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*Article*

# Analysis of a Hybrid Coupled System of $\psi$-Caputo Fractional Derivatives with Generalized Slit-Strips-Type Integral Boundary Conditions and Impulses

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**Abstract:** In the current paper, we analyzed the existence and uniqueness of a solution for a coupled system of impulsive hybrid fractional differential equations involving $\psi$-Caputo fractional derivatives with generalized slit-strips-type integral boundary conditions. We also study the Ulam–Hyers stability for the considered system. For the existence and uniqueness of the solution, we use the Banach contraction principle. With the help of Schaefer's fixed-point theorem and some assumptions, we also obtain at least one solution of the mentioned system. Finally, the main results are verified with an appropriate example.

**Keywords:** coupled system; slit-strips-type integral boundary conditions; $\psi$–Caputo fractional derivative; Schaefer's fixed point theorem; Ulam–Hyers stability; impulsive conditions

**MSC:** 26A33; 34B27; 39B82; 45M10



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## 1. Introduction

Extensive applications and significant contribution leads the popularity of fractional calculus, and operators of arbitrary order gives us more realistic and useful mathematical modeling of many phenomena (see [1–3]). In different fields of science and engineering, fractional-order nonlinear boundary value problems appear with different aspects. In recent studies the main focus is on the existence, uniqueness and stability of nonlinear arbitrary order differential equations with boundary conditions [4–8].

Coupled nonlinear fractional differential equations (FDEs), finds their uses in different applied and scientific models such as disease models [9,10], ecological models [11], synchronization of chaotic systems [12], etc. In recent years Hybrid differential equations of fractional-order is also very important area of research [13–19]. Ahmad et al. studied a new idea of slit-strips type conditions, which has very fundamental uses in acoustics [20] and imaging via strip-detectors [21].

In [22], the authors studied the following problem with slit-strips type condition:

$$\begin{cases} {}^{c}\mathcal{D}_0^{p}x(z) = f_1(z, x(z)),\ m-1 < p \leq m,\ z \in [0,1], m \in N, \\ x(0) = 0, x^{'}(0) = 0, ..., x^{n-2}(0) = 0, \\ x(\xi) = a_1 \displaystyle\int_0^{\eta_1} x(s)ds + a_2 \int_{\xi_1}^{1} x(s)ds,\ 0 < \eta_1 < \xi < \xi_1 < 1, \end{cases}$$

where ${}^{c}\mathcal{D}_0^{p}$ denotes the Caputo fractional derivative (CFD) of order $p$, $f_1$ is a given function and $a_1$,$a_2 \in \mathbb{R}$.

In 2017, Ahmad et al. [23] studied a coupled system of nonlinear FDEs

$$\begin{cases} {}^c\mathcal{D}_0^{\alpha_1}x(z) & = f_1(z,x(z),y(z)),\ z\in[0,1],\ 1<\alpha_1\leq 2,\\ {}^c\mathcal{D}_0^{\beta_1}y(z) & = f_2(z,x(z),y(z)),\ z\in[0,1],\ 1<\beta_1\leq 2, \end{cases}$$

supplemented with the coupled and uncoupled boundary conditions of the form:

$$\begin{aligned} x(0)=0,\quad x(\xi)&=d_1\textstyle\int_0^\eta y(s)ds+d_2\int_{\xi_1}^1 y(s)ds,\ 0<\eta<\xi<\xi_1<1,\\ y(0)=0,\quad y(\xi)&=d_1\textstyle\int_0^\eta x(s)ds+d_2\int_{\xi_1}^1 x(s)ds,\ 0<\eta<\xi<\xi_1<1,\\ x(0)=0,\quad x(\xi)&=d_1\textstyle\int_0^\eta x(s)ds+d_2\int_{\xi_1}^1 x(s)ds,\ 0<\eta<\xi<\xi_1<1,\\ y(0)=0,\quad y(\xi)&=d_1\textstyle\int_0^\eta y(s)ds+d_2\int_{\xi_1}^1 y(s)ds,\ 0<\eta<\xi<\xi_1<1, \end{aligned}$$

where ${}^c\mathcal{D}_0^{\alpha_1}$ and ${}^c\mathcal{D}_0^{\beta_1}$ denote the CFD of order $\alpha_1$ and $\beta_1$ respectively, $f_1,f_2:[0,1]\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ are given continuous functions and $d_1,d_2$ are real constants.

In 2019, Bashir Ahmad et al. [24] studied a coupled system of hybrid nonlinear FDEs

$$\begin{aligned} {}^c\mathcal{D}_0^{\gamma_1}[u(z)-h_1(z,u(z),v(z))]&=\theta_1(z,u(z),v(z)),\ \ z\in[0,1],\ \ 1<\gamma_1\leq 2,\\ {}^c\mathcal{D}_0^{\delta_1}[v(z)-h_2(z,u(z),v(z))]&=\theta_2(z,u(z),v(z)),\ \ z\in[0,1],\ \ 1<\delta_1\leq 2, \end{aligned}$$

equipped with coupled slit-strips type integral boundary conditions:

$$\begin{aligned} u(0)=0,\ \ u(\eta_1)&=\omega_1\textstyle\int_0^{\xi_1}v(s)ds+\omega_2\int_{\xi_2}^1 v(s)ds,\quad 0<\xi_1<\eta_1<\xi_2<1,\\ v(0)=0,\ \ v(\eta_1)&=\omega_1\textstyle\int_0^{\xi_1}u(s)ds+\omega_2\int_{\xi_2}^1 u(s)ds,\quad 0<\xi_1<\eta_1<\xi_2<1, \end{aligned}$$

where ${}^c\mathcal{D}_0^{\gamma_1}$ and ${}^c\mathcal{D}_0^{\delta_1}$ denotes the CFD of orders $\gamma_1$ and $\delta_1$ respectively, $\theta_i,h_i:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions with $h_i(0,u(0),v(0))=0$, $i=1,2$ and $\omega_1$, $\omega_2$ are real constants.

In this article, motivated from the aforementioned work, we study the coupled system of impulsive hybrid FDEs with generalized slit-strips type integral boundary conditions:

$$\begin{cases} {}^c\mathcal{D}_{z_k,z}^{\alpha_1;\psi}[x(z)-f_1(z,x(z),{}^c\mathcal{D}_{z_k,z}^{\alpha_1;\psi}x(z))]=g_1(z,x(z),{}^c\mathcal{D}_{z_k,z}^{\alpha_1;\psi}x(z)),\quad z\in(z_k,z_{k+1}],\quad k=0,1,\ldots,p,\\ {}^c\mathcal{D}_{z_k,z}^{\beta_1;\psi}[y(z)-f_2(z,y(z),{}^c\mathcal{D}_{z_k,z}^{\beta_1;\psi}y(z))]=g_2(z,y(z),{}^c\mathcal{D}_{z_k,z}^{\beta_1;\psi}y(z)),\quad z\in(z_k,z_{k+1}],\quad k=0,1,\ldots,p,\\ x(0)=0,\ \ x(\eta)=a_1\int_{z_k}^{\delta_{2k}}x(\tau)d\tau+a_2\int_{\delta_{2k+1}}^{z_{k+1}}x(\tau)d\tau,\quad z_k<\delta_{2k}<\eta<\delta_{2k+1}<z_{k+1},\\ y(0)=0,\ \ y(\eta)=a_1\int_{z_k}^{\delta_{2k}}y(\tau)d\tau+a_2\int_{\delta_{2k+1}}^{z_{k+1}}y(\tau)d\tau,\quad z_k<\delta_{2k}<\eta<\delta_{2k+1}<z_{k+1},\\ \Delta x(z_k)=x(z_k^+)-x(z_k^-)=I_k(x(t_k)),\quad \Delta x'(z_k)=x'(z_k^+)-x'(z_k^-)=J_k(x(z_k)),\quad k=1,2,\ldots,p,\\ \Delta y(z_k)=y(z_k^+)-y(z_k^-)=I_k^*(y(z_k)),\quad \Delta y'(z_k)=y'(z_k^+)-y'(z_k^-)=J_k^*(y(z_k)),\quad k=1,2,\ldots,p, \end{cases} \tag{1}$$

where ${}^c\mathcal{D}_{z_k,z}^{\alpha_1;\psi}$ and ${}^c\mathcal{D}_{z_k,z}^{\beta_1;\psi}$ denote the $\psi$-CFDs with $\alpha_1,\beta_1\in(1,2]$, and $\mathcal{J}=[0,Z]$ with $Z>0$, $f_1,g_1,f_2,g_2:\mathcal{J}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are given continuous functions with $f_1(0,x(0),{}^c\mathcal{D}_{z_k,z}^{\alpha_1;\psi}x(0))=0$, $f_2(0,y(0),{}^c\mathcal{D}_{z_k,z}^{\beta_1;\psi}y(0))=0$ and $a_1$ , $a_2$ are real constants.

The main novelty of our paper is that, to the best of our knowledge, no one have studied the impulsive systems with slit strip boundary conditions. Here we introduced the generalized form of this boundary condition for each interval of the impulsive systems. In the problem (1), the integral boundary condition describes that the contribution due to finite strips of arbitrary lengths occupying the positions $(z_k,\delta_{2k})$ and $(\delta_{2k+1},z_{k+1})$ on the intervals $(z_k,z_{k+1}]$, $k=0,1,\ldots,p$, is related to the value of the unknown function at a nonlocal point $\eta$ on each impulsive interval $(z_k<\delta_{2k}<\eta<\delta_{2k+1}<z_{k+1})$, $k=0,1,\ldots,p$ located at an arbitrary position in the aperture (slit)—the region of the boundary off the strips. Examples of such boundary conditions include scattering by slits silicon strips detectors for scanned multi-slit X-ray imaging, acoustic impedance of baffled strips radiators, diffraction from an

elastic knife-edge adjacent to a strip, sound fields of infinitely long strips, dielectric-loaded multiple slits in a conducting plane, lattice engineering [25–33].

## 2. Preliminaries and Notations

To overcome the vast number of Definitions of fractional derivatives and integrals [2], we can for instance consider general operators, from which choosing special kernels and some form of differential operator, we obtain the classical fractional integrals and derivatives. For example, for the kernel $k(x,t) = x - t$ and the differential operator $d/dx$, we obtain the Riemann-Liouville fractional derivative, and for $k(x,t) = \ln(\frac{x}{t})$ and the differential $x\frac{d}{dx}$, we obtain the Hadamard fractional derivative. But, in this case, most of the fundamental laws of the derivative operator can not be obtained, due to the arbitrariness of the kernel. The problems that arise from this approach are the natural limitations to the study of the basic properties of the fractional operators. To overcome this issue, Almeida [34] considered the special case when the kernel is of the type $k(x,t) = \psi(x) - \psi(t)$ and the derivative operator is of the form $\frac{1}{\psi'(x)}\frac{d}{dx}$. Although the kernel is still unknown, involving the function $\psi$, he deduced some properties for the fractional operator known as $\psi$-Caputo fractional derivative. Here we recall few definitions and lemmas for $\psi$-Caputo fractional derivative.

**Definition 1** (see [34]). *The left-sided $\psi$-Riemann-Liouville (RL) fractional integral of order $\alpha_1 (> 0)$ for an integrable function $h(z) : [0,1] \to \mathbb{R}$ with respect to another increasing differentiable function $\psi : [a,b] \to \mathbb{R}$ such that $\psi^{'}(z) \neq 0$ for all $z \in [a,b]$ is defined by*

$$\mathcal{I}_{a+}^{\alpha_1;\psi} h(z) = \frac{1}{\Gamma(\alpha_1)} \int_a^z \psi^{'}(s)(\psi(z) - \psi(s))^{\alpha_1 - 1} h(s) ds$$

*where $\Gamma$ is the Euler Gamma function.*

**Definition 2** (see [34]). *Let $n \in N$ and $\psi(z), h(z) \in C^n([a,b],\mathbb{R})$ be two functions such that $\psi(z)$ is increasing and $\psi^{'}(z) \neq 0$ for all $z \in [a,b]$. The left-sided $\psi$-RL fractional derivative of a function $h(z)$ of order $\alpha_1$ is defined by*

$$\begin{aligned} \mathcal{D}_{a+}^{\alpha_1;\psi} h(z) &= \left(\frac{1}{\psi'(z)}\frac{d}{dz}\right)^n \mathcal{I}_{a+}^{n-\alpha_1;\psi} h(z) \\ &= \frac{1}{\Gamma(n-\alpha_1)} \left(\frac{1}{\psi'(z)}\frac{d}{dz}\right)^n \int_a^z \psi^{'}(s)(\psi(z) - \psi(s))^{n-\alpha_1-1} h(s) ds \end{aligned}$$

*where $n = [\alpha_1] + 1$ and $[\alpha_1]$ denotes the integer part of the real number $\alpha_1$.*

**Definition 3** (see [34]). *Let $n - 1 < \alpha_1 < n$ , $n \in N$ and $\psi(z), h(z) \in C^n([a,b],\mathbb{R})$ be two functions such that $\psi(z)$ is increasing and $\psi^{'}(z) \neq 0$ for all $z \in [a,b]$. The left-sided $\psi$-CFD of a function $h(z)$ of order $\alpha_1$ is defined by*

$${}^c\mathcal{D}_{a+}^{\alpha_1;\psi} h(z) = \mathcal{D}_{a+}^{\alpha_1;\psi}\left[h(z) - \sum_{l=0}^{n-1} \frac{h_\psi^{[l](a)}}{l!}(\psi(z) - \psi(a))^l\right],$$

*where $h_\psi^{[l]}(z) = (\frac{1}{\psi'(z)}\frac{d}{dz})^l h(z)$ and $n = [\alpha_1] + 1$ for $\alpha_1 \notin N$, $n = \alpha_1$ for $\alpha_1 \in N$.*

*Further if $h(z) \in C^n([a,b],\mathbb{R})$ and $\alpha_1 \notin N$, then*

$$\begin{aligned} {}^c\mathcal{D}_{a+}^{\alpha_1;\psi} h(z) &= \mathcal{I}_{a+}^{n-\alpha_1;\psi}\left(\frac{1}{\psi'(z)}\frac{d}{dz}\right)^n h(z) \\ &= \frac{1}{\Gamma(n-\alpha_1)} \int_a^z \psi^{'}(s)(\psi(z) - \psi(s))^{n-\alpha_1-1} h_\psi^{[n]}(s) ds. \end{aligned}$$

*Thus if* $\alpha_1 = n \in N$, *then* ${}^c\mathcal{D}_{a+}^{\alpha_1;\psi} h(z) = h_\psi^{[n]}(z)$.

**Lemma 1** (see [34]). *Let* $\alpha_1 > 0$, *and the following holds:*
*If* $h(z) \in C([a,b],\mathbb{R})$, *then* ${}^c\mathcal{D}_{a+}^{\alpha_1;\psi}\mathcal{I}_{a+}^{\alpha_1;\psi} h(z) = h(z)$, $z \in [a,b]$.
*If* $h(z) \in C^n([a,b],\mathbb{R})$, $n-1 < \alpha_1 < n$, *then*

$$\mathcal{I}_{a+}^{\alpha_1;\psi c}\mathcal{D}_{a+}^{\alpha_1;\psi} h(z) = h(z) - \sum_{l=0}^{n-1} c_l(\psi(z) - \psi(a))^l, \quad z \in [a,b],$$

*where* $c_l = \frac{h_\psi^{[l](a)}}{l!}$.

**Lemma 2** (Schaefer's fixed point theorem [35]). *Let* $\tau : E \to E$ *be a completely continuous operator (i.e a map that restricted to any bounded set in E is compact). Let* $S(\tau) = \{x \in E : x = \nu\tau(x)$, *for some* $0 < \nu < 1\}$. *Then either the set* $S(\tau)$ *is unbounded or* $\tau$ *has at least one fixed point.*

**Definition 4** (see [36]). *A differential equation*

$$\frac{dx}{dz} = f(z,x) \tag{2}$$

*is said to be Ulam-Hyers stable, if there exists a constant* $l_f \in \mathbb{R}^+$, *such that for every* $\epsilon' > 0$ *and any solution* $y(z)$ *of the inequality:*

$$|\frac{dy}{dz} - f(z,y)| \leq \epsilon',$$

*there exists a solution* $x_1(z)$ *of* (2), *such that,*

$$|y(z) - x_1(z)| \leq l_f\epsilon'.$$

## 3. Main Results

For $z_k \in \mathcal{J}_k$, such that $0 = z_0 < z_1 < z_2 < \cdots < z_p = Z$ and $\mathcal{J} = \mathcal{J}_0 \cup \mathcal{J}_1 \cup \cdots \cup \mathcal{J}_p$, where $\mathcal{J}_0 = (0, z_1]$, $\mathcal{J}_1 = (z_1, z_2], \ldots, \mathcal{J}_p = [z_p, z_{p+1}]$ and $\mathcal{J}' = \mathcal{J} - \{z_0, z_1, z_2, \cdots, z_p\}$. We define the space $X' = \{x : \mathcal{J} \to \mathbb{R} \mid x \in PC([\mathcal{J},\mathbb{R}])$, such that the right limits $x(z_k^+)$, $x'(z_k^+)$ and left limits $x(z_k^-)$, $x'(z_k^-)$ exists and $\Delta x(z_k) = x(z_k^+) - x(z_k^-)$, $\Delta x'(z_k) = x'(z_k^+) - x'(z_k^-)$, $k = 1,2,\ldots,p\}$. Then clearly, $X'$ is a Banach space equipped with the norm $\|x(z)\| = \max_{z\in\mathcal{J}} |x(z)|$. Similarly, define the space $Y' = \{y : \mathcal{J} \to \mathbb{R} \mid y \in PC([\mathcal{J},\mathbb{R}])$, right limits $y(z_k^+)$, $y'(z_k^+)$ left limits $y(z_k^-)$, $y'(z_k^-)$ exists and $\Delta y(z_k) = y(z_k^+) - y(z_k^-)$, $\Delta y'(z_k) = y'(z_k^+) - y'(z_k^-)$ $k = 1,2,\ldots,p\}$. Then clearly, $Y'$ is a Banach space equipped with the norm $\|y(z)\| = \max_{z\in\mathcal{J}} |y(z)|$.

**Lemma 3.** *The solution* $x(z) \in PC(J,\mathbb{R})$ *of the impulsive FDEs with slit-strips type integral boundary condition*

$$\begin{cases} {}^c\mathcal{D}_{z_k,z}^{\alpha_1;\psi}[x(z) - f_1(z)] = g_1(z),\ z \in \mathcal{J},\ z \neq z_k, \quad 1 < \alpha_1 \leq 2, \quad k = 0,1,\ldots,p, \\ x(0) = 0,\ x(\eta) = a_1\int_{z_k}^{\delta_{2k}} x(\tau)d\tau + a_2\int_{\delta_{2k+1}}^{z_{k+1}} x(\tau)d\tau, \\ \Delta x(z_k) = x(z_k^+) - x(z_k^-) = I_k(x(z_k)),\ \Delta x'(z_k) = x'(z_k^+) - x'(z_k^-) = J_k(x(z_k)),\ k = 1,2,\ldots,p, \end{cases} \tag{3}$$

*is given by*

$$
\begin{aligned}
x(z) \;=\;& f_1(z) + \frac{1}{\Gamma(\alpha_1)} \int_{z_k}^{z} \psi'(s)(\psi(z)-\psi(s))^{\alpha_1-1} g_1(s)ds + \frac{1}{\Delta}\Bigg[ f_1(\eta) + \frac{1}{\Gamma(\alpha_1)} \int_{z_k}^{\eta} \psi'(s)(\psi(\eta)-\psi(s))^{\alpha_1-1} g_1(s)ds \\
&+ \sum_{i=1}^{p}\Bigg( \frac{1}{\Gamma(\alpha_1)} \int_{z_{i-1}}^{z_i} \psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-1} g_1(s)ds \\
&+ \frac{\psi(\eta)-\psi(z_i)}{\psi'(z_i)\Gamma(\alpha_1-1)} \int_{z_{i-1}}^{z_i} \psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-2} g_1(s)ds \\
&+ I_i x(z_i) + \frac{\psi(\eta)-\psi(z_i)}{\psi'(z_i)} J_i x(z_i)\Bigg) - a_1 \int_{z_k}^{\delta_{2k}} \Bigg( f_1(\tau) + \frac{1}{\Gamma(\alpha_1)} \int_{z_k}^{\eta} \psi'(s)(\psi(\tau)-\psi(s))^{\alpha_1-1} g_1(s)ds \\
&+ \sum_{i=1}^{p}\Bigg( \frac{1}{\Gamma(\alpha_1)} \int_{z_{i-1}}^{z_i} \psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-1} g_1(s)ds \\
&+ \frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)\Gamma(\alpha_1-1)} \int_{z_{i-1}}^{z_i} \psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-2} g_1(s)ds \\
&+ I_i x(z_i) + \frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)} J_i x(z_i)\Bigg)\Bigg) d\tau - a_2 \int_{\delta_{2k+1}}^{z_{k+1}} \Bigg( f_1(\tau) + \frac{1}{\Gamma(\alpha_1)} \int_{z_k}^{\eta} \psi'(s)(\psi(\tau)-\psi(s))^{\alpha_1-1} g_1(s)ds \\
&+ \sum_{i=1}^{p}\Bigg( \frac{1}{\Gamma(\alpha_1)} \int_{z_{i-1}}^{z_i} \psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-1} g_1(s)ds \\
&+ \frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)\Gamma(\alpha_1-1)} \int_{z_{i-1}}^{z_i} \psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-2} g_1(s)ds \\
&+ I_i x(z_i) + \frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)} J_i x(z_i)\Bigg)\Bigg) d\tau\Bigg] (\psi(z)-\psi(z_k) \\
&+ \sum_{i=1}^{p}(\psi(z_i)-\psi(z_{i-1})) + \sum_{i=1}^{p} \frac{1}{\Gamma(\alpha_1)} \int_{z_{i-1}}^{z_1} \psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-1} g_1(s)ds \\
&+ \sum_{i=1}^{p} \frac{\psi(z)-\psi(z_i)}{\psi'(z_i)\Gamma(\alpha_1-1)} \int_{z_{i-1}}^{z_1} \psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-2} g_1(s)ds + \sum_{i=1}^{p} I_i x(z_i) \\
&+ \sum_{i=1}^{p} \frac{\psi(z)-\psi(z_i)}{\psi'(z_i)} J_i x(z_i),\ for\ z \in (z_k, z_{k+1}],\ k = 1,2,\ldots,p,
\end{aligned}
$$

*where*

$$
\begin{aligned}
\Delta \;=\;& a_1 \int_{z_k}^{\delta_{2k}} (\psi(\tau)-\psi(z_k) + \sum_{i=1}^{p}(\psi(z_i)-\psi(z_{i-1})))d\tau \\
&+ a_2 \int_{\delta_{2k+1}}^{z_{k+1}} (\psi(\tau)-\psi(z_k) + \sum_{i=1}^{p}(\psi(z_i)-\psi(z_{i-1})))d\tau \\
&- (\psi(\eta)-\psi(z_k) + \sum_{i=1}^{p}(\psi(z_i)-\psi(z_{i-1}))).
\end{aligned}
$$

**Proof.** Let

$$
{}^{c}\mathcal{D}_{z_k,z}^{\alpha_1;\psi}[x(z)-f_1(z)] = g_1(z).
$$

Then applying Lemma 1 to the differential Equation (3), for any $z \in \mathcal{J}_0$, there exist constants $c_0,\ c_1 \in \mathbb{R}$, such that

$$
x(z) = f_1(z) + \frac{1}{\Gamma(\alpha_1)} \int_{z_0}^{z} \psi'(s)(\psi(z)-\psi(s))^{\alpha_1-1} g_1(s)ds + c_0 + c_1(\psi(z)-\psi(z_0)). \tag{4}
$$

Using initial condition $x(0) = 0$, Equation (4) yields that

$$
c_0 = 0.
$$

Therefore, Equation (4) takes the form

$$x(z) = f_1(z) + \frac{1}{\Gamma(\alpha_1)} \int_{z_0}^{z} \psi^{'}(s)(\psi(z) - \psi(s))^{\alpha_1 - 1} g_1(s) ds + c_1(\psi(z) - \psi(z_0)).$$

Furthermore, we obtain

$$x^{'}(z) = f_1^{'}(z) + \frac{1}{\Gamma(\alpha_1 - 1)} \int_{z_0}^{z} \psi^{'}(s)(\psi(z) - \psi(s))^{\alpha_1 - 2} g_1(s) ds + c_1 \psi^{'}(z).$$

For $z \in (z_1, z_2]$, there are $d_0,\ d_1 \in \mathbb{R}$ such that

$$\begin{cases} x(z) = f_1(z) + \frac{1}{\Gamma(\alpha_1)} \int_{z_1}^{z} \psi^{'}(s)(\psi(z) - \psi(s))^{\alpha_1 - 1} g_1(s) ds + d_0 + d_1(\psi(z) - \psi(z_1)), \\ \\ x^{'}(z) = f_1^{'}(z) + \frac{1}{\Gamma(\alpha_1 - 1)} \int_{z_1}^{z} \psi^{'}(s)(\psi(z) - \psi(s))^{\alpha_1 - 2} g_1(s) ds + d_1 \psi^{'}(z). \end{cases}$$

Hence it follows that

$$\begin{cases} x(z_1^-) = f_1(z) + \frac{1}{\Gamma(\alpha_1)} \int_{z_0}^{z_1} \psi^{'}(s)(\psi(z_1) - \psi(s))^{\alpha_1 - 1} g_1(s) ds + c_1(\psi(z_1) - \psi(z_0)), \\ x(z_1^+) = f_1(z) + d_0, \\ x^{'}(z_1^-) = f_1^{'}(z) + \frac{1}{\Gamma(\alpha_1 - 1)} \int_{z_0}^{z_1} \psi^{'}(s)(\psi(z_1) - \psi(s))^{\alpha_1 - 2} g_1(s) ds + c_1 \psi^{'}(z_1), \\ x^{'}(z_1^+) = f_1^{'}(z) + d_1 \psi^{'}(z_1). \end{cases}$$

Using

$$\begin{cases} \Delta x(z_1) = x(z_1^+) - x(z_1^-) = I_1 x(z_1), \\ \Delta x'(z_1) = x'(z_1^+) - x'(z_1^-) = J_1 x(z_1), \end{cases}$$

we obtain

$$\begin{cases} d_0 = \frac{1}{\Gamma(\alpha_1)} \int_{z_0}^{z_1} \psi^{'}(s)(\psi(z_1) - \psi(s))^{\alpha_1 - 1} g_1(s) ds + c_1(\psi(z_1) - \psi(z_0)) + I_1 x(z_1), \\ d_1 = \frac{1}{\psi^{(}z_1)\Gamma(\alpha_1 - 1)} \int_{z_0}^{z_1} \psi^{'}(s)(\psi(z_1) - \psi(s))^{\alpha_1 - 2} g_1(s) ds + c_1 + \frac{1}{\psi^{'}(z_1)} J_1 x(z_1). \end{cases}$$

Thus

$$\begin{aligned} x(z) \quad &= \quad f_1(z) + \frac{1}{\Gamma(\alpha_1)} \int_{z_1}^{z} \psi^{'}(s)(\psi(z) - \psi(s))^{\alpha_1 - 1} g_1(s) ds \\ &\quad + \frac{1}{\Gamma(\alpha_1)} \int_{z_0}^{z_1} \psi^{'}(s)(\psi(z_1) - \psi(s))^{\alpha_1 - 1} g_1(s) ds \\ &\quad + c_1(\psi(z_1 - \psi(z_0)) + I_1 x(z_1) + c_1(\psi(z) - \psi(z_1)) \\ &\quad + \frac{\psi(z) - \psi(z_1)}{\psi^{'}(z_1)\Gamma(\alpha_1 - 1)} \int_{z_0}^{z_1} \psi^{'}(s)(\psi(z_1) - \psi(s))^{\alpha_1 - 2} g_1(s) ds \\ &\quad + \frac{\psi(z) - \psi(z_1)}{\psi^{'}(z_1)} J_1 x(z_1), \ \ z \in (z_1, z_2]. \end{aligned}$$

Similarly, we have

$$\begin{aligned} x(z) \quad = \quad & f_1(z) + \frac{1}{\Gamma(\alpha_1)} \int_{z_k}^{z} \psi^{'}(s)(\psi(z) - \psi(s))^{\alpha_1 - 1} g_1(s) ds + c_1(\psi(z) - \psi(z_k) \\ & + \sum_{i=1}^{p} (\psi(z_i) - \psi(z_{i-1})) + \sum_{i=1}^{p} \frac{1}{\Gamma(\alpha_1)} \int_{z_{i-1}}^{z_1} \psi^{'}(s)(\psi(z_i) - \psi(s))^{\alpha_1 - 1} g_1(s) ds \\ & + \sum_{i=1}^{p} \frac{\psi(z) - \psi(z_i)}{\psi^{'}(z_i)\Gamma(\alpha_1 - 1)} \int_{z_{i-1}}^{z_1} \psi^{'}(s)(\psi(z_i) - \psi(s))^{\alpha_1 - 2} g_1(s) ds + \sum_{i=1}^{p} I_i x(z_i) \\ & + \sum_{i=1}^{p} \frac{\psi(z) - \psi(z_i)}{\psi^{'}(z_i)} J_i x(z_i), \ for \ z \in (z_k, z_{k+1}], \ k = 1, 2, \ldots, p. \end{aligned} \tag{5}$$

Finally, after applying $x(\eta) = a_1 \int_{z_k}^{\delta_{2k}} x(\tau)d\tau + a_2 \int_{\delta_{2k+1}}^{z_{k+1}} x(\tau)d\tau$, to (5) and calculating the value of $c_1$, we obtain

$$\begin{aligned}
c_1 \quad = \quad & \frac{1}{\Delta}\Bigg[f_1(\eta) + \frac{1}{\Gamma(\alpha_1)}\int_{z_k}^{\eta}\psi'(s)(\psi(\eta)-\psi(s))^{\alpha_1-1}g_1(s)ds \\
& + \sum_{i=1}^{p}\Bigg(\frac{1}{\Gamma(\alpha_1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-1}g_1(s)ds \\
& + \frac{\psi(\eta)-\psi(z_i)}{\psi'(z_i)\Gamma(\alpha_1-1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-2}g_1(s)ds \\
& + I_i x(z_i) + \frac{\psi(\eta)-\psi(z_i)}{\psi'(z_i)}J_i x(z_i)\Bigg) - a_1\int_{z_k}^{\delta_{2k}}\Bigg(f_1(\tau) \\
& + \frac{1}{\Gamma(\alpha_1)}\int_{z_k}^{\eta}\psi'(s)(\psi(\tau)-\psi(s))^{\alpha_1-1}g_1(s)ds \\
& + \sum_{i=1}^{p}\Bigg(\frac{1}{\Gamma(\alpha_1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-1}g_1(s)ds \\
& + \frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)\Gamma(\alpha_1-1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-2}g_1(s)ds \\
& + I_i x(z_i) + \frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}J_i x(z_i)\Bigg)\Bigg)d\tau - a_2\int_{\delta_{2k+1}}^{z_{k+1}}\Bigg(f_1(\tau) \\
& + \frac{1}{\Gamma(\alpha_1)}\int_{z_k}^{\eta}\psi'(s)(\psi(\tau)-\psi(s))^{\alpha_1-1}g_1(s)ds \\
& + \sum_{i=1}^{p}\Bigg(\frac{1}{\Gamma(\alpha_1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-1}g_1(s)ds \\
& + \frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)\Gamma(\alpha_1-1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-2}g_1(s)ds \\
& + I_i x(z_i) + \frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}J_i x(z_i)\Bigg)\Bigg)d\tau\Bigg],
\end{aligned}$$

where

$$\begin{aligned}
\Delta \quad = \quad & a_1\int_{z_k}^{\delta_{2k}}(\psi(\tau)-\psi(z_k)+\sum_{i=1}^{p}(\psi(z_i)-\psi(z_{i-1})))d\tau \\
& + a_2\int_{\delta_{2k+1}}^{z_{k+1}}(\psi(\tau)-\psi(z_k)+\sum_{i=1}^{p}(\psi(z_i)-\psi(z_{i-1})))d\tau \\
& -(\psi(\eta)-\psi(z_k)+\sum_{i=1}^{p}(\psi(z_i)-\psi(z_{i-1}))).
\end{aligned}$$

Put the value of $c_1$ in Equation (5), we get Equation (4). □

## 4. Existence and Uniqueness Results for the Problem (1)

In this section we study the existence of solution for the considered problem. Here we consider some hypothesis, which will be used in our results.

$(G_1)$ For each $z \in J$ and $x_1, x_2 \in \mathbb{X}'$, there exist positive constants $M_{f_1}$, $N_{f_1}$ such that

$$|f_1(z, x_1(z), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x_1(z)) - f_1(z, x_2(z), {}^c\mathcal{D}^{\alpha_1}_{z_k,z}x_2(z))| \leq M_{f_1}|x_1(z)-x_2(z)| + N_{f_1}|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x_1(z) - {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x_2(z)|.$$

There exist positive constants $L_{g_1}$, $K_{g_1}$ such that

$$|g_1(z, x_1(z), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z} x_1(z)) - g_1(z, x_2(z), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z} x_2(z))| \leq L_{g_1}|x_1(z) - x_2(z)| + K_{g_1}|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z} x_1(z) - {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z} x_2(z)|.$$

$(G_2)$ For each $z \in J$ and $y_1$,$y_2 \in \mathbb{Y}'$, there exist positive constants $M_{f_2}$, $N_{f_2}$ such that

$$|f_2(z, y_1(z), {}^c\mathcal{D}^{\beta_1;\psi}_{z_k,z} y_1(z)) - f_2(z, y_2(z), {}^c\mathcal{D}^{\beta_1;\psi}_{z_k,z} y_2(z))| \leq M_{f_2}|y_1(z) - y_2(z)| + N_{f_2}|{}^c\mathcal{D}^{\beta_1;\psi}_{z_k,z} y_1(z) - {}^c\mathcal{D}^{\beta_1;\psi}_{z_k,z} y_2(z)|.$$

There exist positive constants $L_{g_2}$, $K_{g_2}$ such that

$$|g_2(z, y_1(z), {}^c\mathcal{D}^{\beta_1;\psi}_{z_k,z} y_1(z)) - g_2(z, y_2(z), {}^c\mathcal{D}^{\beta_1;\psi}_{z_k,z} y_2(z))| \leq L_{g_2}|y_1(z) - y_2(z)| + K_{g_2}|{}^c\mathcal{D}^{\beta_1;\psi}_{z_k,z} y_1(z) - {}^c\mathcal{D}^{\beta_1;\psi}_{z_k,z} y_2(z)|.$$

$(G_3)$ For every $x_1, x_2 \in \mathbb{X}'$ and there exist constants $A_1, A_2 > 0$ such that

$$|I_k(x_1(z_k)) - I_k(x_2(z_k))| \leq A_1|x_1(z_k) - x_2(z_k)|,$$

$$|J_k(x_1(z_k)) - J_k(x_2(z_k))| \leq A_2|x_1(z_k) - x_2(z_k)|.$$

For every $y_1, y_2 \in \mathbb{Y}'$ and there exist constants $A_3, A_4 > 0$ such that

$$|I^*_k(y_1(z_k)) - I^*_k(y_2(z_k))| \leq A_3|y_1(z_k) - y_2(z_k)|,$$

$$|J^*_k(y_1(z_k)) - J^*_k(y_2(z_k))| \leq A_4|y_1(z_k) - y_2(z_k)|.$$

$(G_4)$ There exist constants $\theta_0$, $\theta_1$ and $\theta_2$ such that

$$|f_1(z, x(z), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z} x(z))| \leq \theta_0(z) + \theta_1(z)|x(z)| + \theta_2(z)|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z} x(z)|,$$

with $\sup_{z\in J} \theta_0(z) = \theta^\star_0$, $\sup_{z\in J} \theta_1(z) = \theta^\star_1$ and $\sup_{z\in J} \theta_2(z) = \theta^\star_2$.
There exist constants $\theta_3$, $\theta_4$ and $\theta_5$ such that

$$|g_1(z, x(z), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z} x(z))| \leq \theta_3(z) + \theta_4(z)|x(z)| + \theta_5(z)|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z} x(z)|,$$

with $\sup_{z\in J} \theta_3(z) = \theta^\star_3$, $\sup_{z\in J} \theta_4(z) = \theta^\star_4$ and $\sup_{z\in J} \theta_5(z) = \theta^\star_5$.

$(G_5)$ For each $x(z) \in \mathbb{R}$ there exist constants $A_1, N_1 > 0$ and $A_2, N_2 > 0$ such that the functions $I_k, J_k : \mathbb{R} \to \mathbb{R}$ are continuous and satisfy the inequalities:

$$|I_k x(z_k)| \leq A_1|x(z)| + N_1, \quad |J_k x(z_k)| \leq A_2|x(z)| + N_2,\ k = 1, 2, \ldots, p.$$

For each $y(z) \in \mathbb{Y}'$ there exist constants $A_3, N_3 > 0$ and $A_4, N_4 > 0$ such that the functions $I^*_k, J^*_k : \mathbb{R} \to \mathbb{R}$ are continuous and satisfy the inequalities:

$$|I^*_k y(z_k)| \leq A_3|y(z)| + N_3, \quad |J^*_k v(z_k)| \leq A_4|y(z)| + N_4,\ k = 1, 2, \ldots, p.$$

Let us define an operator $P : \mathbb{X}' \times \mathbb{Y}' \longrightarrow \mathbb{X}' \times \mathbb{Y}'$ such that

$$P(x, y)(z) = (P_1(x, y)(z), P_2(x, y)(z)),$$

where

$$
\begin{aligned}
P_1(x,y)(z) \quad = \quad & f_1(z, x(z), {}^c\mathcal{D}_{z_k,z}^{\alpha_1;\psi} x(z)) \\
& + \frac{1}{\Gamma(\alpha_1)} \int_{z_k}^{z} \psi'(s)(\psi(z) - \psi(s))^{\alpha_1 - 1} g_1(s, x(s), {}^c\mathcal{D}_{z_k,z}^{\alpha_1;\psi} x(s)) ds \\
& + \frac{1}{\Delta}\Bigg[ f_1(\eta, x(\eta), {}^c\mathcal{D}_{z_k,z}^{\alpha_1;\psi} x(\eta)) + \frac{1}{\Gamma(\alpha_1)} \int_{z_k}^{\eta} \psi'(s)(\psi(\eta) - \psi(s))^{\alpha_1 - 1} g_1(s, x(s), {}^c\mathcal{D}_{z_k,z}^{\alpha_1;\psi} x(s)) ds \\
& + \sum_{i=1}^{p} \Bigg( \frac{1}{\Gamma(\alpha_1)} \int_{z_{i-1}}^{z_i} \psi'(s)(\psi(z_i) - \psi(s))^{\alpha_1 - 1} g_1(s, x(s), {}^c\mathcal{D}_{z_k,z}^{\alpha_1;\psi} x(s)) ds \\
& + \frac{\psi(\eta) - \psi(z_i)}{\psi'(z_i)\Gamma(\alpha_1 - 1)} \int_{z_{i-1}}^{z_i} \psi'(s)(\psi(z_i) - \psi(s))^{\alpha_1 - 2} g_1(s, x(s), {}^c\mathcal{D}_{z_k,z}^{\alpha_1;\psi} x(s)) ds \\
& + I_i x(z_i) + \frac{\psi(\eta) - \psi(z_i)}{\psi'(z_i)} J_i x(z_i) \Bigg) - a_1 \int_{z_k}^{\delta_{2k}} \\
& \Bigg( f_1(\tau, x(\tau), {}^c\mathcal{D}_{z_k,z}^{\alpha_1;\psi} x(\tau)) + \frac{1}{\Gamma(\alpha_1)} \int_{z_k}^{\eta} \psi'(s)(\psi(\tau) - \psi(s))^{\alpha_1 - 1} g_1(s, x(s), {}^c\mathcal{D}_{z_k,z}^{\alpha_1;\psi} x(s)) ds \\
& + \sum_{i=1}^{p} \Bigg( \frac{1}{\Gamma(\alpha_1)} \int_{z_{i-1}}^{z_i} \psi'(s)(\psi(z_i) - \psi(s))^{\alpha_1 - 1} g_1(s, x(s), {}^c\mathcal{D}_{z_k,z}^{\alpha_1;\psi} x(s)) ds \\
& + \frac{\psi(\tau) - \psi(z_i)}{\psi'(z_i)\Gamma(\alpha_1 - 1)} \int_{z_{i-1}}^{z_i} \psi'(s)(\psi(z_i) - \psi(s))^{\alpha_1 - 2} g_1(s, x(s), {}^c\mathcal{D}_{z_k,z}^{\alpha_1;\psi} x(s)) ds \\
& + I_i x(z_i) + \frac{\psi(\tau) - \psi(z_i)}{\psi'(z_i)} J_i x(z_i) \Bigg) \Bigg) d\tau - a_2 \int_{\delta_{2k+1}}^{z_{k+1}} \\
& \Bigg( f_1(\tau, x(\tau), {}^c\mathcal{D}_{z_k,z}^{\alpha_1;\psi} x(\tau)) + \frac{1}{\Gamma(\alpha_1)} \int_{z_k}^{\eta} \psi'(s)(\psi(\tau) - \psi(s))^{\alpha_1 - 1} g_1(s, x(s), {}^c\mathcal{D}_{z_k,z}^{\alpha_1;\psi} x(s)) ds \\
& + \sum_{i=1}^{p} \Bigg( \frac{1}{\Gamma(\alpha_1)} \int_{z_{i-1}}^{z_i} \psi'(s)(\psi(z_i) - \psi(s))^{\alpha_1 - 1} g_1(s, x(s), {}^c\mathcal{D}_{z_k,z}^{\alpha_1;\psi} x(s)) ds \\
& + \frac{\psi(\tau) - \psi(z_i)}{\psi'(z_i)\Gamma(\alpha_1 - 1)} \int_{z_{i-1}}^{z_i} \psi'(s)(\psi(z_i) - \psi(s))^{\alpha_1 - 2} g_1(s, x(s), {}^c\mathcal{D}_{z_k,z}^{\alpha_1;\psi} x(s)) ds \\
& + I_i x(z_i) + \frac{\psi(\tau) - \psi(z_i)}{\psi'(z_i)} J_i x(z_i) \Bigg) \Bigg) d\tau \Bigg] (\psi(z) - \psi(z_k)) \\
& + \sum_{i=1}^{p} (\psi(z_i) - \psi(z_{i-1})) + \sum_{i=1}^{p} \frac{1}{\Gamma(\alpha_1)} \int_{z_{i-1}}^{z_1} \psi'(s)(\psi(z_i) - \psi(s))^{\alpha_1 - 1} g_1(s, x(s), {}^c\mathcal{D}_{z_k,z}^{\alpha_1;\psi} x(s)) ds \\
& + \sum_{i=1}^{p} \frac{\psi(z) - \psi(z_i)}{\psi'(z_i)\Gamma(\alpha_1 - 1)} \int_{z_{i-1}}^{z_1} \psi'(s)(\psi(z_i) - \psi(s))^{\alpha_1 - 2} g_1(s, x(s), {}^c\mathcal{D}_{z_k,z}^{\alpha_1;\psi} x(s)) ds + \sum_{i=1}^{p} I_i x(z_i) \\
& + \sum_{i=1}^{p} \frac{\psi(z) - \psi(z_i)}{\psi'(z_i)} J_i x(z_i),
\end{aligned}
$$

$$\begin{aligned}
P_2(x,y)(z) \;\; = \;\; & f_2(z,x(z),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(z)) + \frac{1}{\Gamma(\beta_1)}\int_{z_k}^{z}\psi'(s)(\psi(z)-\psi(s))^{\beta_1-1}g_2(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(s))ds \\
& + \frac{1}{\Delta}\Bigg[f_2(\eta,x(\eta),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(\eta)) + \frac{1}{\Gamma(\beta_1)}\int_{z_k}^{\eta}\psi'(s)(\psi(\eta)-\psi(s))^{\beta_1-1}g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(s))ds \\
& + \sum_{i=1}^{p}\Bigg(\frac{1}{\Gamma(\beta_1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\beta_1-1}g_2(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(s))ds \\
& + \frac{\psi(\eta)-\psi(z_i)}{\psi'(z_i)\Gamma(\beta_1-1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\beta_1-2}g_2(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(s))ds \\
& + I_i^*y(z_i) + \frac{\psi(\eta)-\psi(z_i)}{\psi'(z_i)}J_i^*y(z_i)\Bigg) \\
& - b_1\int_{z_k}^{\delta_{2k}}\Bigg(f_2(\tau,x(\tau),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(\tau)) + \frac{1}{\Gamma(\beta_1)}\int_{z_k}^{\eta}\psi'(s)(\psi(\tau)-\psi(s))^{\beta_1-1}g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(s))ds \\
& + \sum_{i=1}^{p}\Bigg(\frac{1}{\Gamma(\beta_1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\beta_1-1}g_2(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(s))ds \\
& + \frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)\Gamma(\beta_1-1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\beta_1-2}g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(s))ds \\
& + I_i^*y(z_i) + \frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}J_i^*y(z_i)\Bigg)\Bigg)d\tau \\
& - b_2\int_{\delta_{2k+1}}^{z_{k+1}}\Bigg(f_2(\tau,x(\tau),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(\tau)) + \frac{1}{\Gamma(\beta_1)}\int_{z_k}^{\eta}\psi'(s)(\psi(\tau)-\psi(s))^{\beta_1-1}g_2(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(s))ds \\
& + \sum_{i=1}^{p}\Bigg(\frac{1}{\Gamma(\beta_1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\beta_1-1}g_2(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(s))ds \\
& + \frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)\Gamma(\beta_1-1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\beta_1-2}g_2(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(s))ds \\
& + I_i^*y(z_i) + \frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}J_i^*y(z_i)\Bigg)\Bigg)d\tau\Bigg](\psi(z)-\psi(z_k)) \\
& + \sum_{i=1}^{p}(\psi(z_i)-\psi(z_{i-1})) \\
& + \sum_{i=1}^{p}\frac{1}{\Gamma(\beta_1)}\int_{z_{i-1}}^{z_1}\psi'(s)(\psi(z_i)-\psi(s))^{\beta_1-1}g_2(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(s))ds \\
& + \sum_{i=1}^{p}\frac{\psi(z)-\psi(z_i)}{\psi'(z_i)\Gamma(\beta_1-1)}\int_{z_{i-1}}^{z_1}\psi'(s)(\psi(z_i)-\psi(s))^{\beta_1-2}g_2(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(s))ds + \sum_{i=1}^{p}I_i^*y(z_i) \\
& + \sum_{i=1}^{p}\frac{\psi(z)-\psi(z_i)}{\psi'(z_i)}J_i^*y(z_i).
\end{aligned}$$

Our first result is stated as follows.

**Theorem 1.** *Assume that the conditions $(G_1)-(G_3)$ are satisfied, and*

$$Z^* = \max\{Z_1, Z_2\} < 1, \tag{6}$$

*where $Z_1$ and $Z_2$ are provided in the proof, then the coupled system (1) has a unique solution.*

**Proof.** Let $(x,y),(\bar{x},\bar{y}) \in \mathbb{X}' \times \mathbb{Y}'$ then:

$$
\begin{aligned}
& |P_1(x,y)(z) - P_1(\bar{x},\bar{y})(z)| \\
\leq\ & |f_1(z,x(z),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(z)) - f_1(z,\bar{x}(z),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}(z))| \\
& +\frac{1}{\Gamma(\alpha_1)}\int_{z_k}^{z}\psi'(s)(\psi(z)-\psi(s))^{\alpha_1-1}(|g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(s)) - g_1(s,\bar{x}(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}(s))|)ds \\
& +\frac{1}{|\Delta|}\Bigg[|f_1(\eta,x(\eta),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(\eta)) - f_1(\eta,\bar{x}(\eta),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}(\eta))| \\
& +\frac{1}{\Gamma(\alpha_1)}\int_{z_k}^{\eta}\psi'(s)(\psi(\eta)-\psi(s))^{\alpha_1-1}|g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(s)) - g_1(s,\bar{x}(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}(s))|ds \\
& +\sum_{i=1}^{p}\Bigg(\frac{1}{\Gamma(\alpha_1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-1}|g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(s)) - g_1(s,\bar{x}(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}(s))|ds \\
& +\frac{\psi(\eta)-\psi(z_i)}{\psi'(z_i)\Gamma(\alpha_1-1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-2}|g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(s)) - g_1(s,\bar{x}(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}(s))|ds \\
& +|I_ix(z_i) - I_i\bar{x}(z_i)| + \frac{\psi(\eta)-\psi(z_i)}{\psi'(z_i)}|J_ix(z_i) - J_i\bar{x}(z_i)|\Bigg) \\
& +|a_1|\int_{z_k}^{\delta_{2k}}\Bigg(|f_1(\tau,x(\tau),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(\tau)) - f_1(\tau,\bar{x}(\tau),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}(\tau))| + \frac{1}{\Gamma(\alpha_1)} \\
& \times\int_{z_k}^{\eta}\psi'(s)(\psi(\tau)-\psi(s))^{\alpha_1-1}|g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(s)) - g_1(s,\bar{x}(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}(s))|ds \\
& +\sum_{i=1}^{p}\Bigg(\frac{1}{\Gamma(\alpha_1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-1}|g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(s)) - g_1(s,\bar{x}(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}(s))|ds \\
& +\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)\Gamma(\alpha_1-1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-2}|g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(s)) - g_1(s,\bar{x}(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}(s))|ds \\
& +|I_ix(z_i) - I_i\bar{x}(z_i)| + \frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}|J_ix(z_i) - J_i\bar{x}(z_i)|\Bigg)\Bigg)d\tau \\
& +|a_2|\int_{\delta_{2k+1}}^{z_{k+1}}\Bigg(|f_1(\tau,x(\tau),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(\tau)) - f_1(\tau,\bar{x}(\tau),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}(\tau))| + \frac{1}{\Gamma(\alpha_1)} \\
& \times\int_{z_k}^{\eta}\psi'(s)(\psi(\tau)-\psi(s))^{\alpha_1-1}|g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(s)) - g_1(s,\bar{x}(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}(s))|ds \\
& +\sum_{i=1}^{p}\Bigg(\frac{1}{\Gamma(\alpha_1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-1}|g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(s)) - g_1(s,\bar{x}(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}(s))|ds \\
& +\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)\Gamma(\alpha_1-1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-2}|g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(s)) - g_1(s,\bar{x}(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}(s))|ds \\
& +|I_ix(z_i) - I_i\bar{x}(z_i)| + \frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}|J_ix(z_i) - J_i\bar{x}(z_i)|\Bigg)\Bigg)d\tau\Bigg](\psi(z)-\psi(z_k)) \\
& +\sum_{i=1}^{p}\frac{1}{\Gamma(\alpha_1)}\int_{z_{i-1}}^{z_1}\psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-1}|g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(s)) \\
& -g_1(s,\bar{x}(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}(s))|ds \\
& +\sum_{i=1}^{p}\frac{\psi(z)-\psi(z_i)}{\psi'(z_i)\Gamma(\alpha_1-1)}\int_{z_{i-1}}^{z_1}\psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-2}|g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(s)) \\
& -g_1(s,\bar{x}(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}(s))|ds + \sum_{i=1}^{p}|I_ix(z_i) - I_i\bar{x}(z_i)| \\
& +\sum_{i=1}^{p}\frac{\psi(z)-\psi(z_i)}{\psi'(z_i)}|J_ix(z_i) - J_i\bar{x}(z_i)|.
\end{aligned}
$$

$$
\begin{aligned}
|P_1(x,y)(z) - P_1(\bar{x},\bar{y})(z)| \leq\ & M_{f_1}\|x-\bar{x}\| + N_{f_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x - {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\| \\
& + \frac{(\psi(z)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)}(L_{g_1}\|x-\bar{x}\| + K_{g_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x - {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\|) \\
& + \frac{1}{|\Delta|}\Bigg[M_{f_1}\|x-\bar{x}\| + N_{f_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x - {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\| \\
& + \frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)}(L_{g_1}\|x-\bar{x}\| + K_{g_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x - {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\|) \\
& + \sum_{i=1}^{p}\Bigg(\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)}(L_{g_1}\|x-\bar{x}\| + K_{g_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x - {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\|) + \\
& + \frac{(\psi(\eta)-\psi(z_i))(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\psi'(z_i)\Gamma(\alpha_1)}(L_{g_1}\|x-\bar{x}\| \\
& + K_{g_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x - {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\|) + A_1|x(z_i)-\bar{x}(z_i)| + \frac{\psi(\eta)-\psi(z_i)}{\psi'(z_i)}A_2|x(z_i)-\bar{x}(z_i)|\Bigg) \\
& + |a_1|\int_{z_k}^{\delta_{2k}}\Bigg(M_{f_1}\|x-\bar{x}\| + N_{f_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x - {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\| \\
& + \frac{(\psi(\tau)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)}(L_{g_1}\|x-\bar{x}\| + K_{g_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x - {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\|) \\
& + \sum_{i=1}^{p}\Bigg(\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)}(L_{g_1}\|x-\bar{x}\| + K_{g_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x - {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\|) \\
& + \frac{(\psi(\tau)-\psi(z_i))(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\psi'(z_i)\Gamma(\alpha_1)}(L_{g_1}\|x-\bar{x}\| + K_{g_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x - {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\|) \\
& + A_1|x(z_1)-\bar{x}(z_i)| + \frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}A_2|x(z_i)-\bar{x}(z_i)|\Bigg)\Bigg)d\tau \\
& + |a_2|\int_{\delta_{2k+1}}^{z_{k+1}}\Bigg((M_{f_1}\|x-\bar{x}\| + N_{f_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x - {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\|) \\
& \frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1)}(L_{g_1}\|x-\bar{x}\| + K_{g_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x - {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\|) \\
& + \sum_{i=1}^{p}\Bigg(\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)}(L_{g_1}\|x-\bar{x}\| + K_{g_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x - {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\|) \\
& + \frac{(\psi(\tau)-\psi(z_i))(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\psi'^{(z_i)\Gamma(\alpha_1)}}(L_{g_1}\|x-\bar{x}\| + K_{g_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x - {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\|) \\
& + A_1|x(z_i)-\bar{x}(z_i)| + \frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}A_2\|x-\bar{x}\|\Bigg)\Bigg)d\tau\Bigg]|\psi(z)-\psi(z_k)| \\
& + \sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha}}{\Gamma(\alpha_1+1)}(L_{g_1}\|x-\bar{x}\| \\
& + K_{g_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x - {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\|) + \sum_{i=1}^{p}\frac{\psi(z)-\psi(z_i)(\psi(z_i)-\psi(z_{i-1}))^{\alpha-1}}{\psi'(z_i)\Gamma(\alpha_1)}(L_{g_1}\|x-\bar{x}\| \\
& + K_{g_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x - {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\|) + \sum_{i=1}^{p}A_1|x(z_i)-\bar{x}(z_i)| \\
& + \sum_{i=1}^{p}\frac{\psi(z)-\psi(z_i)}{\psi'(z_i)}A_2|x(z_i)-\bar{x}(z_i)|
\end{aligned}
$$

$$
\begin{aligned}
|P_1(x,y)(z) - P_1(\bar{x},\bar{y})(z)| \leq\ & M_{f_1}\|x-\bar{x}\| + N_{f_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x - {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\| + \\
& + \frac{(\psi(z)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)} L_{g_1}\|x-\bar{x}\| + \frac{(\psi(z)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)} K_{g_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x - {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\|) \\
& + \Bigg( \frac{M_{f_1}}{|\Delta|}\|x-\bar{x}\| + \frac{N_{f_1}}{|\Delta|}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x - {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\| \\
& + \frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|} L_{g_1}\|x-\bar{x}\| + \frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|} K_{g_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x - {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\| \\
& + \sum_{i=1}^{p} \frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|} L_{g_1}\|x-\bar{x}\| + \sum_{i=1}^{p} \frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|} K_{g_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x - {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\| \\
& + \sum_{i=1}^{p} \frac{\psi(\eta)-\psi(z_i)(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\psi'(z_i)\Gamma(\alpha_1)|\Delta|} L_{g_1}\|x-\bar{x}\| \\
& + \sum_{i=1}^{p} \frac{\psi(\eta)-\psi(z_i)(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\psi'(z_i)\Gamma(\alpha_1)|\Delta|} K_{g_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x - {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\| \\
& + \sum_{i=1}^{p} \frac{A_1}{|\Delta|}\|x-\bar{x}\| + \sum_{i=1}^{p} \frac{\psi(\eta)-\psi(z_i)}{\psi'(z_i)|\Delta|} A_2\|x-\bar{x}\| \\
& + \frac{|a_1|}{|\Delta|}(\delta_{2k}-z_k)M_{f_1}\|x-\bar{x}\| + \frac{|a_1|}{|\Delta|}(\delta_{2k}-z_k)N_{f_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x - {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\| \\
& + \frac{(\psi(\tau)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|} |a_1|(\delta_{2k}-z_k)L_{g_1}\|x-\bar{x}\| \\
& + \frac{(\psi(\tau)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|} |a_1|(\delta_{2k}-z_k)K_{g_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x - {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\| \\
& + \sum_{i=1}^{p} \frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|} |a_1|(\delta_{2k}-z_k)L_{g_1}\|x-\bar{x}\| \\
& + \sum_{i=1}^{p} \frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|} |a_1|(\delta_{2k}-z_k)K_{g_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x - {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\| \\
& + \frac{(\psi(\tau)-\psi(z_i))(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\psi'(z_i)\Gamma(\alpha_1)|\Delta|} |a_1|(\delta_{2k}-z_k)L_{g_1}\|x-\bar{x}\| \\
& + \frac{(\psi(\tau)-\psi(z_i))(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\psi'(z_i)\Gamma(\alpha_1)|\Delta|} |a_1|(\delta_{2k}-z_k)K_{g_1} \\
& \times \|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x - {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\| \\
& + p\frac{A_1}{|\Delta|}|a_1|(\delta_{2k}-z_k)\|x-\bar{x}\| \\
& + \sum_{i=1}^{p} \frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)|\Delta|} |a_1|(\delta_{2k}-z_k)A_2\|x-\bar{x}\| \\
& + \frac{|a_2|}{|\Delta|}(z_{k+1}-\delta_{2k+1})M_{f_1}\|x-\bar{x}\| \\
& + \frac{|a_2|(z_{k+1}-\delta_{2k+1})}{|\Delta|} N_{f_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x - {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\| \\
& + \frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1)|\Delta|} |a_2|(z_{k+1}-\delta_{2k+1})L_{g_1}\|x-\bar{x}\|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1)|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1})K_{g_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x-{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\| \\
&+\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1})L_{g_1}\|x-\bar{x}\| \\
&+\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1})K_{g_1} \\
&\times\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x-{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\| \\
&+p\frac{A_1}{|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1})\|x-\bar{x}\| \\
&+\left.\sum_{i=1}^{p}\frac{A_2}{|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1})\|x-\bar{x}\|\right)\bigg(\psi(z)-\psi(z_k)\bigg) \\
&+\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha}}{\Gamma(\alpha_1+1)}L_{g_1}\|x-\bar{x}\| \\
&+\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)}K_{g_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x-{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\| \\
&+\sum_{i=1}^{p}\frac{\psi(z)-\psi(z_i)(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\psi'(z_i)\Gamma(\alpha_1)}L_{g_1}\|x-\bar{x}\| \\
&+\sum_{i=1}^{p}\frac{\psi(z)-\psi(z_i)(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\psi'(z_i)\Gamma(\alpha_1)}K_{g_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x-{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\| \\
&+pA_1\|x-\bar{x}\|+A_2\sum_{i=1}^{p}\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}\|x-\bar{x}\|.
\end{aligned}
$$

$$
\begin{aligned}
|P_1(x,y)(z)-P_1(\bar{x},\bar{y})(z)| \leq\ & \Bigg(M_{f_1}+\frac{(\psi(z)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)}L_{g_1} \\
&+\Bigg(\frac{M_{f_1}}{|\Delta|}+\frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|}L_{g_1} \\
&+\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|}L_{g_1}+ \\
&+\sum_{i=1}^{p}\frac{\psi(\eta)-\psi(z_i)(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\psi'(z_i)\Gamma(\alpha_1)|\Delta|}L_{g_1} \\
&+p\frac{A_1}{|\Delta|}+\sum_{i=1}^{p}\frac{\psi(\eta)-\psi(z_i)}{\psi'(z_i)|\Delta|}A_2 \\
&+\frac{|a_1|}{|\Delta|}(\delta_{2k}-z_k)M_{f_1} \\
&+\frac{(\psi(\tau)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|}|a_1|(\delta_{2k}-z_k)L_{g_1} \\
&+\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|}|a_1|(\delta_{2k}-z_k)L_{g_1} \\
&+\frac{(\psi(\tau)-\psi(z_i))(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\psi'(z_i)\Gamma(\alpha_1)|\Delta|}|a_1|(\delta_{2k}-z_k)L_{g_1}+p\frac{A_1}{|\Delta|}|a_1|(\delta_{2k}-z_k) \\
&+\sum_{i=1}^{p}\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)|\Delta|}|a_1|(\delta_{2k}-z_k)A_2+\frac{|a_2|}{|\Delta|}(z_{k+1}-\delta_{2k+1})M_{f_1}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1)|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1})L_{g_1}\\
&+\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1})L_{g_1}\\
&+p\frac{A_1}{|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1})\\
&+\sum_{i=1}^{p}\frac{A_2}{|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1})\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}\Bigg)\Bigg(\psi(z)-\psi(z_k)\Bigg)\\
&+\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha}}{\Gamma(\alpha_1+1)}L_{g_1}\\
&+\sum_{i=1}^{p}\frac{\psi(z)-\psi(z_i)(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\psi'(z_i)\Gamma(\alpha_1)}L_{g_1}\\
&+pA_1+A_2\sum_{i=1}^{p}\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}\Bigg)\Big\|x-\bar{x}\Big\|\\
&+\Bigg(N_{f_1}+\frac{(\psi(z)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)}K_{g_1}\\
&+\Bigg(\frac{N_{f_1}}{|\Delta|}+\frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|}K_{g_1}+\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|}K_{g_1}\\
&+\sum_{i=1}^{p}\frac{\psi(\eta)-\psi(z_i)(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\psi'(z_i)\Gamma(\alpha_1)|\Delta|}K_{g_1}+\frac{|a_1|}{|\Delta|}(\delta_{2k}-z_k)N_{f_1}\\
&+\frac{(\psi(\tau)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|}|a_1|(\delta_{2k}-z_k)K_{g_1}\\
&+\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|}|a_1|(\delta_{2k}-z_k)K_{g_1}\\
&+\frac{(\psi(\tau)-\psi(z_i))(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\psi'(z_i)\Gamma(\alpha_1)|\Delta|}|a_1|(\delta_{2k}-z_k)K_{g_1}+\frac{|a_2|(z_{k+1}-\delta_{2k+1})}{|\Delta|}N_{f_1}\\
&+\frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1)|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1})K_{g_1}\\
&+\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1})K_{g_1}\Bigg)\Bigg(\psi(z)-\psi(z_k)\Bigg)\\
&+\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)}K_{g_1}\\
&+\sum_{i=1}^{p}\frac{\psi(z)-\psi(z_i)(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\psi'(z_i)\Gamma(\alpha_1)}K_{g_1}\Bigg)\Big\|{}^{c}\mathcal{D}_{z_k,z}^{\alpha_1;\psi}x-{}^{c}\mathcal{D}_{z_k,z}^{\alpha_1;\psi}\bar{x}\Big\|.
\end{aligned}
$$

Let

$$
\begin{aligned}
\Omega_1 \quad = \quad & M_{f_1} + \frac{(\psi(z) - \psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1 + 1)} L_{g_1} \\
& + \Bigg( \frac{M_{f_1}}{|\Delta|} + \frac{(\psi(\eta) - \psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1 + 1)|\Delta|} L_{g_1} \\
& + \sum_{i=1}^{p} \frac{(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1 + 1)|\Delta|} L_{g_1} + \\
& + \sum_{i=1}^{p} \frac{\psi(\eta) - \psi(z_i)(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1 - 1}}{\psi'(z_i)\Gamma(\alpha_1)|\Delta|} L_{g_1} \\
& + p\frac{A_1}{|\Delta|} + \sum_{i=1}^{p} \frac{\psi(\eta) - \psi(z_i)}{\psi'(z_i)|\Delta|} A_2 \\
& + \frac{|a_1|}{|\Delta|}(\delta_{2k} - z_k) M_{f_1} \\
& + \frac{(\psi(\tau) - \psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1 + 1)|\Delta|} |a_1|(\delta_{2k} - z_k) L_{g_1} \\
& + \sum_{i=1}^{p} \frac{(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1 + 1)|\Delta|} |a_1|(\delta_{2k} - z_k) L_{g_1} \\
& + \frac{(\psi(\tau) - \psi(z_i))(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1 - 1}}{\psi'(z_i)\Gamma(\alpha_1)|\Delta|} |a_1|(\delta_{2k} - z_k) L_{g_1} + p\frac{A_1}{|\Delta|}|a_1|(\delta_{2k} - z_k) \\
& + A_2 \sum_{i=1}^{p} \frac{\psi(\tau) - \psi(z_i)}{\psi'(z_i)|\Delta|} |a_1|(\delta_{2k} - z_k) + \frac{|a_2|}{|\Delta|}(z_{k+1} - \delta_{2k+1}) M_{f_1} \\
& + \frac{(\psi(\eta) - \psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1)|\Delta|} |a_2|(z_{k+1} - \delta_{2k+1}) L_{g_1} \\
& + \sum_{i=1}^{p} \frac{(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1 + 1)|\Delta|} |a_2|(z_{k+1} - \delta_{2k+1}) L_{g_1} \\
& + p\frac{A_1}{|\Delta|}|a_2|(z_{k+1} - \delta_{2k+1}) \\
& + \frac{A_2}{|\Delta|}|a_2|(z_{k+1} - \delta_{2k+1}) \sum_{i=1}^{p} \frac{\psi(\tau) - \psi(z_i)}{\psi'(z_i)} \Bigg) \Bigg( \psi(z) - \psi(z_k) \Bigg) \\
& + \sum_{i=1}^{p} \frac{(\psi(z_i) - \psi(z_{i-1}))^{\alpha}}{\Gamma(\alpha_1 + 1)} L_{g_1} \\
& + \sum_{i=1}^{p} \frac{\psi(z) - \psi(z_i)(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1 - 1}}{\psi'(z_i)\Gamma(\alpha_1)} L_{g_1} \\
& + pA_1 + A_2 \sum_{i=1}^{p} \frac{\psi(\tau) - \psi(z_i)}{\psi'(z_i)} \Bigg)
\end{aligned}
$$

and

$$\begin{aligned}
\Omega_2 \quad = \quad & N_{f_1} + \frac{(\psi(z)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)} K_{g_1} \\
& + \Bigg( \frac{N_{f_1}}{|\Delta|} + \frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|} K_{g_1} + \sum_{i=1}^{p} \frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|} K_{g_1} \\
& + \sum_{i=1}^{p} \frac{\psi(\eta)-\psi(z_i)(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\psi'(z_i)\Gamma(\alpha_1)|\Delta|} K_{g_1} + \frac{|a_1|}{|\Delta|}(\delta_{2k}-z_k) N_{f_1} \\
& + \frac{(\psi(\tau)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|} |a_1|(\delta_{2k}-z_k) K_{g_1} \\
& + \sum_{i=1}^{p} \frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|} |a_1|(\delta_{2k}-z_k) K_{g_1} \\
& + \frac{(\psi(\tau)-\psi(z_i))(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\psi'(z_i)\Gamma(\alpha_1)|\Delta|} |a_1|(\delta_{2k}-z_k) K_{g_1} + \frac{|a_2|(z_{k+1}-\delta_{2k+1})}{|\Delta|} N_{f_1} \\
& + \frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1)|\Delta|} |a_2|(z_{k+1}-\delta_{2k+1}) K_{g_1} \\
& + \sum_{i=1}^{p} \frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|} |a_2|(z_{k+1}-\delta_{2k+1}) K_{g_1} \Bigg) \Bigg( \psi(z)-\psi(z_k) \Bigg) \\
& + \sum_{i=1}^{p} \frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)} K_{g_1} \\
& + \sum_{i=1}^{p} \frac{\psi(z)-\psi(z_i)(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\psi'(z_i)\Gamma(\alpha_1)} K_{g_1}.
\end{aligned}$$

Thus we have,

$$\|P_1(x,y)-P_1(\bar{x},\bar{y})\| \quad \leq \quad \Omega_1\|x-\bar{x}\| + \Omega_2\|{}^{c}\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x - {}^{c}\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\|.$$

But

$$\|{}^{c}\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x - {}^{c}\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\| \quad \leq \quad \frac{\frac{(\psi(z)-\psi(z_k))^{1-\alpha_1}}{\Gamma(\alpha_1)}(\frac{1}{\psi'(z)}\frac{d}{dz})M_{f_1} + L_{g_1}}{1-(\frac{(\psi(z)-\psi(z_k))^{1-\alpha_1}}{\Gamma(\alpha_1)}(\frac{1}{\psi'(z)}\frac{d}{dz})N_{f_1} + K_{g_1})}\|x-\bar{x}\|.$$

Then,

$$\|P_1(x,y)-P_1(\bar{x},\bar{y})\| \quad \leq \quad \Bigg(\Omega_1 + \Omega_2\frac{\frac{(\psi(z)-\psi(z_k))^{1-\alpha_1}}{\Gamma(\alpha_1)}(\frac{1}{\psi'(z)}\frac{d}{dz})M_{f_1} + L_{g_1}}{1-(\frac{(\psi(z)-\psi(z_k))^{1-\alpha_1}}{\Gamma(\alpha_1)}(\frac{1}{\psi'(z)}\frac{d}{dz})N_{f_1} + K_{g_1})}\Bigg)\|x-\bar{x}\|.$$

Provided that,

$$\frac{(\psi(z)-\psi(z_k))^{1-\alpha_1}}{\Gamma(\alpha_1)}(\frac{1}{\psi'(z)}\frac{d}{dz})N_{f_1} + K_{g_1} < 1.$$

Similarly,

$$\|P_2(x,y)-P_2(\bar{x},\bar{y})\| \quad \leq \quad \Bigg(\Omega_3 + \Omega_4\frac{\frac{(\psi(z)-\psi(z_k))^{1-\beta_1}}{\Gamma(\beta_1)}(\frac{1}{\psi'(z)}\frac{d}{dz})M_{f_2} + L_{g_2}}{1-(\frac{(\psi(z)-\psi(z_k))^{1-\beta_1}}{\Gamma(\beta_1)}(\frac{1}{\psi'(z)}\frac{d}{dz})N_{f_2} + K_{g_2})}\Bigg)\|y-\bar{y}\|.$$

Provided that,

$$\frac{(\psi(z)-\psi(z_k))^{1-\beta_1}}{\Gamma(\beta_1)}\left(\frac{1}{\psi'(z)}\frac{d}{dz}\right)N_{f_2}+K_{g_2}<1.$$

Where

$$\begin{aligned}
\Omega_3 \;=\;& M_{f_2}+\frac{(\psi(z)-\psi(z_k))^{\beta_1}}{\Gamma(\beta_2+1)}L_{g_2} \\
&+\Bigg(\frac{M_{f_2}}{|\Delta|}+\frac{(\psi(\eta)-\psi(z_k))^{\beta_2}}{\Gamma(\beta_2+1)|\Delta|}L_{g_2} \\
&+\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\beta_2}}{\Gamma(\beta_2+1)|\Delta|}L_{g_2}+ \\
&+\sum_{i=1}^{p}\frac{\psi(\eta)-\psi(z_i)(\psi(z_i)-\psi(z_{i-1}))^{\beta_2-1}}{\psi'(z_i)\Gamma(\beta_2)|\Delta|}L_{g_2} \\
&+p\frac{A_1}{|\Delta|}+\sum_{i=1}^{p}\frac{\psi(\eta)-\psi(z_i)}{\psi'(z_i)|\Delta|}A_2 \\
&+\frac{|a_2|}{|\Delta|}(\delta_{2k}-z_k)M_{f_2} \\
&+\frac{(\psi(\tau)-\psi(z_k))^{\beta_2}}{\Gamma(\beta_2+1)|\Delta|}|a_2|(\delta_{2k}-z_k)L_{g_2} \\
&+\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\beta_2}}{\Gamma(\beta_2+1)|\Delta|}|a_2|(\delta_{2k}-z_k)L_{g_2} \\
&+\frac{(\psi(\tau)-\psi(z_i))(\psi(z_i)-\psi(z_{i-1}))^{\beta_2-1}}{\psi'(z_i)\Gamma(\beta_2)|\Delta|}|a_2|(\delta_{2k}-z_k)L_{g_2}+\frac{A_1}{|\Delta|}|a_2|(\delta_{2k}-z_k) \\
&+\sum_{i=1}\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)|\Delta|}|a_2|(\delta_{2k}-z_k)A_2+\frac{|a_2|}{|\Delta|}(z_{k+1}-\delta_{2k+1})M_{f_2} \\
&+\frac{(\psi(\eta)-\psi(z_k))^{\beta_2}}{\Gamma(\beta_2)|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1})L_{g_2} \\
&+\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\beta_2}}{\Gamma(\beta_2+1)|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1})L_{g_2} \\
&+\frac{A_1}{|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1}) \\
&+\frac{A_2}{|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1})\sum_{i=1}^{p}\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}\Bigg)\Bigg(\psi(z)-\psi(z_k)\Bigg) \\
&+\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha}}{\Gamma(\beta_2+1)}L_{g_2} \\
&+\sum_{i=1}^{p}\frac{\psi(z)-\psi(z_i)(\psi(z_i)-\psi(z_{i-1}))^{\beta_2-1}}{\psi'(z_i)\Gamma(\beta_2)}L_{g_2} \\
&+pA_1+A_2\sum_{i=1}^{p}\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}\Bigg)
\end{aligned}$$

$$
\begin{aligned}
\Omega_4 \;=\;& N_{f_2}+\frac{(\psi(z)-\psi(z_k))^{\beta_2}}{\Gamma(\beta_2+1)}K_{g_2} \\
&+\Bigg(\frac{N_{f_2}}{|\Delta|}+\frac{(\psi(\eta)-\psi(z_k))^{\beta_2}}{\Gamma(\beta_2+1)|\Delta|}K_{g_2}+\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\beta_2}}{\Gamma(\beta_2+1)|\Delta|}K_{g_2} \\
&+\sum_{i=1}^{p}\frac{\psi(\eta)-\psi(z_i)(\psi(z_i)-\psi(z_{i-1}))^{\beta_2-1}}{\psi'(z_i)\Gamma(\beta_2)|\Delta|}K_{g_2}+\frac{|a_2|}{|\Delta|}(\delta_{2k}-z_k)N_{f_2} \\
&+\frac{(\psi(\tau)-\psi(z_k))^{\beta_2}}{\Gamma(\beta_2+1)|\Delta|}|a_2|(\delta_{2k}-z_k)K_{g_2} \\
&+\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\beta_2}}{\Gamma(\beta_2+1)|\Delta|}|a_2|(\delta_{2k}-z_k)K_{g_2} \\
&+\frac{(\psi(\tau)-\psi(z_i))(\psi(z_i)-\psi(z_{i-1}))^{\beta_2-1}}{\psi'(z_i)\Gamma(\beta_2)|\Delta|}|a_2|(\delta_{2k}-z_k)K_{g_2}+\frac{|a_2|(z_{k+1}-\delta_{2k+1})}{|\Delta|}N_{f_2} \\
&+\frac{(\psi(\eta)-\psi(z_k))^{\beta_2}}{\Gamma(\beta_2)|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1})K_{g_2} \\
&+\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\beta_2}}{\Gamma(\beta_2+1)|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1})K_{g_2}\Bigg)\Big(\psi(z)-\psi(z_k)\Big) \\
&+\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\beta_2}}{\Gamma(\beta_2+1)}K_{g_2} \\
&+\sum_{i=1}^{p}\frac{\psi(z)-\psi(z_i)(\psi(z_i)-\psi(z_{i-1}))^{\beta_2-1}}{\psi'(z_i)\Gamma(\beta_2)}K_{g_2}.
\end{aligned}
$$

Let

$$
Z_1 \;=\; \Omega_1+\Omega_2\frac{\frac{(\psi(z)-\psi(z_k))^{1-\alpha_1}}{\Gamma(\alpha_1)}(\frac{1}{\psi'(z)}\frac{d}{dz})M_{f_1}+L_{g_1}}{1-(\frac{(\psi(z)-\psi(z_k))^{1-\alpha_1}}{\Gamma(\alpha_1)}(\frac{1}{\psi'(z)}\frac{d}{dz})N_{f_1}+K_{g_1})}.
$$

Then we obtain

$$
\|P_1(x,y)-P_1(\bar{x},\bar{y})\| \le Z_1\|x-\bar{x}\|.
$$

Similarly,

$$
\|P_2(x,y)-P_2(\bar{x},\bar{y})\| \le Z_2\|y-\bar{y}\|,
$$

where

$$
Z_2 \;=\; \Omega_3+\Omega_4\frac{\frac{(\psi(z)-\psi(z_k))^{1-\beta_1}}{\Gamma(\beta_1)}(\frac{1}{\psi'(z)}\frac{d}{dz})M_{f_2}+L_{g_2}}{1-(\frac{(\psi(z)-\psi(z_k))^{1-\beta_1}}{\Gamma(\beta_1)}(\frac{1}{\psi'(z)}\frac{d}{dz})N_{f_2}+K_{g_2})}.
$$

As it is assumed that

$$
\max\{Z_1,Z_2\}=Z^*<1. \tag{7}
$$

So we have

$$
\|P(x,y)-P(\bar{x},\bar{y})\| \le Z^*(\|x-\bar{x}\|+\|y-\bar{y}\|).
$$

Then from above inequality we can say that $P$ is a contraction mapping and by Banach contraction principle $P$ has a unique fixed point. □

**Theorem 2.** *Assume that the conditions* $(G_1)$–$(G_5)$ *are satisfied, then the coupled system* (1) *has at least one solution.*

**Proof.** To prove that the coupled system (1) has at least one solution, we use the Schaefer's fixed point theorem. As $f_1, g_1, I, J$ are continuous functions, so $P_1$ is continuous. Also from the continuity of $f_2, g_2$ and $I^*, J^*$ the operator $P_2$ is continuous. This shows that $P$ is continuous.

Consider a set:

$$Q_r = \{(x,y) \in \mathbb{X}^{'} \times \mathbb{Y}^{'} : \|(x,y)\| \leq r\}.$$

For any $z \in [0, Z]$, we have

$$\begin{aligned}
|P_1(x,y)(z)| \leq\ & |f_1(z, x(z), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z} x(z))| + \frac{1}{\Gamma(\alpha_1)} \int_{z_k}^{z} \psi^{'}(s)(\psi(z) - \psi(s))^{\alpha_1 - 1} |g_1(s, x(s), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z} x(s))| ds \\
& + \frac{1}{|\Delta|}\Bigg[ |f_1(\eta, x(\eta), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z} x(\eta))| + \frac{1}{\Gamma(\alpha_1)} \int_{z_k}^{\eta} \psi^{'}(s)(\psi(\eta) - \psi(s))^{\alpha_1 - 1} |g_1(s, x(s), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z} x(s))| ds \\
& + \sum_{i=1}^{p} \Bigg( \frac{1}{\Gamma(\alpha_1)} \int_{z_{i-1}}^{z_i} \psi^{'}(s)(\psi(z_i) - \psi(s))^{\alpha_1 - 1} |g_1(s, x(s), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z} x(s))| ds \\
& + \frac{\psi(\eta) - \psi(z_i)}{\psi^{'}(z_i)\Gamma(\alpha_1 - 1)} \int_{z_{i-1}}^{z_i} \psi^{'}(s)(\psi(z_i) - \psi(s))^{\alpha_1 - 2} |g_1(s, x(s), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z} x(s))| ds \\
& + |I_i x(z_i)| + \frac{\psi(\eta) - \psi(z_i)}{\psi^{'}(z_i)} |J_i x(z_i)| \Bigg) \\
& + |a_1| \int_{z_k}^{\delta_{2k}} \Bigg( |f_1(\tau, x(\tau), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,\tau} x(\tau))| + \frac{1}{\Gamma(\alpha_1)} \int_{z_k}^{\eta} \psi^{'}(s)(\psi(\tau) - \psi(s))^{\alpha_1 - 1} |g_1(s, x(s), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z} x(s))| ds \\
& + \sum_{i=1}^{p} \Bigg( \frac{1}{\Gamma(\alpha_1)} \int_{z_{i-1}}^{z_i} \psi^{'}(s)(\psi(z_i) - \psi(s))^{\alpha_1 - 1} |g_1(s, x(s), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z} x(s))| ds \\
& + \frac{\psi(\tau) - \psi(z_i)}{\psi^{'}(z_i)\Gamma(\alpha_1 - 1)} \int_{z_{i-1}}^{z_i} \psi^{'}(s)(\psi(z_i) - \psi(s))^{\alpha_1 - 2} |g_1(s, x(s), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z} x(s))| ds \\
& + |I_i x(z_i)| + \frac{\psi(\tau) - \psi(z_i)}{\psi^{'}(z_i)} |J_i x(z_i)| \Bigg) \Bigg) d\tau \\
& + |a_2| \int_{\delta_{2k+1}}^{z_{k+1}} \Bigg( |f_1(\tau, x(\tau), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,\tau} x(\tau))| + \frac{1}{\Gamma(\alpha_1)} \int_{z_k}^{\eta} \psi^{'}(s)(\psi(\tau) - \psi(s))^{\alpha_1 - 1} |g_1(s, x(s), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z} x(s))| ds \\
& + \sum_{i=1}^{p} \Bigg( \frac{1}{\Gamma(\alpha_1)} \int_{z_{i-1}}^{z_i} \psi^{'}(s)(\psi(z_i) - \psi(s))^{\alpha_1 - 1} |g_1(s, x(s), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z} x(s))| ds \\
& + \frac{\psi(\tau) - \psi(z_i)}{\psi^{'}(z_i)\Gamma(\alpha_1 - 1)} \int_{z_{i-1}}^{z_i} \psi^{'}(s)(\psi(z_i) - \psi(s))^{\alpha_1 - 2} |g_1(s, x(s), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z} x(s))| ds \\
& + |I_i x(z_i|) + \frac{\psi(\tau) - \psi(z_i)}{\psi^{'}(z_i)} |J_i x(z_i|) \Bigg) \Bigg) d\tau \Bigg] (\psi(z) - \psi(z_k)) \\
& + \sum_{i=1}^{p} (\psi(z_i) - \psi(z_{i-1})) + \sum_{i=1}^{p} \Bigg( \frac{1}{\Gamma(\alpha_1)} \int_{z_{i-1}}^{z_1} \psi^{'}(s)(\psi(z_i) - \psi(s))^{\alpha_1 - 1} |g_1(s, x(s), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z} x(s))| ds \\
& + \frac{\psi(z) - \psi(z_i)}{\psi^{'}(z_i)\Gamma(\alpha_1 - 1)} \int_{z_{i-1}}^{z_1} \psi^{'}(s)(\psi(z_i) - \psi(s))^{\alpha_1 - 2} g_1(s) ds + |I_i x(z_i)| \\
& + \frac{\psi(z) - \psi(z_i)}{\psi^{'}(z_i)} |J_i x(z_i)| \Bigg)
\end{aligned}$$

$$
\begin{aligned}
\leq\ & \theta_0(z)+\theta_1(z)|x(z)|+\theta_2(z)|{}^c\mathcal{D}^{\alpha_1,\psi}_{z_k,z}x(z)|+\frac{(\psi(z)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3(z)+\theta_4(z)|x(z)|+\theta_5(z)|{}^c\mathcal{D}^{\alpha_1,\psi}_{z_k,z}x(z)|)\\
&+\frac{1}{|\Delta|}\bigg[\theta_0(\eta)+\theta_1(\eta)|x(\eta)|+\theta_2(\eta)|{}^c\mathcal{D}^{\alpha_1,\psi}_{z_k,\eta}x(\eta)|\\
&+\frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3(\eta)+\theta_4(\eta)|x(\eta)|+\theta_5(\eta)|{}^c\mathcal{D}^{\alpha_1,\psi}_{z_k,\eta}x(\eta)|)\\
&+\sum_{i=1}^{p}\bigg(\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3(z_i)+\theta_4(z_i)|x(z_i)|+\theta_5(z_i)|{}^c\mathcal{D}^{\alpha_1,\psi}_{z_{i-1},z_i}x(z_i)|)\\
&+\frac{\psi(\eta)-\psi(z_i)}{\psi'(z_i)}\times\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\Gamma(\alpha_1)}(\theta_3(z_i)+\theta_4(z_i)|x(z_i)|+\theta_5(z_i)|{}^c\mathcal{D}^{\alpha_1,\psi}_{z_{i-1},z_i}x(z_i)|)\\
&+A_1|x(z)|+N_1+\frac{\psi(\eta)-\psi(z_i)}{\psi'(z_i)}(A_2|x(z)|+N_2)\bigg)\\
&+|a_1|\int_{z_k}^{\delta_{2k}}\bigg(\theta_0(\tau)+\theta_1(\tau)|x(\eta)|+\theta_2(\tau)|{}^c\mathcal{D}^{\alpha_1,\psi}_{z_k,\tau}x(\tau)|\\
&+\frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3(\eta)+\theta_4(\eta)|x(\eta)|+\theta_5(\eta)|{}^c\mathcal{D}^{\alpha_1,\psi}_{z_k,\eta}x(\eta)|)\\
&+\sum_{i=1}^{p}\bigg(\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3(z_i)+\theta_4(z_i)|x(z_i)|+\theta_5(z_i)|{}^c\mathcal{D}^{\alpha_1,\psi}_{z_{i-1},z_i}x(z_i)|)\\
&+\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}\times\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\Gamma(\alpha_1)}(\theta_3(z_i)+\theta_4(z_i)|x(z_i)|+\theta_5(z_i)|{}^c\mathcal{D}^{\alpha_1,\psi}_{z_{i-1},z_i}x(z_i)|)\\
&+A_1|x(z)|+N_1+\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}(A_2|x(z)|+N_2)\bigg)\bigg)d\tau\\
&+|a_2|\int_{\delta_{2k+1}}^{z_{k+1}}\bigg(\theta_0(\tau)+\theta_1(\tau)|x(\eta)|+\theta_2(\tau)|{}^c\mathcal{D}^{\alpha_1,\psi}_{z_k,\tau}x(\tau)|\\
&+\frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3(\eta)+\theta_4(\eta)|x(\eta)|+\theta_5(\eta)|{}^c\mathcal{D}^{\alpha_1,\psi}_{z_k,\eta}x(\eta)|)\\
&+\sum_{i=1}^{p}\bigg(\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3(z_i)+\theta_4(z_i)|x(z_i)|+\theta_5(z_i)|{}^c\mathcal{D}^{\alpha_1,\psi}_{z_{i-1},z_i}x(z_i)|)\\
&+\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}\times\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\Gamma(\alpha_1)}(\theta_3(z_i)+\theta_4(z_i)|x(z_i)|+\theta_5(z_i)|{}^c\mathcal{D}^{\alpha_1,\psi}_{z_{i-1},z_i}x(z_i)|)\\
&+A_1|x(z)|+N_1+\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}(A_2|x(z)|+N_2)\bigg)\bigg)d\tau\bigg](\psi(z)-\psi(z_k))\\
&+\sum_{i=1}^{p}(\psi(z_i)-\psi(z_{i-1}))+\sum_{i=1}^{p}\bigg(\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3(z_i)+\theta_4(z_i)|x(z_i)|+\theta_5(z_i)|{}^c\mathcal{D}^{\alpha_1,\psi}_{z_{i-1},z_i}x(z_i)|)\\
&+\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}\times\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\Gamma(\alpha_1)}(\theta_3(z_i)+\theta_4(z_i)|x(z_i)|+\theta_5(z_i)|{}^c\mathcal{D}^{\alpha_1,\psi}_{z_{i-1},z_i}x(z_i)|)\\
&+A_1|x(z)|+N_1+\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}(A_2|x(z)|+N_2)\bigg)
\end{aligned}
$$

From $(G_4)$ and (3), we have

$$
\begin{aligned}
|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(z)| \quad\leq\quad & \frac{\theta_3^*+\frac{(\psi(z)-\psi(z_k))^{1-\alpha_1}}{\Gamma(\alpha_1)}\theta_0^*}{1-(\theta_5^*+\frac{(\psi(z)-\psi(z_k))^{1-\alpha_1}}{\Gamma(\alpha_1)}\theta_2^*)}\\
&+\frac{(\theta_4^*+\frac{(\psi(z)-\psi(z_k))^{1-\alpha_1}}{\Gamma(\alpha_1)}\theta_1^*)}{1-(\theta_5^*+\frac{(\psi(z)-\psi(z_k))^{1-\alpha_1}}{\Gamma(\alpha_1)}\theta_2^*)}|x(z)|
\end{aligned}
$$

Let

$$\mathcal{A}_1 = \frac{\theta_3^* + \frac{(\psi(z)-\psi(z_k))^{1-\alpha_1}}{\Gamma(\alpha_1)}\theta_0^*}{1-(\theta_5^* + \frac{(\psi(z)-\psi(z_k))^{1-\alpha_1}}{\Gamma(\alpha_1)}\theta_2^*)}$$

and

$$\mathcal{A}_2 = \frac{(\theta_4^* + \frac{(\psi(z)-\psi(z_k))^{1-\alpha_1}}{\Gamma(\alpha_1)}\theta_1^*)}{1-(\theta_5^* + \frac{(\psi(z)-\psi(z_k))^{1-\alpha_1}}{\Gamma(\alpha_1)}\theta_2^*)}.$$

Then

$$|{}^c\mathcal{D}_{z_k,z}^{\alpha_1;\psi}x(z)| \leq \mathcal{A}_1 + \mathcal{A}_2|x(z)|. \tag{8}$$

Let $|x(z)| \leq r_1$, then we get

$$|{}^c\mathcal{D}_{z_k,z}^{\alpha_1;\psi}x(z)| \leq \mathcal{A}_1 + \mathcal{A}_2 r_1.$$

Using the above estimates, we get

$$\begin{aligned}
\leq\ & \theta_0^* + \theta_1^* r_1 + \theta_2^*(\mathcal{A}_1 + \mathcal{A}_2 r_1) + \frac{(\psi(z)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3^* + \theta_4^* r_1 + \theta_5^*(\mathcal{A}_1 + \mathcal{A}_2 r_1)) \\
& + \frac{1}{|\Delta|}\bigg[\theta_0^* + \theta_1^* r_1 + \theta_2^*(\mathcal{A}_1 + \mathcal{A}_2 r_1) \\
& + \frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3^* + \theta_4^* r_1 + \theta_5^*(\mathcal{A}_1 + \mathcal{A}_2 r_1)) \\
& + \sum_{i=1}^{p}\bigg(\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3^* + \theta_4^* r_1 + \theta_5^*(\mathcal{A}_1 + \mathcal{A}_2 r_1) \\
& + \frac{\psi(\eta)-\psi(z_i)}{\psi'(z_i)} \times \frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\Gamma(\alpha_1)}(\theta_3^* + \theta_4^* r_1 + \theta_5^*(\mathcal{A}_1 + \mathcal{A}_2 r_1)) \\
& + A_1 r_1 + N_1 + \frac{\psi(\eta)-\psi(z_i)}{\psi'(z_i)}(A_2 r_1 + N_2)\bigg) \\
& + |a_1|\int_{z_k}^{\delta_{2k}}\bigg(\theta_0^* + \theta_1^* r_1 + \theta_2^*(\mathcal{A}_1 + \mathcal{A}_2 r_1) \\
& + \frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3^* + \theta_4^* r_1 + \theta_5^*(\mathcal{A}_1 + \mathcal{A}_2 r_1)) \\
& + \sum_{i=1}^{p}\bigg(\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3^* + \theta_4^* r_1 + \theta_5^*(\mathcal{A}_1 + \mathcal{A}_2 r_1)) \\
& + \frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)} \times \frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\Gamma(\alpha_1)}(\theta_3^* + \theta_4^* r_1 + \theta_5^*(\mathcal{A}_1 + \mathcal{A}_2 r_1)) \\
& + A_1 r_1 + N_1 + \frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}(A_2 r_1 + N_2)\bigg)\bigg)d\tau
\end{aligned}$$

$$
\begin{aligned}
&+|a_2|\int_{\delta_{2k+1}}^{z_{k+1}}\bigg(\theta_0^*+\theta_1^*r_1+\theta_2^*(\mathcal{A}_1+\mathcal{A}_2r_1)\\
&+\frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3^*+\theta_4^*r_1+\theta_5^*(\mathcal{A}_1+\mathcal{A}_2r_1))\\
&+\sum_{i=1}^{p}\bigg(\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3^*+\theta_4^*r_1+\theta_5^*(\mathcal{A}_1+\mathcal{A}_2r_1))\\
&+\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}\times\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\Gamma(\alpha_1)}(\theta_3^*+\theta_4^*r_1+\theta_5^*(\mathcal{A}_1+\mathcal{A}_2r_1))\\
&+A_1r_1+N_1+\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}(A_2r_1+N_2)\bigg)\bigg)d\tau\bigg](\psi(z)-\psi(z_k))\\
&+\sum_{i=1}^{p}(\psi(z_i)-\psi(z_{i-1}))+\sum_{i=1}^{p}\bigg(\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3^*+\theta_4^*r_1+\theta_5^*(\mathcal{A}_1+\mathcal{A}_2r_1))\\
&+\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}\times\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\Gamma(\alpha_1)}(\theta_3^*+\theta_4^*r_1+\theta_5^*(\mathcal{A}_1+\mathcal{A}_2r_1))\\
&+A_1r_1+N_1+\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}(A_2r_1+N_2)\bigg).
\end{aligned}
$$

Therefore,

$$\|P_1(x,y)(z)\|_{\mathbb{X}'}\le F_1.$$

In the same way, we can prove that

$$\|P_2(x,y)(z)\|_{\mathbb{Y}'}\le F_2.$$

Let $\max\{F_1,F_2\}=F$. Then we have

$$\|P(x,y)(z)\|_{\mathbb{U}'\times\mathbb{Y}'}\le F.$$

The above inequality shows that the operator $P$ is bounded. Now we need to show that the operator $P$ is equicontinuous. For this let $\omega_1,\omega_2\in J_k$ such that $\omega_1<\omega_2$ where $k=0,1,2,\dots,p$.

Let $(x,y)\in Q_r$, and then we have

$$
\begin{aligned}
&\Big|P_1(x,y)(\omega_2) - P_1(x,y)(\omega_1)\Big| \\
\le\ & \Big|f_1(\omega_2, x(\omega_2), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(\omega_2)) - f_1(\omega_1, x(\omega_1), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(\omega_1))\Big| \\
&+\frac{1}{\Gamma(\alpha_1)}\Big|\int_{z_k}^{\omega_2}\psi'(s)(\psi(\omega_2)-\psi(s))^{\alpha_1-1}g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(s))ds \\
&-\int_{z_k}^{\omega_1}\psi'(s)(\psi(\omega_1)-\psi(s))^{\alpha_1-1}g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(s))ds\Big| \\
&+\frac{1}{\Delta}\Big[\Big|f_1(\eta,x(\eta),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(\eta)) + \frac{1}{\Gamma(\alpha_1)}\int_{z_k}^{\eta}\psi'(s)(\psi(\eta)-\psi(s))^{\alpha_1-1}g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(s))ds \\
&+\sum_{i=1}^{p}\Big(\frac{1}{\Gamma(\alpha_1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-1}g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(s))ds \\
&+\frac{\psi(\eta)-\psi(z_i)}{\psi'(z_i)\Gamma(\alpha_1-1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-2}g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(s))ds \\
&+I_ix(z_i)+\frac{\psi(\eta)-\psi(z_i)}{\psi'(z_i)}J_ix(z_i)\Big)\Big| \\
&+|a_1|\int_{z_k}^{\delta_{2k}}|\Big|\Big(f_1(\tau)+\frac{1}{\Gamma(\alpha_1)}\int_{z_k}^{\eta}\psi'(s)(\psi(\tau)-\psi(s))^{\alpha_1-1}g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(s))ds \\
&+\sum_{i=1}^{p}\Big(\frac{1}{\Gamma(\alpha_1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-1}g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(s))ds \\
&+\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)\Gamma(\alpha_1-1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-2}g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(s))ds \\
&+I_ix(z_i)+\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}J_ix(z_i)\Big)\Big)\Big|d\tau \\
&+|a_2|\int_{\delta_{2k+1}}^{z_{k+1}}\Big|\Big(f_1(\tau)+\frac{1}{\Gamma(\alpha_1)}\int_{z_k}^{\eta}\psi'(s)(\psi(\tau)-\psi(s))^{\alpha_1-1}g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(s))ds \\
&+\sum_{i=1}^{p}\Big(\frac{1}{\Gamma(\alpha_1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-1}g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(s))ds \\
&+\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)\Gamma(\alpha_1-1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-2}g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(s))ds \\
&+I_ix(z_i)+\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}J_ix(z_i)\Big)\Big)\Big|d\tau\Big]\Big||((\psi(\omega_2)-\psi(z_k))-(\psi(\omega_1)-\psi(z_k)))| \\
&+\sum_{i=1}^{p}\frac{|((\psi(\omega_2)-\psi(z_i))-(\psi(\omega_1)-\psi(z_i)))|}{\psi'(z_i)\Gamma(\alpha_1-1)}\int_{z_{i-1}}^{z_i}|\psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-2}g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(s))|ds \\
&+\sum_{i=1}^{p}\frac{|((\psi(\omega_2)-\psi(z_i))-(\psi(\omega_1)-\psi(z_i)))|}{\psi'(z_i)}J_ix(z_i).
\end{aligned}
$$

From above inequality, if $\omega_1 \to \omega_2$, we deduce that

$$|P_1(x,y)(\omega_2) - P_1(x,y)(\omega_1)| \to 0.$$

In the same way we can prove that

$$|P_2(x,y)(\omega_2) - P_2(x,y)(\omega_1)| \to 0.$$

Hence, by the Arzila-Ascoli theorem $P_1$ and $P_2$ are completely continuous. This shows that $P$ is completely continuous.

Now let us define a set:

$$G = \{(x,y) \in \mathbb{X}^{'} \times \mathbb{Y}^{'}; (x,y) = \lambda P(x,y); 0 < \lambda < 1\}.$$

we prove that the set $G$ is bounded.

For $z \in J$ and $(x,y) \in G$ then $(x,y) = \lambda P(x,y)$ i.e. $x(z) = \lambda P_1(x,y)$ and $y(z) = \lambda P_2(x,y)$. Now

$$\begin{aligned}
&|x(z)| = |\lambda P_1(x,y)| \\
\leq\ & \lambda\Bigg\{ |f_1(z, x(z), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z} x(z))| \\
&+\frac{1}{\Gamma(\alpha_1)} \int_{z_k}^{z} \psi^{'}(s)(\psi(z)-\psi(s))^{\alpha_1-1}|g_1(s, x(s), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z} x(s))|ds \\
&+\frac{1}{|\Delta|}\Bigg[|f_1(\eta, x(\eta), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z} x(\eta))| + \frac{1}{\Gamma(\alpha_1)} \int_{z_k}^{\eta} \psi^{'}(s)(\psi(\eta)-\psi(s))^{\alpha_1-1}|g_1(s, x(s), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z} x(s))|ds \\
&+\sum_{i=1}^{p}\Bigg(\frac{1}{\Gamma(\alpha_1)}\int_{z_{i-1}}^{z_i} \psi^{'}(s)(\psi(z_i)-\psi(s))^{\alpha_1-1}|g_1(s, x(s), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z} x(s))|ds \\
&+\frac{\psi(\eta)-\psi(z_i)}{\psi^{'}(z_i)\Gamma(\alpha_1-1)}\int_{z_{i-1}}^{z_i} \psi^{'}(s)(\psi(z_i)-\psi(s))^{\alpha_1-2}|g_1(s, x(s), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z} x(s))|ds \\
&+I_i x(z_i) + \frac{\psi(\eta)-\psi(z_i)}{\psi^{'}(z_i)} J_i x(z_i)\Bigg) \\
&+|a_1|\int_{z_k}^{\delta_{2k}}\Bigg(|f_1(\tau, x(\tau), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,\tau} x(\tau))| + \frac{1}{\Gamma(\alpha_1)} \int_{z_k}^{\eta} \psi^{'}(s)(\psi(\tau)-\psi(s))^{\alpha_1-1}|g_1(s, x(s), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z} x(s))|ds \\
&+\sum_{i=1}^{p}\Bigg(\frac{1}{\Gamma(\alpha_1)}\int_{z_{i-1}}^{z_i} \psi^{'}(s)(\psi(z_i)-\psi(s))^{\alpha_1-1}|g_1(s, x(s), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z} x(s))|ds \\
&+\frac{\psi(\tau)-\psi(z_i)}{\psi^{'}(z_i)\Gamma(\alpha_1-1)}\int_{z_{i-1}}^{z_i} \psi^{'}(s)(\psi(z_i)-\psi(s))^{\alpha_1-2}|g_1(s, x(s), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z} x(s))|ds \\
&+I_i x(z_i) + \frac{\psi(\tau)-\psi(z_i)}{\psi^{'}(z_i)} J_i x(z_i)\Bigg)\Bigg)d\tau \\
&+|a_2|\int_{\delta_{2k+1}}^{z_{k+1}}\Bigg(|f_1(\tau, x(\tau), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,\tau} x(\tau))| + \frac{1}{\Gamma(\alpha_1)} \int_{z_k}^{\eta} \psi^{'}(s)(\psi(\tau)-\psi(s))^{\alpha_1-1}|g_1(s, x(s), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z} x(s))|ds \\
&+\sum_{i=1}^{p}\Bigg(\frac{1}{\Gamma(\alpha_1)}\int_{z_{i-1}}^{z_i} \psi^{'}(s)(\psi(z_i)-\psi(s))^{\alpha_1-1}|g_1(s, x(s), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z} x(s))|ds \\
&+\frac{\psi(\tau)-\psi(z_i)}{\psi^{'}(z_i)\Gamma(\alpha_1-1)}\int_{z_{i-1}}^{z_i} \psi^{'}(s)(\psi(z_i)-\psi(s))^{\alpha_1-2}|g_1(s, x(s), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z} x(s))|ds \\
&+|I_i x(z_i|) + \frac{\psi(\tau)-\psi(z_i)}{\psi^{'}(z_i)} |J_i x(z_i|)\Bigg)\Bigg)d\tau\Bigg](\psi(z)-\psi(z_k)) \\
&+\sum_{i=1}^{p}(\psi(z_i)-\psi(z_{i-1})) + \sum_{i=1}^{p}\Bigg(\frac{1}{\Gamma(\alpha_1)}\int_{z_{i-1}}^{z_1} \psi^{'}(s)(\psi(z_i)-\psi(s))^{\alpha_1-1}|g_1(s, x(s), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z} x(s))|ds \\
&+\frac{\psi(z)-\psi(z_i)}{\psi^{'}(z_i)\Gamma(\alpha_1-1)}\int_{z_{i-1}}^{z_1} \psi^{'}(s)(\psi(z_i)-\psi(s))^{\alpha_1-2} g_1(s)ds + |I_i x(z_i)| \\
&+\frac{\psi(z)-\psi(z_i)}{\psi^{'}(z_i)}|J_i x(z_i)|\Bigg)\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
\leq\ & \lambda\Bigg\{\theta_0(z)+\theta_1(z)|x(z)|+\theta_2(z)|{}^c\mathcal{D}^{\alpha_1,\psi}_{z_k,z}x(z)|+\frac{(\psi(z)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3(z)+\theta_4(z)|x(z)|+\theta_5(z)|{}^c\mathcal{D}^{\alpha_1,\psi}_{z_k,z}x(z)|) \\
& +\frac{1}{|\Delta|}\Bigg[\theta_0(\eta)+\theta_1(\eta)|x(\eta)|+\theta_2(\eta)|{}^c\mathcal{D}^{\alpha_1,\psi}_{z_k,\eta}x(\eta)| \\
& +\frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3(\eta)+\theta_4(\eta)|x(\eta)|+\theta_5(\eta)|{}^c\mathcal{D}^{\alpha_1,\psi}_{z_k,\eta}x(\eta)|) \\
& +\sum_{i=1}^{p}\Bigg(\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3(z_i)+\theta_4(z_i)|x(z_i)|+\theta_5(z_i)|{}^c\mathcal{D}^{\alpha_1,\psi}_{z_{i-1},z_i}x(z_i)|) \\
& +\frac{\psi(\eta)-\psi(z_i)}{\psi'(z_i)}\times\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\Gamma(\alpha_1)}(\theta_3(z_i)+\theta_4(z_i)|x(z_i)|+\theta_5(z_i)|{}^c\mathcal{D}^{\alpha_1,\psi}_{z_{i-1},z_i}x(z_i)|) \\
& +A_1|x(z)|+N_1+\frac{\psi(\eta)-\psi(z_i)}{\psi'(z_i)}(A_2|x(z)|+N_2)\Bigg) \\
& +|a_1|\int_{z_k}^{\delta_{2k}}\Bigg(\theta_0(\tau)+\theta_1(\tau)|x(\tau)|+\theta_2(\tau)|{}^c\mathcal{D}^{\alpha_1,\psi}_{z_k,\tau}x(\tau)| \\
& +\frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3(\eta)+\theta_4(\eta)|x(\eta)|+\theta_5(\eta)|{}^c\mathcal{D}^{\alpha_1,\psi}_{z_k,\eta}x(\eta)|) \\
& +\sum_{i=1}^{p}\Bigg(\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3(z_i)+\theta_4(z_i)|x(z_i)|+\theta_5(z_i)|{}^c\mathcal{D}^{\alpha_1,\psi}_{z_{i-1},z_i}x(z_i)|) \\
& +\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}\times\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\Gamma(\alpha_1)}(\theta_3(z_i)+\theta_4(z_i)|x(z_i)|+\theta_5(z_i)|{}^c\mathcal{D}^{\alpha_1,\psi}_{z_{i-1},z_i}x(z_i)|) \\
& +A_1|x(z)|+N_1+\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}(A_2|x(z)|+N_2)\Bigg)\Bigg)d\tau \\
& +|a_2|\int_{\delta_{2k+1}}^{z_{k+1}}\Bigg(\theta_0(\tau)+\theta_1(\tau)|x(\eta)|+\theta_2(\tau)|{}^c\mathcal{D}^{\alpha_1,\psi}_{z_k,\tau}x(\tau)| \\
& +\frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3(\eta)+\theta_4(\eta)|x(\eta)|+\theta_5(\eta)|{}^c\mathcal{D}^{\alpha_1,\psi}_{z_k,\eta}x(\eta)|) \\
& +\sum_{i=1}^{p}\Bigg(\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3(z_i)+\theta_4(z_i)|x(z_i)|+\theta_5(z_i)|{}^c\mathcal{D}^{\alpha_1,\psi}_{z_{i-1},z_i}x(z_i)|) \\
& +\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}\times\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\Gamma(\alpha_1)}(\theta_3(z_i)+\theta_4(z_i)|x(z_i)|+\theta_5(z_i)|{}^c\mathcal{D}^{\alpha_1,\psi}_{z_{i-1},z_i}x(z_i)|) \\
& +A_1|x(z)|+N_1+\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}(A_2|x(z)|+N_2)\Bigg)\Bigg)d\tau\Bigg](\psi(z)-\psi(z_k)) \\
& +\sum_{i=1}^{p}(\psi(z_i)-\psi(z_{i-1}))+\sum_{i=1}^{p}\Bigg(\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3(z_i)+\theta_4(z_i)|x(z_i)|+\theta_5(z_i)|{}^c\mathcal{D}^{\alpha_1,\psi}_{z_{i-1},z_i}x(z_i)|) \\
& +\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}\times\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\Gamma(\alpha_1)}(\theta_3(z_i)+\theta_4(z_i)|x(z_i)|+\theta_5(z_i)|{}^c\mathcal{D}^{\alpha_1,\psi}_{z_{i-1},z_i}x(z_i)|) \\
& +A_1|x(z)|+N_1+\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}(A_2|x(z)|+N_2)\Bigg)\Bigg\}
\end{aligned}
$$

Using (4.3), we have

$$
\begin{aligned}
|x(z)| \quad \leq \quad & \lambda\Bigg\{\theta_0^* + \theta_1^*|x(z)| + \theta_2^*(\mathcal{A}_1 + \mathcal{A}_2|x(z)|) + \frac{(\psi(z)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3^* + \theta_4^* r_1 + \theta_5^*(\mathcal{A}_1 + \mathcal{A}_2|x(z)|)) \\
& + \frac{1}{|\Delta|}\Bigg[\theta_0^* + \theta_1^*|x(\eta)| + \theta_2^*(\mathcal{A}_1 + \mathcal{A}_2|x(\eta))| \\
& + \frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3^* + \theta_4^* r_1 + \theta_5^*((\mathcal{A}_1 + \mathcal{A}_2|x(\eta)|))) \\
& + \sum_{i=1}^{p}\Bigg(\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3^* + \theta_4^*|x(\eta)|| + \theta_5^*((\mathcal{A}_1 + \mathcal{A}_2|x(\eta)|))) \\
& + \frac{\psi(\eta)-\psi(z_i)}{\psi'(z_i)} \times \frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\Gamma(\alpha_1)}(\theta_3^* + \theta_4^* x(\eta) + \theta_5^*((\mathcal{A}_1 + \mathcal{A}_2|x(\eta)|))) \\
& + A_1 x(z) + N_1 + \frac{\psi(\eta)-\psi(z_i)}{\psi'(z_i)}(A_2 x(z) + N_2)\Bigg) \\
& + |a_1|\int_{z_k}^{\delta_{2k}}\Bigg(\theta_0^* + \theta_1^*|x(\tau)| + \theta_2^*(\mathcal{A}_1 + \mathcal{A}_2|x(\tau)|) \\
& + \frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3^* + \theta_4^*|x(\eta)| + \theta_5^*((\mathcal{A}_1 + \mathcal{A}_2|x(\eta)|))) \\
& + \sum_{i=1}^{p}\Bigg(\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3^* + \theta_4^*|x(\eta)| + \theta_5^*((\mathcal{A}_1 + \mathcal{A}_2|x(\eta)|))) \\
& + \frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)} \times \frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\Gamma(\alpha_1)}(\theta_3^* + \theta_4^*|x(\eta)| + \theta_5^*((\mathcal{A}_1 + \mathcal{A}_2|x(\eta)|))) \\
& + A_1|x(z)| + N_1 + \frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}(A_2|x(z)| + N_2)\Bigg)\Bigg)d\tau \\
& + |a_2|\int_{\delta_{2k+1}}^{z_{k+1}}\Bigg(\theta_0^* + \theta_1^*|x(\tau)| + \theta_2^*(\mathcal{A}_1 + \mathcal{A}_2|x(\tau)|) \\
& + \frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3^* + \theta_4^*|x(\eta)| + \theta_5^*((\mathcal{A}_1 + \mathcal{A}_2|x(\eta)|))) \\
& + \sum_{i=1}^{p}\Bigg(\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3^* + \theta_4^*|x(\eta)| + \theta_5^*((\mathcal{A}_1 + \mathcal{A}_2|x(\eta)|))) \\
& + \frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)} \times \frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\Gamma(\alpha_1)}(\theta_3^* + \theta_4^*|x(\eta)| + \theta_5^*((\mathcal{A}_1 + \mathcal{A}_2|x(\eta)|))) \\
& + A_1|x(z)| + N_1 + \frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}(A_2|x(z)| + N_2)\Bigg)\Bigg)d\tau\Bigg](\psi(z)-\psi(z_k)) \\
& + \sum_{i=1}^{p}(\psi(z_i)-\psi(z_{i-1})) + \sum_{i=1}^{p}\Bigg(\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3^* + \theta_4^*|x(\eta)| + \theta_5^*((\mathcal{A}_1 + \mathcal{A}_2|x(\eta)|))) \\
& + \frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)} \times \frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\Gamma(\alpha_1)}(\theta_3^* + \theta_4^*|x(\eta)| + \theta_5^*((\mathcal{A}_1 + \mathcal{A}_2|x(\eta)|))) \\
& + A_1|x(z)| + N_1 + \frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}(A_2|x(z)| + N_2)\Bigg)\Bigg\}.
\end{aligned}
$$

By further simplification, we get that

$$
\begin{aligned}
|x(z)| \quad \leq \quad & \lambda\Bigg\{\theta_0^* + \theta_1^*|x(z)| + \theta_2^*(\mathcal{A}_1 + \mathcal{A}_2|x(z)|) + \frac{(\psi(z) - \psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1 + 1)}(\theta_3^* + \theta_4^* r_1 + \theta_5^*(\mathcal{A}_1 + \mathcal{A}_2|x(z)|)) \\
& + \frac{1}{|\Delta|}\Bigg[\theta_0^* + \theta_1^*|x(\eta)| + \theta_2^*((\mathcal{A}_1 + \mathcal{A}_2|x(\eta)|)) \\
& + \frac{(\psi(\eta) - \psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1 + 1)}(\theta_3^* + \theta_4^* r_1 + \theta_5^*(\mathcal{A}_1 + \mathcal{A}_2|x(\eta)|)) \\
& + \sum_{i=1}^{p}\Bigg(\frac{(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1 + 1)}(\theta_3^* + \theta_4^*|x(\eta)|| + \theta_5^*(\mathcal{A}_1 + \mathcal{A}_2|x(\eta)|)) \\
& + \frac{\psi(\eta) - \psi(z_i)}{\psi'(z_i)} \times \frac{(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1 - 1}}{\Gamma(\alpha_1)}(\theta_3^* + \theta_4^* x(\eta) + \theta_5^*(\mathcal{A}_1 + \mathcal{A}_2|x(\eta)|)) \\
& + A_1 x(z) + N_1 + \frac{\psi(\eta) - \psi(z_i)}{\psi'(z_i)}(A_2|x(z)| + N_2)\Bigg) \\
& + |a_1|\int_{z_k}^{\delta_{2k}}\Bigg(\theta_0^* + \theta_1^*|x(\tau)| + \theta_2^*\mathcal{A}_1 + \mathcal{A}_2|x(\tau)| \\
& + \frac{(\psi(\eta) - \psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1 + 1)}(\theta_3^* + \theta_4^*|x(\eta)| + \theta_5^*(\mathcal{A}_1 + \mathcal{A}_2|x(\eta)|)) \\
& + \sum_{i=1}^{p}\Bigg(\frac{(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1 + 1)}(\theta_3^* + \theta_4^*|x(\eta)| + \theta_5^*(\mathcal{A}_1 + \mathcal{A}_2|x(\eta)|)) \\
& + \frac{\psi(\tau) - \psi(z_i)}{\psi'(z_i)} \times \frac{(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1 - 1}}{\Gamma(\alpha_1)}(\theta_3^* + \theta_4^*|x(\eta)| + \theta_5^*(\mathcal{A}_1 + \mathcal{A}_2|x(\eta)|)) \\
& + A_1|x(z)| + N_1 + \frac{\psi(\tau) - \psi(z_i)}{\psi'(z_i)}(A_2|x(z)| + N_2)\Bigg)\Bigg)d\tau \\
& + |a_2|\int_{\delta_{2k+1}}^{z_{k+1}}\Bigg(\theta_0^* + \theta_1^*|x(\tau)| + \theta_2^*\mathcal{A}_1 + \mathcal{A}_2|x(\tau)| \\
& + \frac{(\psi(\eta) - \psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1 + 1)}(\theta_3^* + \theta_4^*|x(\eta)| + \theta_5^*(\mathcal{A}_1 + \mathcal{A}_2|x(\eta)|)) \\
& + \sum_{i=1}^{p}\Bigg(\frac{(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1 + 1)}(\theta_3^* + \theta_4^*|x(\eta)| + \theta_5^*(\mathcal{A}_1 + \mathcal{A}_2|x(\eta)|)) \\
& + \frac{\psi(\tau) - \psi(z_i)}{\psi'(z_i)} \times \frac{(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1 - 1}}{\Gamma(\alpha_1)}(\theta_3^* + \theta_4^*|x(\eta)| + \theta_5^*(\mathcal{A}_1 + \mathcal{A}_2|x(\eta)|)) \\
& + A_1|x(z)| + N_1 + \frac{\psi(\tau) - \psi(z_i)}{\psi'(z_i)}(A_2|x(z)| + N_2)\Bigg)\Bigg)d\tau\Bigg](\psi(z) - \psi(z_k)) \\
& + \sum_{i=1}^{p}(\psi(z_i) - \psi(z_{i-1})) + \sum_{i=1}^{p}\Bigg(\frac{(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1 + 1)}(\theta_3^* + \theta_4^*|x(\eta)| + \theta_5^*(\mathcal{A}_1 + \mathcal{A}_2|x(\eta)|)) \\
& + \frac{\psi(\tau) - \psi(z_i)}{\psi'(z_i)} \times \frac{(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1 - 1}}{\Gamma(\alpha_1)}(\theta_3^* + \theta_4^*|x(\eta)| + \theta_5^*(\mathcal{A}_1 + \mathcal{A}_2|x(\eta)|)) \\
& + A_1|x(z)| + N_1 + \frac{\psi(\tau) - \psi(z_i)}{\psi'(z_i)}(A_2|x(z)| + N_2)\Bigg)\Bigg\}.
\end{aligned}
$$

Let us assume that $\mathcal{M}^* < 1$, where

$$\begin{aligned}
\mathcal{M}^* \quad = \quad & \lambda\Bigg[\theta_1^* + \theta_2^*\mathcal{A}_2 + \frac{(\psi(z) - \psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_4^* + \theta_5^*\mathcal{A}_2) \\
& + \frac{1}{|\Delta|}\Bigg[\theta_1^* + \theta_2^*\mathcal{A}_2 + \frac{(\psi(\eta) - \psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_4^* + \theta_5^*\mathcal{A}_2) \\
& + \sum_{i=1}^{p}\Bigg(\frac{(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_4^* + \theta_5^*\mathcal{A}_2) \\
& + \frac{(\psi(\eta) - \psi(z_i))(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1-1}}{\psi'(z_i)\Gamma(\alpha_1)}(\theta_4^* + \theta_5^*\mathcal{A}_2) \\
& + A_1 + \frac{\psi(\eta) - \psi(z_i)}{\psi'(z_i)}(A_2)\Bigg) \\
& + |a_1|\int_{z_k}^{\delta_{2k}}\Bigg(\theta_1^*|x(\tau)| + \theta_2^*\mathcal{A}_2|x(\tau)| \\
& + \frac{(\psi(\eta) - \psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_4^* + \theta_5^*\mathcal{A}_2) \\
& + \sum_{i=1}^{p}\Bigg(\frac{(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_4^* + \theta_5^*\mathcal{A}_2) \\
& + \frac{\psi(\tau) - \psi(z_i)}{\psi'(z_i)} \times \frac{(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1-1}}{\Gamma(\alpha_1)}(\theta_4^* + \theta_5^*\mathcal{A}_2) \\
& + A_1 + \frac{\psi(\tau) - \psi(z_i)}{\psi'(z_i)}A_2\Bigg)\Bigg)(\delta_{2k} - z_k) \\
& + |a_2|\int_{\delta_{2k+1}}^{z_{k+1}}\Bigg(\theta_1^*|x(\tau)| + \theta_2^*\mathcal{A}_2|x(\tau)| \\
& + \frac{(\psi(\eta) - \psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_4^* + \theta_5^*\mathcal{A}_2) \\
& + \sum_{i=1}^{p}\Bigg(\frac{(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_4^* + \theta_5^*\mathcal{A}_2) \\
& + \frac{\psi(\tau) - \psi(z_i)}{\psi'(z_i)} \times \frac{(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1-1}}{\Gamma(\alpha_1)}(\theta_4^* + \theta_5^*\mathcal{A}_2) \\
& + A_1 + \frac{\psi(\tau) - \psi(z_i)}{\psi'(z_i)}(A_2)\Bigg)\Bigg)(z_{k+1} - \delta_{2k+1})\Bigg](\psi(z) - \psi(z_k)) \\
& + \sum_{i=1}^{p}(\psi(z_i) - \psi(z_{i-1})) + \sum_{i=1}^{p}\Bigg(\frac{(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_4^* + \theta_5^*\mathcal{A}_2) \\
& + \frac{\psi(\tau) - \psi(z_i)}{\psi'(z_i)} \times \frac{(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1-1}}{\Gamma(\alpha_1)}(\theta_4^* + \theta_5^*\mathcal{A}_2) \\
& + A_1 + \frac{\psi(\tau) - \psi(z_i)}{\psi'(z_i)}A_2\Bigg)\Bigg].
\end{aligned}$$

Then we have

$$
\begin{aligned}
\|x(z)\| \leq\ & \frac{\lambda}{1-\mathcal{M}^*}\Bigg\{\theta_0^* + \theta_2^*\mathcal{A}_1 + \frac{(\psi(z)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3^* + \theta_5^*\mathcal{A}_1) \\
& + \frac{1}{|\Delta|}\Bigg[\theta_0^* + \theta_2^*(\mathcal{A}_1) \\
& + \frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3^* + \theta_5^*\mathcal{A}_1 \\
& + \sum_{i=1}^{p}\Bigg(\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3^* + \theta_5^*\mathcal{A}_1) \\
& + \frac{\psi(\eta)-\psi(z_i)}{\psi'(z_i)} \times \frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\Gamma(\alpha_1)}(\theta_3^* + \theta_4^* x(\eta) + \theta_5^*\mathcal{A}_1) \\
& + N_1 + \frac{\psi(\eta)-\psi(z_i)}{\psi'(z_i)}N_2\Bigg) \\
& + |a_1|\Bigg(\theta_0^* + \theta_2^*\mathcal{A}_1 \\
& + \frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3^* + \theta_5^*(\mathcal{A}_1)) \\
& + \sum_{i=1}^{p}\Bigg(\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3^* + \theta_5^*\mathcal{A}_1) \\
& + \frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)} \times \frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\Gamma(\alpha_1)}(\theta_3^* + \theta_5^*\mathcal{A}_1) \\
& + N_1 + \frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}N_2\Bigg)\Bigg)\Bigg(\delta_{2k} - z_k\Bigg) \\
& + |a_2|\Bigg(\theta_0^* + \theta_2^*\mathcal{A}_1 \\
& + \frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3^* + \theta_5^*\mathcal{A}_1) \\
& + \sum_{i=1}^{p}\Bigg(\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3^* + \theta_5^*\mathcal{A}_1) \\
& + \frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)} \times \frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\Gamma(\alpha_1)}(\theta_3^* + \theta_5^*\mathcal{A}_1) \\
& + N_1 + \frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}N_2\Bigg)\Bigg)\Bigg(z_{k+1} - \delta_{2k+1}\Bigg)\Bigg](\psi(z)-\psi(z_k)) \\
& + \sum_{i=1}^{p}(\psi(z_i)-\psi(z_{i-1})) + \sum_{i=1}^{p}\Bigg(\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\theta_3^* + \theta_5^*\mathcal{A}_1) \\
& + \frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)} \times \frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\Gamma(\alpha_1)}(\theta_3^* + \theta_5^*\mathcal{A}_1) \\
& + N_1 + \frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}N_2\Bigg)\Bigg\}.
\end{aligned}
$$

Thus there exists a positive constant $\mathcal{F}_1$, such that

$$\|x\|_{\mathbb{X}'} \leq \mathcal{F}_1.$$

In the same way, we can prove that there exists $\mathcal{F}_1$, such that

$$\|y\|_{\mathbb{Y}'} \leq \mathcal{F}_2.$$

Let $\max\{\digamma_1, \digamma_2\} = \digamma$. Then we have

$$\|(x,y)\|_{\mathbb{U}' \times \mathbb{Y}'} \leq \digamma.$$

Thus the set $G$ is bounded, and by the Schaefer's fixed point theorem, the operator $P$ has at least one fixed point, i.e., the problem (1) has at least one solution. □

## 5. Ulam's Stability Results

Using Definition 4, in this section we give the Ulam-Hyers stability of the problem (3).

**Theorem 3.** *If assumptions $(G_1)$, $(G_2)$ are satisfied, then the coupled system (1) is Ulam-Hyers stable.*

**Proof.** Let $(x, y) \in \mathbb{X}' \times \mathbb{Y}'$ be an approximate solution of the inequality:

$$\begin{cases} |{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}[x(z) - f_1(z, x(z), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(z))] - g_1(z, x(z), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(z))| < \epsilon_1, \;\; z \in (z_k, z_{k+1}], \\ |\Delta x(z_k) - I_k(x(z_k))| < \epsilon_1, \;\; k = 1, 2, \ldots, p, \\ |\Delta x'(z_k) - J_k(x(z_k))| < \epsilon_1. \\ |{}^c\mathcal{D}^{\beta_1;\psi}_{z_k,z}[y(z) - f_2(z, y(z), {}^c\mathcal{D}^{\beta_1;\psi}_{z_k,z}y(z))] - g_2(z, y(z), {}^c\mathcal{D}^{\beta_1;\psi}_{z_k,z}y(z))| < \epsilon_2, \\ |\Delta y(z_k) - I^*_k(y(z_k))| < \epsilon_2, \\ |\Delta y'(z_k) - J^*_k(y(z_k))| < \epsilon_2, \;\; k = 1, 2, \ldots, p. \end{cases} \tag{9}$$

From the inequality (9), we have

$$\begin{cases} {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}[x(z) - f_1(z, x(z), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(z))] = g_1(z, x(z), {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(z)) + \varphi(z), \\ z \in (z_k, z_{k+1}], \;\; k = 0, 1, 2, \ldots, p. \\ \Delta x(z_k) = I_k(x(z_k)) + \varphi_k(z), \;\; k = 1, 2, \ldots, p, \\ \Delta x'(z_k) = J_k(x(z_k)) + \varphi_k(z). \\ {}^c\mathcal{D}^{\beta_1;\psi}_{z_k,z}[y(z) - f_2(z, y(z), {}^c\mathcal{D}^{\beta_1;\psi}_{z_k,z}y(z))] = g_2(z, y(z), {}^c\mathcal{D}^{\beta_1;\psi}_{z_k,z}y(z)) + \phi(z), \\ z \in (z_k, z_{k+1}], \;\; k = 0, 1, 2, \ldots, p, \\ \Delta y(z_k) = I^*_k(y(z_k)) + \phi_k(z), \;\;\; k = 1, 2, \ldots, p, \\ \Delta y'(z_k) = J^*_k(y(z_k)) + \phi_k(z). \end{cases} \tag{10}$$

The $x$ and $y$ parts of the solution $(x, y)$ of problem (10) are equivalent to

$$x(z)=\begin{cases} f_1(z,x(z),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(z))+\frac{1}{\Gamma(\alpha_1)}\int_{z_k}^{z}\psi'(s)(\psi(z)-\psi(s))^{\alpha_1-1}g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,s}x(s))ds\\ +\frac{1}{\Gamma(\alpha_1)}\int_{z_k}^{z}\psi'(s)(\psi(z)-\psi(s))^{\alpha_1-1}\varphi(s)ds\\ +\frac{1}{\Delta}\bigg[f_1(\eta,x(\eta),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(\eta))+\frac{1}{\Gamma(\alpha_1)}\int_{z_k}^{\eta}\psi'(s)(\psi(\eta)-\psi(s))^{\alpha_1-1}\\ \times g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,s}x(s))ds+\frac{1}{\Gamma(\alpha_1)}\int_{z_k}^{\eta}\psi'(s)(\psi(\eta)-\psi(s))^{\alpha_1-1}\varphi(s)ds\\ +\sum_{i=1}^{p}\bigg(\frac{1}{\Gamma(\alpha_1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-1}g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,s}x(s))ds+\frac{1}{\Gamma(\alpha_1)}\\ \times\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-1}\varphi(s)ds+\frac{\psi(\eta)-\psi(z_i)}{\psi'(z_i)\Gamma(\alpha_1-1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-2}\\ \times g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,s}x(s))ds+\frac{\psi(\eta)-\psi(z_i)}{\psi'(z_i)\Gamma(\alpha_1-1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-2}\varphi(s)ds\\ +I_ix(z_i)+\varphi_i(z(i))+\frac{\psi(\eta)-\psi(z_i)}{\psi'(z_i)}J_ix(z_i)+\frac{\psi(\eta)-\psi(z_i)}{\psi'(z_i)}\varphi(z_i)\bigg)\\ -a_1\int_{z_k}^{\delta_{2k}}\bigg(f_1(\tau,x(\tau),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(\tau))+\frac{1}{\Gamma(\alpha_1)}\int_{z_k}^{\tau}\psi'(s)(\psi(\tau)-\psi(s))^{\alpha_1-1}g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,s}x(s))ds\\ +\frac{1}{\Gamma(\alpha_1)}\int_{z_k}^{\tau}\psi'(s)(\psi(\tau)-\psi(s))^{\alpha_1-1}\varphi(s)ds+\sum_{i=1}^{p}\bigg(\frac{1}{\Gamma(\alpha_1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-1}\\ g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,s}x(s))ds+\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)\Gamma(\alpha_1-1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-2}g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,s}x(s))ds\\ +I_ix(z_i)+\varphi_i(z(i))+\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}J_ix(z_i)+\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}\varphi(z_i)\bigg)\bigg)d\tau\\ -a_2\int_{\delta_{2k+1}}^{z_{k+1}}\bigg(f_1(\tau,x(\tau),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(\tau))+\frac{1}{\Gamma(\alpha_1)}\int_{z_k}^{\eta}\psi'(s)(\psi(\tau)-\psi(s))^{\alpha_1-1}g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,s}x(s))ds\\ +\frac{1}{\Gamma(\alpha_1)}\int_{z_k}^{\eta}\psi'(s)(\psi(\tau)-\psi(s))^{\alpha_1-1}\varphi(s)ds\\ +\sum_{i=1}^{p}\bigg(\frac{1}{\Gamma(\alpha_1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-1}g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,s}x(s))ds\\ +\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)\Gamma(\alpha_1-1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-2}g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,s}x(s))ds\\ +I_ix(z_i)+\varphi_i(z(i))+\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}J_ix(z_i)+\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}\varphi(z_i)\bigg)\bigg)d\tau\bigg](\psi(z)-\psi(z_k))\\ +\sum_{i=1}^{p}(\psi(z_i)-\psi(z_{i-1}))+\sum_{i=1}^{p}\frac{1}{\Gamma(\alpha_1)}\int_{z_{i-1}}^{z_1}\psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-1}g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,s}x(s))ds\\ +\sum_{i=1}^{p}\frac{\psi(z)-\psi(z_i)}{\psi'(z_i)\Gamma(\alpha_1-1)}\int_{z_{i-1}}^{z_1}\psi'(s)(\psi(z_i)-\psi(s))^{\alpha_1-2}g_1(s,x(s),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,s}x(s))ds+\sum_{i=1}^{p}I_ix(z_i)\\ +\sum_{i=1}^{p}\varphi_i(z(i))+\sum_{i=1}^{p}\frac{\psi(z)-\psi(z_i)}{\psi'(z_i)}J_ix(z_i)+\sum_{i=1}^{p}\frac{\psi(z)-\psi(z_i)}{\psi'(z_i)}\varphi_i(z(i)), \end{cases} \tag{11}$$

$$y(z)=\begin{cases} f_2(z,x(z),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(z))+\frac{1}{\Gamma(\beta_1)}\int_{z_k}^{z}\psi'(s)(\psi(z)-\psi(s))^{\beta_1-1}g_2(s,y(s),{}^c\mathcal{D}^{\beta_1;\psi}_{z_k,s}y(s))ds\\ +\frac{1}{\Gamma(\beta_1)}\int_{z_k}^{z}\psi'(s)(\psi(z)-\psi(s))^{\beta_1-1}\phi(s)ds\\ +\frac{1}{\Delta}\Big[f_2(\eta,x(\eta),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(\eta))+\frac{1}{\Gamma(\beta_1)}\int_{z_k}^{\eta}\psi'(s)(\psi(\eta)-\psi(s))^{\beta_1-1}g_2(s,y(s),{}^c\mathcal{D}^{\beta_1;\psi}_{z_k,s}y(s))ds\\ +\frac{1}{\Gamma(\beta_1)}\int_{z_k}^{\eta}\psi'(s)(\psi(\eta)-\psi(s))^{\beta_1-1}\phi(s)ds\\ +\sum_{i=1}^{p}\Big(\frac{1}{\Gamma(\beta_1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\beta_1-1}g_2(s,y(s),{}^c\mathcal{D}^{\beta_1;\psi}_{z_k,s}y(s))ds\\ +\frac{1}{\Gamma(\beta_1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\beta_1-1}\phi(s)ds\\ +\frac{\psi(\eta)-\psi(z_i)}{\psi'(z_i)\Gamma(\beta_1-1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\beta_1-2}g_2(s,y(s),{}^c\mathcal{D}^{\beta_1;\psi}_{z_k,s}y(s))ds\\ +\frac{\psi(\eta)-\psi(z_i)}{\psi'(z_i)\Gamma(\beta_1-1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\beta_1-2}\phi(s)ds\\ +I_i^*y(z_i)+\phi_i(z_(i))+\frac{\psi(\eta)-\psi(z_i)}{\psi'(z_i)}J_i^*y(z_i)+\frac{\psi(\eta)-\psi(z_i)}{\psi'(z_i)}\phi(z_i)\Big)\\ -a_1\int_{z_k}^{\delta_{2k}}\Big(f_2(\tau,x(\tau),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(\tau))+\frac{1}{\Gamma(\beta_1)}\int_{z_k}^{\tau}\psi'(s)(\psi(\tau)-\psi(s))^{\beta_1-1}g_2(s,y(s),{}^c\mathcal{D}^{\beta_1;\psi}_{z_k,s}y(s))ds\\ +\frac{1}{\Gamma(\beta_1)}\int_{z_k}^{\tau}\psi'(s)(\psi(\tau)-\psi(s))^{\beta_1-1}\phi(s)ds\\ +\sum_{i=1}^{p}\Big(\frac{1}{\Gamma(\beta_1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\beta_1-1}g_2(s,y(s),{}^c\mathcal{D}^{\beta_1;\psi}_{z_k,s}y(s))ds\\ +\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)\Gamma(\beta_1-1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\beta_1-2}g_2(s,y(s),{}^c\mathcal{D}^{\beta_1;\psi}_{z_k,s}y(s))ds\\ +I_i^*y(z_i)+\phi_i(z_(i))+\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}J_i^*y(z_i)+\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}\phi(z_i)\Big)\Big)d\tau-a_2\int_{\delta_{2k+1}}^{z_{k+1}}\Big(f_2(\tau,x(\tau),{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x(\tau))\\ +\frac{1}{\Gamma(\beta_1)}\int_{z_k}^{\eta}\psi'(s)(\psi(\tau)-\psi(s))^{\beta_1-1}g_2(s,y(s),{}^c\mathcal{D}^{\beta_1;\psi}_{z_k,s}y(s))ds\\ \frac{1}{\Gamma(\beta_1)}\int_{z_k}^{\eta}\psi'(s)(\psi(\tau)-\psi(s))^{\beta_1-1}\phi(s)ds\\ +\sum_{i=1}^{p}\Big(\frac{1}{\Gamma(\beta_1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\beta_1-1}g_2(s,y(s),{}^c\mathcal{D}^{\beta_1;\psi}_{z_k,s}y(s))ds\\ +\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)\Gamma(\beta_1-1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\beta_1-2}g_2(s,y(s),{}^c\mathcal{D}^{\beta_1;\psi}_{z_k,s}y(s))ds\\ +I_i^*y(z_i)+\phi_i(z_i)+\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}J_i^*y(z_i)+\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}\phi(z_i)\Big)\Big)d\tau\Big](\psi(z)-\psi(z_k))\\ +\sum_{i=1}^{p}(\psi(z_i)-\psi(z_{i-1}))+\sum_{i=1}^{p}\frac{1}{\Gamma(\beta_1)}\int_{z_{i-1}}^{z_i}\psi'(s)(\psi(z_i)-\psi(s))^{\beta_1-1}g_2(s,y(s),{}^c\mathcal{D}^{\beta_1;\psi}_{z_k,s}y(s))ds\\ +\sum_{i=1}^{p}\frac{\psi(z)-\psi(z_i)}{\psi'(z_i)\Gamma(\beta_1-1)}\int_{z_{i-1}}^{z_1}\psi'(s)(\psi(z_i)-\psi(s))^{\beta_1-2}g_2(s,y(s),{}^c\mathcal{D}^{\beta_1;\psi}_{z_k,s}y(s))ds+\sum_{i=1}^{p}I_i^*y(z_i)+\sum_{i=1}^{p}\phi_i(z_(i))\\ +\sum_{i=1}^{p}\frac{\psi(z)-\psi(z_i)}{\psi'(z_i)}J_i^*y(z_i)+\sum_{i=1}^{p}\frac{\psi(z)-\psi(z_i)}{\psi'(z_i)}\phi(z_i), \end{cases} \tag{12}$$

Now let $(x,y)$ be the solution of (3) and $(x_1,y_1)$ be the solution of (5.2), then

$$
\begin{aligned}
|x(z)-\bar{x}(z)| \leq\ & M_{f_1}\|x-\bar{x}\| + N_{f_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x - {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\| + \frac{(\psi(z)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)}L_{g_1}\|x-\bar{x}\| \\
& + \frac{(\psi(z)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)}K_{g_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x - {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\| \\
& + \frac{(\psi(z)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)}|\varphi(s)| \\
& + \Bigg(\frac{M_{f_1}}{|\Delta|}\|x-\bar{x}\| + \frac{N_{f_1}}{|\Delta|}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x - {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\| \\
& + \frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|}L_{g_1}\|x-\bar{x}\| + \frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|}K_{g_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x - {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\| \\
& + \frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{|\Delta|\Gamma(\alpha_1+1)}|\varphi(s)| \\
& + \sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|}L_{g_1}\|x-\bar{x}\| \\
& + \sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|}K_{g_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x - {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\| + \sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{|\Delta|\Gamma(\alpha_1+1)}|\varphi(s)| \\
& + \sum_{i=1}^{p}\frac{\psi(\eta)-\psi(z_i)(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\psi'(z_i)\Gamma(\alpha_1)|\Delta|}L_{g_1}\|x-\bar{x}\| \\
& + \sum_{i=1}^{p}\frac{\psi(\eta)-\psi(z_i)(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\psi'(z_i)\Gamma(\alpha_1)|\Delta|}K_{g_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x - {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\| \\
& \sum_{i=1}^{p}\frac{\psi(\eta)-\psi(z_i)(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\psi'(z_i)\Gamma(\alpha_1)|\Delta|}|\phi(z)| \\
& + \sum_{i=1}^{p}\frac{A_1}{|\Delta|}|x(z_i)-\bar{x}(z_i)| + \sum_{i=1}^{p}\frac{1}{|\Delta|}|\phi(z)| + \sum_{i=1}^{p}\frac{\psi(\eta)-\psi(z_i)}{\psi'(z_i)|\Delta|}A_2|x(z_i)-\bar{x}(z_i)| \\
& + \sum_{i=1}^{p}\frac{\psi(\eta)-\psi(z_i)}{\psi'(z_i)|\Delta|}|\phi(z)| \\
& + \frac{|a_1|}{|\Delta|}(\delta_{2k}-z_k)M_{f_1}\|x-\bar{x}\| + \frac{|a_1|}{|\Delta|}(\delta_{2k}-z_k)N_{f_1} \\
& + |{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x - {}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\| + \frac{|a_1|}{|\Delta|}(\delta_{2k}-z_k)|\phi(z)| \\
& + \frac{(\psi(\tau)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|}|a_1|(\delta_{2k}-z_k)L_{g_1}\|x-\bar{x}\|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(\psi(\tau)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|}|a_1|(\delta_{2k}-z_k)K_{g_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x-{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\| \\
&+\frac{(\psi(\tau)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|}|a_1|(\delta_{2k}-z_k)|\phi(z)| \\
&+\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|}|a_1|(\delta_{2k}-z_k)L_{g_1}\|x-\bar{x}\| \\
&+\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|}|a_1|(\delta_{2k}-z_k)K_{g_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x-{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\| \\
&+\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|}|a_1|(\delta_{2k}-z_k)|\phi(z)| \\
&+\sum_{i=1}^{p}\frac{(\psi(\tau)-\psi(z_i))(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\psi'(z_i)\Gamma(\alpha_1)|\Delta|}|a_1|(\delta_{2k}-z_k)L_{g_1}\|x-\bar{x}\| \\
&+\sum_{i=1}^{p}\frac{(\psi(\tau)-\psi(z_i))(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\psi'(z_i)\Gamma(\alpha_1)|\Delta|}|a_1|(\delta_{2k}-z_k)K_{g_1} \\
&\times\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x-{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\| \\
&+\sum_{i=1}^{p}\frac{(\psi(\tau)-\psi(z_i))(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\psi'(z_i)\Gamma(\alpha_1)|\Delta|}|a_1|(\delta_{2k}-z_k)|\phi(z)| \\
&+\sum_{i=1}^{p}\frac{A_1}{|\Delta|}|a_1|(\delta_{2k}-z_k)|x(z_i)-\bar{x}(z_i)|+\sum_{i=1}^{p}\frac{1}{|\Delta|}|a_1|(\delta_{2k}-z_k)|\phi(z)| \\
&+\sum_{i=1}^{p}\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)|\Delta|}|a_1|(\delta_{2k}-z_k)A_2|x(z_i)-\bar{x}(z_i)| \\
&+\sum_{i=1}^{p}\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)|\Delta|}|a_1|(\delta_{2k}-z_k)|\phi(z)| \\
&+\frac{|a_2|}{|\Delta|}(z_{k+1}-\delta_{2k+1})M_{f_1}\|x-\bar{x}\| \\
&+\frac{|a_2|(z_{k+1}-\delta_{2k+1})}{|\Delta|}N_{f_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x-{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\|+\frac{|a_2|}{|\Delta|}(z_{k+1}-\delta_{2k+1})|\phi(z)| \\
&+\frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1)|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1})L_{g_1}\|x-\bar{x}\|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1)|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1})K_{g_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x-{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\| \\
&+\frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1)|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1})|\phi(z)| \\
&+\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1})L_{g_1}\|x-\bar{x}\| \\
&+\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1})K_{g_1} \\
&\times\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x-{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\| \\
&+\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1})|\phi(z)| \\
&+\sum_{i=1}^{p}\frac{A_1}{|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1})|x(z_i)-\bar{x}(z_i)|+\sum_{i=1}^{p}\frac{1}{|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1})|\phi(z_i)| \\
&+\sum_{i=1}^{p}A_2\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1})\|x-\bar{x}\| \\
&+\sum_{i=1}^{p}\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1})|\phi(z_i)|\Bigg)\Bigg(\psi(z)-\psi(z_k)\Bigg) \\
&+\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha}}{\Gamma(\alpha_1+1)}L_{g_1}\|x-\bar{x}\| \\
&+\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)}K_{g_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x-{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\| \\
&+\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)}|\phi(z)| \\
&+\sum_{i=1}^{p}\frac{\psi(z)-\psi(z_i)(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\psi'(z_i)\Gamma(\alpha_1)}L_{g_1}\|x-\bar{x}\| \\
&+\sum_{i=1}^{p}\frac{\psi(z)-\psi(z_i)(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\psi'(z_i)\Gamma(\alpha_1)}K_{g_1}\|{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}x-{}^c\mathcal{D}^{\alpha_1;\psi}_{z_k,z}\bar{x}\| \\
&+\sum_{i=1}^{p}\frac{\psi(z)-\psi(z_i)(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\psi'(z_i)\Gamma(\alpha_1)}|\phi(z)| \\
&+pA_1\|x-\bar{x}\|+p|\phi(z)|+\sum_{i=1}^{p}A_2\frac{\psi(z)-\psi(z_i)}{\psi'(z_i)}\|x-\bar{x}\|+\sum_{i=1}^{p}\frac{\psi(z)-\psi(z_i)}{\psi'(z_i)}|\phi(z)|.
\end{aligned}
$$

$$
\begin{aligned}
|x(z)-\bar{x}(z)| \quad \leq \quad & \Bigg(M_{f_1}+\frac{(\psi(z)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)}L_{g_1} \\
& +\Bigg(\frac{M_{f_1}}{|\Delta|}+\frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|}L_{g_1} \\
& +\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|}L_{g_1}+ \\
& +\sum_{i=1}^{p}\frac{\psi(\eta)-\psi(z_i)(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\psi'(z_i)\Gamma(\alpha_1)|\Delta|}L_{g_1} \\
& +\sum_{i=1}^{p}\frac{A_1}{|\Delta|}+\sum_{i=1}^{p}\frac{\psi(\eta)-\psi(z_i)}{\psi'(z_i)|\Delta|}A_2 \\
& +\frac{|a_1|}{|\Delta|}(\delta_{2k}-z_k)M_{f_1} \\
& +\frac{(\psi(\tau)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|}|a_1|(\delta_{2k}-z_k)L_{g_1} \\
& +\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|}|a_1|(\delta_{2k}-z_k)L_{g_1} \\
& +\frac{(\psi(\tau)-\psi(z_i))(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\psi'(z_i)\Gamma(\alpha_1)|\Delta|}|a_1|(\delta_{2k}-z_k)L_{g_1}+p\frac{A_1}{|\Delta|}|a_1|(\delta_{2k}-z_k) \\
& +\sum_{i=1}^{p}\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)|\Delta|}|a_1|(\delta_{2k}-z_k)A_2+\frac{|a_2|}{|\Delta|}(z_{k+1}-\delta_{2k+1})M_{f_1} \\
& +\frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1)|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1})L_{g_1} \\
& +\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1})L_{g_1} \\
& +p\frac{A_1}{|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1}) \\
& +\frac{A_2}{|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1})\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}\Bigg)\Big(\psi(z)-\psi(z_k)\Big) \\
& \sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha}}{\Gamma(\alpha_1+1)}L_{g_1} \\
& +\sum_{i=1}^{p}\frac{\psi(z)-\psi(z_i)(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\psi'(z_i)\Gamma(\alpha_1)}L_{g_1} \\
& +pA_1+A_2\sum_{i=1}^{p}\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)}\Bigg)\Big\|x-\bar{x}\Big\| \\
& +\Bigg(N_{f_1}+\frac{(\psi(z)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)}K_{g_1}
\end{aligned}
$$

$$
\begin{aligned}
&+\Bigg(\frac{N_{f_1}}{|\Delta|}+\frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|}K_{g_1}+\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|}K_{g_1}\\
&+\sum_{i=1}^{p}\frac{\psi(\eta)-\psi(z_i)(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\psi'(z_i)\Gamma(\alpha_1)|\Delta|}K_{g_1}+\frac{|a_1|}{|\Delta|}(\delta_{2k}-z_k)N_{f_1}\\
&+\frac{(\psi(\tau)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|}|a_1|(\delta_{2k}-z_k)K_{g_1}\\
&+\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|}|a_1|(\delta_{2k}-z_k)K_{g_1}\\
&+\frac{(\psi(\tau)-\psi(z_i))(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\psi'(z_i)\Gamma(\alpha_1)|\Delta|}|a_1|(\delta_{2k}-z_k)K_{g_1}+\frac{|a_2|(z_{k+1}-\delta_{2k+1})}{|\Delta|}N_{f_1}\\
&+\frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1)|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1})K_{g_1}\\
&+\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1})K_{g_1}\Bigg)\Big(\psi(z)-\psi(z_k)\Big)\\
&+\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)}K_{g_1}\\
&+\sum_{i=1}^{p}\frac{\psi(z)-\psi(z_i)(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\psi'(z_i)\Gamma(\alpha_1)}K_{g_1}\Bigg)\Big\|{}^{c}\mathcal{D}_{z_k,z}^{\alpha_1;\psi}x-{}^{c}\mathcal{D}_{z_k,z}^{\alpha_1;\psi}\bar{x}\Big\|\\
&+\Bigg(\frac{(\psi(z)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)}+\frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{|\Delta|\Gamma(\alpha_1+1)}+\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{|\Delta|\Gamma(\alpha_1+1)}\\
&+\sum_{i=1}^{p}\frac{\psi(\eta)-\psi(z_i)(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\psi'(z_i)\Gamma(\alpha_1)|\Delta|}\\
&+\sum_{i=1}^{p}\frac{\psi(\eta)-\psi(z_i)}{\psi'(z_i)|\Delta|}+\frac{|a_1|}{|\Delta|}(\delta_{2k}-z_k)\\
&+\frac{(\psi(\tau)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|}|a_1|(\delta_{2k}-z_k)\\
&+\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|}|a_1|(\delta_{2k}-z_k)\\
&+\sum_{i=1}^{p}\frac{(\psi(\tau)-\psi(z_i))(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\psi'(z_i)\Gamma(\alpha_1)|\Delta|}|a_1|(\delta_{2k}-z_k)\\
&+\sum_{i=1}^{p}\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)|\Delta|}|a_1|(\delta_{2k}-z_k)+\frac{|a_2|}{|\Delta|}(z_{k+1}-\delta_{2k+1})\\
&+\frac{(\psi(\eta)-\psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1)|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1})\\
&+\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1})+\sum_{i=1}^{p}\frac{1}{|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1})\\
&+\sum_{i=1}^{p}\frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1})\Bigg)\Big(\psi(z)-\psi(z_k)\Big)\\
&+\sum_{i=1}^{p}\frac{(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1+1)}\\
&+\sum_{i=1}^{p}\frac{\psi(z)-\psi(z_i)(\psi(z_i)-\psi(z_{i-1}))^{\alpha_1-1}}{\psi'(z_i)\Gamma(\alpha_1)}+\sum_{i=1}^{p}\frac{\psi(z)-\psi(z_i)}{\psi'(z_i)}.\Bigg)\epsilon_1
\end{aligned}
$$

Assume that,

$$
\begin{aligned}
\Omega_5 \quad = \quad & M_{f_1} + \frac{(\psi(z) - \psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1 + 1)} L_{g_1} \\
& + \Bigg( \frac{M_{f_1}}{|\Delta|} + \frac{(\psi(\eta) - \psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1 + 1)|\Delta|} L_{g_1} \\
& + \sum_{i=1}^{p} \frac{(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1 + 1)|\Delta|} L_{g_1} + \\
& + \sum_{i=1}^{p} \frac{\psi(\eta) - \psi(z_i)(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1 - 1}}{\psi'(z_i)\Gamma(\alpha_1)|\Delta|} L_{g_1} \\
& + \sum_{i=1}^{p} \frac{A_1}{|\Delta|} + \sum_{i=1}^{p} \frac{\psi(\eta) - \psi(z_i)}{\psi'(z_i)|\Delta|} A_2 \\
& + \frac{|a_1|}{|\Delta|} (\delta_{2k} - z_k) M_{f_1} \\
& + \frac{(\psi(\tau) - \psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1 + 1)|\Delta|} |a_1| (\delta_{2k} - z_k) L_{g_1} \\
& + \sum_{i=1}^{p} \frac{(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1 + 1)|\Delta|} |a_1| (\delta_{2k} - z_k) L_{g_1} \\
& + \frac{(\psi(\tau) - \psi(z_i))(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1 - 1}}{\psi'(z_i)\Gamma(\alpha_1)|\Delta|} |a_1| (\delta_{2k} - z_k) L_{g_1} + p \frac{A_1}{|\Delta|} |a_1| (\delta_{2k} - z_k) \\
& + \sum_{i=1}^{p} \frac{\psi(\tau) - \psi(z_i)}{\psi'(z_i)|\Delta|} |a_1| (\delta_{2k} - z_k) A_2 + \frac{|a_2|}{|\Delta|} (z_{k+1} - \delta_{2k+1}) M_{f_1} \\
& + \frac{(\psi(\eta) - \psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1)|\Delta|} |a_2| (z_{k+1} - \delta_{2k+1}) L_{g_1} \\
& + \sum_{i=1}^{p} \frac{(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1 + 1)|\Delta|} |a_2| (z_{k+1} - \delta_{2k+1}) L_{g_1} \\
& + p \frac{A_1}{|\Delta|} |a_2| (z_{k+1} - \delta_{2k+1}) \\
& + \frac{A_2}{|\Delta|} |a_2| (z_{k+1} - \delta_{2k+1}) \sum_{i=1}^{p} \frac{\psi(\tau) - \psi(z_i)}{\psi'(z_i)} \Bigg) \Bigg( \psi(z) - \psi(z_k) \Bigg) \\
& + \sum_{i=1}^{p} \frac{(\psi(z_i) - \psi(z_{i-1}))^{\alpha}}{\Gamma(\alpha_1 + 1)} L_{g_1} \\
& + \sum_{i=1}^{p} \frac{\psi(z) - \psi(z_i)(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1 - 1}}{\psi'(z_i)\Gamma(\alpha_1)} L_{g_1} \\
& + pA_1 + A_2 \sum_{i=1}^{p} \frac{\psi(\tau) - \psi(z_i)}{\psi'(z_i)}.
\end{aligned}
$$

$$
\begin{aligned}
\Omega_6 \quad = \quad & N_{f_1} + \frac{(\psi(z) - \psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1 + 1)} K_{g_1} \\
& + \Bigg( \frac{N_{f_1}}{|\Delta|} + \frac{(\psi(\eta) - \psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1 + 1)|\Delta|} K_{g_1} + \sum_{i=1}^{p} \frac{(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1 + 1)|\Delta|} K_{g_1} \\
& + \sum_{i=1}^{p} \frac{\psi(\eta) - \psi(z_i)(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1 - 1}}{\psi'(z_i)\Gamma(\alpha_1)|\Delta|} K_{g_1} + \frac{|a_1|}{|\Delta|}(\delta_{2k} - z_k) N_{f_1} \\
& + \frac{(\psi(\tau) - \psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1 + 1)|\Delta|} |a_1|(\delta_{2k} - z_k) K_{g_1} \\
& + \sum_{i=1}^{p} \frac{(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1 + 1)|\Delta|} |a_1|(\delta_{2k} - z_k) K_{g_1} \\
& + \frac{(\psi(\tau) - \psi(z_i))(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1 - 1}}{\psi'(z_i)\Gamma(\alpha_1)|\Delta|} |a_1|(\delta_{2k} - z_k) K_{g_1} + \frac{|a_2|(z_{k+1} - \delta_{2k+1})}{|\Delta|} N_{f_1} \\
& + \frac{(\psi(\eta) - \psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1)|\Delta|} |a_2|(z_{k+1} - \delta_{2k+1}) K_{g_1} \\
& + \sum_{i=1}^{p} \frac{(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1 + 1)|\Delta|} |a_2|(z_{k+1} - \delta_{2k+1}) K_{g_1} \Bigg) \Big( \psi(z) - \psi(z_k) \Big) \\
& + \sum_{i=1}^{p} \frac{(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1 + 1)} K_{g_1} \\
& + \sum_{i=1}^{p} \frac{\psi(z) - \psi(z_i)(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1 - 1}}{\psi'(z_i)\Gamma(\alpha_1)} K_{g_1}.
\end{aligned}
$$

$$
\begin{aligned}
\Omega_7 \quad = \quad & \frac{(\psi(z) - \psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1 + 1)} + \frac{(\psi(\eta) - \psi(z_k))^{\alpha_1}}{|\Delta|\Gamma(\alpha_1 + 1)} + \sum_{i=1}^{p} \frac{(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1}}{|\Delta|\Gamma(\alpha_1 + 1)} \\
& \sum_{i=1}^{p} \frac{\psi(\eta) - \psi(z_i)(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1 - 1}}{\psi'(z_i)\Gamma(\alpha_1)|\Delta|} \\
& + \sum_{i=1}^{p} \frac{\psi(\eta) - \psi(z_i)}{\psi'(z_i)|\Delta|} + \frac{|a_1|}{|\Delta|}(\delta_{2k} - z_k) \\
& + \frac{(\psi(\tau) - \psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1 + 1)|\Delta|} |a_1|(\delta_{2k} - z_k) \\
& + \sum_{i=1}^{p} \frac{(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1 + 1)|\Delta|} |a_1|(\delta_{2k} - z_k) \\
& + \sum_{i=1}^{p} \frac{(\psi(\tau) - \psi(z_i))(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1 - 1}}{\psi'(z_i)\Gamma(\alpha_1)|\Delta|} |a_1|(\delta_{2k} - z_k) \\
& + \sum_{i=1}^{p} \frac{\psi(\tau) - \psi(z_i)}{\psi'(z_i)|\Delta|} |a_1|(\delta_{2k} - z_k) + \frac{|a_2|}{|\Delta|}(z_{k+1} - \delta_{2k+1}) \\
& + \frac{(\psi(\eta) - \psi(z_k))^{\alpha_1}}{\Gamma(\alpha_1)|\Delta|} |a_2|(z_{k+1} - \delta_{2k+1}) \\
& + \sum_{i=1}^{p} \frac{(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1 + 1)|\Delta|} |a_2|(z_{k+1} - \delta_{2k+1}) + \sum_{i=1}^{p} \frac{1}{|\Delta|} |a_2|(z_{k+1} - \delta_{2k+1}) \\
& + \sum_{i=1}^{p} \frac{\psi(\tau) - \psi(z_i)}{\psi'(z_i)|\Delta|} |a_2|(z_{k+1} - \delta_{2k+1}) \Bigg) \Big( \psi(z) - \psi(z_k) \Big) \\
& + \sum_{i=1}^{p} \frac{(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1}}{\Gamma(\alpha_1 + 1)} \\
& + \sum_{i=1}^{p} \frac{\psi(z) - \psi(z_i)(\psi(z_i) - \psi(z_{i-1}))^{\alpha_1 - 1}}{\psi'(z_i)\Gamma(\alpha_1)} + \sum_{i=1}^{p} \frac{\psi(z) - \psi(z_i)}{\psi'(z_i)}.
\end{aligned}
$$

$$
\begin{aligned}
\Omega_8 \quad = \quad & M_{f_2} + \frac{(\psi(z) - \psi(z_k))^{\beta_1}}{\Gamma(\beta_1+1)} L_{g_2} \\
& + \Bigg( \frac{M_{f_2}}{|\Delta|} + \frac{(\psi(\eta) - \psi(z_k))^{\beta_1}}{\Gamma(\beta_1+1)|\Delta|} L_{g_2} \\
& + \sum_{i=1}^{p} \frac{(\psi(z_i) - \psi(z_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)|\Delta|} L_{g_2} + \\
& + \sum_{i=1}^{p} \frac{\psi(\eta) - \psi(z_i)(\psi(z_i) - \psi(z_{i-1}))^{\beta_1 - 1}}{\psi'(z_i)\Gamma(\beta_1)|\Delta|} L_{g_2} \\
& + \sum_{i=1}^{p} \frac{A_1}{|\Delta|} + \sum_{i=1}^{p} \frac{\psi(\eta) - \psi(z_i)}{\psi'(z_i)|\Delta|} A_2 \\
& + \frac{|a_1|}{|\Delta|} (\delta_{2k} - z_k) M_{f_2} \\
& + \frac{(\psi(\tau) - \psi(z_k))^{\beta_1}}{\Gamma(\beta_1+1)|\Delta|} |a_1| (\delta_{2k} - z_k) L_{g_2} \\
& + \sum_{i=1}^{p} \frac{(\psi(z_i) - \psi(z_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)|\Delta|} |a_1| (\delta_{2k} - z_k) L_{g_2} \\
& + \frac{(\psi(\tau) - \psi(z_i))(\psi(z_i) - \psi(z_{i-1}))^{\beta_1 - 1}}{\psi'(z_i)\Gamma(\beta_1)|\Delta|} |a_1| (\delta_{2k} - z_k) L_{g_2} + p \frac{A_1}{|\Delta|} |a_1| (\delta_{2k} - z_k) \\
& + \sum_{i=1}^{p} \frac{\psi(\tau) - \psi(z_i)}{\psi'(z_i)|\Delta|} |a_1| (\delta_{2k} - z_k) A_2 + \frac{|a_2|}{|\Delta|} (z_{k+1} - \delta_{2k+1}) M_{f_2} \\
& + \frac{(\psi(\eta) - \psi(z_k))^{\beta_1}}{\Gamma(\beta_1)|\Delta|} |a_2| (z_{k+1} - \delta_{2k+1}) L_{g_2} \\
& + \sum_{i=1}^{p} \frac{(\psi(z_i) - \psi(z_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)|\Delta|} |a_2| (z_{k+1} - \delta_{2k+1}) L_{g_2} \\
& + p \frac{A_1}{|\Delta|} |a_2| (z_{k+1} - \delta_{2k+1}) \\
& + \frac{A_2}{|\Delta|} |a_2| (z_{k+1} - \delta_{2k+1}) \sum_{i=1}^{p} \frac{\psi(\tau) - \psi(z_i)}{\psi'(z_i)} \Bigg) \Bigg( \psi(z) - \psi(z_k) \Bigg) \\
& \sum_{i=1}^{p} \frac{(\psi(z_i) - \psi(z_{i-1}))^{\alpha}}{\Gamma(\beta_1+1)} L_{g_2} \\
& + \sum_{i=1}^{p} \frac{\psi(z) - \psi(z_i)(\psi(z_i) - \psi(z_{i-1}))^{\beta_1 - 1}}{\psi'(z_i)\Gamma(\beta_1)} L_{g_2} \\
& + p A_1 + A_2 \sum_{i=1}^{p} \frac{\psi(\tau) - \psi(z_i)}{\psi'(z_i)}.
\end{aligned}
$$

$$
\begin{aligned}
\Omega_9 \quad = \quad & N_{f_2} + \frac{(\psi(z) - \psi(z_k))^{\beta_1}}{\Gamma(\beta_1 + 1)} K_{g_2} \\
& + \Bigg( \frac{N_{f_2}}{|\Delta|} + \frac{(\psi(\eta) - \psi(z_k))^{\beta_1}}{\Gamma(\beta_1 + 1)|\Delta|} K_{g_2} + \sum_{i=1}^{p} \frac{(\psi(z_i) - \psi(z_{i-1}))^{\beta_1}}{\Gamma(\beta_1 + 1)|\Delta|} K_{g_2} \\
& + \sum_{i=1}^{p} \frac{\psi(\eta) - \psi(z_i)(\psi(z_i) - \psi(z_{i-1}))^{\beta_1 - 1}}{\psi'(z_i)\Gamma(\beta_1)|\Delta|} K_{g_2} + \frac{|a_1|}{|\Delta|}(\delta_{2k} - z_k) N_{f_2} \\
& + \frac{(\psi(\tau) - \psi(z_k))^{\beta_1}}{\Gamma(\beta_1 + 1)|\Delta|} |a_1|(\delta_{2k} - z_k) K_{g_2} \\
& + \sum_{i=1}^{p} \frac{(\psi(z_i) - \psi(z_{i-1}))^{\beta_1}}{\Gamma(\beta_1 + 1)|\Delta|} |a_1|(\delta_{2k} - z_k) K_{g_2} \\
& + \frac{(\psi(\tau) - \psi(z_i))(\psi(z_i) - \psi(z_{i-1}))^{\beta_1 - 1}}{\psi'(z_i)\Gamma(\beta_1)|\Delta|} |a_1|(\delta_{2k} - z_k) K_{g_2} + \frac{|a_2|(z_{k+1} - \delta_{2k+1})}{|\Delta|} N_{f_2} \\
& + \frac{(\psi(\eta) - \psi(z_k))^{\beta_1}}{\Gamma(\beta_1)|\Delta|} |a_2|(z_{k+1} - \delta_{2k+1}) K_{g_2} \\
& + \sum_{i=1}^{p} \frac{(\psi(z_i) - \psi(z_{i-1}))^{\beta_1}}{\Gamma(\beta_1 + 1)|\Delta|} |a_2|(z_{k+1} - \delta_{2k+1}) K_{g_2} \Bigg) \Big( \psi(z) - \psi(z_k) \Big) \\
& + \sum_{i=1}^{p} \frac{(\psi(z_i) - \psi(z_{i-1}))^{\beta_1}}{\Gamma(\beta_1 + 1)} K_{g_2} \\
& + \sum_{i=1}^{p} \frac{\psi(z) - \psi(z_i)(\psi(z_i) - \psi(z_{i-1}))^{\beta_1 - 1}}{\psi'(z_i)\Gamma(\beta_1)} K_{g_2}.
\end{aligned}
$$

$$
\begin{aligned}
\Omega_{10} \quad = \quad & \frac{(\psi(z)-\psi(z_k))^{\beta_1}}{\Gamma(\beta_1+1)} + \frac{(\psi(\eta)-\psi(z_k))^{\beta_1}}{|\Delta|\Gamma(\beta_1+1)} + \sum_{i=1}^{p} \frac{(\psi(z_i)-\psi(z_{i-1}))^{\beta_1}}{|\Delta|\Gamma(\beta_1+1)} \\
& \sum_{i=1}^{p} \frac{\psi(\eta)-\psi(z_i)(\psi(z_i)-\psi(z_{i-1}))^{\beta_1-1}}{\psi'(z_i)\Gamma(\beta_1)|\Delta|} \\
& + \sum_{i=1}^{p} \frac{\psi(\eta)-\psi(z_i)}{\psi'(z_i)|\Delta|} + \frac{|a_1|}{|\Delta|}(\delta_{2k}-z_k) \\
& + \frac{(\psi(\tau)-\psi(z_k))^{\beta_1}}{\Gamma(\beta_1+1)|\Delta|}|a_1|(\delta_{2k}-z_k) \\
& + \sum_{i=1}^{p} \frac{(\psi(z_i)-\psi(z_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)|\Delta|}|a_1|(\delta_{2k}-z_k) \\
& + \sum_{i=1}^{p} \frac{(\psi(\tau)-\psi(z_i))(\psi(z_i)-\psi(z_{i-1}))^{\beta_1-1}}{\psi'(z_i)\Gamma(\beta_1)|\Delta|}|a_1|(\delta_{2k}-z_k) \\
& + \sum_{i=1}^{p} \frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)|\Delta|}|a_1|(\delta_{2k}-z_k) + \frac{|a_2|}{|\Delta|}(z_{k+1}-\delta_{2k+1}) \\
& + \frac{(\psi(\eta)-\psi(z_k))^{\beta_1}}{\Gamma(\beta_1)|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1}) \\
& + \sum_{i=1}^{p} \frac{(\psi(z_i)-\psi(z_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1}) + \sum_{i=1}^{p} \frac{1}{|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1}) \\
& + \sum_{i=1}^{p} \frac{\psi(\tau)-\psi(z_i)}{\psi'(z_i)|\Delta|}|a_2|(z_{k+1}-\delta_{2k+1})\Bigg)\Bigg(\psi(z)-\psi(z_k)\Bigg) \\
& + \sum_{i=1}^{p} \frac{(\psi(z_i)-\psi(z_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)} \\
& + \sum_{i=1}^{p} \frac{\psi(z)-\psi(z_i)(\psi(z_i)-\psi(z_{i-1}))^{\beta_1-1}}{\psi'(z_i)\Gamma(\beta_1)} + \sum_{i=1}^{p} \frac{\psi(z)-\psi(z_i)}{\psi'(z_i)}.
\end{aligned}
$$

$$
\|x-\bar{x}\| \quad \leq \quad \Omega_5\|x-\bar{x}\| + \Omega_6\|{}^c\mathcal{D}_{z_k,z}^{\alpha_1;\psi}x - {}^c\mathcal{D}_{z_k,z}^{\alpha_1;\psi}\bar{x}(z)\| + \Omega_7\epsilon_1.
$$

Since

$$
\|{}^c\mathcal{D}_{z_k,z}^{\alpha_1;\psi}x(z) - {}^c\mathcal{D}_{z_k,z}^{\alpha_1;\psi}\bar{x}(z)\| \quad \leq \quad \frac{\frac{(\psi(z)-\psi(z_k))^{1-\alpha_1}}{\Gamma(\alpha_1)}\frac{1}{\psi'(z)}\frac{d}{dz}M_{f_1} + L_{g_1}}{(1-\frac{(\psi(z)-\psi(z_k))^{1-\alpha_1}}{\Gamma(\alpha_1)}\frac{1}{\psi'(z)}\frac{d}{dz}N_{f_1} - K_{g_1})}\|x-\bar{x}\|.
$$

So we have,

$$
\begin{aligned}
\|x-\bar{x}\| \quad \leq \quad & \Omega_5\|x-\bar{x}\| + \Omega_6(\frac{\frac{(\psi(z)-\psi(z_k))^{1-\alpha_1}}{\Gamma(\alpha_1)}\frac{1}{\psi'(z)}\frac{d}{dz}M_{f_1} + L_{g_1}}{(1-\frac{(\psi(z)-\psi(z_k))^{1-\alpha_1}}{\Gamma(\alpha_1)}\frac{1}{\psi'(z)}\frac{d}{dz}N_{f_1} - K_{g_1})} \\
& \times\|x-\bar{x}\| + \Omega_7\epsilon_1,
\end{aligned}
$$

$$
\|x-\bar{x}\| \quad \leq \quad \frac{\Omega_7}{1-(\Omega_5+\Omega_6\frac{\frac{(\psi(z)-\psi(z_k))^{1-\alpha_1}}{\Gamma(\alpha_1)}\frac{1}{\psi'(z)}\frac{d}{dz}M_{f_1}+L_{g_1}}{(1-\frac{(\psi(z)-\psi(z_k))^{1-\alpha_1}}{\Gamma(\alpha_1)}\frac{1}{\psi'(z)}\frac{d}{dz}N_{f_1}-K_{g_1})})}\epsilon_1.
$$

Where we assumed that

$$\Omega_5+\Omega_6\frac{\frac{(\psi(z)-\psi(z_k))^{1-\alpha_1}}{\Gamma(\alpha_1)}\frac{1}{\psi'(z)}\frac{d}{dz}M_{f_1}+L_{g_1}}{(1-\frac{(\psi(z)-\psi(z_k))^{1-\alpha_1}}{\Gamma(\alpha_1)}\frac{1}{\psi'(z)}\frac{d}{dz}N_{f_1}-K_{g_1})}<1.$$

Similarly we have,

$$\|y-\bar{y}\| \leq \frac{\Omega_{10}}{1-(\Omega_8+\Omega_9\frac{\frac{(\psi(z)-\psi(z_k))^{1-\beta_1}}{\Gamma(\beta_1)}\frac{1}{\psi'(z)}\frac{d}{dz}M_{f_2}+L_{g_2}}{(1-\frac{(\psi(z)-\psi(z_k))^{1-\beta_1}}{\Gamma(\beta_1)}\frac{1}{\psi'(z)}\frac{d}{dz}N_{f_2}-K_{g_2})})}\epsilon_2$$

where we assumed that

$$\Omega_8+\Omega_9\frac{\frac{(\psi(z)-\psi(z_k))^{1-\beta_1}}{\Gamma(\beta_1)}\frac{1}{\psi'(z)}\frac{d}{dz}M_{f_2}+L_{g_2}}{(1-\frac{(\psi(z)-\psi(z_k))^{1-\beta_1}}{\Gamma(\beta_1)}\frac{1}{\psi'(z)}\frac{d}{dz}N_{f_2}-K_{g_2})}<1.$$

Let

$$\max\{\epsilon_1,\epsilon_2\}=\epsilon,$$

then

$$\begin{aligned}\|x-\bar{x}\|+\|y-\bar{y}\| \leq\ & \frac{\Omega_7}{1-\Omega_5-\Omega_6\frac{\frac{(\psi(z)-\psi(z_k))^{1-\alpha_1}}{\Gamma(\alpha_1)}\frac{1}{\psi'(z)}\frac{d}{dz}M_{f_1}+L_{g_1}}{(1-\frac{(\psi(z)-\psi(z_k))^{1-\alpha_1}}{\Gamma(\alpha_1)}\frac{1}{\psi'(z)}\frac{d}{dz}N_{f_1}-K_{g_1})}}\epsilon \\ &+\frac{\Omega_{10}}{1-\Omega_8-\Omega_9\frac{\frac{(\psi(z)-\psi(z_k))^{1-\beta_1}}{\Gamma(\beta_1)}\frac{1}{\psi'(z)}\frac{d}{dz}M_{f_2}+L_{g_2}}{(1-\frac{(\psi(z)-\psi(z_k))^{1-\beta_1}}{\Gamma(\beta_1)}\frac{1}{\psi'(z)}\frac{d}{dz}N_{f_2}-K_{g_2})}}\epsilon.\end{aligned}$$

$$\begin{aligned}\|x-\bar{x}\|+\|y-\bar{y}\| \leq\ & \Bigg[\frac{\Omega_7}{1-\Omega_5-\Omega_6\frac{\frac{(\psi(z)-\psi(z_k))^{1-\alpha_1}}{\Gamma(\alpha_1)}\frac{1}{\psi'(z)}\frac{d}{dz}M_{f_1}+L_{g_1}}{(1-\frac{(\psi(z)-\psi(z_k))^{1-\alpha_1}}{\Gamma(\alpha_1)}\frac{1}{\psi'(z)}\frac{d}{dz}N_{f_1}-K_{g_1})}} \\ &+\frac{\Omega_{10}}{1-\Omega_8-\Omega_9\frac{\frac{(\psi(z)-\psi(z_k))^{1-\beta_1}}{\Gamma(\beta_1)}\frac{1}{\psi'(z)}\frac{d}{dz}M_{f_2}+L_{g_2}}{(1-\frac{(\psi(z)-\psi(z_k))^{1-\beta_1}}{\Gamma(\beta_1)}\frac{1}{\psi'(z)}\frac{d}{dz}N_{f_2}-K_{g_2})}}\Bigg]\epsilon.\end{aligned}$$

Let

$$\begin{aligned}\Theta =\ & [\frac{\Omega_7}{1-\Omega_5-\Omega_6\frac{\frac{(\psi(z)-\psi(z_k))^{1-\alpha_1}}{\Gamma(\alpha_1)}\frac{1}{\psi'(z)}\frac{d}{dz}M_{f_1}+L_{g_1}}{(1-\frac{(\psi(z)-\psi(z_k))^{1-\alpha_1}}{\Gamma(\alpha_1)}\frac{1}{\psi'(z)}\frac{d}{dz}N_{f_1}-K_{g_1})}} \\ &+\frac{\Omega_{10}}{1-\Omega_8-\Omega_9\frac{\frac{(\psi(z)-\psi(z_k))^{1-\beta_1}}{\Gamma(\beta_1)}\frac{1}{\psi'(z)}\frac{d}{dz}M_{f_2}+L_{g_2}}{(1-\frac{(\psi(z)-\psi(z_k))^{1-\beta_1}}{\Gamma(\beta_1)}\frac{1}{\psi'(z)}\frac{d}{dz}N_{f_2}-K_{g_2})}}].\end{aligned}$$

Hence, we have
$$\|(x,y)-(\bar{x},\bar{y})\|_{\mathbb{X}'\times\mathbb{Y}'} \leq \Theta\epsilon.$$
Thus system (1) is Ulam-Hyers stable. □

## 6. Example

In this portion, we discuss an example related to our main results.

**Example 1.**

$$\begin{aligned}
{}^{c}\mathcal{D}^{\frac{5}{3};x}[x(z)-(\frac{\cos z|x(z)|}{450(1+|x(z)|)}+\frac{|{}^{c}\mathcal{D}^{\frac{5}{3};x}x(z)|}{460(1+|{}^{c}\mathcal{D}^{\frac{5}{3};x}x(z)|)})] &= \frac{3}{7}+\frac{4+|x(z)|}{1600(1+|x(z)|)} \\
&\quad+\frac{|{}^{c}\mathcal{D}^{\frac{5}{3};x}x(z)|}{1650+|{}^{c}\mathcal{D}^{\frac{5}{3};x}x(z)|}, \\
&\quad z\in[0,1],\ z\neq\frac{4}{5}.
\end{aligned}$$

$$\begin{aligned}
{}^{c}\mathcal{D}^{\frac{5}{3};y}[y(z)-(\frac{\cos z|y(z)|}{450(1+|y(z)|)}+\frac{|{}^{c}\mathcal{D}^{\frac{5}{3};y}y(z)|}{460(1+|{}^{c}\mathcal{D}^{\frac{5}{3};y}y(z)|)})] &= \frac{5}{7}+\frac{|\sin y(z)|}{1200(1+|\sin y(z)|)} \\
&\quad+\frac{1}{1250}|\cos {}^{c}\mathcal{D}^{\frac{5}{3};y}y(z)|. \\
&\quad z\in[0,1], z\neq\frac{5}{6}.
\end{aligned}$$

$$x(0)=0,\quad x(\frac{1}{4})=\int_0^{\frac{1}{4}}x(\tau)d\tau+\int_{\frac{1}{3}}^{\frac{4}{5}}x(\tau)d\tau.$$
$$y(0)=0,\quad y(\frac{1}{4})=\int_0^{\frac{1}{4}}y(\tau)d\tau+\int_{\frac{1}{3}}^{\frac{5}{6}}y(\tau)d\tau.$$
$$I_1x(\frac{4}{5})=\frac{1}{230+|x(z)|},\quad J_1x(\frac{4}{5})=\frac{1}{260+|{}^{c}\mathcal{D}^{\frac{5}{3};x}x(z)|}.$$
$$I_1^*y(\frac{5}{6})=\frac{1}{250+|y(z)|},\quad J_1^*y(\frac{5}{6})=\frac{1}{300+|{}^{c}\mathcal{D}^{\frac{5}{3};y}y(z)|}.$$

*We see in the proposed problem, that* $\alpha_1=\beta_1=\frac{5}{3}$, *and* $z_j\neq\frac{4}{5}$, *where* $j = 1, 2, \ldots, 60$.

*For* $z\in[0,1]$ *and* $x_1(z),x_2(z),y_1(z),y_2(z)\in\mathbb{R}$, *we have,*

$$|f_1(z,x_1(z),{}^{c}\mathcal{D}^{\frac{5}{3};x}x_1(z))-f_1(z,x_2(z),{}^{c}\mathcal{D}^{\frac{5}{3};x}x_2(z))|\leq\frac{1}{450}|x_1(z)-x_2(z)|+\frac{1}{460}|{}^{c}\mathcal{D}^{\frac{5}{3};x}x_1(z)-{}^{c}\mathcal{D}^{\frac{5}{3};x}x_2(z)|.$$

$$|g_1(z,x_1(z),{}^{c}\mathcal{D}^{\frac{5}{3};x}x_1(z))-g_1(z,x_2(z),{}^{c}\mathcal{D}^{\frac{5}{3};x}x_2(z))|\leq\frac{1}{1600}|x_1(z)-x_2(z)|+\frac{1}{1650}|{}^{c}\mathcal{D}^{\frac{5}{3};x}x_1(z)-{}^{c}\mathcal{D}^{\frac{5}{3};x}x_2(z)|.$$

$$|f_2(z,y_1(z),{}^{c}\mathcal{D}^{\frac{5}{3};y}y_1(z))-f_2(z,y_2(z),{}^{c}\mathcal{D}^{\frac{5}{3};y}y_2(z))|\leq\frac{1}{450}|y_1(z)-y_2(z)|+\frac{1}{460}|{}^{c}\mathcal{D}^{\frac{5}{3};y}y_1(z)-{}^{c}\mathcal{D}^{\frac{5}{3};y}y_2(z)|.$$

$$|g_2(z,y_1(z),{}^{c}\mathcal{D}^{\frac{5}{3};y}y_1(z))-g_2(z,y_2(z),{}^{c}\mathcal{D}^{\frac{5}{3};y}y_2(z))|\leq\frac{1}{1200}|y_1(z)-y_2(z)|+\frac{1}{1250}|{}^{c}\mathcal{D}^{\frac{5}{3};y}y_1(z)-{}^{c}\mathcal{D}^{\frac{5}{3};y}y_2(z)|.$$

$$|Ix_1(z)(z_i) - Ix_2(z)(z_i)| \leq \frac{1}{130}|x_1(z) - x_2(z)|, \ |Jx_1(z)(z_i) - Jx_2(z)(z_i)| \leq \frac{1}{160}|x_1(z) - x_2(z)|.$$

$$|I^*y_1(z)(z_j) - I^*y_1(z)(z_j)| \leq \frac{1}{150}|y_1(z) - y_2(z)|, \ |J^*y_1(z)(z_j) - J^*y_2(z)(z_j)| \leq \frac{1}{200}|y_1(z) - y_2(z)|.$$

*From above inequalities, we obtain that* $M_{f_1} = \frac{1}{450}, N_{f_1} = \frac{1}{460}, L_{g_1} = \frac{1}{1600}, K_{g_1} = \frac{1}{1650}, M_{f_2} = \frac{1}{450}, N_{f_2} = \frac{1}{460}, L_{g_2} = \frac{1}{1200}, K_{g_2} = \frac{1}{1250}, \mu_1 = \frac{1}{130}, \mu_2 = \frac{1}{160}, \mu_3 = \frac{1}{150}, \mu_4 = \frac{1}{200}$.

*Since*

$$\begin{aligned} Z_1 &= (\Omega_0 M_{f_1} + \Omega_1 L_{g_1} + \Omega_2) \\ &\quad + (\Omega_0 N_{f_1} + \Omega_1 K_{g_1} + \Omega_2)\Bigg(\frac{\frac{(\psi(z)-\psi(z_k))^{1-\alpha_1}}{\Gamma(\alpha_1)}(\frac{1}{\psi'(z)}\frac{d}{dz})M_{f_1} + L_{g_1}}{1 - (\frac{(\psi(z)-\psi(z_k))^{1-\alpha_1}}{\Gamma(\alpha_1)}(\frac{1}{\psi'(z)}\frac{d}{dz})N_{f_1} + K_{g_1})}\Bigg), \end{aligned}$$

$$\begin{aligned} Z_2 &= (\Omega_0 M_{f_2} + \Omega_3 L_{g_2} + \Omega_4) \\ &\quad + (\Omega_0 N_{f_2} + \Omega_3 K_{g_2} + \Omega_4)\Bigg(\frac{\frac{(\psi(z)-\psi(z_k))^{1-\beta_1}}{\Gamma(\beta_1)}(\frac{1}{\psi'(z)}\frac{d}{dz})M_{f_2} + L_{g_2}}{1 - (\frac{(\psi(z)-\psi(z_k))^{1-\beta_1}}{\Gamma(\beta_1)}(\frac{1}{\psi'(z)}\frac{d}{dz})N_{f_2} + K_{g_2})}\Bigg). \end{aligned}$$

*On calculating* $Z_1$ *and* $Z_2$ *we have* $Z_1 = 0.40370658619 < 1$ *and* $Z_2 = 0.51479638 < 1$. *Then* $\max\{Z_1, Z_2\} < 1$, *and the coupled system* (1) *has unique solution.*

*Also on calculating* $\Theta = 8.92827998857$, *and* $\epsilon = 0.00214$, *we get* $\Theta\epsilon = 0.01910651917 > 0$. *Therefore, the coupled system* (1) *is Ulam-Hyers stable.*

## 7. Conclusion

In this article, we studied the existence and uniqueness property of the coupled system of impulsive hybrid fractional differential equations with slit-strips integral boundary conditions. We utilized the Schaefer's fixed point theorem for the existence of at least one solution of the problem (1). For the uniqueness of the solution of problem (1) we used Banach contraction principle. Also we studied the Ulam-Hyers stability of the proposed problem (1). Finally the we provide an example for the support of the results. Our obtained results can be utilized in the impulsive problems involving the scattering by slits silicon strips detectors for scanned multi-slit X-ray imaging, the acoustic impedance of baffled strips radiators, diffraction from an elastic knife-edge adjacent to a strip, sound fields of infinitely long strips, dielectric-loaded multiple slits in a conducting plane, lattice engineering. The Ulam-Hyers stability means that for any approximation in specific region we will get to the exact distinction, so the obtained results can be utilized in numerical analysis and approximation theory of the related impulsive problems.

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