial functional differential equations with state-dependent delay. *Nonlin. Anal.* **2006**, *7*, 510–519. [CrossRef]
18. Benchohra, M.; Bouazzaoui, F.; Karapinar, E.; Salim, A. Controllability of second order functional random differential equations with delay. *Mathematics* **2022**, *16*, 1120. [CrossRef]
19. Diop, M.A.; Ezzinbi, K.; Ly, M.P. Nonlocal problems for integrodifferential equation via resolvent operators and optimal control. *Differ. Incl. Control Optim.* **2022**, *42*, 5–25. [CrossRef]
20. Dieye, M.; Diop, M.A.; Ezzinbi, K.; Hmoyed, H. On the existence of mild solutions for nonlocal impulsive integrodifferential equations in Banach spaces. *Matematiche* **2019**, *74*, 13–34.
21. Ezzinbi, K.; Ghnimi, S.; Taoudi, M.A. Existence results forsome nonlocal partial integrodifferential equations withoutcompactness or equicontinuity. *J. Fixed Point Theo. Appl.* **2019**, *21*, 53. [CrossRef]
22. Blot, J.; Buse, C.; Cieutat, P. Local attractivity in nonautonomous semilinear evolution equations. *Nonauton. Dyn. Syst.* **2014**, *1*, 72–82. [CrossRef]
23. Salim, A.; Abbas, S.; Benchohra, M.; Karapinar, E. Global stability results for Volterra-Hadamard random partial fractional integral equations. *Rend. Circ. Mat. Palermo* **2022**, *2*, 1–13. [CrossRef]
24. Laledj, N.; Salim, A.; Lazreg, J.E.; Abbas, S.; Ahmad, B.; Benchohra, M. On implicit fractional q-difference equations: Analysis and stability. *Math. Methods Appl. Sci.* **2022** , 1–23. [CrossRef]
25. Heris, A.; Salim, A.; Benchohra, M.; Karapinar, E. Fractional partial random differential equations with infinite delay. *Results Phys.* **2022**, *37*, 105557. [CrossRef]
26. Salim, A.; Abbas, S.; Benchohra, M.; Karapinar, E. A Filippov's theorem and topological structure of solution sets for fractional q-difference inclusions. *Dynam. Syst. Appl.* **2022**, *31*, 17–34. [CrossRef]
27. Yosida, K. *Functional Analysis*; Springer: Berlin, Germany, 1980; Volume 6.

28. Desch, W.; Grimmer, R.C.; Schappacher, W. Some considerations for linear integrodiffferential equations. *J. Math. Anal. Appl.* **1984**, *104*, 219–234. [CrossRef]
29. Grimmer, R.C. Resolvent opeators for integral equations in a Banach space. *Trans. Amer. Math. Soc.* **1982**, *273*, 333–349. [CrossRef]
30. Grimmer, R.; Pritchard, A.J. Analytic resolvent operators for integral equations in a Banach space. *J. Differ. Equ.* **1983**, *50*, 234–259. [CrossRef]
31. Banaś, J.; Goebel, K. Measure of Noncompactness in Banach Spaces. In *Lecture Notes in Pure and Applied Math*; Marcel Dekker: New York, NY, USA, 1980; Volume 60.
32. Dudek, S. Fixed point theorems in Fréchet algebras and Fréchet spaces and applications to nonlinear integral equations. *Appl. Anal. Disc. Math.* **2017**, *11*, 340–357. [CrossRef]
33. Mönch, H. Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces. *Nonlinear Anal.* **1980**, *4*, 985–999. [CrossRef]
34. Horvath, C. Measure of non-compactness and multivalued mappings in complete metric topological spaces. *J. Math. Anal. Appl.* **1985**, *108*, 403–408. [CrossRef]
35. Kirk, W.A.; Sims, B. *Handbook of Metric Fixed Point Theory*; Springer-Science + Business Media, B.V.: Dordrecht, The Netherlands, 2001.
36. Dhage, B.C.; Lakshmikantham, V. On global existence and attractivity results for nonlinear functional integral equations. *Nonlinear Anal.* **2010**, *72*, 2219–2227. [CrossRef]
37. Banaś, J. Measures of noncompactness in the space of continuous tempered functions. *Demonstratio Math.* **1981**, *14*, 127–133.
38. Arab, R.; Allahyari, R.; Haghighi, A.S. Construction of a measure of noncompactness on $BC(\Omega)$ and its application to volterra integral equations. *Mediterr. J. Math.* **2016**, *13*, 1197–1210. [CrossRef]
39. Hino, Y.; Murakami, S.; Naito, T. Functional-differential equations with infinite delay. In *Lecture Notes in Mathematics*; Stahy, S., Ed.; Springer: Berlin, Germany, 1991; Volume 1473.
40. Hernandez, E.; Sakthivel, R.; Tanaka, A. Existence results for impulsive evolution differential equations with state-dependent delay. *Electron. J. Differ. Equ.* **2008**, *2008*, 1–11.
41. Darbo, G. Punti uniti in transformazioni a condominio non compatto. *Rend. Sem. Math. Univ. Padova* **1955**, *24*, 84–92.
42. Engel, K.J.; Nagel, R. *One-Parameter Semigroups for Linear Evolution Equations*; Springer: New York, NY, USA, 2000.
43. Diop, A.; Diop, M.A.; Diallo, O.; Traoré, M.B. Local attractivity for integro-differential equations with noncompact semigroups. *Nonauton. Dyn. Syst.* **2020**, *7*, 102–117. [CrossRef]