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Properties of Hadamard Fractional Integral and Its Application

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Abstract: We begin by introducing some function spaces $L_c^p(\mathbb{R}^+), X_c^p(J)$ made up of integrable functions with exponent or power weights defined on infinite intervals, and then we investigate the properties of Mellin convolution operators mapping on these spaces, next, we derive some new boundedness and continuity properties of Hadamard integral operators mapping on $X_c^p(J)$ and $X^p(J)$. Based on this, we investigate a class of boundary value problems for Hadamard fractional differential equations with the integral boundary conditions and the disturbance parameters, and obtain uniqueness results for positive solutions to the boundary value problem under some weaker conditions.

Keywords: hadamard fractional integral operator; boundary value problem; infinite interval; uniqueness

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1. Introduction

Due to its extensive and sustainable development in theory and applications, particularly in various branches of applied sciences including physics, electronics, mechanics, engineering, biology, etc., fractional calculus has attracted a lot of interest in recent decades.

Apart from the most well-known Riemann–Liouville and Caputo fractional integral and derivative, there are other definitions of fractional integrals and derivatives, such as the Hadamard fractional integral and derivative. The fundamental distinction between the Hadamard integral and the Riemann–Liouville or Caputo integral is the type of kernel used; the Hadamard integral contains a logarithmic function, which was first developed by Hadamard in 1892 ([1]), whereas the Riemann–Liouville integral uses a power function.

Another distinction is that Hadamard fractional calculus is more suitable for describing phenomena unrelated to dilation on the semi-axis, while Riemann–Liouville fractional calculus is better appropriate to describe abnormal convection and diffusion phenomena. In igneous rocks, there is a creep phenomenon in the rheology and super slow kinetics. The Lomnitz logarithmic creep law describes it, and Hadamard fractional calculus can more clearly illuminate its mathematical underpinnings ([2–4]). In addition, Hadamard fractional calculus can also be used to describe a wide variety of material mechanics issues, including fracture analysis ([5]).

The properties of Hadamard fractional calculus, including the semigroup property, Mellin transformation formula of Hadamard fractional calculus, the boundedness of Hadamard fractional integrals have been investigated in [6–15]. The consideration of the boundedness, continuity, and compactness of integral operators in earlier work has primarily focused on the expansion of integrable, continuous, or Hölder' continuous functions that are defined on finite intervals. There are not many conclusions about the outcomes of Hadamard integral operators for integrable or continuous functions on infinite intervals.

The boundary value problems of fractional differential equations on infinite intervals have been extensively studied by a large number of researchers in recent decades, see [16–32]. Among them, there are also many studies on the boundary value problem of Hadamard fractional differential equations on infinite intervals, see [25–32]. In [26], the following boundary value problem of Hadamard fractional integro-differential equations on infinite domain was considered

$$\begin{cases} {}^H D^\alpha u(t) + f(t, u(t), {}^H I^\gamma u(t), {}^H D^{\alpha-1}u(t)) = 0,\ 1 < \alpha < 2,\ t \in (1, \infty), \\ u(1) = 0,\ {}^H D^{\alpha-1}u(\infty) = \sum\limits_{i=1}^{m} \lambda_i {}^H I^{\beta_i} u(\eta), \end{cases}$$

where ${}^H D^\alpha$ denotes Hadamard fractional derivative of order α, ${}^H I^{(\cdot)}$ is the Hadamard fractional integral, $\eta \in (1, \infty), \gamma, \beta_i, \lambda_i \geq 0 (i = 1, 2, \ldots, m.)$ and $\Gamma(\alpha) > \sum\limits_{i=1}^{m} \frac{\lambda_i \Gamma(\alpha)}{\Gamma(\alpha+\beta_i)} (\log \eta)^{\alpha+\beta_i-1}$. $f(t, u, v, w) : [1, \infty) \times \mathbb{R}^3 \to \mathbb{R}^+$ is nondecreasing with respect to u, v, w. By using monotone iterative technique, the existence of positive solutions was obtained, meanwhile the positive minimal and maximal solutions and two explicit monotone iterative sequences which converge to the extremal solutions were acquired.

Similarly, Wang et al. [27] used monotone iterative technique to investigate a new class of boundary value problems of one-dimensional lower-order nonlinear Hadamard fractional differential equations and nonlocal multipoint discrete and Hadamard integral boundary conditions

$$\begin{cases} {}^H D^q x(t) + \sigma(t) f(t, x(t)) = 0,\ 2 < q \leq 3,\ t \in (1, +\infty), \\ x(1) = x'(1) = 0, {}^H D^{q-1}x(\infty) = a {}^H I^\beta x(\xi) + b \sum\limits_{i=1}^{m-2} \alpha_i x(\eta_i), \end{cases}$$

where ${}^H D^q$ denotes Hadamard fractional derivative of order q, $\beta > 0, 1 < \xi < \eta_1 < \eta_2 < \ldots < +\infty, a, b \in R, \alpha_i > 0 (i = 1, 2, \ldots, m-2)$. $\sigma : [1, \infty) \to [0, \infty)$ and $0 < \int_1^\infty \sigma(s) \frac{ds}{s} < \infty$. $f \in C([1, \infty) \times [0, \infty), [0, \infty))$.

In [28], by making use of a fixed point theorem for generalized concave operators, the existence and uniqueness of positive solutions for a new class of Hadamard fractional differential equations on infinite intervals is established

$$\begin{cases} {}^H D^\nu x(t) + b(t) f(t, x(t)) + c(t) = 0,\ 1 < \nu < 2,\ t \in (1, \infty), \\ x(1) = 0,\ {}^H D^{\nu-1}x(\infty) = \sum\limits_{i=1}^{m} \gamma_i {}^H I^{\beta_i} x(\eta), \end{cases}$$

where ${}^H D^\nu$ denotes the Hadamard fractional derivative of order ν, $\beta_i, \gamma_i \geq 0 (i = 1, 2, \ldots, m)$, $\eta \in (1, \infty)$ and $\Gamma(\nu) > \sum\limits_{i=1}^{m} \frac{\gamma_i \Gamma(\nu)}{\Gamma(\nu+\beta_i)} (\log \eta)^{\nu+\beta_i-1}$. $b, c \in C([1, \infty), [0, \infty)), f : [1, \infty) \times [0, \infty) \to [0, \infty)$ is continuous. By making use of a fixed point theorem for generalized concave operators, the existence and uniqueness of positive solutions was established.

In [30], utilizing the generalized Avery–Henderson fixed point theorem, Zhang and Ni presented a new result on the existence of positive solutions for Hadamard-type fractional differential equation with more general boundary conditions on infinite interval as follows

$$\begin{cases} {}^H D_{1+}^\alpha x(t) + a(t) f(t, x(t)) = 0,\ 2 < \alpha < 3,\ t \in (1, +\infty), \\ x(1) = x'(1) = 0,\ {}^H D_{1+}^{\alpha-1} x(+\infty) = \sum\limits_{i=1}^{m} \alpha_i {}^H I_{1+}^{\beta_i} x(\eta) + \rho \sum\limits_{j=1}^{n} \sigma_j x(\xi_j), \end{cases}$$

where ${}^H D_{1+}^\alpha$ is the Hadamard-type fractional derivative of order α. $\beta_i > 0 (i = 1, 2, \ldots, m)$; $1 < \eta < \xi_1 < \xi_2 < \ldots < \xi_n < +\infty; \rho, \alpha_i, \sigma_j \geq 0 (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$ and $\Gamma(\alpha) - \sum\limits_{i=1}^{m} \alpha_i \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta_i)} (\ln \eta)^{\alpha+\beta_i-1} - \rho \sum\limits_{j=1}^{n} \sigma_j (\ln \xi_j)^{\alpha-1} > 0$. $a : [1, \infty) \to [0, \infty)$ and $0 < \int_1^\infty a(s) \frac{ds}{s} < \infty$. $f : [1, \infty) \times [0, \infty) \to [0, \infty)$ and $f(t, 0) \not\equiv 0$ on any subinterval of $[1, \infty)$.

Inspired by these above studies and other relevant references, this paper will study the existence of unique solution for the following Hadamard fractional differential equation

$$\mathcal{D}^{\alpha}_{1+}u(t) + f(t, u(t), \mathcal{J}^{\beta}_{1+}u(t)) = 0, \quad 1 < t < +\infty, \tag{1}$$

supplemented with integral boundary conditions and disturbance parameters

$$\begin{cases} \mathcal{D}^{\alpha-3}_{1+}u(1) = \lambda_3 + \mu_3 \int_1^{+\infty} g_3(s)u(s)\frac{ds}{s}, \\ \mathcal{D}^{\alpha-2}_{1+}u(1) = \lambda_2 + \mu_2 \int_1^{+\infty} g_2(s)u(s)\frac{ds}{s}, \\ \mathcal{D}^{\alpha-1}_{1+}u(+\infty) = \lambda_1 + \mu_1 \int_1^{+\infty} g_1(s)u(s)\frac{ds}{s}, \end{cases} \tag{2}$$

where $2 < \alpha < 3, \beta > 0$. $\mu_i, \lambda_i \geq 0 (i = 1, 2, 3)$ and at least one of these parameters is positive. $f : J \times (\mathbb{R}^+)^3 \to \mathbb{R}^+, g_i : J \to \mathbb{R}^+(i = 1, 2, 3), J = (1, +\infty), \mathbb{R}^+ = [0, \infty)$ and f may be singular at $t = 1$. $\mathcal{D}^{\alpha}_{1+}$ denotes Hadamard fractional derivative of order α, $\mathcal{J}^{\beta}_{1+}$ denotes the left-sided Hadamard fractional integral of order β.

Compared to the existing papers, the new insights presented in this paper can be summed up as follows: first, some new properties of Hadamard fractional integral operator acting on functions defined on infinite intervals are presented, and these properties are demonstrated in Section 3; second, the function f may be singular at $t = 1$ and it should be mentioned that the boundary value problem takes non-zero values at the initial point and has the boundary values containing integrals and disturbance parameters ; and finally, with the help of some of the results from Sections 3 and 4, the existence of unique solution for the boundary value problem of Hadamard fractional differential equation is obtained in two different function spaces. A special point to mention is that we prove that there exists a unique positive solution to the boundary value problem on the space $X^p_c(J)$ under weaker conditions.

The remainder of this paper is structured as follows. Section 2 includes some mapping properties of the Mellin convolution operator among function spaces. By utilizing these properties, in Section 3, we obtain the boundedness and continuity of Hadamard integral operator in the spaces $X^p_c(J)$ and $X^p(J)$. Section 4 gives some auxiliary results that will be used in the study of the Hadamard fractional boundary value problem. In Sections 5 and 6, uniqueness results of Hadamard fractional boundary value problem are acquired in two different function spaces. In Section 7, we give a conclusion to summarize the core and highlights of the whole paper.

2. Auxiliary Results

In this section, we present auxiliary results needed in our proofs later. From here on, for a positive real number β, we use ϑ_β to denote the function defined on J or $J_1 = [1, \infty)$ by $\vartheta_\beta(t) = (\ln t)^{\beta-1}/\Gamma(\beta)$.

The left-sided Hadamard fractional integral of order $\alpha (\alpha > 0)$ on the infinite interval J has the form

$$(\mathcal{J}^{\alpha}_{1+}f)(x) = \frac{1}{\Gamma(\alpha)} \int_1^x \left(\ln \frac{x}{t}\right)^{\alpha-1} f(t)\frac{dt}{t}.$$

This definition is just a formal one. Obviously, the rationality of the definition lies in the selection of appropriate functions for the existence of the integral, that is, the selection of functions to ensure that the integral is convergent. In some literature, we can see some results, not only can we see what kind of function can guarantee the existence of fractional integrals, but also get the boundedness of integral operators, see ([6–15]).

Let us introduce some function spaces. One is the space of p- integrable functions defined on J, i.e., $L^p(J)(1 \leq p \leq \infty)$, whose norm is defined by

$$\|f\|_{L^p(J)} = \left(\int_1^{\infty} |f(t)|^p dt\right)^{\frac{1}{p}} (1 \leq p < \infty), \ \|f\|_{L^\infty(J)} = ess \sup_{t \in J} |f(t)|.$$

If the p- Lebesgue integrable function is defined on $\mathbb{R}^+$, we denote the Lebesgue integrable function space as $L^p(\mathbb{R}^+)$, whose norm is written as $\|f\|_{L^p(\mathbb{R}^+)}$.

The other is the space with exponential weight consisting of those real-valued Lebesgue measurable functions $f : \mathbb{R}^+ \to \mathbb{R}$ denoted by $L^p_c(\mathbb{R}^+)(1 \le p \le \infty, c \in \mathbb{R})$ by defining the norm

$$\begin{cases} \|f\|_{L^p_c(\mathbb{R}^+)} = \left(\int_0^\infty |e^{ct}f(t)|^p dt\right)^{\frac{1}{p}}, 1 \le p < \infty, c \in \mathbb{R}, \\ \|f\|_{L^\infty_c(\mathbb{R}^+)} = ess \sup\limits_{t\in\mathbb{R}_+} |e^{ct}f(t)|, \ c \in \mathbb{R}. \end{cases} \tag{3}$$

The space $L^p_c(1 \le p \le \infty, c \in \mathbb{R})$ was defined in [7], where it contained those complex-valued Lebesgue measurable functions f defined on $\mathbb{R}$ with $\|f\|_{L^p_c} < \infty$.

The third is the weighted L^p- space with the power weight, which is denoted by $X^p_c(J)(c \in \mathbb{R}, 1 \le p \le \infty)$ and consists of those Lebesgue measurable functions f on J for which $\|f\|_{X^p_c} < \infty$, where

$$\begin{cases} \|f\|_{X^p_c(J)} = \left(\int_1^\infty |t^c f(t)|^p \frac{dt}{t}\right)^{\frac{1}{p}}, \ 1 \le p < \infty, c \in \mathbb{R}, \\ \|f\|_{X^\infty_c(J)} = ess \sup\limits_{t\in J}[t^c|f(t)|], \ c \in \mathbb{R}. \end{cases} \tag{4}$$

The space $X^p_c(1 \le p \le \infty, c \in \mathbb{R})$, first defined in [6], is composed of complex-valued Lebesgue measurable functions f on finite intervals $[a, b]$ satisfying $\|f\|_{X^p_c} < \infty$. In particular, for $c = 0$, we denote $X^p_0(J) = X^p(J)$ and the norm is defined as

$$\|f\|_{X^p(J)} = \left(\int_1^\infty |f(t)|^p \frac{dt}{t}\right)^{\frac{1}{p}} (1 \le p < \infty), \ \|f\|_{X^\infty(J)} = ess \sup_{t\in J} |f(t)|. \tag{5}$$

As for the relation between those function spaces mentioned above and Lebesgue integrable functions space $L^p(J)$, from the norm relation between each other, namely $\|f\|_{X^p_c(J)} \le \|f\|_{X^p(J)} \le \|f\|_{L^p(J)}(c < 0, 1 \le p \le \infty)$, we can conclude that

$$L^p(J) \subseteq X^p(J) \subseteq X^p_c(J), \ c < 0, 1 \le p \le \infty. \tag{6}$$

For $\alpha > 0, 1 \le r < \infty$, we have

$$\int_1^\infty \vartheta^r_\alpha(t)\frac{dt}{t} = \frac{1}{(\Gamma(\alpha))^r}\int_0^\infty (\ln t)^{(\alpha-1)r}\frac{dt}{t} = \frac{1}{(\Gamma(\alpha))^r}\int_0^\infty s^{(\alpha-1)r}\frac{ds}{s}.$$

The integral $\int_0^\infty s^{(\alpha-1)r}\frac{ds}{s}$ is never finite for $\alpha > 0, 1 \le r < \infty$. For $\alpha > 0, r = \infty$, we know $ess \sup\limits_{t\in J}[|\vartheta_\alpha(t)|] = \infty$. Hence $\vartheta_\alpha \notin X^r(J)(\alpha > 0, 1 \le r \le \infty)$. From inclusion relation (6), we further deduce $\vartheta_\alpha \notin L^r(J)(\alpha > 0, 1 \le r \le \infty)$. Consider another integral

$$\begin{aligned} \int_1^\infty |t^c\vartheta_\alpha(t)|^r\frac{dt}{t} &= \frac{1}{(\Gamma(\alpha))^r}\int_0^\infty e^{crs}s^{(\alpha-1)r}ds \\ &= \frac{1}{(\Gamma(\alpha))^r}\left[\int_0^1 e^{crs}s^{(\alpha-1)r}ds + \int_1^\infty e^{crs}s^{(\alpha-1)r}ds\right]. \end{aligned}$$

By convergence discriminant of the integral, if $\alpha \ge 1, 1 \le r < \infty, c < 0$, we know the improper integral $\int_0^\infty e^{crs}s^{(\alpha-1)r}ds$ is convergent. If $0 < \alpha < 1, c < 0$, $\int_0^1 e^{crs}s^{(\alpha-1)r}ds$ is convergent under the condition $1 \le r < \frac{1}{1-\alpha}$ and divergent under the condition $\frac{1}{1-\alpha} \le r < \infty$, meanwhile, $\int_1^\infty e^{crs}s^{(\alpha-1)r}ds$ is convergent for any $1 \le r < \infty$. To sum up, when $0 < \alpha < 1, c < 0$, we have

$$\int_1^\infty |t^c\vartheta_\alpha(t)|^r\frac{dt}{t} < \infty(1 \le r < \frac{1}{1-\alpha}) \ and \ \int_1^\infty |t^c\vartheta_\alpha(t)|^r\frac{dt}{t} = \infty(\frac{1}{1-\alpha} \le r < \infty).$$

In addition,

$$ess\sup_{t\in J}[t^c|\vartheta_\alpha(t)|]\begin{cases} < \infty, \ \alpha \geq 1, c < 0, \\ = \infty, \ 0 < \alpha < 1, c < 0. \end{cases}$$

In summary, the subordinate inclusion relationships of the function $\vartheta_\alpha(t)$ with spaces $X_c^r(J)(c<0)$, $X^r(J)$ and $L^r(J)$ respectively are presented in the following Table 1.

Table 1. Inclusion relation.

Condition	$X_c^r(J)(c<0)$	$X^r(J)$	$L^r(J)$
$\alpha \geq 1, 1 \leq r \leq \infty$	$\in$	$\notin$	$\notin$
$0<\alpha<1, 1 \leq r < \frac{1}{1-\alpha}$	$\in$	$\notin$	$\notin$
$0<\alpha<1, \frac{1}{1-\alpha} \leq r \leq \infty$	$\notin$	$\notin$	$\notin$

It is known that the classical Riemann–Liouville fractional integral of order $\alpha > 0$ on the half-axis $\mathbb{R}^+$ is defined by

$$(\mathcal{I}_{0+}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)dt, \ \ (x>0).$$

In order to establish the connection between Riemann–Liouville fractional integral and Hadamard fractional integral, we have to introduce an elementary operator. For a real-valued function $f(x)$ defined almost everywhere on $\mathbb{R}^+$, the operator A is defined as follows:

$$(Af)(x) = f(e^x).$$

Then for a function g defined almost everywhere J, its inverse A^{-1} has the form

$$(A^{-1}g)(x) = g(\ln x).$$

Using these two operators, we establish the connection between Hadamard fractional integral and Riemann–Liouville fractional integral, which can be shown by the relation:

$$(\mathcal{J}_{1+}^\alpha f)(x) = (A^{-1}\mathcal{I}_{0+}^\alpha Af)(x). \tag{7}$$

Remark 1. *The aforementioned operators A, A^{-1} are indicated in part in [7].*

Theorem 1 ([7])**.** *Let $c \in \mathbb{R}$ and $1 \leq p \leq \infty$.*

$$\begin{cases} A: \ X_c^p(J) \to L_c^p(\mathbb{R}^+) \ \ or \ \ X_c^p(\mathbb{R}^+) \to L_c^p(\mathbb{R}), \\ A^{-1}: L_c^p(\mathbb{R}^+) \to X_c^p(J) \ \ or \ \ L_c^p(\mathbb{R}) \to X_c^p(\mathbb{R}^+) \end{cases}$$

and A, A^{-1} are isometric isomorphism, that is $\|Af\|_{L_c^p} = \|f\|_{X_c^p}$, $\|A^{-1}f\|_{X_c^p} = \|f\|_{L_c^p}$.

It is obvious that the following Corollary holds when $c = 0$.

Corollary 1. *Assume $1 \leq p \leq \infty$.*
(1) A is isometric isomorphism of $X^p(J)$ onto $L^p(\mathbb{R}^+)$, and $\|Af\|_{L^p(\mathbb{R}^+)} = \|f\|_{X^p(J)}$.
(2) A^{-1} is isometric isomorphism of $L^p(\mathbb{R}^+)$ onto $X^p(J)$, and $\|Af\|_{X^p(J)} = \|f\|_{L^p(\mathbb{R}^+)}$.

Let h and f be real-valued functions defined on J, then Mellin convolution product, written as $h * f$, is the function defined by

$$(h*f)(x) := \int_1^x h\left(\frac{x}{t}\right) f(t)\frac{dt}{t}. \tag{8}$$

Remark 2. *The definition of Mellin convolution product $h * f$ of functions f and h is a little different form the definition in [7], where the integral interval is from 1 to ∞. Moreover, we can also obtain the relation $h * f = f * h$ from the definition.*

Remark 3. *According to the definition of Mellin convolution product (8), we rewrite the integral definition as*

$$(\mathcal{J}_{1+}^{\alpha} f)(x) = (\vartheta_\alpha * f)(x). \tag{9}$$

In view of Hölder's inequality, we get the following mapping properties (Theorem 2 (1) and (2)) of the Mellin convolution operator in X_c^p and, just like Young's inequality ([33,34]), we derive a similar result (Theorem 2 (3)), which is described but not proved in the literature [7].

Theorem 2. *Assume $c \in \mathbb{R}, 1 \le p, q, r \le \infty$, let $1 - \frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. Suppose $h \in X_c^r(J), f \in X_c^p(J)$. Then $h * f \in X_c^q(J)$ and $\|h * f\|_{X_c^q(J)} \le \|h\|_{X_c^r(J)} \|f\|_{X_c^p(J)}$. Specifically,*
(1) If $r = 1, q = p$, then

$$\|h * f\|_{X_c^p(J)} \le \|h\|_{X_c^1(J)} \|f\|_{X_c^p(J)} (1 \le p \le \infty).$$

(2) If $r = p'$ (p' be the exponent conjugate to p) and $q = \infty$, then

$$\|h * f\|_{X_c^\infty(J)} \le \|h\|_{X_c^{p'}(J)} \|f\|_{X_c^p(J)}.$$

(3) If $1 < r < p', \max\{p, r\} < q < \infty$ and $p < \infty$, then

$$\|h * f\|_{X_c^q(J)} \le \|h\|_{X_c^r(J)} \|f\|_{X_c^p(J)}.$$

Proof of Theorem 2. The proof of the conclusions (1) and (2) is similar to the proof in [7], then the process is omitted.

(3) Put $s = \frac{pq}{q-p}, t = \frac{qr}{q-r}$, then $\frac{1}{s} + \frac{1}{t} + \frac{1}{q} = 1$. By the generalized Hölder's inequality, for $h \in X_c^r(J), f \in X_c^p(J)$, we have

$$\begin{aligned}
&|(h * f)(x)| \\
\le& \int_1^x \left| y^{-c} h\left(\frac{x}{y}\right) y^c f(y) \right| \frac{dy}{y} \\
=& \int_1^x \frac{\left| y^{-c} h\left(\frac{x}{y}\right)\right|^{1-\frac{r}{q}}}{y^{\frac{1}{t}}} \frac{|y^c f(y)|^{1-\frac{p}{q}}}{y^{\frac{1}{s}}} \frac{\left| y^{-c} h\left(\frac{x}{y}\right)\right|^{\frac{r}{q}} |y^c f(y)|^{\frac{p}{q}}}{y^{\frac{1}{q}}} dy \\
\le& \left(\int_1^x \left| y^{-c} h\left(\frac{x}{y}\right)\right|^{t(1-\frac{r}{q})} \frac{dy}{y} \right)^{\frac{1}{t}} \left(\int_1^x |y^c f(y)|^{s(1-\frac{p}{q})} \frac{dy}{y} \right)^{\frac{1}{s}} \\
& \left(\int_1^x \left| y^{-c} h\left(\frac{x}{y}\right)\right|^{r} |y^c f(y)|^p \frac{dy}{y} \right)^{\frac{1}{q}} \\
\le& x^{\frac{-cr}{t}} (\|h\|_{X_c^r(J)})^{\frac{r}{t}} (\|f\|_{X_c^p}(J))^{\frac{p}{s}} \left(\int_1^x \left| y^{-c} h\left(\frac{x}{y}\right)\right|^{r} |y^c f(y)|^p \frac{dy}{y} \right)^{\frac{1}{q}},
\end{aligned}$$

then

$$
\begin{aligned}
&\|h*f\|^q_{X^q_c(J)}\\
\leq&(\|h\|_{X^r_c(J)})^{\frac{qr}{t}}(\|f\|_{X^p_c(J)})^{\frac{qp}{s}}\int_1^\infty x^{cq}x^{\frac{-qcr}{t}}\int_1^x\left|y^{-c}h\left(\frac{x}{y}\right)\right|^r|y^cf(y)|^p\frac{dy}{y}\frac{dx}{x}\\
\leq&(\|h\|_{X^r_c(J)})^{\frac{qr}{t}}(\|f\|_{X^p_c(J)})^{\frac{qp}{s}}\int_1^\infty\left(\int_y^\infty x^{cr}\left|h\left(\frac{x}{y}\right)\right|^r\frac{dx}{x}\right)y^{cp-cq}|f(y)|^p\frac{dy}{y}\\
\leq&(\|h\|_{X^r_c(J)})^{\frac{qr}{t}}(\|f\|_{X^p_c(J)})^{\frac{qp}{s}}\int_1^\infty|u^ch(u)|^r\frac{du}{u}\int_1^\infty|y^cf(y)|^p\frac{dy}{y}\\
=&(\|h\|_{X^r_c(J)})^q(\|f\|_{X^p_c(J)})^p.
\end{aligned}
$$

Hence, $\|h*f\|_{X^q_c(J)}\leq\|h\|_{X^r_c(J)}\|f\|_{X^p_c(J)}$. □

Following the same technique, similar results are obtained in $X^p(J)$ and $L^p(J)$.

Corollary 2. *Assume $1\leq p,q,r\leq\infty$, let $1+\frac{1}{q}=\frac{1}{p}+\frac{1}{r}$. Suppose $h\in X^r(J), f\in X^p(J)$. Then $h*f\in X^q(J)$ and $\|h*f\|_{X^q(J)}\leq\|h\|_{X^r(J)}\|f\|_{X^p(J)}$.*

Corollary 3. *Assume $1\leq p,q,r\leq\infty$, let $1+\frac{1}{q}=\frac{1}{p}+\frac{1}{r}$. Suppose $h\in L^r(J), f\in L^p(J)$. Then $h*f\in L^q(J)$ and $\|h*f\|_{L^q(J)}\leq\|h\|_{L^r(J)}\|f\|_{L^p(J)}$.*

There is a well-known fact that the Riemann–Liouville fractional integral $\mathcal{I}^\alpha_{0+}$ is a Laplace convolution operator of the form

$$
\mathcal{I}^\alpha_{0+}f(x)=\int_0^x\frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}f(t)dt=(k\circledast f)(x),
$$

where $k(x)=\frac{x^{\alpha-1}}{\Gamma(\alpha)}$, $(k\circledast f)(x)=\int_0^xk(x-t)f(t)dt$. It is obvious that the Laplace convolution operator has similar properties to the Mellin convolution operator.

Theorem 3. *Let $1\leq p\leq\infty, c\in\mathbb{R}$.*
(a) Suppose $k\in L^1_c(\mathbb{R}^+), f\in L^p_c(\mathbb{R}^+)$, then $k\circledast f\in L^p_c(\mathbb{R}^+)$ and the following estimate holds:

$$
\|k\circledast f\|_{L^p_c(\mathbb{R}^+)}\leq\|k\|_{L^1_c(\mathbb{R}^+)}\|f\|_{L^p_c(\mathbb{R}^+)}.
$$

(b) Let $1\leq q<\infty, \frac{1}{r}=1+\frac{1}{q}-\frac{1}{p}$. If $k\in L^r_c(\mathbb{R}^+), f\in L^p_c(\mathbb{R}^+)$, then $k\circledast f$ is bounded from $L^p_c(\mathbb{R}^+)$ to $L^q_c(\mathbb{R}^+)$ and

$$
\|k\circledast f\|_{L^q_c(\mathbb{R}^+)}\leq\|k\|_{L^r_c(\mathbb{R}^+)}\|f\|_{L^p_c(\mathbb{R}^+)}.
$$

Applying Theorem 3 to Riemann–Liouville fractional integrals, we will get the properties of Riemann–Liouville fractional integrals as follows.

Theorem 4. *Let $1\leq p,q\leq\infty, c<0$.*
(1) $\mathcal{I}^\alpha_{0+}:L^p_c(\mathbb{R}^+)\to L^p_c(\mathbb{R}^+)$ is bounded, and $\|\mathcal{I}^\alpha_{0+}f\|_{L^p_c(\mathbb{R}^+)}\leq|c|^{-\alpha}\|f\|_{L^p_c(\mathbb{R}^+)}$.
(2) If $1\leq p\leq q\leq\infty$ and $\alpha>\frac{1}{p}-\frac{1}{q}$. Then the operator $\mathcal{I}^\alpha_{0+}$ is bounded from $L^p_c(\mathbb{R}^+)$ into $L^q_c(\mathbb{R}^+)$, and $\|\mathcal{I}^\alpha_{0+}f\|_{L^q_c(\mathbb{R}^+)}\leq(|c|r)^{-\alpha+1-\frac{1}{r}}\frac{\Gamma[(\alpha-1)r+1]^{\frac{1}{r}}}{\Gamma(\alpha)}\|f\|_{L^p_c(\mathbb{R}^+)}$.

3. Properties of Hadamard Fractional Integrals

In this section, we first establish the following boundedness property of Hadamard integral operator in the spaces $X^p_c(J)$ and $X^p(J)$.

Property 1. *Let $1 \le p \le \infty, \alpha > 0, c < 0$. Then the integral operator $\mathcal{J}_{1+}^{\alpha}$ is bounded in $X_c^p(J)$ and*

$$\|\mathcal{J}_{1+}^{\alpha} f\|_{X_c^p(J)} \le C\|f\|_{X_c^p(J)}, \;\; C = |c|^{-\alpha}.$$

Proof of Property 1. Let us first take the case where $p = 1$. For any $f \in X_c^1(J)$, Utilizing Fubini's theorem and appropriate variable substitution, we have

$$\begin{aligned}
\|\mathcal{J}_{1+}^{\alpha} f\|_{X_c^1(J)} &\le \frac{1}{\Gamma(\alpha)} \int_1^{\infty} x^c \int_1^x \left(\ln \frac{x}{t}\right)^{\alpha-1} |f(t)| \frac{dt}{t} \frac{dx}{x} \\
&= \frac{1}{\Gamma(\alpha)} \int_1^{\infty} \left(\int_t^{\infty} x^c \left(\ln \frac{x}{t}\right)^{\alpha-1} \frac{dx}{x} \right) |f(t)| \frac{dt}{t} \\
&= \frac{1}{\Gamma(\alpha)} \int_1^{\infty} t^c |f(t)| \frac{dt}{t} \int_0^{\infty} e^{cy} y^{\alpha-1} dy \\
&= |c|^{-\alpha} \|f\|_{X_c^1(J)}.
\end{aligned}$$

If $1 < p < \infty$, by (9), Hadamard integral operator is considered as the Mellin convolution operation, on the basis of the generalized Minkowski inequality, we derive the following inequality

$$\begin{aligned}
\Gamma(\alpha)\|\mathcal{J}_{1+}^{\alpha} f\|_{X_c^p(J)} &= \left[\int_1^{\infty} x^{cp} \left| \int_1^x (\ln t)^{\alpha-1} f\left(\frac{x}{t}\right) \frac{dt}{t} \right|^p \frac{dx}{x} \right]^{\frac{1}{p}} \\
&\le \left[\int_1^{\infty} x^{cp} \left(\int_1^x (\ln t)^{\alpha-1} \left| f\left(\frac{x}{t}\right) \right| \frac{dt}{t} \right)^p \frac{dx}{x} \right]^{\frac{1}{p}} \\
&\le \int_1^{\infty} (\ln t)^{\alpha-1} \frac{dt}{t} \left(\int_t^{\infty} x^{cp} \left| f\left(\frac{x}{t}\right) \right|^p \frac{dx}{x} \right)^{\frac{1}{p}}.
\end{aligned}$$

Making variable substitution $x = ty$, the above integral is then transformed into

$$\int_t^{\infty} x^{cp} \left| f\left(\frac{x}{t}\right) \right|^p \frac{dx}{x} = t^{cp} \left(\int_1^{\infty} y^{cp} |f(y)|^p \frac{dy}{y} \right).$$

Hence, we further derive

$$\begin{aligned}
\Gamma(\alpha)\|\mathcal{J}_{1+}^{\alpha} f\|_{X_c^p(J)} &\le \int_1^{\infty} t^c (\ln t)^{\alpha-1} \frac{dt}{t} \left(\int_1^{\infty} y^{cp} |f(y)|^p \frac{dy}{y} \right)^{\frac{1}{p}} \\
&= \Gamma(\alpha)|c|^{-\alpha} \|f\|_{X_c^p(J)},
\end{aligned}$$

then

$$\|\mathcal{J}_{1+}^{\alpha} f\|_{X_c^p(J)} \le |c|^{-\alpha} \|f\|_{X_c^p(J)}.$$

If $p = \infty$, for almost $x \in J$, we get

$$\begin{aligned}
|x^c \mathcal{J}_{1+}^{\alpha} f(x)| &\le \frac{\|f\|_{X_c^{\infty}(J)}}{\Gamma(\alpha)} x^c \int_1^x \left(\ln \frac{x}{t}\right)^{\alpha-1} t^{-c} \frac{dt}{t} \\
&\le \frac{\|f\|_{X_c^{\infty}(J)}}{\Gamma(\alpha)} \int_0^{\infty} y^{\alpha-1} e^{cy} dy \\
&= |c|^{-\alpha} \|f\|_{X_c^{\infty}(J)}.
\end{aligned}$$

This completes the proof of the property. □

Remark 4. *The theorem has been formulated as a special case in [7,11], where the following two methods have been used to prove it. From Table 1 we know that $\vartheta_\alpha \in X_c^1$, the norm can be computed* *directly* $\|\vartheta_\alpha\|_{X_c^1} = |c|^{-\alpha}$. *By* (9), *applying Theorem* 2 *(1) to Hadamard integral operator, we get Property* 1. *In fact,*

$$\|\mathcal{J}_{1+}^{\alpha} f\|_{X_c^p} = \|\vartheta_\alpha * f\| \le \|\vartheta_\alpha\|_{X_c^1}\|f\|_{X_c^p} = |c|^{-\alpha}\|f\|_{X_c^p} = C\|f\|_{X_c^p}.$$

There is another way to prove it. From (7), *using Theorem* 1 *and Theorem* 4 *repeatedly, for any* $f \in X_c^p(J)$, *we will get* $\mathcal{J}_{1+}^{\alpha} f \in X_c^p(J)$.

Property 2. *Suppose* $1 \le p \le q \le \infty, \alpha > \frac{1}{p} - \frac{1}{q}, c < 0$, *let* $\frac{1}{r} = \frac{1}{q} - \frac{1}{p} + 1$. *Then* $\mathcal{J}_{1+}^{\alpha} : X_c^p(J) \to X_c^q(J)$ *is bounded and*

$$\|\mathcal{J}_{1+}^{\alpha} f\|_{X_c^q(J)} \le C_1\|f\|_{X_c^p(J)}, \ C_1 = (|c|r)^{-\alpha+1-\frac{1}{r}}\frac{\Gamma[(\alpha-1)r+1]^{\frac{1}{r}}}{\Gamma(\alpha)}.$$

Remark 5. *As with Property* 1, *we can obtain Property* 2 *in two ways. One is applying the properties of Mellin convolution operator, i.e., Theorem* 2 *(3), to* (9). *The other is using an operator-theoretic approach, according to* (7), *combining Theorem* 1 *and Theorem* 4, *we confirm that Property* 2 *is established.*

The conclusion in Theorem 9 in [7], *left a little problem with the constant* C_1, *the constant* C_1 *is the norm of* ϑ_α, *that is* $C_1 = \|\vartheta_\alpha\|_{X_c^r} = (|c|r)^{-\alpha+1-\frac{1}{r}}\frac{\Gamma[(\alpha-1)r+1]^{\frac{1}{r}}}{\Gamma(\alpha)}$. *So the exponent should be* $-\alpha + 1 - \frac{1}{r}$ *not* $-\alpha + \frac{1}{r}$.

Property 3. *Let* $1 \le p < \infty, 1 < q < \infty, \nu < \mu < 0$ *and* $\alpha p > 1, \alpha - 1 + \frac{1}{q} > 0$. *Then the operator* $\mathcal{J}_{1+}^{\alpha}$ *is bounded from* $X_\mu^p(J)$ *into* $X_\nu^q(J)$, *there holds the estimate* $\|\mathcal{J}_{1+}^{\alpha} f\|_{X_\nu^q(J)} \le C\|f\|_{X_\mu^p(J)}$, *where C is a constant.*

Proof of Property 3. First we consider the case $1 < p < \infty$. Then the exponent p' conjugate to p satisfies $1 < p' < \infty$. Using Hölder's inequality, for almost $x \in J$ we have

$$\begin{aligned} |(\mathcal{J}_{1+}^{\alpha} f)(x)| &\le \frac{1}{\Gamma(\alpha)}\int_1^x (\ln\frac{x}{t})^{\alpha-1} t^{-\mu} t^{\mu} |f(t)|\frac{dt}{t} \\ &\le \frac{\|f\|_{X_\mu^p(J)}}{\Gamma(\alpha)}\left(\int_1^x (\ln\frac{x}{t})^{(\alpha-1)p'} t^{-\mu p'}\frac{dt}{t}\right)^{\frac{1}{p'}}. \end{aligned} \tag{10}$$

By the change of variable $t = xe^{-y}$, the above integral in (10) gives the following estimate

$$\begin{aligned} &\left(\int_1^\infty x^{\nu q}|\mathcal{J}_{1+}^{\alpha} f(x)|^q\frac{dx}{x}\right)^{\frac{1}{q}} \\ \le&\frac{\|f\|_{X_\mu^p(J)}}{\Gamma(\alpha)}\left[\int_1^\infty x^{\nu q}\left(\int_1^x (\ln\frac{x}{t})^{(\alpha-1)p'} t^{-\mu p'}\frac{dt}{t}\right)^{\frac{q}{p'}}\frac{dx}{x}\right]^{\frac{1}{q}} \\ \le&\frac{\|f\|_{X_\mu^p(J)}}{\Gamma(\alpha)}\left[\int_1^\infty x^{\nu q}\left(x^{-\mu p'}\int_0^{\ln x} y^{(\alpha-1)p'} e^{\mu p' y} dy\right)^{\frac{q}{p'}}\frac{dx}{x}\right]^{\frac{1}{q}} \\ \le&\frac{\|f\|_{X_\mu^p(J)}}{\Gamma(\alpha)}\left(\int_1^\infty x^{(\nu-\mu)q-1}dx\right)^{\frac{1}{q}}\left(\int_0^\infty y^{(\alpha-1)p'} e^{\mu p' y} dy\right)^{\frac{1}{p'}}. \end{aligned}$$

Let $\mu p' y = -s$, from the definition of the gamma function we can obtain

$$\left(\int_0^\infty y^{(\alpha-1)p'}e^{\mu p' y}dy\right)^{\frac{1}{p'}} =\left(\frac{1}{(|\mu|p')^{(\alpha-1)p'+1}}\int_0^\infty s^{(\alpha-1)p'}e^{-s}ds\right)^{\frac{1}{p'}}$$
$$=\frac{(\Gamma((\alpha-1)p'+1))^{\frac{1}{p'}}}{(|\mu|p')^{\alpha-\frac{1}{p}}}.$$

Since $\nu<\mu, q>1$, the integral $\int_1^\infty x^{(\nu-\mu)q-1}dx$ is convergent. Hence, there exists a constant C_1 such that

$$\|\mathcal{J}^\alpha_{1+}f\|_{X^q_\nu(J)}\le C_1\|f\|_{X^p_\mu(J)},$$

where C_1 is a positive constant related only to μ,ν,p,q and α.

If $p=1$, for almost $x\in J$, applying Hölder's inequality again, we obtain

$$|(\mathcal{J}^\alpha_{1+}f)(x)|\le\frac{1}{\Gamma(\alpha)}\int_1^x|t^\mu f(t)|^{\frac{1}{q'}}t^{-\frac{\mu}{q'}}|f(t)|^{\frac{1}{q}}\left(\ln\frac{x}{t}\right)^{\alpha-1}\frac{dt}{t}$$
$$\le\frac{1}{\Gamma(\alpha)}\left(\int_1^x|t^\mu f(t)|\frac{dt}{t}\right)^{\frac{1}{q'}}\left(\int_1^x t^{-\frac{\mu q}{q'}}|f(t)|\left(\ln\frac{x}{t}\right)^{(\alpha-1)q}\frac{dt}{t}\right)^{\frac{1}{q}}$$
$$\le\frac{\|f\|^{\frac{1}{q'}}_{X^1_\mu(J)}}{\Gamma(\alpha)}\left(\int_1^x t^{-\frac{\mu q}{q'}}|f(t)|\left(\ln\frac{x}{t}\right)^{(\alpha-1)q}\frac{dt}{t}\right)^{\frac{1}{q}}.$$

It follows from Fubini's theorem that

$$\|\mathcal{J}^\alpha_{1+}f\|_{X^q_\nu(J)}\le\frac{\|f\|^{\frac{1}{q'}}_{X^1_\mu(J)}}{\Gamma(\alpha)}\left[\int_1^\infty x^{\nu q}\int_1^x t^{-\frac{\mu q}{q'}}|f(t)|\left(\ln\frac{x}{t}\right)^{(\alpha-1)q}\frac{dt}{t}\frac{dx}{x}\right]^{\frac{1}{q}}$$
$$=\frac{\|f\|^{\frac{1}{q'}}_{X^1_\mu(J)}}{\Gamma(\alpha)}\left[\int_1^\infty\left(\int_t^\infty x^{\nu q}\left(\ln\frac{x}{t}\right)^{(\alpha-1)q}\frac{dx}{x}\right)t^{-\frac{\mu q}{q'}}|f(t)|\frac{dt}{t}\right]^{\frac{1}{q}}.$$

Substituting the variable $x=te^y$, we get

$$\int_t^\infty x^{\nu q}\left(\ln\frac{x}{t}\right)^{(\alpha-1)q}\frac{dx}{x}=t^{\nu q}\int_0^\infty e^{\nu q y}y^{(\alpha-1)q}dy=t^{\nu q}\frac{\Gamma((\alpha-1)q+1)}{(|\nu|q)^{(\alpha-1)q+1}}.$$

As a result,

$$\|\mathcal{J}^\alpha_{1+}f\|_{X^q_\nu(J)}\le\frac{\|f\|^{\frac{1}{q'}}_{X^1_\mu(J)}}{\Gamma(\alpha)}\frac{(\Gamma((\alpha-1)q+1))^{\frac{1}{q}}}{(|\nu|q)^{\alpha-1+\frac{1}{q}}}\left[\int_1^\infty t^{\nu q}t^{-\frac{\mu q}{q'}}|f(t)|\frac{dt}{t}\right]^{\frac{1}{q}}$$
$$\le\frac{(\Gamma((\alpha-1)q+1))^{\frac{1}{q}}}{\Gamma(\alpha)(|\nu|q)^{\alpha-1+\frac{1}{q}}}\|f\|^{\frac{1}{q'}}_{X^1_\mu(J)}\left(\int_1^\infty t^\mu|f(t)|\frac{dt}{t}\right)^{\frac{1}{q}}$$
$$=C_2\|f\|_{X^1_\mu(J)},$$

where $C_2=\frac{(\Gamma((\alpha-1)q+1))^{\frac{1}{q}}}{\Gamma(\alpha)(|\nu|q)^{\alpha-1+\frac{1}{q}}}$.

Set $C=\max\{C_1,C_2\}$, for $1\le p<\infty$, the conclusion $\|\mathcal{J}^\alpha_{1+}f\|_{X^q_\nu(J)}\le C\|f\|_{X^p_\mu(J)}$ always holds. □

Property 4. *Let $1\le p\le\infty, 1\le q\le\infty, \alpha>0$. Then $\mathcal{J}^\alpha_{1+}$ is bounded from $X^p(J)$ into $X^q(J)$ if and only if $0<\alpha<\frac{1}{p}<1, q=\frac{p}{1-\alpha p}$, meanwhile, there holds the following estimate*

$$\|\mathcal{J}^\alpha_{1+}f\|_{X^q(J)}\le C_2\|f\|_{X^p(J)},$$

where C_2 is certain unspecified positive constant.

Proof of Property 4. In literature [7,12], there is only conclusion about the sufficiency of Property 4, but the corresponding result of necessity is also valid. Now, we prove the necessity. For any $f \in L^p(\mathbb{R}^+)$, by Corollary 1, we know $A : X^p(J) \to L^p(\mathbb{R}^+)$ is isometric isomorphism, then there exists a function $\varphi \in X^p(J)$ such that $A\varphi = f$. According to the condition, $\mathcal{J}^\alpha_{1+}\varphi \in X^q(J)$, that is $A^{-1}\mathcal{I}^\alpha_{0+}A\varphi \in X^q(J)$. Using Corollary 1 again, we have $A(A^{-1}\mathcal{I}^\alpha_{0+}A\varphi) \in L^q(\mathbb{R}^+)$. Thereupon, $A(A^{-1}\mathcal{I}^\alpha_{0+}A\varphi) = \mathcal{I}^\alpha_{0+}A\varphi = \mathcal{I}^\alpha_{0+}f \in L^q(\mathbb{R}^+)$. In view of Hardy-Littlewood theorem ([11] Lemma 2.11) with limiting exponent, we get the conclusion. □

Remark 6. *By Corollary 2, although the mapping property of Mellin convolution is just like Theorem 2 in the space X^q, since $\vartheta_\alpha \notin X^r$ from Table 1, Property 4 cannot be inferred from this.*

Remark 7. *Given the inclusion relation of the function spaces (6), combining the above properties we can directly deduce some other mapping properties of the Hadamard integral operator, such as $\mathcal{J}^\alpha_{1+} : L^p(J) \to X^p_c(J)$ or $\mathcal{J}^\alpha_{1+} : X^p(J) \to X^p_c(J)$ is bounded for $1 \le p \le \infty, c < 0$. If $0 < \alpha < \frac{1}{p} < 1, q = \frac{p}{1-\alpha p}$, then $\mathcal{J}^\alpha_{1+} : L^p(J) \to X^q(J)$ is bounded and so on.*

The above properties reveal that the boundedness of Hadamard fractional integral is available in the space of integrable functions. The latter two properties present that Hadamard integral operator can be mapped to a certain class of weighted continuous function spaces and the operator is continuous.

Property 5. *Let $1 \le p \le \infty, \alpha p > 1$, then the operator $\mathcal{J}^\alpha_{1+} : X^p(J) \to C_{\alpha-\frac{1}{p},\ln}(J_1)$ is continuous, where $C_{\alpha-\frac{1}{p},\ln}(J_1) = \{\Phi \in C(J_1) \mid \sup_{x\in J_1} \frac{|\Phi(x)|}{1+(\ln x)^{\alpha-\frac{1}{p}}} < \infty\}$ is a Banach space endowed with the norm $\|\Phi\|_{\alpha-\frac{1}{p}} = \sup_{x\in J_1} \frac{|\Phi(x)|}{1+(\ln x)^{\alpha-\frac{1}{p}}}$.*

Proof of Property 5. First, we will show the conclusion holds for $1 < p \le \infty$. For any $f \in X^p(J)$, we first prove the continuity of the function $\mathcal{J}^\alpha_{1+}f$. Selecting two elements x_1, x_2 from J_1 satisfying $1 \le x_1 < x_2 < \infty$, using Hölder's inequality we find

$$\begin{aligned}
&\Gamma(\alpha)|\mathcal{J}^\alpha_{1+}f(x_2) - \mathcal{J}^\alpha_{1+}f(x_1)| \\
\le&\|f\|_{X^p(J)}\left[\left(\int_{x_1}^{x_2}\left(\ln\frac{x_2}{t}\right)^{(\alpha-1)p'}\frac{dt}{t}\right)^{\frac{1}{p'}} + \left(\int_1^{x_1}\left|\left(\ln\frac{x_2}{t}\right)^{\alpha-1} - \left(\ln\frac{x_1}{t}\right)^{\alpha-1}\right|^{p'}\frac{dt}{t}\right)^{\frac{1}{p'}}\right] \\
=&\|f\|_{X^p(J)}(I_1 + I_2).
\end{aligned}$$

Now we estimate these two integrals I_1 and I_2 respectively.

$$I_1 = ((\alpha-1)p'+1)^{-\frac{1}{p'}}\left(\ln\frac{x_2}{x_1}\right)^{\alpha-\frac{1}{p}}, \tag{11}$$

and from the inequality $(x-y)^q \le x^q - y^q, (0 \le y \le x, q \ge 1)$ we know

$$\begin{aligned}
I_2 \le&\left(\int_1^{x_1}\left|\left(\ln\frac{x_2}{t}\right)^{(\alpha-1)p'} - \left(\ln\frac{x_1}{t}\right)^{(\alpha-1)p'}\right|\frac{dt}{t}\right)^{\frac{1}{p'}} \\
\le&((\alpha-1)p'+1)^{-\frac{1}{p'}}\left(\left|(\ln x_2)^{(\alpha-1)p'+1} - (\ln x_1)^{(\alpha-1)p'+1} - \left(\ln\frac{x_2}{x_1}\right)^{(\alpha-1)p'+1}\right|\right)^{\frac{1}{p'}}.
\end{aligned}$$

If $0 < \alpha \le 1$, then $0 < (\alpha-1)p' + 1 < 1$. Hence,

$$\left(\ln \frac{x_2}{x_1}\right)^{(\alpha-1)p'+1} \ge (\ln x_2)^{(\alpha-1)p'+1} - (\ln x_1)^{(\alpha-1)p'+1} > 0.$$

The inequality gives that

$$I_2 \le ((\alpha-1)p'+1)^{-\frac{1}{p'}} \left(\ln \frac{x_2}{x_1}\right)^{\alpha-\frac{1}{p}}. \tag{12}$$

If $\alpha \ge 1$, then $(\ln x_2)^{(\alpha-1)p'+1} - (\ln x_1)^{(\alpha-1)p'+1} \ge \left(\ln \frac{x_2}{x_1}\right)^{(\alpha-1)p'+1}$. This implies that

$$I_2 \le ((\alpha-1)p'+1)^{-\frac{1}{p'}} \left((\ln x_2)^{(\alpha-1)p'+1} - (\ln x_1)^{(\alpha-1)p'+1}\right)^{\frac{1}{p'}}. \tag{13}$$

Collecting the estimates (11)–(13) for I_1, I_2, we take the limit $|x_1 - x_2| \to 0$, then we have $|\mathcal{J}^{\alpha}_{1+}f(x_1) - \mathcal{J}^{\alpha}_{1+}f(x_2)| \to 0$. Thus, the conclusion $\mathcal{J}^{\alpha}_{1+}f \in C(J_1)$ holds. For $x \in J_1$, using Hölder's inequality, we know

$$|\mathcal{J}^{\alpha}_{1+}f(x)| \le \frac{\|f\|_{X^p(J)}}{\Gamma(\alpha)}((\alpha-1)p'+1)^{-\frac{1}{p'}}(\ln x)^{\alpha-\frac{1}{p}},$$

therefore

$$\|\mathcal{J}^{\alpha}_{1+}f\|_{\alpha-\frac{1}{p}} = \sup_{x\in J_1} \frac{|\mathcal{J}^{\alpha}_{1+}f(x)|}{1+(\ln x)^{\alpha-\frac{1}{p}}} \le \frac{\|f\|_{X^p(J)}}{\Gamma(\alpha)}((\alpha-1)p'+1)^{-\frac{1}{p'}} < \infty.$$

This shows that $\mathcal{J}^{\alpha}_{1+}f(x) \in C_{\alpha-\frac{1}{p},\ln}(J_1)$.

Let $f_n \to f$ in $X^p(J)$, the following inequality is obtained by Hölder's inequality,

$$\|\mathcal{J}^{\alpha}_{1+}f_n - \mathcal{J}^{\alpha}_{1+}f\|_{\alpha-\frac{1}{p}} \le \frac{\|f_n - f\|_{X^p(J)}}{\Gamma(\alpha)}((\alpha-1)p'+1)^{-\frac{1}{p'}},$$

which implies that $\mathcal{J}^{\alpha}_{1+}f_n \to \mathcal{J}^{\alpha}_{1+}f$ in $C_{\alpha-\frac{1}{p},\ln}(J_1)$.

When $p = 1$, from $\alpha p > 1$ we know $\alpha > 1$. Suppose $f \in X^1(J)$, then

$$|\mathcal{J}^{\alpha}_{1+}f(t)| \le \frac{(\ln t)^{\alpha-1}}{\Gamma(\alpha)} \int_1^{\infty} |f(t)| \frac{dt}{t} = \frac{\|f\|_{X^1(J)}}{\Gamma(\alpha)}(\ln t)^{\alpha-1},\ t \in J_1.$$

Consequently,

$$\|\mathcal{J}^{\alpha}_{1+}f\|_{\alpha-1} = \sup_{x\in J_1} \frac{|\mathcal{J}^{\alpha}_{1+}f(x)|}{1+(\ln x)^{\alpha-1}} \le \frac{\|f\|_{X^1(J)}}{\Gamma(\alpha)}.$$

Choose arbitrary sequence $\{f_n\}$ with $f_n \to f$ in $X^1(J)$, from the same deduction method, we obtain $\mathcal{J}^{\alpha}_{1+}f_n \to \mathcal{J}^{\alpha}_{1+}f$ in $C_{\alpha-1,\ln}(J_1)$. □

Property 6. *Let $1 \le p \le \infty, c < 0, \alpha p > 1$. Then the integral operator $\mathcal{J}^{\alpha}_{1+} : X^p_c(J) \to C_{c,\alpha-\frac{1}{p},\ln}(J_1)$ is continuous, where*

$$C_{c,\alpha-\frac{1}{p},\ln}(J_1) = \{\Phi(x) \in C(J_1) \mid \sup_{x\in J_1} \frac{|x^c\Phi(x)|}{1+(\ln x)^{\alpha-\frac{1}{p}}} < \infty\}$$

is a Banach space endowed with the norm $\|\Phi\|_{c,\alpha-\frac{1}{p}} = \sup_{x\in J_1} \frac{|x^c\Phi(x)|}{1+(\ln x)^{\alpha-\frac{1}{p}}}$. *Furthermore,* $\mathcal{J}^{\alpha}_{1+}f(1) = 0$, $\forall f \in X^p_c(J)$.

Proof of Property 6. When $1 < p \leq \infty$, for any $x \in J_1, f \in X^p_c(J)$, we have

$$\begin{aligned}
|\mathcal{J}^{\alpha}_{1+}f(x)| \leq & \frac{\|f\|_{X^p_c(J)}}{\Gamma(\alpha)} \left(\int_1^x (\ln t)^{(\alpha-1)p'} \left(\frac{x}{t}\right)^{-cp'} \frac{dt}{t} \right)^{\frac{1}{p'}} \\
= & \frac{x^{-c}\|f\|_{X^p_c(J)}}{\Gamma(\alpha)} \left(\int_0^{\ln x} s^{(\alpha-1)p'} e^{cp's} ds \right)^{\frac{1}{p'}} \\
\leq & \frac{x^{-c}\|f\|_{X^p_c(J)}}{\Gamma(\alpha)} \left(\int_0^{\infty} s^{(\alpha-1)p'} e^{cp's} ds \right)^{\frac{1}{p'}} \\
\leq & \frac{x^{-c}\|f\|_{X^p_c(J)}}{\Gamma(\alpha)} \frac{\Gamma((\alpha-1)p'+1)^{\frac{1}{p'}}}{(|c|p')^{\alpha-\frac{1}{p}}},
\end{aligned}$$

it follows that $x^c|\mathcal{J}^{\alpha}_{1+}f(x)| \leq \frac{\Gamma((\alpha-1)p'+1)^{\frac{1}{p'}}}{\Gamma(\alpha)(|c|p')^{\alpha-\frac{1}{p}}}\|f\|_{X^p_c(J)} < \infty$, that is to say $\mathcal{J}^{\alpha}_{1+}f$ is well defined. Meanwhile we obtain

$$|\mathcal{J}^{\alpha}_{1+}f(x)| \leq \frac{x^{-c}\|f\|_{X^p_c(J)}}{\Gamma(\alpha)} \left(\int_0^{\ln x} s^{(\alpha-1)p'} ds \right)^{\frac{1}{p'}} \leq \frac{x^{-c}\|f\|_{X^p_c(J)}}{\Gamma(\alpha)} \frac{(\ln x)^{\alpha-\frac{1}{p}}}{((\alpha-1)p'+1)^{\frac{1}{p'}}}. \quad (14)$$

Now we will show that $\mathcal{J}^{\alpha}_{1+}f \in C(J_1)$ for any $f \in X^p_c(J)$. For any x_1, x_2 with $1 \leq x_1 < x_2 < \infty$, one has

$$\begin{aligned}
& \Gamma(\alpha)|\mathcal{J}^{\alpha}_{1+}f(x_2) - \mathcal{J}^{\alpha}_{1+}f(x_1)| \\
\leq & \int_{x_1}^{x_2} \left(\ln\frac{x_2}{t}\right)^{\alpha-1} |f(t)|\frac{dt}{t} + \int_1^{x_1} \left| \left(\ln\frac{x_2}{t}\right)^{\alpha-1} - \left(\ln\frac{x_1}{t}\right)^{\alpha-1} \right| |f(t)|\frac{dt}{t} \\
\leq & \|f\|_{X^p_c(J)} \left[\left(\int_{x_1}^{x_2} \left(\ln\frac{x_2}{t}\right)^{(\alpha-1)p'} t^{-cp'}\frac{dt}{t} \right)^{\frac{1}{p'}} + \left(\int_1^{x_1} \left| \left(\ln\frac{x_2}{t}\right)^{\alpha-1} \right.\right.\right. \\
& \left.\left.\left. - \left(\ln\frac{x_1}{t}\right)^{\alpha-1} \right|^{p'} t^{-cp'}\frac{dt}{t} \right)^{\frac{1}{p'}} \right] \\
= & \|f\|_{X^p_c(J)}(I_1 + I_2). \qquad (15)
\end{aligned}$$

For the first integral I_1, substituting the variable $s = \frac{\ln x_2 - \ln t}{\ln x_2 - \ln x_1}$, we get

$$\begin{aligned}
I_1 = & x_2^{-c}\left(\ln\frac{x_2}{x_1}\right)^{\alpha-\frac{1}{p}} \left(\int_0^1 s^{(\alpha-1)p'} \left(\frac{x_2}{x_1}\right)^{csp'} ds \right)^{\frac{1}{p'}} \\
\leq & \frac{x_2^{-c}}{((\alpha-1)p'+1)^{\frac{1}{p'}}} \left(\ln\frac{x_2}{x_1}\right)^{\alpha-\frac{1}{p}}. \qquad (16)
\end{aligned}$$

We consider the integral I_2 for two cases. If $\alpha \geq 1$, let $t = x_1 e^{-u}$, then

$$
\begin{aligned}
I_2 \le& \left(\int_1^{x_1} \left(\left(\ln\frac{x_2}{t}\right)^{(\alpha-1)p'} - \left(\ln\frac{x_1}{t}\right)^{(\alpha-1)p'}\right) t^{-cp'}\frac{dt}{t}\right)^{\frac{1}{p'}} \\
=& x_1^{-c}\left(\int_0^{\ln x_1}\left(\left(\ln\frac{x_2}{x_1}+u\right)^{(\alpha-1)p'} - u^{(\alpha-1)p'}\right)e^{cup'}du\right)^{\frac{1}{p'}} \\
\le& x_1^{-c}\left(\int_0^{\ln x_1}\left(\left(\ln\frac{x_2}{x_1}+u\right)^{(\alpha-1)p'} - u^{(\alpha-1)p'}\right)du\right)^{\frac{1}{p'}} \\
=& \frac{x_1^{-c}}{((\alpha-1)p'+1)^{\frac{1}{p'}}}\left((\ln x_2)^{(\alpha-1)p'+1} - (\ln x_1)^{(\alpha-1)p'+1} - \left(\ln\frac{x_2}{x_1}\right)^{(\alpha-1)p'+1}\right)^{\frac{1}{p'}} \\
\le& \frac{x_1^{-c}}{((\alpha-1)p'+1)^{\frac{1}{p'}}}\left((\ln x_2)^{(\alpha-1)p'+1} - (\ln x_1)^{(\alpha-1)p'+1}\right)^{(\alpha-1)p'+1}.
\end{aligned} \tag{17}
$$

Similarly, if $0<\alpha<1$, we have

$$
\begin{aligned}
I_2 \le& \frac{x_1^{-c}}{((\alpha-1)p'+1)^{\frac{1}{p'}}}\left(\left(\ln\frac{x_2}{x_1}\right)^{(\alpha-1)p'+1} - (\ln x_2)^{(\alpha-1)p'+1} + (\ln x_1)^{(\alpha-1)p'+1}\right)^{\frac{1}{p'}} \\
\le& \frac{x_1^{-c}}{((\alpha-1)p'+1)^{\frac{1}{p'}}}\left(\ln\frac{x_2}{x_1}\right)^{\alpha-\frac{1}{p}}.
\end{aligned} \tag{18}
$$

Let's substitute (16)–(18) into (15), if $|x_2-x_1|\to 0$, then $|\mathcal{J}^\alpha_{1+}f(x_1) - \mathcal{J}^\alpha_{1+}f(x_2)|\to 0$. From (14), we know

$$
\sup_{x\in J_1}\frac{|x^c\mathcal{J}^\alpha_{1+}f(x)|}{1+(\ln x)^{\alpha-\frac{1}{p}}} \le \frac{\|f\|_{X^p_c(J)}}{\Gamma(\alpha)((\alpha-1)p'+1)^{\frac{1}{p'}}} < \infty.
$$

Next, we will show that $\mathcal{J}^\alpha_{1+} : X^p_c(J)\to C_{c,\alpha-\frac{1}{p},\ln}(J_1)$ is continuous. Let $\{f_n\}$ be convergent sequence in $X^p_c(J)$ and $\|f_n - f\|_{X^p_c(J)}\to 0$. According to the initial conclusion of the proof, we get

$$
\|\mathcal{J}^\alpha_{1+}f_n - \mathcal{J}^\alpha_{1+}f\|_{c,\alpha-\frac{1}{p}} \le \frac{\|f_n-f\|_{X^p_c(J)}}{\Gamma(\alpha)}\frac{\Gamma((\alpha-1)p'+1)^{\frac{1}{p'}}}{(|c|p')^{\alpha-\frac{1}{p}}},
$$

then it comes to the conclusion that the operator $\mathcal{J}^\alpha_{1+}$ is continuous. From (14), let $x\to 1$, the conclusion $\mathcal{J}^\alpha_{1+}f(1)=0$ is satisfied.

When $p=1$, then $\alpha>1$. Suppose $f\in X^1_c(J)$, then

$$
\left|\frac{x^c\mathcal{J}^\alpha_{1+}f(x)}{1+(\ln x)^{\alpha-1}}\right| \le \frac{(\ln x)^{\alpha-1}}{\Gamma(\alpha)(1+(\ln x)^{\alpha-1})}\int_1^x x^c t^{-c}|t^c f(t)|\frac{dt}{t} \le \frac{\|f\|_{X^1_c(J)}}{\Gamma(\alpha)},\ t\in J_1.
$$

Hence,

$$
\|\mathcal{J}^\alpha_{1+}f\|_{c,\alpha-1} \le \frac{\|f\|_{X^1_c(J)}}{\Gamma(\alpha)} < \infty.
$$

Choose arbitrary sequence $\{f_n\}$ with $f_n\to f$ in $X^1_c(J)$, likewise, we obtain $\mathcal{J}^\alpha_{1+}f_n\to\mathcal{J}^\alpha_{1+}f$ in $C_{c,\alpha-1,\ln}(J_1)$.

This completes the proof. □

Remark 8. *In addition to focusing on the mapping properties of Hadamard integral operator between function spaces of the same type, we are more interested in the properties of the integral operator between spaces of different types, especially whether the integral operator has good results* *on the mapping from larger spaces to smaller spaces, as in properties 5 and 6 above. This point will be left for further thinking.*

4. Preliminary Results

In this section, we recall some related lemmas and give some auxiliary results that will be used in the study of Hadamard fractional boundary value problem on the infinite interval.

The left-sided Hadamard fractional derivative of order $\alpha(\alpha > 0)$ is defined by

$$(\mathcal{D}_{1+}^{\alpha} f)(x) = \delta^n (\mathcal{J}_{1+}^{n-\alpha} f)(x) = (x\frac{d}{dx})^n \frac{1}{\Gamma(n-\alpha)} \int_1^x (\ln\frac{x}{t})^{n-\alpha-1} f(t)\frac{dt}{t},$$

where $\delta = xD, D = \frac{d}{dx}$ is so-called $\delta-$derivative, $n-1 < \alpha < n, n = [\alpha]+1$.

When $\alpha = m \in \mathbb{N}$, then $(\mathcal{D}_{1+}^{\alpha} f)(x) = (\delta^m f)(x), x > 1$.

If $\alpha < 0$, we also denote $\mathcal{D}_{1+}^{\alpha} = \mathcal{J}_{1+}^{-\alpha}$.

Lemma 1 ([11]). *Let $\alpha, \beta > 0, 1 \le p \le \infty, c < 0$. Then for $u \in X_c^p(J)$,*
(1) $\mathcal{J}_{1+}^{\alpha}\mathcal{J}_{1+}^{\beta} u = \mathcal{J}_{1+}^{\alpha+\beta} u$.
(2) $\mathcal{D}_{1+}^{\alpha}\mathcal{J}_{1+}^{\alpha} u(t) = u(t)$, for almost $t \in J$.
(3) $\mathcal{D}_{1+}^{\alpha}\mathcal{J}_{1+}^{\beta} u(t) = \mathcal{J}_{1+}^{\beta-\alpha} u(t)$, provided that $\beta > \alpha$.

Lemma 2. *If $\beta, \gamma > 0$, then*
(1) $\mathcal{J}_{1+}^{\gamma}\vartheta_{\beta}(t) = \vartheta_{\beta+\gamma}(t)$.
(2) $\mathcal{D}_{1+}^{\gamma}\vartheta_{\beta}(t) = \vartheta_{\beta-\gamma}(t)$, provided that $\beta > \gamma$.
(3) $\mathcal{D}_{1+}^{\gamma}\vartheta_{\gamma-j+1}(t) = 0$, for $n-1 < \gamma < n, j = 1, 2, \ldots, n$.

Lemma 3 ([11]). *For $\alpha > 0$, the equality $\mathcal{D}_{1+}^{\alpha} u(t) = 0$ valid if and only if,*

$$u(t) = \sum_{j=1}^{n} c_j (\ln t)^{\alpha-j},$$

where n is the smallest integer greater than or equal to α and $c_j (j = 1, 2, \ldots, n)$ are arbitrary constants.

In view of Lemmas 1 and 3, it is easy to deduce the following lemma.

Lemma 4. *Let $1 \le p \le \infty, c < 0$. If $\mathcal{D}_{1+}^{\alpha} u \in X_c^p(J)$, then*

$$\mathcal{J}_{1+}^{\alpha}(\mathcal{D}_{1+}^{\alpha} u)(t) = u(t) + c_1(\ln t)^{\alpha-1} + c_2(\ln t)^{\alpha-2} + \ldots + c_n(\ln t)^{\alpha-n},$$

where $c_i \in \mathbb{R}$ $(i = 1, 2, \ldots, n), n = [\alpha]+1$.

Now we introduce a linear space

$$X = \left\{ x : J \to \mathbb{R} \;\middle|\; x(t) \in C(J),\ \sup_{t\in J_1} \frac{(\ln t)^{3-\alpha}|x(t)|}{1+(\ln t)^2} < \infty \right\},$$

then X is a Banach space with respect to the norm $\|x\|_X = \sup\limits_{t\in J_1}\left|\frac{(\ln t)^{3-\alpha}x(t)}{1+(\ln t)^2}\right|$.

We establish the inclusion relationship of the spaces X and $X_c^P(J)$.

Lemma 5. *Let $2 < \alpha < 3, 1 \le p < \frac{1}{3-\alpha}, c < 0$. Then $X \subseteq X_c^p(J)$ and there exists a constant M_1 related only to α, c and p, such that $\|u\|_{X_c^p(J)} \le M_1\|u\|_X$.*

Proof of Lemma 5. For any $u \in X$,

$$\begin{aligned}\|u\|_{X_c^p(J)} \leq & \|u\|_X \left(\int_1^{\infty} t^{cp} \left(\frac{1+(\ln t)^2}{(\ln t)^{3-\alpha}} \right)^p \frac{dt}{t} \right)^{\frac{1}{p}} \\ \leq & 2\|u\|_X \left(\int_0^{\infty} e^{cpx} x^{(\alpha-3)p} dx + \int_0^{\infty} e^{cpx} x^{(\alpha-1)p} dx \right)^{\frac{1}{p}}.\end{aligned}$$

Obviously, the integral $\int_0^{\infty} e^{cpx} x^{(\alpha-1)p} dx$ is convergent. Since

$$\int_0^{\infty} e^{cpx} x^{(\alpha-3)p} dx = \int_0^1 e^{cpx} x^{(\alpha-3)p} dx + \int_1^{\infty} e^{cpx} x^{(\alpha-3)p} dx,$$

these two integrals are convergent under the condition $2 < \alpha < 3, c < 0, 1 \leq p < \frac{1}{3-\alpha}$. Therefore, such constant M_1 exists and guarantees the conclusion $\|u\|_{X_c^p(J)} \leq M_1 \|u\|_X$ holds. □

Lemma 6. *Assume that* $2 < \alpha < 3, \alpha+\beta-3 > 0, c < 0$ *and* $p > 1$ *satisfies* $\beta p > 1, (3-\alpha)p < 1$. *Then*
(1). $\mathcal{J}_{1+}^{\beta} : X_c^p(J) \to C_{c,\beta-\frac{1}{p},\ln}(J_1)$ *is continuous.*
(2). $\mathcal{J}_{1+}^{\beta} : X \to C_{\alpha+\beta-1,\ln}(J_1)$ *is bounded and there exists a positive constant* C *such that*

$$\|\mathcal{J}_{1+}^{\beta} u\|_{\alpha+\beta-1,\ln}\| \leq C\|u\|_X.$$

Proof of Lemma 6. Under the assumptions $\beta p > 1, c < 0, p > 1$, the first claim (1) can be derived directly from Property 6.

For any $u \in X, t \in J_1$, we know

$$\begin{aligned}|\mathcal{J}_{1+}^{\beta} u(t)| \leq & \frac{\|u\|_X}{\Gamma(\beta)} \int_1^t \left(\ln \frac{t}{s} \right)^{\beta-1} \frac{1+(\ln s)^2}{(\ln s)^{3-\alpha}} \frac{ds}{s} \\ = & \frac{\|u\|_X}{\Gamma(\beta)} \left(B(\alpha-2,\beta)(\ln t)^{\alpha+\beta-3} + B(\alpha,\beta)(\ln t)^{\alpha+\beta-1} \right) \\ = & r(t)\|u\|_X < \infty.\end{aligned} \tag{19}$$

Since $u \in X$, from Lemma 5, then $u \in X_c^p(J)$, it gives that $\mathcal{J}_{1+}^{\beta} u \in C(J_1)$ from (1). By (19), we further deduce

$$\begin{aligned}\frac{|\mathcal{J}_{1+}^{\beta} u(t)|}{1+(\ln t)^{\alpha+\beta-1}} \leq & \frac{r(t)\|u\|_x}{1+(\ln t)^{\alpha+\beta-1}} \\ = & \left(\frac{B(\alpha-2,\beta)}{\Gamma(\beta)} \frac{(\ln t)^{\alpha+\beta-3}}{1+(\ln t)^{\alpha+\beta-1}} + \frac{B(\alpha,\beta)}{\Gamma(\beta)} \frac{(\ln t)^{\alpha+\beta-1}}{1+(\ln t)^{\alpha+\beta-1}} \right) \|u\|_x.\end{aligned}$$

Since $\alpha+\beta-3 > 0$, we know by normal calculation that there must be a positive constant C_0 such that

$$\sup_{t \in J_1} \frac{(\ln t)^{\alpha+\beta-3}}{1+(\ln t)^{\alpha+\beta-1}} \leq C_0 \text{ and } \sup_{t \in J_1} \frac{(\ln t)^{\alpha+\beta-1}}{1+(\ln t)^{\alpha+\beta-1}} \leq 1.$$

Set $C = \frac{B(\alpha-2,\beta)C_0}{\Gamma(\beta)} + \frac{B(\alpha,\beta)}{\Gamma(\beta)}$, then

$$\|\mathcal{J}_{1+}^{\beta} u\|_{\alpha+\beta-1,\ln} = \sup_{t \in J_1} \frac{|\mathcal{J}_{1+}^{\beta} u(t)|}{1+(\ln t)^{\alpha+\beta-1}} \leq C\|u\|_X < \infty.$$

□

In order to present the fixed point theorem which we will use in our study, we introduce a family of functions $\mathcal{F}$, which contains a serious of functions $\psi : (0,\infty) \to \mathbb{R}$ satisfying
(i) ψ is strictly increasing.
(ii) For any sequence $\{t_n\} \subset (0,\infty)$, we have

$$\lim_{n\to\infty} t_n = 0 \iff \lim_{n\to\infty} \psi(t_n) = -\infty.$$

(iii) There exists $\theta \in (0,1)$ such that

$$\lim_{t\to 0+} t^{\theta}\psi(t) = 0.$$

Choose $0 < \theta < 1$, let $\psi_1(t) = -t^{-\theta}, t \in (0,\infty)$, then $\psi_1(t) \in \mathcal{F}$. Again, for example, $\psi_2(t) = \ln t, \psi_3(t) = \ln(t) + t, \ldots$ These are examples of functions belonging to class $\mathcal{F}$.

Next, we present the above announced fixed point theorem which appears in [35].

Theorem 5. *Let (X,d) be a complete metric space and $T : X \to X$ a mapping such that there exist $\tau > 0$ and $\psi \in \mathcal{F}$ satisfying, for any $x, y \in X$ with $d(Tx,Ty) > 0$,*

$$\tau + \psi(d(Tx,Ty)) \le \psi(d(x,y)).$$

Then T has a unique fixed point in X.

For further analysis, we introduce the following denotations:

$$G_\alpha(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} (\ln t)^{\alpha-1} - (\ln \frac{t}{s})^{\alpha-1}, 1 \le s \le t < +\infty, \\ (\ln t)^{\alpha-1}, \quad\quad 1 \le t \le s < +\infty, \end{cases} \tag{20}$$

$$\Phi(t) = \sum_{j=1}^{3} \lambda_j \vartheta_{\alpha-j+1}(t), \tag{21}$$

$$l_{ij} = \int_1^\infty g_i(t)\vartheta_{\alpha-j+1}(t)\frac{dt}{t},\ i,j = 1,2,3, \tag{22}$$

$$A = \begin{pmatrix} 1-\mu_1 l_{11} & -\mu_2 l_{12} & -\mu_3 l_{13} \\ -\mu_1 l_{21} & 1-\mu_2 l_{22} & -\mu_3 l_{23} \\ -\mu_1 l_{31} & -\mu_2 l_{32} & 1-\mu_3 l_{33} \end{pmatrix}, \tag{23}$$

$$A^{-1} = \frac{A^*}{|A|} = \frac{1}{|A|}\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}, \text{ if } |A| \ne 0, \tag{24}$$

where A^{-1} is inverse matrix of A, $A^*, |A|$ denote the adjoint matrix and determinant of matrix A, A_{ij} is the algebraic cofactor ($i,j = 1,2,3$).

$$\varphi_i(t) = \frac{1}{|A|}\sum_{j=1}^{3} \mu_j A_{ij}\vartheta_{\alpha-j+1}(t),\ i = 1,2,3, \tag{25}$$

$$\omega_j = \lambda_j + \frac{1}{|A|}\left(\sum_{i,k=1}^{3} \lambda_k l_{ik} A_{ij}\right)\mu_j,\ j = 1,2,3. \tag{26}$$

Now we list some basic assumptions as follows.

($\mathbf{C_1}$) $2 < \alpha < 3, \alpha + \beta - 3 > 0, c < 0$ and $p > 1$ satisfies $\beta p > 1, (3-\alpha)p < 1$. p' is conjugate exponent of p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$.

($\mathbf{C_2}$) $g_j : J \to \mathbb{R}^+$, and $l_{g_j} = \left(\int_1^\infty |g_j(s)|^{p'} s^{-cp'} \frac{ds}{s}\right)^{\frac{1}{p'}} < \infty, j = 1,2,3.$

($\mathbf{C_3}$) $A_{ij} > 0$ and $|A| > 0$.

Lemma 7. *Suppose the conditions* ($\mathbf{C_1}$), ($\mathbf{C_2}$) *hold and* $|A| \neq 0$. *Let* $h \in X^1(J)$, *then the unique solution of Hadamard fractional differential equation*

$$\mathcal{D}_{1+}^{\alpha} u(t) + h(t) = 0, \tag{27}$$

subjected to the same condition (2) *can be expressed by*

$$u(t) = \int_1^{+\infty} G_\alpha(t,s)h(s)\frac{ds}{s} + \sum_{j=1}^{3} \mu_j \vartheta_{\alpha-j+1}(t) \int_1^{+\infty} g_j(s)u(s)\frac{ds}{s} + \Phi(t), \tag{28}$$

and the solution can be further expressed as

$$u(t) = \int_1^{\infty} G_\alpha(t,s)h(s)\frac{ds}{s} + \sum_{i=1}^{3} \varphi_i(t) \int_1^{\infty} g_i^*(s)h(s)\frac{ds}{s} + \sum_{j=1}^{3} \omega_j \vartheta_{\alpha-j+1}(t), \tag{29}$$

where

$$g_i^*(s) = \int_1^{\infty} G_\alpha(t,s)g_i(t)\frac{dt}{t},\ i = 1,2,3, \tag{30}$$

$G_\alpha(t,s), \varphi_i(t), \omega_j$ *are defined in* (20), (25) *and* (26) *respectively.*

Proof of Lemma 7. Due to Lemma 4, applying the Hadamard fractional integral operator $\mathcal{J}_{1+}^{\alpha}$ to both sides of (27), we have

$$u(t) = -\mathcal{J}_{1+}^{\alpha} h(t) + c_1(\ln t)^{\alpha-1} + c_2(\ln t)^{\alpha-2} + c_3(\ln t)^{\alpha-3},$$

where $c_i \in \mathbb{R}(i = 1,2,3)$ are arbitrary constants. By Lemma 2, we have

$$\mathcal{D}_{1+}^{\alpha-3} u(t) = -\mathcal{J}_{1+}^{3} h(t) + c_1\frac{\Gamma(\alpha)}{\Gamma(3)}(\ln t)^2 + c_2\frac{\Gamma(\alpha-1)}{\Gamma(2)}(\ln t) + c_3\frac{\Gamma(\alpha-2)}{\Gamma(1)}.$$

From $\mathcal{D}_{1+}^{\alpha-3} u(1) = \lambda_3 + \mu_3 \int_1^{+\infty} g_3(s)u(s)\frac{ds}{s}$, we have

$$c_3 = \frac{\lambda_3}{\Gamma(\alpha-2)} + \frac{\mu_3}{\Gamma(\alpha-2)} \int_1^{+\infty} g_3(s)u(s)\frac{ds}{s}.$$

Similarly, the boundary condition $\mathcal{D}_{1+}^{\alpha-2} u(1) = \lambda_2 + \mu_2 \int_1^{+\infty} g_2(s)u(s)\frac{ds}{s}$ implies that

$$c_2 = \frac{\lambda_2}{\Gamma(\alpha-1)} + \frac{\mu_2}{\Gamma(\alpha-1)} \int_1^{+\infty} g_2(s)u(s)\frac{ds}{s},$$

and the boundary condition $\mathcal{D}_{1+}^{\alpha-1} u(+\infty) = \lambda_1 + \mu_1 \int_1^{+\infty} g_1(s)u(s)\frac{ds}{s}$ gives

$$c_1 = \frac{\lambda_1}{\Gamma(\alpha)} + \frac{\mu_1}{\Gamma(\alpha)} \int_1^{+\infty} g_1(s)u(s)\frac{ds}{s} + \frac{1}{\Gamma(\alpha)} \int_1^{+\infty} h(s)\frac{ds}{s}.$$

Consequently, the solution of Equation (27) subjected to the condition (2) is

$$
\begin{aligned}
u(t) &= -\mathcal{J}_{1+}^{\alpha} h(t) + c_1(\ln t)^{\alpha-1} + c_2(\ln t)^{\alpha-2} + c_3(\ln t)^{\alpha-3} \\
&= \int_1^{+\infty} G_\alpha(t,s)h(s)\frac{ds}{s} + \sum_{j=1}^{3} \mu_j \vartheta_{\alpha-j+1}(t) \int_1^{+\infty} g_j(s)u(s)\frac{ds}{s} + \Phi(t).
\end{aligned} \tag{31}
$$

Multiplying both sides of (31) by $\frac{g_i(t)}{t}(i = 1,2,3)$ and integrating from 1 to ∞, then we get

$$
\begin{aligned}
\int_1^{\infty} g_i(t)u(t)\frac{dt}{t} &= \int_1^{\infty} g_i(t)\left(\int_1^{\infty} G_\alpha(t,s)h(s)\frac{ds}{s}\right)\frac{dt}{t} + \sum_{j=1}^{3}\lambda_j \int_1^{\infty} g_i(t)\vartheta_{\alpha-j+1}(t)\frac{dt}{t} \\
&\quad + \sum_{j=1}^{3}\mu_j\left(\int_1^{\infty} g_i(t)\vartheta_{\alpha-j+1}(t)\frac{dt}{t}\right)\int_1^{\infty} g_j(s)u(s)\frac{ds}{s} \\
&= \int_1^{\infty}\left(\int_1^{\infty} G_\alpha(t,s)g_i(t)\frac{dt}{t}\right)h(s)\frac{ds}{s} + \sum_{j=1}^{3}\lambda_j l_{ij} + \sum_{j=1}^{3}\mu_j l_{ij}\int_1^{\infty} g_j(s)u(s)\frac{ds}{s} \\
&= \int_1^{\infty} g_i^*(s)h(s)\frac{ds}{s} + \sum_{j=1}^{3}\lambda_j l_{ij} + \sum_{j=1}^{3}\mu_j l_{ij}\int_1^{\infty} g_j(s)u(s)\frac{ds}{s},\ i = 1,2,3,
\end{aligned} \tag{32}
$$

where $l_{ij}, g_i^*(s)$ are given in (22) and (30) respectively. For convenience, we denote

$$
g_j = \int_1^{\infty} g_j(t)u(t)\frac{dt}{t}, \quad G_j = \sum_{k=1}^{3}\lambda_k l_{jk} + \int_1^{\infty} g_j^*(s)h(s)\frac{ds}{s},\ j = 1,2,3.
$$

Then (32) can be rewritten as

$$
A\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = \begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix}.
$$

Since $|A| \neq 0$, the matrix equation is solvable and the solution is uniquely expressed as

$$
\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = A^{-1}\begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix} = \frac{A^*}{|A|}\begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix} = \frac{1}{|A|}\begin{pmatrix} \sum_{i=1}^{3} A_{i1}G_i \\ \sum_{i=1}^{3} A_{i2}G_i \\ \sum_{i=1}^{3} A_{i3}G_i \end{pmatrix}.
$$

That is to say, $g_j = \frac{1}{|A|}\sum_{i=1}^{3} A_{ij}G_i,\ j = 1,2,3$. Substituting g_1, g_2 and g_3 into (31), we infer

$$
\begin{aligned}
u(t) &= \int_1^{\infty} G_\alpha(t,s)h(s)\frac{ds}{s} + \sum_{j=1}^{3}\mu_j\frac{\vartheta_{\alpha-j+1}(t)}{|A|}\sum_{i=1}^{3}A_{ij}G_i + \sum_{j=1}^{3}\lambda_j\vartheta_{\alpha-j+1}(t) \\
&= \int_1^{\infty} G_\alpha(t,s)h(s)\frac{ds}{s} + \sum_{j=1}^{3}\frac{\mu_j\vartheta_{\alpha-j+1}(t)}{|A|}\sum_{i=1}^{3}A_{ij}\left(\sum_{k=1}^{3}\lambda_k l_{ik} + \int_1^{\infty} g_i^*(s)h(s)\frac{ds}{s}\right) \\
&\quad + \sum_{j=1}^{3}\lambda_j\vartheta_{\alpha-j+1}(t) \\
&= \int_1^{\infty} G_\alpha(t,s)h(s)\frac{ds}{s} + \sum_{i=1}^{3}\varphi_i(t)\int_1^{\infty} g_i^*(s)h(s)\frac{ds}{s} + \sum_{j=1}^{3}\omega_j\vartheta_{\alpha-j+1}(t).
\end{aligned}
$$

□

5. Uniqueness Results of Hadamard BVP in $X_c^p(J)$

For the readers' convenience, in this section, we list the assumptions needed in the proof of the existence of unique solution.

$(\mathbf{H_1})$ $f(t,u,v): J\times\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ and $f(t,0,0)\in X^1(J)$.

$(\mathbf{H_2})$ There exist two nonnegative functions $a(t), b(t)$ defined on J such that for any $t\in J$ and $u_i, v_i (i=1,2)\in\mathbb{R}$,

$$|f(t,u_1,v_1)-f(t,u_2,v_2)|\le a(t)|u_1-u_2|+b(t)|v_1-v_2|,$$

where a,b satisfy $l_a=\int_1^\infty a^{p'}(t)t^{-cp'}\frac{dt}{t}<\infty$, and

$$l_b=\left(\Gamma(\beta)\left((\beta-1)p'+1\right)^{\frac{1}{p'}}\right)^{-1}\int_1^\infty b(t)(\ln t)^{\beta-\frac{1}{p}}t^{-c}\frac{dt}{t}<\infty.$$

Let

$$d_j=\left(\int_1^\infty t^{cp}\vartheta^p_{\alpha-j+1}(t)\frac{dt}{t}\right)^{\frac{1}{p}}(j=1,2,3), \tag{33}$$

by Table 1, we know $0<d_j<\infty$.

According to the representation of $G_\alpha(t,s)$ in (20), the following estimate holds

$$0\le G_\alpha(t,s)\le\vartheta_\alpha(t),\ \forall t,s\in J_1. \tag{34}$$

It follows from (33) and Table 1 that $\Phi(t)\in X_c^p(J)$, hence

$$0\le l_\Phi=\left(\int_1^\infty t^{cp}\Phi^p(t)\frac{dt}{t}\right)^{\frac{1}{p}}\le 3(\lambda_1d_1+\lambda_2d_2+\lambda_3d_3)<\infty. \tag{35}$$

Theorem 6. *Assume that conditions $(\mathbf{C_1})$, $(\mathbf{C_2})$, $(\mathbf{H_1})$ and $(\mathbf{H_2})$ hold. Then the boundary value problem (1), (2) has a unique positive solution in $X_c^p(J)$ provided that $d_1(l_a+l_b)+\sum\limits_{j=1}^{3}\mu_jd_jl_{g_j}<1$.*

Proof of Theorem 6. We consider an operator on $X_c^p(J)$ as follows

$$Tu(t)=\int_1^{+\infty}G_\alpha(t,s)f(s,u(s),\mathcal{J}^\beta_{1+}u(s))\frac{ds}{s}+\sum_{j=1}^{3}\mu_j\vartheta_{\alpha-j+1}(t)\int_1^{+\infty}g_j(s)u(s)\frac{ds}{s}+\Phi(t).$$

According to Lemma 7, the fixed point of T is the solution of the boundary value problem (1), (2). It suffices to show that the operator T has a unique fixed point.

Step 1. For any $u\in X_c^p(J)$, we first show that $f(t,u(t),\mathcal{J}^\beta_{1+}u(t)), g_j(t)u(t)\in X^1(J)$. Given the assumptions, by Hölder's inequality, we have

$$\begin{aligned}
&\int_1^\infty|f(t,u(t),\mathcal{J}^\beta_{1+}u(t))|\frac{dt}{t}\\
\le&\int_1^\infty a(t)|u(t)|\frac{dt}{t}+\int_1^\infty b(t)|\mathcal{J}^\beta_{1+}u(t)|\frac{dt}{t}+\int_1^\infty|f(t,0,0)|\frac{dt}{t}\\
\le&\|u\|_{X_c^p(J)}\left(\int_1^\infty a^{p'}(t)t^{-cp'}\frac{dt}{t}\right)^{\frac{1}{p'}}+\frac{\|u\|_{X_c^p(J)}}{\Gamma(\beta)}\int_1^\infty b(t)\left(\int_1^t\left(\ln\frac{t}{s}\right)^{(\beta-1)p'}s^{-cp'}\frac{ds}{s}\right)^{\frac{1}{p'}}\frac{dt}{t}\\
&+\int_1^\infty|f(t,0,0)|\frac{dt}{t}\\
=&l_a\|u\|_{X_c^p(J)}+\frac{\|u\|_{X_c^p(J)}}{\Gamma(\beta)}\int_1^\infty b(t)(\ln t)^{\beta-\frac{1}{p}}\left(\int_0^1(1-\tau)^{(\beta-1)p'}t^{-cp'\tau}d\tau\right)^{\frac{1}{p'}}\frac{dt}{t}\\
&+\int_1^\infty|f(t,0,0)|\frac{dt}{t}\\
\le&(l_a+l_b)\|u\|_{X_c^p(J)}+\int_1^\infty|f(t,0,0)|\frac{dt}{t}<\infty.
\end{aligned}$$

Similarly, by **(C$_2$)**, using Hölder's inequality, we get

$$\int_1^\infty |g_j(s)u(s)|\frac{ds}{s} \le l_{g_j}\|u\|_{X_c^p(J)} < \infty.$$

Set $\Omega = \{u \in X_c^p(J) \big| \, u(t) \ge 0\}$. Notice that Ω is a closed set of $X_c^p(J)$, if we define a metric $d(x,y) = \left(\int_1^\infty t^{cp}|x(t)-y(t)|^p \frac{dt}{t}\right)^{\frac{1}{p}}$ on it, then (Ω, d) is also a complete metric space.

Step 2. $T : \Omega \to \Omega$ is well defined. According to Minkowski's inequality, for any $u \in \Omega$,

$$\begin{aligned}
&\left(\int_1^\infty t^{cp}|Tu(t)|^p\frac{dt}{t}\right)^{\frac{1}{p}} \\
\le& \left[\int_1^\infty t^{cp}\left(\int_1^\infty G_\alpha(t,s)|f(s,u(s),\mathcal{J}_{1+}^\beta u(s))|\frac{ds}{s}\right)^p\frac{dt}{t}\right]^{\frac{1}{p}} \\
&+\sum_{j=1}^3 \mu_j\int_1^\infty |g_j(s)u(s)|\frac{ds}{s}\left[\int_1^\infty t^{cp}\vartheta^p_{\alpha-j+1}(t)\frac{dt}{t}\right]^{\frac{1}{p}} + \left[\int_1^\infty t^{cp}\Phi^p(t)\frac{dt}{t}\right]^{\frac{1}{p}} \\
\le& \left[d_1\int_1^\infty |f(s,u(s),\mathcal{J}_{1+}^\beta u(s))|\frac{ds}{s} + \sum_{j=1}^3 \mu_j d_j\int_1^\infty |g_j(s)u(s)|\frac{ds}{s} + l_\Phi\right],
\end{aligned}$$

where $d_j(j=1,2,3)$ is defined in (33). Combining those two estimates in Step 1, we get

$$\|Tu\|_{X_c^p(J)} < \infty,$$

hence $Tu \in X_c^p(J)$. Notice that $G_\alpha \ge 0$ (see (34)) and $f, g_j(j=1,2,3)$ are nonnegative, it is obvious that $Tu(t) \ge 0$ and $Tu \in \Omega$.

Step 3. We will show that T has a unique fixed point in Ω. Following the proof method in Step 1, for any $x, y \in \Omega$, we have

$$\begin{aligned}
d(Tx,Ty) =& \left(\int_1^\infty t^{cp}|Tx(t)-Ty(t)|\frac{dt}{t}\right)^{\frac{1}{p}} \\
\le& \left[\int_1^\infty t^{cp}\left(\int_1^\infty G_\alpha(t,s)|f(s,x(s),\mathcal{J}_{1+}^\beta x(s)) - f(s,y(s),\mathcal{J}_{1+}^\beta y(s))|\frac{ds}{s}\right)^p\frac{dt}{t}\right]^{\frac{1}{p}} \\
&+\sum_{j=1}^3 \mu_j\int_1^\infty |g_j(s)|x(s)-y(s)|\frac{ds}{s}\left[\int_1^\infty t^{cp}\vartheta^p_{\alpha-j+1}(t)\frac{dt}{t}\right]^{\frac{1}{p}} \\
\le& \left[d_1\int_1^\infty |f(s,x(s),\mathcal{J}_{1+}^\beta x(s)) - f(s,y(s),\mathcal{J}_{1+}^\beta y(s))|\frac{ds}{s}\right. \\
&\left.+\sum_{j=1}^3 \mu_j d_j\int_1^\infty |g_j(s)||x(s)-y(s)|\frac{ds}{s}\right] \\
\le& \left[d_1(l_a+l_b) + \sum_{j=1}^3 \mu_j d_j l_{g_j}\right]d(x,y).
\end{aligned}$$

Let $e^\tau = \left[d_1(l_a+l_b) + \sum_{j=1}^3 \mu_j d_j l_{g_j}\right]^{-1}$, $\psi(t) = \ln t$. It follows that $\tau > 0$ from the condition and

$$\psi(d(Tx,Ty)) + \tau \le \psi(d(x,y)).$$

Therefore, the conditions appearing in Theorem 5 are satisfied and the operator T has a unique fixed point u in Ω. Since at least one of these coefficients $\mu_i, \lambda_i(i=1,2,3)$ is positive in the expression of operator T, the unique solution u must be positive. □

Example 1. *Consider the following boundary value problem*

$$\begin{cases} \mathcal{D}_{1+}^{2.8}u(t) + f(t, u(t), \mathcal{J}_{1+}^{0.9}u(t)) = 0, \\ \mathcal{D}_{1+}^{-0.2}u(1) = 2.1 + 0.1\int_1^\infty (\ln s)^{-0.05}s^{-0.8}u(s)\frac{ds}{s}, \\ \mathcal{D}_{1+}^{0.8}u(1) = 5.8 + 0.005\int_1^\infty (\ln s)^{-0.1}s^{-1}u(s)\frac{ds}{s}, \\ \mathcal{D}_{1+}^{1.8}u(1) = 8.8 + 0.025\int_1^\infty (\ln s)^{0.5}s^{-1.2}u(s)\frac{ds}{s}. \end{cases} \tag{36}$$

Let $\alpha = 2.8, \beta = 0.9, p = \frac{5}{4}, p' = 5, c = -\frac{3}{5}$. *Then* $(\mathbf{C_1})$ *is satisfied. Set* $g_1(t) = (\ln t)^{-0.05}t^{-0.8}, g_2(t) = (\ln t)^{-0.1}t^{-1}, g_3(t) = (\ln t)^{0.5}t^{-1.2}$, *then* $g_j : J \to \mathbb{R}^+$ *and*

$$l_{g_1} = \left(\int_1^\infty t^3 t^{-4}(\ln t)^{-0.25}\frac{dt}{t}\right)^{\frac{1}{5}} = (\Gamma(0.75))^{\frac{1}{5}} \approx 1.0416,$$

$l_{g_2} = \left(\frac{\Gamma(0.5)}{\sqrt{2}}\right)^{\frac{1}{5}} \approx 1.0462, l_{g_3} = \left(\frac{\Gamma(3.5)}{3^{3.5}}\right)^{\frac{1}{5}} \approx 0.5893$. *It implies that the condition* $(\mathbf{C_2})$ *holds.*

Let $f(t) = (\ln t)^{-0.5}t^{-1} + (\ln t)^{-0.16}t^{-0.8}\left|\sin\frac{x}{16t}\right| + \frac{1}{20}(\ln t)^{-0.6}t^{-1.5}\left|\sin\frac{y+1}{t}\right|, (t, x, y) \in J \times \mathbb{R}^+ \times \mathbb{R}^+$. *From this function expression, we have* $f(t, 0, 0) = (\ln t)^{-0.5}t^{-1} + \frac{1}{20}(\ln t)^{-0.6} t^{-1.5}\left|\sin\frac{1}{t}\right|$. *By the fact that* $\int_1^\infty (\ln t)^{-0.5}t^{-1}\frac{dt}{t} = \Gamma(0.5) < \infty$ *and*

$$\int_1^\infty (\ln t)^{-0.6}t^{-1.5}\left|\sin\frac{1}{t}\right|\frac{dt}{t} \le \int_1^\infty (\ln t)^{-0.6}t^{-1.5}\frac{dt}{t} = \frac{\Gamma(0.4)}{1.5^{0.4}} < \infty,$$

we know $\int_1^\infty f(t, 0, 0)\frac{dt}{t} < \infty$. *In addition, for* $(t, x_1, y_1) \in J \times \mathbb{R}^+ \times \mathbb{R}^+$ *and* $(t, x_2, y_2) \in J \times \mathbb{R}^+ \times \mathbb{R}^+$, *we infer the following conclusion*

$$\begin{aligned} &|f(t, x_1, y_1) - f(t, x_2, y_2)| \\ \le&(\ln t)^{-0.16}t^{-0.8}\left|\left|\sin\frac{x_1}{16t}\right| - \left|\sin\frac{x_2}{16t}\right|\right| + \frac{1}{20}(\ln t)^{-0.6}t^{-1.5}\left|\left|\sin\frac{y_1+1}{t}\right| - \left|\sin\frac{y_2+1}{t}\right|\right| \\ \le&(\ln t)^{-0.16}t^{-0.8}\left|\sin\frac{x_1}{16t} - \sin\frac{x_2}{16t}\right| + \frac{1}{20}(\ln t)^{-0.6}t^{-1.5}\left|\sin\frac{y_1+1}{t} - \sin\frac{y_2+1}{t}\right| \\ \le&\frac{1}{16}(\ln t)^{-0.16}t^{-0.8}|x_1 - x_2| + \frac{1}{20}(\ln t)^{-0.6}t^{-1.5}|y_1 - y_2|. \end{aligned}$$

Let $a(t) = \frac{1}{16}(\ln t)^{-0.16}t^{-0.8}, b(t) = \frac{1}{20}(\ln t)^{-0.6}t^{-1.5}$, *by calculation, we have*

$$\begin{aligned} l_a &= \frac{1}{16}\left(\int_1^\infty t^3 t^{-4}(\ln t)^{-0.8}\frac{dt}{t}\right)^{\frac{1}{5}} = \frac{1}{16}(\Gamma(0.2))^{0.2} \approx 0.0848, \\ l_b &= \left(\Gamma(0.9)(0.5)^{\frac{1}{5}}\right)^{-1}\int_1^\infty b(t)(\ln t)^{0.1}t^{0.6}\frac{dt}{t} = \frac{\Gamma(0.5)}{20(0.5)^{\frac{1}{5}}\Gamma(0.9)\sqrt{0.9}} \approx 0.1002. \end{aligned}$$

According to (33), *we get*

$$d_1 = \frac{1}{\Gamma(2.8)}\left(\int_1^\infty t^{-0.75}(\ln t)^{2.25}\frac{dt}{t}\right)^{\frac{4}{5}} = \frac{1}{\Gamma(2.8)}\left(\frac{\Gamma(3.25)}{(0.75)^{3.25}}\right)^{0.8} \approx 2.6619,$$

similarly, $d_2 \approx 1.6998, d_3 \approx 1.2005$. *Synthesizing the above formulas, one has* $d_1(l_a + l_b) + \sum_{j=1}^3 \mu_j d_j l_{g_j} \approx 0.7571 < 1$. *Therefore, all these conditions in Theorem* 6 *are satisfied, and the boundary value problem* (36) *has a unique positive solution in* $X_{-0.6}^{1.25}(J)$.

6. Uniqueness Results of Hadamard BVP in X

In this section, we will use the following conditions

$(\mathbf{H_2})'$ There exist nonnegative functions $c(t), d(t), \eta_i(t)(i = 1, 2)$ defined on J such that for any $t \in J$ and $u_i, v_i (i = 1, 2) \in \mathbb{R}$,

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \le c(t)\frac{|u_1 - u_2|}{(1 + \eta_1(t)|u_1 - u_2|^{\theta})^{\frac{1}{\theta}}} + d(t)\frac{|v_1 - v_2|}{(1 + \eta_2(t)|v_1 - v_2|^{\theta})^{\frac{1}{\theta}}},$$

where $c, d, \eta_i(i = 1, 2)$ satisfy

$$l_c = \int_1^{\infty} c(t)\frac{1 + (\ln t)^2}{(\ln t)^{3-\alpha}}\frac{dt}{t} < \infty,\ \ l_d = \int_1^{\infty} d(t)r(t)\frac{dt}{t} < \infty,$$

$$\inf_{t\in J}\eta_1(t)\left(\frac{1 + (\ln t)^2}{(\ln t)^{3-\alpha}}\right)^{\theta} > \tau_1 > 0,\ \ \inf_{t\in J}\eta_2(t)r^{\theta}(t) > \tau_2 > 0,$$

$0 < \theta < 1$ is a constant, $r(t)$ is defined in (19).

$(\mathbf{H_2})''$ There exist two nonnegative functions $a_1(t), b_1(t)$ defined on J such that for any $t \in J$ and $u_i, v_i \in \mathbb{R}(i = 1, 2)$,

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \le a(t)|u_1 - u_2| + b(t)|v_1 - v_2|,$$

where a_1, b_1 satisfy $l_{a_1} = \int_1^{\infty} a_1(t)\frac{1+(\ln t)^2}{(\ln t)^{3-\alpha}}\frac{dt}{t} < \infty, l_{b_1} = \int_1^{\infty} b_1(t)r(t)\frac{dt}{t} < \infty$.

If we introduce a metric $d(x, y) = \sup_{t\in J_1}\left|\frac{(\ln t)^{3-\alpha}|x(t)-y(t)|}{1+(\ln t)^2}\right|$ on the space X, let $P = \{u \in X|\ u(t) \ge 0, t \in J\}$. Then (P, d) is a complete metric space.

Based on (29), we define another one operator on the space P,

$$Au(t) = \int_1^{\infty} G(t, s)f(s, u(s), \mathcal{J}_{1+}^{\beta}u(s))\frac{ds}{s} + \sum_{j=1}^{3}\omega_j\vartheta_{\alpha-j+1}(t), \tag{37}$$

where $G(t, s) = G_\alpha(t, s) + \sum\limits_{i=1}^{3}\varphi_i(t)g_i^*(s)$.

Lemma 8. *Assume that conditions* $(\mathbf{C_1})$–$(\mathbf{C_3})$ *hold. For any* $t \in J, s \in J_1, G(t, s)$ *is nonnegative and has the following estimate*

$$\sup_{t,s\in J_1}\frac{(\ln t)^{3-\alpha}G(t, s)}{1 + (\ln t)^2} \le l_G,\ \ l_G \text{ is a positive constant.}$$

Proof of Lemma 8. From (22) and condition $(\mathbf{C_2})$, we have

$$0 \le l_{ij} \le \left(\int_1^{\infty}|g_j(s)|^{p'}s^{-cp'}\frac{ds}{s}\right)^{\frac{1}{p'}}\|\vartheta_{\alpha-j+1}\|_{X_c^p(J)} < \infty.$$

Due to this inequality and (30), we infer

$$0 \le g_i^*(s) \le \int_1^{\infty} g_i(t)\vartheta_\alpha(t)\frac{dt}{t} = l_{i1} < \infty.$$

Furthermore, for any $s \in J_1$, we deduce that the following limit exists

$$\lim_{t\to 1+}\frac{(\ln t)^{3-\alpha}G(t, s)}{1 + (\ln t)^2} = g_3^*(s) < \infty.$$

Considering the above limit, combining condition $(\mathbf{C_3})$ and (25), there must be a positive constant l_G such that

$$\sup_{t,s\in J_1}\frac{(\ln t)^{3-\alpha}G(t,s)}{1+(\ln t)^2}\leq \sup_{t,s\in J_1}\left[\frac{(\ln t)^{3-\alpha}\vartheta_\alpha(t)}{1+(\ln t)^2}+\sum_{i=1}^{3}\frac{(\ln t)^{3-\alpha}\varphi_i(t)}{1+(\ln t)^2}g_i^*(s)\right]<l_G<\infty.$$

□

Theorem 7. *Assume that conditions $(\mathbf{C_1})$–$(\mathbf{C_3})$, $(\mathbf{H_1})$ and $(\mathbf{H_2})'$ hold. Then the boundary value problem (1), (2) has a unique positive solution in P provided that $l_G(l_c+l_d)\leq 1$.*

Proof of Theorem 7. According to Lemma 7, the fixed point of A is the solution of the boundary value problem (1), (2). It suffices to show that the operator A has a unique fixed point.

Now we will prove that

$$f(s,u(s),\mathcal{J}_{1+}^{\beta}u(s))\in X^1(J),\ \forall u\in P. \tag{38}$$

In fact, in view of $(\mathbf{H_2})'$, we have

$$\begin{aligned}\int_1^\infty f(s,u(s),\mathcal{J}_{1+}^{\beta}u(s))\frac{ds}{s}\leq&\int_1^\infty c(s)\frac{u(s)}{(1+\eta_1(s)(u(s))^\theta)^{\frac{1}{\theta}}}\frac{ds}{s}\\&+\int_1^\infty d(s)\frac{\mathcal{J}_{1+}^{\beta}u(s)}{(1+\eta_2(s)(\mathcal{J}_{1+}^{\beta}u(s))^\theta)^{\frac{1}{\theta}}}\frac{ds}{s}+\int_1^\infty f(s,0,0)\frac{ds}{s}\\=&I_1+I_2+I_3.\end{aligned}$$

To make further estimates of the integral I_1,I_2, let us first introduce a function $\rho(x)=\frac{x}{(1+\tau x^\theta)^{\frac{1}{\theta}}}(\tau>0)$. It has been proved that ρ is increasing in $\mathbb{R}^+$ (see [16] Lemma 4). Using $(\mathbf{H_2})'$ again, we get

$$\begin{aligned}0\leq I_1\leq&\int_1^\infty c(s)\frac{1+(\ln s)^2}{(\ln s)^{3-\alpha}}\frac{\frac{(\ln s)^{3-\alpha}}{1+(\ln s)^2}u(s)}{\left(1+\tau_1\left(\frac{(\ln s)^{3-\alpha}}{1+(\ln s)^2}u(s)\right)^\theta\right)^{\frac{1}{\theta}}}\frac{ds}{s}\\\leq&\frac{l_c\|u\|_X}{(1+\tau_1\|u\|_X^\theta)^{\frac{1}{\theta}}}<\infty.\end{aligned}$$

From (19), following the same technique, we deduce that $0\leq I_2\leq\frac{l_d\|u\|_X}{(1+\tau_2\|u\|_X^\theta)^{\frac{1}{\theta}}}<\infty$.

By $(\mathbf{H_1})$ we know $0\leq I_3<\infty$. The conclusion $\int_1^\infty f(s,u(s),\mathcal{J}_{1+}^{\beta}u(s))<\infty$ is drawn by summing up the above inequalities.

Next we show that $A:P\to P$ is well defined. Let us modify and rewrite the expression of the operator A as follows

$$\begin{aligned}Au(t)=&\vartheta_\alpha(t)\int_1^\infty f(s,u(s),\mathcal{J}_{1+}^{\beta}u(s))\frac{ds}{s}-\mathcal{J}_{1+}^{\alpha}f(t,u(t),\mathcal{J}_{1+}^{\beta}u(t))\\&+\sum_{i=1}^{3}\varphi_i(t)\int_1^\infty g_i^*(s)f(s,u(s),\mathcal{J}_{1+}^{\beta}u(s))\frac{ds}{s}+\sum_{j=1}^{3}\omega_j\vartheta_{\alpha-j+1}(t).\end{aligned}$$

It is obvious that $\vartheta_{\alpha-j+1}(t)\in C(J)(j=1,2,3)$, then it follows that $\varphi_i(t)\in C(J)$ $(i=1,2,3)$ from (25). According to Property 5 and (38), we get $\mathcal{J}_{1+}^{\alpha}f(t,u(t),\mathcal{J}_{1+}^{\beta}u(t))\in C(J_1)$.

To sum up, we know $Au(t) \in C(J)$. Since $G(t,s) \geq 0$ and $\omega_j \geq 0$ from (26) and $(\mathbf{C_3})$, then $Au(t) \geq 0$. Furthermore, from Lemma 8 and (38), we have

$$\sup_{t \in J_1} \frac{(\ln t)^{3-\alpha} Au(t)}{1+(\ln t)^2} \leq l_G \int_1^\infty f(s, u(s), \mathcal{J}_{1+}^{\beta} u(s)) \frac{ds}{s} + \sum_{j=1}^{3} \frac{\omega_j}{\Gamma(\alpha - j + 1)} < \infty.$$

This proves that A applies P into itself.

At last, we check the other conditions in Theorem 5 are satisfied. Let $\tau = \min\{\tau_1, \tau_2\}$, similarly, for any $x, y \in P$ with $d(Tx, Ty) > 0$, we have

$$\begin{aligned} d(Ax, Ay) \leq & l_G \left(l_c \frac{d(x,y)}{(1+\tau_1 d^\theta(x,y))^{\frac{1}{\theta}}} + l_d \frac{d(x,y)}{(1+\tau_2 d^\theta(x,y))^{\frac{1}{\theta}}} \right) \\ \leq & l_G (l_c + l_d) \frac{d(x,y)}{(1+\tau d^\theta(x,y))^{\frac{1}{\theta}}} \\ \leq & \frac{d(x,y)}{(1+\tau d^\theta(x,y))^{\frac{1}{\theta}}}. \end{aligned} \tag{39}$$

From this , it follows

$$\tau - \frac{1}{d^\theta(Ax, Ay)} \leq -\frac{1}{d^\theta(x,y)}.$$

Choose $\psi(t) = -t^{-\theta}$, then the inequality is rewritten as $\tau + \psi(d(Ax, Ay)) \leq \psi(d(x,y))$, which shows that all the conditions in Theorem 5 hold. Hence, the operator A has a unique fixed point in P, and this means that BVP (1), (2) has a unique positive solution in P. □

Theorem 8. *Assume that conditions* $(\mathbf{C_1})$–$(\mathbf{C_3})$, $(\mathbf{H_1})$ *and* $(\mathbf{H_2})''$ *hold. Then the boundary value problem* (1), (2) *has a unique positive solution in P provided that* $l_G(l_{a_1} + l_{b_1}) < 1$.

Proof of Theorem 8. As the proof in Theorem 7, we know $A : P \to P$ is well defined. Moreover, from $(\mathbf{H_2})''$ we have

$$\begin{aligned} d(Ax, Ay) \leq & l_G \int_1^\infty \left(a_1(s)|x(s) - y(s)| + b_1(s)|\mathcal{J}_{1+}^{\beta} x(s) - \mathcal{J}_{1+}^{\beta} y(s)| \right) \frac{ds}{s} \\ \leq & l_G (l_{a_1} + l_{b_1}) d(x,y). \end{aligned}$$

By Banach's fixed point theorem, we know that the operator A has a unique fixed point on P. Therefore, BVP (1), (2) has a unique positive solution in P. □

Example 2. *Consider the following boundary value problem*

$$\begin{cases} \mathcal{D}_{1+}^{2.6} u(t) + f(t, u(t), \mathcal{J}_{1+}^{0.6} u(t)) = 0, \\ \mathcal{D}_{1+}^{-0.4} u(1) = 0.1 \int_1^\infty (\ln s)^{1.6} s^{-5} u(s) \frac{ds}{s}, \\ \mathcal{D}_{1+}^{0.6} u(1) = 6 + 0.8 \int_1^\infty (\ln s)^{0.4} s^{-6} u(s) \frac{ds}{s}, \\ \mathcal{D}_{1+}^{1.6} u(1) = 12 + 0.5 \int_1^\infty (\ln s)^{-0.1} s^{-10} u(s) \frac{ds}{s}. \end{cases} \tag{40}$$

Let $\alpha = 2.6, \beta = 0.6, p = p' = 2, c = -\frac{1}{2}$. *Then* $(\mathbf{C_1})$ *is satisfied. Set*

$$g_1(t) = (\ln t)^{-0.1} t^{-10}, g_2(t) = (\ln t)^{0.4} t^{-6}, g_3(t) = (\ln t)^{1.6} t^{-5},$$

then $g_j : J \to \mathbb{R}^+$. *Applying the formula* $l_{g_j} = \left(\int_1^\infty |g_j(s)|^{p'} s^{-cp'} \frac{ds}{s} \right)^{\frac{1}{p'}}$ *we have*

$$l_{g_1} = \left(\frac{\Gamma(0.8)}{19^{0.8}} \right)^{\frac{1}{2}} \approx 0.3324, l_{g_2} = \left(\frac{\Gamma(1.8)}{11^{1.8}} \right)^{\frac{1}{2}} \approx 0.1116, l_{g_3} = \left(\frac{\Gamma(4.2)}{9^{4.2}} \right)^{\frac{1}{2}} \approx 0.0276,$$

it implies that the condition $(\mathbf{C_2})$ *holds. According to* (22), *let* $i, j = 1,2,3$ *we calculate in turn*

$$l_{13} \approx 0.3763, \quad l_{12} \approx 0.0314, \quad l_{11} \approx 0.0029,$$

$$l_{23} \approx 0.1119, \quad l_{22} \approx 0.0311, \quad l_{21} \approx 0.0065,$$

$$l_{33} \approx 0.0214, \quad l_{32} \approx 0.0157, \quad l_{31} \approx 0.0063.$$

Let $\mu_1 = 0.5, \mu_2 = 0.8, \mu_3 = 0.1$, *substituting them into* (23), *one has*

$$A = \begin{pmatrix} 0.9995 & -0.0251 & -0.0376 \\ -0.0033 & 0.9751 & -0.0112 \\ -0.0032 & -0.0126 & 0.9979 \end{pmatrix}.$$

It follows that $|A| \approx 0.9722$ *and*

$$A^{-1} \approx \begin{pmatrix} 0.9729 & 0.0255 & 0.0369 \\ 0.0033 & 0.9973 & 0.0113 \\ 0.0032 & 0.0127 & 0.9745 \end{pmatrix}.$$

Therefore, $(\mathbf{C_3})$ *is satisfied. By* (33), *we have*

$$d_1 = \frac{(\Gamma(4.2))^{\frac{1}{2}}}{\Gamma(2.6)} \approx 1.9475, d_2 = \frac{(\Gamma(2.2))^{\frac{1}{2}}}{\Gamma(1.6)} \approx 0.625, d_3 = \frac{(\Gamma(0.2))^{\frac{1}{2}}}{\Gamma(0.6)} \approx 1.439.$$

Let

$$f(t,x,y) = c(t)\frac{x}{(1+\eta_1(t)x^{0.5})^2} + d(t)\frac{y}{(1+\eta_2(t)y^{0.5})^2} + e(t),$$

where $c(t) = \frac{1}{10}t^{-80}(\ln t)^{-0.2}$, $d(t) = \frac{1}{5}t^{-16}(\ln t)^{-0.7}$, $e(t) = \frac{1}{t^2}\left|\sin\frac{1}{t-1}\right|$, $\eta_1(t) = \left[1 + \left(\frac{(\ln t)^{0.4}}{1+(\ln t)^2}\right)^{0.5}\right]\left(1+\frac{1}{t}\right)^{0.5}, \eta_2(t) = 1 + \left(\frac{(\ln t)^{-0.2}}{1+(\ln t)^2}\right)^{0.5}$. *Let* $z(t) = \frac{t}{(1+at^{0.5})^2}$, $t \in (0,\infty), a > 0$ *is a constant, from Lemma 4 in [16], we know* $|z(t) - z(s)| \le z(|t-s|)$. *Thus, we can derive the inequality relation for* f *in* $(\mathbf{H_2})'$. *Furthermore,*

$$\inf_{t\in J}\eta_1(t)\left(\frac{1+(\ln t)^2}{(\ln t)^{3-\alpha}}\right)^{\theta} > \inf_{t\in J}\left(1+\frac{1}{t}\right)^{0.5} = 1,$$

$$\inf_{t\in J}\eta_2(t)r^{\theta}(t) \ge \inf_{t\in J}\frac{\Gamma(0.6)}{\Gamma(1.2)}\frac{1}{1+(\ln t)^2} = \frac{\Gamma(0.6)}{\Gamma(1.2)},$$

$$l_c = \frac{1}{10}\left(\frac{\Gamma(0.4)}{10^{0.4}} + \frac{\Gamma(2.4)}{10^{2.4}}\right) \approx 0.0888, l_d = \frac{1}{5}\left(\frac{\Gamma(0.6)}{\Gamma(1.2)}\frac{\Gamma(0.5)}{16^{0.5}} + \frac{\Gamma(2.6)}{\Gamma(3.2)}\frac{\Gamma(2.5)}{16^{2.5}}\right) \approx 0.144.$$

From Lemma 8, we choose $l_G = \frac{1}{\Gamma(\alpha)} + \sum_{i=1}^{3}\frac{l_{i1}}{|A|}\sum_{j=1}^{3}\frac{\mu_j A_{ij}}{\Gamma(\alpha-j+1)} \approx 0.7068$, *then,* $l_G(l_c + l_d) \approx 0.1645 < 1$. *All these conditions in Theorem 7 are satisfied, hence the boundary value problem* (40) *has a unique positive solution.*

Remark 9. *After the derivation, we can see that Example 2 satisfies the conditions of both Theorems 6 and 7, which means that Example 2 can also lead to the conclusion of Theorem 6. On the other hand, Example 1 does not satisfy the conditions of Theorem 7.*

7. Conclusions

The main purpose of this thesis is to study the boundedness and continuity of Hadamrd integral operators on weighted integrable function spaces, which consists of functions defined in infinite intervals. While the previous articles focused more on the properties of Hadamard integral operators on the spaces of functions defined in finite intervals, this increases the difficulty. Based on these properties, we discuss a class of boundary value problems with integral boundary values and interference parameters, and obtain the uniqueness of the problem in two different spaces. Theorem 6 reveals that under weaker conditions, we may obtain the uniqueness of solution for the problem (1) and (2) on the weighted integral space $X_c^p(J)$, whereas Theorem 7 shows that the unique continuous solution of the boundary value problem is achieved on the weighted continuous function space X.

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