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Article

On the Application of Multi-Dimensional Laplace Decomposition Method for Solving Singular Fractional Pseudo-Hyperbolic Equations

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Abstract: In this work, the exact and approximate solution for generalized linear, nonlinear, and coupled systems of fractional singular M-dimensional pseudo-hyperbolic equations are examined by using the multi-dimensional Laplace Adomian decomposition method (M-DLADM). In particular, some two-dimensional illustrative examples are provided to confirm the efficiency and accuracy of the present method.

Keywords: double Laplace; three-dimensional Laplace transforms (3-DLT); inverse 3-DLT; pseudo-hyperbolic equation; coupled pseudo-hyperbolic equation; decomposition methods

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1. Introduction

Fractional calculus has attracted much attention from many researchers due to its applications in physical sciences, engineering problems and finance. The partial differential equations (PDE) having fractional order have also attracted much attention. This is mostly due to their frequent appearance in many applications in fluid mechanics, viscoelastic, biology, engineering, and physics. There are many methods to solve the PDEs with fractional order, however, most of them do not have an exact analytical solution, so there are some numerical techniques to obtain approximate solutions. We can list some of these as the Adomian decomposition method (ADM), homotopy analysis method (HAM), variational iteration method(VAM), and homotopy perturbation method(HPM), see [1]. In [2], the authors discussed the solutions of linear time-fractional differential equations. The system of second-order singularly perturbed delay differential equations of convection–diffusion type problem is studied by using Runge–Kutta methods and hybrid finite difference method, see [3]. Similarly, in [4], an efficient meshless approach for approximating the nonlinear fractional fourth-order diffusion model in the Riemann–Liouville sense was described. The error estimates and convergence rate of a three-level explicit time-split MacCormack scheme for solving the two-dimensional nonlinear unsteady advection–diffusion equation with constant coefficients have been considered to see [5].

Among the PDEs, the parabolic and hyperbolic equations appeared very often in applied mathematics, see for example [6–9], due to a wide range of applications. Similarly, the solution of the fractional diffusion equation has been obtained by using the Adomian decomposition method(ADM) as well as the series expansion method by some authors, see [10,11]. Further, there are several studies in the literature, which are associated with the qualities and applications of fractional derivatives, see [12–14]. There are also many works on the pseudo-parabolic equation since they represent diverse physical operations in the study of various problems such as hydrodynamics, thermodynamics, filtration theory, etc. At the same time, there are some studies, on the existence of solutions, see [15]. In general, the nonlinear equations including either ordinary or partial differential types in real-life problems are so far very difficult to solve either theoretically or numerically (since they require complicated computations). Currently, many researchers made some suggestions on exact solutions to the one-dimensional coupled parabolic equation, see the details in [16,17]. On the other side, the convergence of the Adomian method (AD) was discussed by several researchers, we may refer the readers to see [18–21], in [22], the modified method is applied to accelerate the convergence of the series solution for coupled pseudo-parabolic equations.

In recent years, the 3-dimensional Laplace Adomian decomposition (3-DLADM) has been applied to many problems, such as to solve regular and singular coupled Burgers' equations, see [23]. Similarly, in [24], singular pseudo-parabolic equations were studied. Later, very recently, the q-modified Laplace transform method was introduced and applied to solve the homogeneous and non-homogeneous Mboctara partial differential equations, see [25]. In the present study, we expand the definition of the Laplace transform to the multi-dimensional Laplace transform and some related theorems are proved. Later, we generalize the linear and nonlinear pseudo-hyperbolic equations. Further, we apply the multi-Laplace transform to solve the generalized singular fractional pseudo-hyperbolic equations and we provide three different examples in order to check the present methods.

2. Some Basic Ideas of the Multi-Dimensional Laplace Transforms (n+1)-DLT

First of all, we recall the multi-dimensional Laplace transform:

Definition 1. *Let $g : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ be a piecewise continuous function on the product of intervals*

$$[0,\infty)^{n+1} = [0,\infty) \times [0,\infty) \times \ldots \times [0,\infty) \times [0,\infty),$$

then the $(n+1)$-DLT is defined by

$$\begin{aligned} L_m \ldots L_t(g(x_1,x_2,x_3,\ldots,x_n,t)) &= G(p_1,p_2,p_3,\ldots,p_n,s) \\ &= \int_0^\infty \ldots \int_0^\infty \int_0^\infty e^{-p_1x_1-p_2x_2\ldots-p_nx_n-st} g(x_1,x_2,x_3,\ldots,x_n,t)dx_1dx_2\ldots dx_n dt, \end{aligned}$$

where the symbol $L_m \ldots L_t$ indicate to (n+1)-DLT and $p_1,p_2,p_3,\ldots,p_n,s \in \mathbb{C}$. The inverse of $G(p_1,p_2,p_3,\ldots,s)$ is determined by

$$\begin{aligned} L_m^{-1} \ldots L_s^{-1}[G(p_1,p_2,p_3,\ldots,s)] &= g(x_1,x_2,x_3,\ldots,x_n,t) \\ &= \frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty} e^{p_1x_1}dp_1 \ldots \frac{1}{2\pi i}\int_{c_n-i\infty}^{c_n+i\infty} e^{p_nx_n}dp_n \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} e^{st}G(p_1,p_2,p_3,\ldots,p_n,s)ds, \end{aligned}$$

where $L_m^{-1} \ldots L_s^{-1}$ indicates the inverse respect to $p_1,p_2,p_3,\ldots,p_n$ and s.

2.1. Existence Condition for the Multi Laplace Transform

The function $f(x_1,x_2,x_3,\ldots,x_n,t)$ is said to be multi-dimensional exponential order for $\forall a_i, b > 0$ on $for all x_i in[0,\infty)$, $t \in (0,\infty)$, if there exists a positive constant K such that for all i, $x_i > X$ and $t > T$

$$|f(x_1,x_2,x_3,\ldots,x_n,t)| \le Ke^{a_1x_1+a_2x_2+,a_3x_3+\ldots+a_nx_n+bt},$$

where $f(x_1,x_2,x_3,\ldots,x_n,t)$ can be written as

$$f(x_1,x_2,x_3,\ldots,x_n,t) = O(e^{a_1x_1+a_2x_2+,a_3x_3+\ldots+a_nx_n+bt}),$$

as $x_i \to \infty$ for all i and $t \to \infty$, or

$$\begin{aligned} & \lim_{\substack{x_1,x_2,x_3,\ldots,x_n\to\infty \\ t\to\infty}} e^{-\alpha_1 x_1-\alpha_2 x_2-\alpha_3 x_3-\ldots-\alpha_n x_n-\alpha t}|f(x_1,x_2,x_3,\ldots,x_n,t)| \\ = {} & k \lim_{\substack{x_1,x_2,x_3,\ldots,x_n\to\infty \\ t\to\infty}} e^{-(\alpha_1-a_1)x_1-(\alpha_2-a_2)x_2-(\alpha_3-a_3)x_3-\ldots-(\alpha_n-a_n)x_n-(\alpha-b)t} = 0, \\ \alpha_1 > {} & a_1, \alpha_2 > a_2, \alpha_3 > a_3 \ldots \alpha_n > a_n, \alpha > b, \end{aligned}$$

that is f is simply called an exponential order as for all i, $x_i \to \infty$, and does not grow faster than $Ke^{a_1x_1+a_2x_2+,a_3x_3+\ldots+a_nx_n+bt}$ as $t \to \infty$.

Theorem 1. *If a function $f(x_1,x_2,x_3,\ldots,x_n,t)$ is a continuous function in every finite intervals $(0,X_1)$, $(0,X_2)$, $(0,X_3)$, $(0,X_4)$,…,$(0,X_n)$ and $(0,T)$ and of exponential order then the double Laplace transform of f exists for all $p_1,p_2,p_3,\ldots,p_n$ and s provided $Re(p_i) > a_i$ for each $1 \le i \le n$ and $Re(s) > b$.*

Proof. Since we have

$$\begin{aligned} |F(p_1,p_2,\ldots,p_n,s)| &= \left|\int_0^\infty \ldots \int_0^\infty \int_0^\infty e^{-p_1x_1-p_2x_2-\ldots-p_nx_n-st} f(x_1,x_2,x_3,\ldots,x_n,t)dx_1dx_2\ldots dx_ndt\right| \\ \le K\int_0^\infty \ldots \int_0^\infty & e^{-(p_1-a_1)x_1-(p_2-a_2)x_2-(p_3-a_3)x_3-\ldots-(p_n-a_n)x_n-(s-b)t}dx_1dx_2\ldots dx_ndt \\ &= \frac{K}{(p_1-a_1)(p_2-a_2)(p_3-a_3)\ldots(p_n-a_n)(s-b)}. \end{aligned}$$

For $Re(p_1) > a_1$, $Re(p_2) > a_2$, $Re(p_3) > a_3 \ldots$, $Re(p_n) > a_n$ and $Re(s) > b$. □

In particular, two and three-dimensional Laplace transforms are defined as:

Definition 2. *Let $g(x,y)$ be a continuous function then the 2-DLT of g is given*

$$L_2[g(x,y)] = G(\rho,\sigma) = \int_0^\infty \int_0^\infty e^{-\rho x-\sigma y} g(x,y)dx\,dy,$$

where $x,y > 0$, L_2 indicates to 2-DLT and ρ,σ are complex values.

Definition 3 ([26]). *Let $g(x,y,t)$ be a piecewise continuous function on the interval $[0,\infty)^3$ of exponential order and consider $a,b,c \in \mathbb{R}$ where $\dfrac{|g(x,y,t)|}{e^{ax+by+ct}} < \infty$. Then 3-DLT is defined by*

$$L_3(g(x,y,t)) = G(\rho,\sigma,s) = \int_0^\infty \int_0^\infty \int_0^\infty e^{-\rho x-\sigma y-st} g(x,y,t)dtdydx,$$

where, the symbol L_3 indicates the 3-DLT and $\rho,\sigma,s \in \mathbb{C}$. Then, the inverse of $G(\rho,\sigma,s)$ is determined by

$$L_3^{-1}[G(\rho,\sigma,s)] = g(x,y,t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{\rho x}d\rho \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\sigma y}d\sigma \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} e^{st}G(\rho,\sigma,s)ds,$$

where L_3^{-1} indicates to inverse 3-DLT with respect to ρ,σ and s.

Furthermore 3-DLT of the derivatives $\psi_x(x,y,t)$ and $\psi_t(x,y,t)$ are presented by

$$\begin{aligned} L_3[\psi_x(x,y,t)] &= \rho\Psi(\rho,\sigma,s) - \Psi(0,\sigma,s) \\ L_3[\psi_t(x,y,t)] &= s\psi(\rho,\sigma,s) - \Psi(\rho,\sigma,0). \end{aligned} \tag{1}$$

In the next, we recall Mittag–Leffer function in two parameters which will play a significant role in this work.

2.2. Mittag–Leffler Function (MLf)

The Mittag–Leffler function of one parameter is established by

$$\Xi_{\eta}(\tau) = \sum_{i=0}^{\infty} \frac{\tau^{i}}{\Gamma(\eta i + 1)}, \quad \tau \in \mathbb{C}, Re(\eta) > 0, \tag{2}$$

similarly, two parameters is determined by

$$\Xi_{\eta,\gamma}(\tau) = \sum_{i=0}^{\infty} \frac{\tau^{i}}{\Gamma(\eta i + \gamma)}, \quad \tau \in \mathbb{C}, Re(\eta) > 0, \tag{3}$$

see [27].

If we set $\eta = 1$ in Equation (3) we obtain Equation (2). It appears from Equation (3) that

$$\Xi_{1,1}(\tau) = \sum_{i=0}^{\infty} \frac{\tau^{i}}{\Gamma(i+1)} = \sum_{i=0}^{\infty} \frac{\tau^{i}}{i!} = e^{\tau}, \tag{4}$$

$$\Xi_{1,2}(\tau) = \sum_{k=0}^{\infty} \frac{\tau^{i}}{\Gamma(i+2)} = \sum_{i=0}^{\infty} \frac{\tau^{i}}{(i+1)!} = \frac{1}{\tau} \sum_{k=0}^{\infty} \frac{\tau^{i+1}}{(i+1)} = \frac{e^{\tau} - 1}{\tau}, \tag{5}$$

and

$$\Xi_{1,3}(\tau) = \sum_{i=0}^{\infty} \frac{\tau^{i}}{\Gamma(i+3)} = \sum_{i=0}^{\infty} \frac{\tau^{i}}{(i+2)!} = \frac{1}{\tau^{2}} \sum_{i=0}^{\infty} \frac{\tau^{i+2}}{(i+2)} = \frac{e^{\tau} - 1 - 1}{\tau^{2}}, \tag{6}$$

hence, in general

$$\Xi_{1,m}(\tau) = \frac{1}{\tau^{m-1}} \left[e^{\tau} - \sum_{i=0}^{m-2} \frac{\tau^{i}}{i!} \right]. \tag{7}$$

Differentiation of the MLf is represented by

$$\frac{d^{n}}{dt^{n}} \left[t^{\eta-1} \Xi_{\zeta,\eta}(t^{\zeta}) \right] = t^{\eta-n-1} \Xi_{\zeta,\eta-n}(t^{\eta}) \tag{8}$$

for more details, see [28].

Next, we provide the 3-DLT of MLfs are helpful in this research

$$\begin{aligned}
L_3\left[x^{2} t^{\zeta} \Xi_{1,\zeta+1}(t)\right] &= \frac{2!}{\rho^{3} \sigma s^{\zeta}(s-1)}, \\
L_3\left[t^{\zeta} \Xi_{1,\zeta+1}(t)\right] &= \frac{1}{\rho \sigma s^{\zeta}(s-1)}, \\
L_3\left[t^{2\zeta} \Xi_{1,2\zeta+1}(t)\right] &= \frac{1}{\rho \sigma s^{2\zeta}(s-1)}, \\
L_3\left[t^{\zeta-1} \Xi_{1,\zeta}(t)\right] &= \frac{1}{\rho \sigma s^{\zeta-1}(s-1)}, \\
L_3\left[t^{2\zeta-1} \Xi_{1,2\zeta}(\lambda t)\right] &= \frac{1}{\rho \sigma s^{2\zeta-1}(s-\lambda)}.
\end{aligned} \tag{9}$$

In the same way, the 3-DLT of two-parameter MLfs

$$L_3\left[t^{\eta-1} \Xi_{\zeta,\eta}(\pm \lambda t^{\zeta})\right] = \frac{s^{\zeta-\eta}}{\rho \sigma (s^{\zeta} - \lambda)}. \tag{10}$$

Theorem 2. *Let f be a piecewise continuous function on $[0,\infty)\times[0,\infty)\times[0,\infty)\times\ldots\times[0,\infty)\times[0,\infty)$ (n+1)-DLT of the partial derivatives of order $\alpha-th$ $\prod_{i=1}^{n} x_i\frac{\partial^\alpha\psi}{\partial t^\alpha}$ and $\prod_{i=1}^{n} x_i f(x_1,x_2,x_3,\ldots,x_n,t)$, are given by*

$$L_m L_t\left[\prod_{i=1}^{n} x_i\frac{\partial^\alpha}{\partial t^\alpha}\psi(x_1,x_2,x_3,\ldots,x_n,t)\right]=(-1)^n\frac{\partial^i}{\prod_{i=1}^{n}\partial p_i}[\Lambda], \tag{11}$$

where

$$\Lambda=s^\alpha\Psi(p_1,p_2,p_3,\ldots,p_n,s)-\sum_{i=0}^{n-1}s^{\alpha-1-i}L_m\left[\frac{\partial^i}{\partial t^i}\psi(x_1,x_2,x_3,\ldots,x_n,0)\right],$$

and

$$L_m L_t\left[\prod_{i=1}^{n} x_i f(x_1,x_2,x_3,\ldots,x_n,t)\right]=(-1)^n\frac{\partial^i}{\prod_{i=1}^{n}\partial p_i}[F(p_1,p_2,p_3,\ldots,p_n,s)]. \tag{12}$$

Proof. On using the definition of n-DLT for $\frac{\partial^\alpha\psi}{\partial t^\alpha}$, we obtain

$$L_m L_t\left(\frac{\partial^\alpha\psi}{\partial t^\alpha}\right)=\int_0^\infty\ldots\int_0^\infty\int_0^\infty e^{-p_1x_1-p_2x_2\ldots-p_nx_n-st}\frac{\partial^\alpha\psi}{\partial t^\alpha}(x_1,x_2,x_3,\ldots,x_n,t)dx_1dx_2\ldots dx_ndt, \tag{13}$$

and taking partial derivatives $\frac{\partial^i}{\prod_{i=1}^{n}\partial p_i}$ on both sides of Equation (13), we have

$$\frac{\partial^i}{\prod_{i=1}^{n}\partial p_i}\left(L_m L_t\left(\frac{\partial^\alpha\psi}{\partial t^\alpha}\right)\right)=\int_0^\infty e^{-st}\frac{\partial^\alpha\psi}{\partial t^\alpha}\left(\int_0^\infty\ldots\int_0^\infty\frac{\partial^i}{\prod_{i=1}^{n}\partial p_i}\left(e^{-p_1x_1-p_2x_2\ldots-p_nx_n}dx_1dx_2\ldots dx_n\right)\right)dt. \tag{14}$$

The integral inside bracket determined by

$$\int_0^\infty\ldots\int_0^\infty\frac{\partial^i}{\prod_{i=1}^{n}\partial p_i}\left(e^{-p_1x_1-p_2x_2\ldots-p_nx_n}dx_1dx_2\ldots dx_n\right)=(-1)^n\int_0^\infty\ldots\int_0^\infty(\Delta)dx_1dx_2\ldots dx_n, \tag{15}$$

where $\Delta=\prod_{i=1}^{n}x_ie^{-p_1x_1-p_2x_2\ldots-p_nx_n}$, hence, we find that

$$\begin{aligned}\frac{\partial^i}{\prod_{i=1}^{n}\partial p_i}\left(L_m L_t\left(\frac{\partial^\alpha\psi}{\partial t^\alpha}\right)\right)&=(-1)^n\int_0^\infty\ldots\int_0^\infty\int_0^\infty\prod_{i=1}^{n}x_ie^{-p_1x_1-p_2x_2\ldots-p_nx_n-st}\frac{\partial^\alpha\psi}{\partial t^\alpha}dx_1dx_2\ldots dx_ndt\\&=(-1)^nL_mL_t\left[\prod_{i=1}^{n}x_i\frac{\partial^\alpha}{\partial t^\alpha}\psi(x_1,x_2,x_3,\ldots,x_n,t)\right],\end{aligned} \tag{16}$$

now substituting the Equations (14) and (15) into Equation (16), we achieved

$$L_m L_t\left[\prod_{i=1}^{n} x_i\frac{\partial^\alpha}{\partial t^\alpha}\psi(x_1,x_2,x_3,\ldots,x_n,t)\right]=(-1)^n\frac{\partial^i}{\prod_{i=1}^{n}\partial p_i}[\Lambda].$$

Similarly, one can obtain Equation (12). □

In particular at $n = 2$, we have

$$L_2 L_t\left[xy\frac{\partial^\alpha \psi}{\partial t^\alpha}\right] = \frac{\partial^2}{\partial p_1 \partial p_2}\left[s^\alpha \Psi(p_1, p_2, s) - s^{\alpha-1}\Psi(p_1, p_2, 0)\right], \tag{17}$$

and

$$L_2 L_t[xyf(x, y, t)] = \frac{\partial^2}{\partial p_1 \partial p_2}(L_2 L_t[f(x, y, t)]). \tag{18}$$

3. Singular m-D fractional Pseudo-Hyperbolic Equation

In this unit, the $(m+1)$-D Laplace-Adomian Decomposition Method is addressed for the solution of $m-$dimensional pseudo-hyperbolic equation.

Problem 1. *The $(m+1$-DLADM), is a useful method to solve linear singular m-dimensional pseudo-hyperbolic equation.*

We consider, $0 < \alpha \leq 1$, a general form of fractional singular m-D pseudo-hyperbolic equation.

$$D_{tt}^\alpha \psi = \sum_{i=1}^{m} \frac{1}{x_i}\frac{\partial}{\partial x_i}\left(x_i \frac{\partial}{\partial x_i}\psi\right) + \sum_{i=1}^{m} \frac{1}{x_i}\frac{\partial^2}{\partial x_i \partial t}\left(x_i \frac{\partial}{\partial x_i}\psi\right) + f(x_1, x_2, \ldots, x_m, t), \tag{19}$$

with condition

$$\begin{aligned} \psi(x_1, x_2, \ldots, x_m, 0) &= f_1(x_1, x_2, \ldots, x_m) \\ \psi_t(x_1, x_2, \ldots, x_m, 0) &= f_2(x_1, x_2, \ldots, x_m), \end{aligned} \tag{20}$$

where, $\sum_{i=1}^{m} \frac{1}{x_i}\frac{\partial}{\partial x_i}\left(x_i \frac{\partial}{\partial x_i}\psi\right)$ is defined as Bessel's operator and $f(x_1, x_2, \ldots, x_m, t)$, $f_1(x_1, x_2, \ldots, x_m)$ and $f_2(x_1, x_2, \ldots, x_m)$ are given functions. Now the objective is to solve the Equation (19), then we have the following steps:

Step 1: First, we multiply the both sides of Equation (19) by $\sum_{i=1}^{m} x_i$.

$$\begin{aligned} \prod_{j=1}^{m} x_j D_{tt}^\alpha \psi &= \sum_{i=1}^{m} \prod_{\substack{j=1 \\ j\neq i}}^{m} x_j \frac{\partial}{\partial x_i}\left(x_i \frac{\partial}{\partial x_i}\psi\right) \\ &\quad + \sum_{i=1}^{m} \prod_{\substack{j=1 \\ j\neq i}}^{m} x_j \frac{\partial^2}{\partial x_i \partial t}\left(x_i \frac{\partial}{\partial x_i}\psi\right) \\ &\quad + \prod_{j=1}^{m} x_j (f(x_1, x_2, \ldots, x_m, t)). \end{aligned} \tag{21}$$

Step 2: By implementing Equations (17), (18) and (1) in the previous step and m-DLT for condition, we obtain

$$\begin{aligned}\frac{\partial^m}{\partial p_1 \partial p_2 \ldots \partial p_m}\Psi(p_1,p_2,\ldots,p_m,s) = {} & \frac{1}{s}\frac{\partial^m}{\partial p_1 \partial p_2 \ldots \partial p_m}L_m[f_1(x_1,x_2,\ldots,x_m)] \\ & + \frac{1}{s^2}\frac{\partial^m}{\partial p_1 \partial p_2 \ldots \partial p_m}L_m[f_2(x_1,x_2,\ldots,x_m)] \\ & + \frac{1}{s^{2\alpha}}\frac{\partial^m}{\partial p_1 \partial p_2 \ldots \partial p_m}(L_m L_t[f(x_1,x_2,\ldots,x_m,t)]) \\ & + \frac{1}{s^{2\alpha}}L_m L_t\left[\sum_{i=1}^{m}\prod_{\substack{j=1\\ j\neq i}}^{m} x_j \frac{\partial}{\partial x_i}\left(x_i\frac{\partial}{\partial x_i}\psi\right)\right] \\ & + \frac{1}{s^{2\alpha}}L_m L_t\left[\sum_{i=1}^{m}\prod_{\substack{j=1\\ j\neq i}}^{m} x_j \frac{\partial^2}{\partial x_i \partial t}\left(x_i\frac{\partial}{\partial x_i}\psi\right)\right].\end{aligned} \tag{22}$$

Step 3: The integration of Equation (22), from 0 to p_1, 0 to p_2, …, 0 to p_m with respect to $p_1, p_2, \ldots, p_m$, respectively, we obtain

$$\begin{aligned}\Psi(p_1,p_2,\ldots,p_m,s) = {} & \frac{1}{s}\int_0^{p_1}\cdots\int_0^{p_m}\left(\frac{\partial^m}{\partial p_1 \partial p_2 \ldots \partial p_m}L_m[f_1(x_1,x_2,\ldots,x_m)]\right)dp_1\ldots dp_m \\ & + \frac{1}{s^2}\int_0^{p_1}\cdots\int_0^{p_m}\left(\frac{\partial^m}{\partial p_1 \partial p_2 \ldots \partial p_m}L_m[f_2(x_1,x_2,\ldots,x_m)]\right)dp_1\ldots dp_m \\ & + \frac{1}{s^{2\alpha}}\int_0^{p_1}\cdots\int_0^{p_m}\left(\frac{\partial^m}{\partial p_1 \partial p_2 \ldots \partial p_m}L_m L_t[f(x_1,x_2,\ldots,x_m,t)]\right)dp_1\ldots dp_m \\ & + \frac{1}{s^{2\alpha}}\int_0^{p_1}\cdots\int_0^{p_m}\left(L_m L_t\left[\sum_{i=1}^{m}\prod_{\substack{j=1\\ j\neq i}}^{m} x_j \frac{\partial}{\partial x_i}\left(x_i\frac{\partial}{\partial x_i}\psi\right)\right]\right)dp_1\ldots dp_m \\ & + \frac{1}{s^{2\alpha}}\int_0^{p_1}\cdots\int_0^{p_m}\left(L_m L_t\left[\sum_{i=1}^{m}\prod_{\substack{j=1\\ j\neq i}}^{m} x_j \frac{\partial^2}{\partial x_i \partial t}\left(x_i\frac{\partial}{\partial x_i}\psi\right)\right]\right)dp_1\ldots dp_m.\end{aligned} \tag{23}$$

Step 4: Now, the series solution of the singular m-D pseudo-hyperbolic equation follows:

$$\psi(x_1,x_2,\ldots,x_m,t) = \sum_{n=0}^{\infty}\psi_n(x_1,x_2,\ldots,x_m,t). \tag{24}$$

Step 5: Working with the 3-DLT both sides of Equation (23) and apply Equation (24), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\psi_n(x_1,x_2,\ldots,x_m,t) = {} & f_1(x_1,x_2,\ldots,x_m) + f_2(x_1,x_2,\ldots,x_m)t \\ & + L_m^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\cdots\int_0^{p_m}\left(\frac{\partial^m}{\partial p_1 \partial p_2 \ldots \partial p_m}L_m L_t[f(x_1,x_2,\ldots,x_m,t)]\right)dp_1\ldots dp_m\right] \\ & + L_m^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\cdots\int_0^{p_m}\left(L_m L_t\left[\sum_{i=1}^{m}\prod_{\substack{j=1\\ j\neq i}}^{m} x_j \frac{\partial}{\partial x_i}\left(x_i\frac{\partial}{\partial x_i}\psi\right)\right]\right)dp_1\ldots dp_m\right] \\ & + L_m^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\cdots\int_0^{p_m}\left(L_m L_t\left[\sum_{i=1}^{m}\prod_{\substack{j=1\\ j\neq i}}^{m} x_j \frac{\partial^2}{\partial x_i \partial t}\left(x_i\frac{\partial}{\partial x_i}\psi\right)\right]\right)dp_1\ldots dp_m\right],\end{aligned}$$

in view of the first approximation,

$$\begin{aligned}\psi_0 &= f_1(x_1, x_2, \ldots, x_m) + f_2(x_1, x_2, \ldots, x_m)t \\ &+L_m^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\ldots\int_0^{p_m}\left(\frac{\partial^m}{\partial p_1 \partial p_2 \ldots \partial p_m}L_m L_t[f(x_1, x_2, \ldots, x_m, t)]\right)dp_1\ldots dp_m\right],\end{aligned} \tag{25}$$

and the remaining components ψ_{n+1}, $n \geq 0$, are denoted by

$$\psi_{n+1} = L_m^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\ldots\int_0^{p_m}\left(L_m L_t\left[\sum_{i=1}^{m}\prod_{\substack{j=1\\ j\neq i}}^{m} x_j \frac{\partial}{\partial x_i}\left(x_i \frac{\partial}{\partial x_i}\psi\right)\right]\right)dp_1\ldots dp_m\right]$$
$$+ L_m^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\ldots\int_0^{p_m}\left(L_m L_t\left[\sum_{i=1}^{m}\prod_{\substack{j=1\\ j\neq i}}^{m} x_j \frac{\partial^2}{\partial x_i \partial t}\left(x_i \frac{\partial}{\partial x_i}\psi\right)\right]\right)dp_1\ldots dp_m\right]. \tag{26}$$

Here, we consider the inverse $(m+1)$-DLT respect to $p_1, p_2, \ldots, p_m$ and s of Equations (25) and (26) to be exist. Next we display an application at $m = 2$.

Example 1. *Singular 2-D pseudo-hyperbolic equation is given by:*

$$\begin{aligned}D_{tt}^{\alpha}\psi &= \frac{1}{x_1}\frac{\partial}{\partial x_1}(x_1\psi_{x_1}) + \frac{1}{x_2}\frac{\partial}{\partial x_2}(x_2\psi_{x_2}) \\ &+ \frac{1}{x_1}\frac{\partial^2}{\partial x_1 \partial t}(x_1\psi_{x_1}) + \frac{1}{x_2}\frac{\partial^2}{\partial x_2 \partial t}(x_2\psi_{x_2}) - u \\ &0 \leq x_1, x_2, t < \infty, \;\; 0 < \alpha \leq 1,\end{aligned} \tag{27}$$

subject to

$$\psi(x_1, x_2, 0) = 0, \quad \psi_t(x_1, x_2, 0) = x_1^2 - x_2^2. \tag{28}$$

By applying previous steps, Theorem 1 and 3-DLT for Equation (27), we compute:

$$\psi_0 = \left(x_1^2 - x_2^2\right)t, \tag{29}$$

and

$$\begin{aligned}\psi_{n+1} &= L_2^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}\left(L_2 L_t\left[x_2\frac{\partial}{\partial x_1}(x_1\psi_{nx_1})\right]\right)dp_1 dp_2\right] \\ &+L_2^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}\left(L_2 L_t\left[x_1\frac{\partial}{\partial x_2}(x_2\psi_{nx_2})\right]\right)dp_1 dp_2\right] \\ &+L_2^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}\left(L_2 L_t\left[x_2\frac{\partial^2}{\partial x_1 \partial t}(x_1\psi_{nx_1})\right]\right)dp_1 dp_2\right] \\ &+L_2^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}\left(L_2 L_t\left[x_1\frac{\partial^2}{\partial x_2 \partial t}(x_2\psi_{nx_2})\right]\right)dp_1 dp_2\right] \\ &-L_2^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}(L_2 L_t[x_1 x_2 u_n])dp_1 dp_2\right],\end{aligned} \tag{30}$$

according to the (3-DLADM) we obtain the following components: at $n = 0$

$$
\begin{aligned}
\psi_1 &= L_2^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}\left(L_2L_t\left[x_2\frac{\partial}{\partial x_1}(x_1\psi_{0x_1}) + x_1\frac{\partial}{\partial x_2}(x_2\psi_{0x_2})\right]\right)dp_1dp_2\right] \\
&+ L_2^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}\left(L_2L_t\left[x_2\frac{\partial^2}{\partial x_1\partial t}(x_1\psi_{0x_1}) + x_1\frac{\partial^2}{\partial x_2\partial t}(x_2\psi_{0x_2})\right]\right)dp_1dp_2\right] \\
&\quad -L_2^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}(L_2L_t[x_1x_2u_0])dp_1dp_2\right] \\
&= L_2^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}\left(L_2L_t\left[-\left(x_1^3x_2 - x_1x_2^3\right)t\right]\right)dp_1dp_2\right] \\
&= L_2^{-1}L_s^{-1}\left[-\frac{2}{p_1^3p_2s^{2\alpha+2}} + \frac{2}{p_1p_2^3s^{2\alpha+2}}\right] \\
\psi_1 &= -\frac{x_1^2t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{x_2^2t^{2\alpha+1}}{\Gamma(2\alpha+2)}.
\end{aligned}
$$

In the same way, we receive that at $n = 1$

$$
\begin{aligned}
\psi_2 &= L_2^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}\left(L_2L_t\left[x_2\frac{\partial}{\partial x_1}(x_1\psi_{1x_1}) + x_1\frac{\partial}{\partial x_2}(x_2\psi_{1x_2})\right]\right)dp_1dp_2\right] \\
&\quad +L_2^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}\left(L_2L_t\left[x_2\frac{\partial^2}{\partial x_1\partial t}(x_1\psi_{1x_1}) + x_1\frac{\partial^2}{\partial x_2\partial t}(x_2\psi_{1x_2})\right]\right)dp_1dp_2\right] \\
&\quad -L_2^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}(L_2L_t[x_1x_2u_1])dp_1dp_2\right] \\
&= L_2^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}\left(L_2L_t\left[\frac{x_1^3x_2t^{2\alpha+1}}{\Gamma(2\alpha+2)} - \frac{x_1x_2^3t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right]\right)dp_1dp_2\right] \\
&= L_2^{-1}L_s^{-1}\left[\frac{2}{p_1^3p_2s^{4\alpha+2}} - \frac{2}{p_1p_2^3s^{4\alpha+2}}\right] \\
\psi_2 &= \frac{x_1^2t^{4\alpha+1}}{\Gamma(4\alpha+2)} - \frac{x_2^2t^{4\alpha+1}}{\Gamma(4\alpha+2)}
\end{aligned}
$$

at $n = 2$

$$
\begin{aligned}
\psi_3 &= L_2^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}\left(L_2L_t\left[x_2\frac{\partial}{\partial x_1}(x_1\psi_{2x_1}) + x_1\frac{\partial}{\partial x_2}(x_2\psi_{2x_2})\right]\right)dp_1dp_2\right] \\
&\quad +L_2^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}\left(L_2L_t\left[x_2\frac{\partial^2}{\partial x_1\partial t}(x_1\psi_{2x_1}) + x_1\frac{\partial^2}{\partial x_2\partial t}(x_2\psi_{2x_2})\right]\right)dp_1dp_2\right] \\
&\quad -L_2^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}(L_2L_t[x_1x_2u_2])dp_1dp_2\right] \\
&= L_2^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}\left(L_2L_t\left[-\frac{x_1^3x_2t^{4\alpha+1}}{\Gamma(4\alpha+2)} + \frac{x_1x_2^3t^{4\alpha+1}}{\Gamma(4\alpha+2)}\right]\right)dp_1dp_2\right] \\
&= L_2^{-1}L_s^{-1}\left[-\frac{2}{p_1^3p_2s^{4\alpha+2}} + \frac{2}{p_1p_2^3s^{4\alpha+2}}\right] \\
\psi_3 &= -\frac{x_1^2t^{6\alpha+1}}{\Gamma(6\alpha+2)} + \frac{x_2^2t^{6\alpha+1}}{\Gamma(6\alpha+2)}.
\end{aligned}
$$

By adding the all terms, we have

$$\psi(x,y,t) = \psi_0 + \psi_1 + \psi_2 + \psi_3 + \dots$$

Thus, the approximate solution of Equation (27), is given by

$$\begin{aligned} \psi(x_1,x_2,t) &= \left(x_1^2 - x_2^2\right)t - \left(x_1^2 - x_2^2\right)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \\ &\quad + \left(x_1^2 - x_2^2\right)\frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)} - \left(x_1^2 - x_2^2\right)\frac{t^{6\alpha+1}}{\Gamma(6\alpha+2)} + \dots \end{aligned}$$

By using $\alpha = 1$, the approximation solution becomes

$$\begin{aligned} \psi(x,y,t) &= \left(x_1^2 - x_2^2\right)\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} - \dots\right) \\ \psi(x_1,x_2,t) &= \left(x_1^2 - x_2^2\right)\sin t. \end{aligned}$$

Problem 2. *Consider the the following nonlinear singular m-D pseudo-hyperbolic equation*

$$\begin{aligned} D_{tt}^{\alpha}\psi &= \sum_{i=1}^{2}\frac{1}{x_i}\frac{\partial}{\partial x_i}\left(x_i\frac{\partial\psi}{\partial x_i}\right) + \sum_{i=1}^{2}\frac{1}{x_i}\frac{\partial^2}{\partial x_i\partial t}\left(x_i\frac{\partial\psi}{\partial x_i}\right) \\ &\quad + x_2\psi\frac{\partial\psi}{\partial x_1} + x_1\psi\frac{\partial\psi}{\partial x_2} + f(x_1,x_2,t), \end{aligned} \tag{31}$$

subject to

$$\begin{aligned} \psi(x_1,x_2,0) &= f_1(x_1,x_2) \\ \psi_t(x_1,x_2,0) &= f_2(x_1,x_2). \end{aligned} \tag{32}$$

Then, the first approximation is as follows

$$\psi_0 = f_1(x_1,x_2) + f_2(x_1,x_2)t + L_2^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}\left(\frac{\partial^2}{\partial p_1\partial p_2}(L_2L_t[f(x_1,x_2,t)])\right)dp_1dp_2\right] \tag{33}$$

and the rest terms is given by

$$\begin{aligned} \psi_{n+1} &= L_2^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}\left(L_2L_t\left[x_2\frac{\partial}{\partial x_1}\left(x_1\frac{\partial\psi_n}{\partial x_1}\right) + x_1\frac{\partial}{\partial x_2}\left(x_2\frac{\partial\psi_n}{\partial x_2}\right)\right]\right)dp_1dp_2\right] \\ &+ L_2^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}\left(L_2L_t\left[x_2\frac{\partial^2}{\partial x_1\partial t}\left(x_1\frac{\partial\psi_n}{\partial x_1}\right) + x_1\frac{\partial^2}{\partial x_2\partial t}\left(x_2\frac{\partial\psi_n}{\partial x_2}\right)\right]\right)dp_1dp_2\right] \\ &+ L_3^{-1}\left[\frac{1}{s^{\alpha}}\int_0^{p_1}\int_0^{p_2}\left(L_3\left[x_1x_2^2A_n + x_1^2x_2\,B_n\right]\right)dp_1dp_2\right], \end{aligned} \tag{34}$$

where nonlinear terms A_n and B_n are decomposed as

$$A_n = \sum_{n=0}^{\infty}\psi_n\frac{\partial\psi_n}{\partial x_1},\quad B_n = \sum_{n=0}^{\infty}\psi_n\frac{\partial\psi_n}{\partial x_2}. \tag{35}$$

The nonlinear terms $\psi\frac{\partial\psi}{\partial x_1}$ *and* $\psi\frac{\partial\psi}{\partial x_2}$ *are denoted by*

$$\begin{aligned}
A_0 &= \psi_0\frac{\partial\psi_0}{\partial x_1}\\
A_1 &= \psi_0\frac{\partial\psi_1}{\partial x_1}+\psi_1\frac{\partial\psi_0}{\partial x_1},\\
A_2 &= \psi_0\frac{\partial\psi_2}{\partial x_1}+\psi_1\frac{\partial\psi_1}{\partial x_1}+\psi_2\frac{\partial\psi_0}{\partial x_1},\\
A_3 &= \psi_0\frac{\partial\psi_3}{\partial x_1}+\psi_1\frac{\partial\psi_2}{\partial x_1}+\psi_2\frac{\partial\psi_1}{\partial x_1}+\psi_3\frac{\partial\psi_0}{\partial x_1}.
\end{aligned}\tag{36}$$

and

$$\begin{aligned}
B_0 &= \psi_0\frac{\partial\psi_0}{\partial x_2}\\
B_1 &= \psi_0\frac{\partial\psi_1}{\partial x_2}+\psi_1\frac{\partial\psi_0}{\partial x_2},\\
B_2 &= \psi_0\frac{\partial\psi_2}{\partial x_2}+\psi_1\frac{\partial\psi_1}{\partial x_2}+\psi_2\frac{\partial\psi_0}{\partial x_2},\\
B_3 &= \psi_0\frac{\partial\psi_3}{\partial x_2}+\psi_1\frac{\partial\psi_2}{\partial x_2}+\psi_2\frac{\partial\psi_1}{\partial x_2}+\psi_3\frac{\partial\psi_0}{\partial x_2}.
\end{aligned}\tag{37}$$

Next, we provide the following illustrative example.

Example 2. *Consider the nonlinear pseudo-hyperbolic equation*

$$\begin{aligned}
D_{tt}^{\alpha}\psi &= \sum_{i=1}^{2}\frac{1}{x_i}\frac{\partial}{\partial x_i}\left(x_i\frac{\partial\psi}{\partial x_i}\right)+\sum_{i=1}^{2}\frac{1}{x_i}\frac{\partial^2}{\partial x_i\partial t}\left(x_i\frac{\partial\psi}{\partial x_i}\right)\\
&\quad +x_2\psi\frac{\partial\psi}{\partial x_1}+x_1\psi\frac{\partial\psi}{\partial x_2}+\left(x^2-y^2\right)e^{-t}\\
0 &\le x_1,x_2,t<\infty,\quad 0<\alpha\le 1,
\end{aligned}\tag{38}$$

subject to

$$\psi(x_1,x_2,0)=x_1^2-x_2^2,\ \ \psi_t(x_1,x_2,0)=-\left(x_1^2-x_2^2\right).\tag{39}$$

By using the mentioned method and Theorem 1 we have:

$$\psi_0=\left(x_1^2-x_2^2\right)-\left(x_1^2-x_2^2\right)t+\left(x_1^2-x_2^2\right)t^{2\alpha}\Xi_{1,2\alpha+1}(-t),\tag{40}$$

and

$$\begin{aligned}
\psi_{n+1} &= L_2^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}\left(L_2L_t\left[x_2\frac{\partial}{\partial x_1}\left(x_1\frac{\partial\psi_n}{\partial x_1}\right)+x_1\frac{\partial}{\partial x_2}\left(x_2\frac{\partial\psi_n}{\partial x_2}\right)\right]\right)dp_1dp_2\right]\\
&+L_2^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}\left(L_2L_t\left[x_2\frac{\partial^2}{\partial x_1\partial t}\left(x_1\frac{\partial\psi_n}{\partial x_1}\right)+x_1\frac{\partial^2}{\partial x_2\partial t}\left(x_2\frac{\partial\psi_n}{\partial x_2}\right)\right]\right)dp_1dp_2\right]\\
&+L_2^{-1}L_s^{-1}\left[\frac{1}{s^{\alpha}}\int_0^{p_1}\int_0^{p_2}\left(L_2L_t\left[x_1x_2^2A_n+x_1^2x_2\ B_n\right]\right)dp_1dp_2\right],
\end{aligned}\tag{41}$$

where A_n and B_n are defined in Equations (36) and (37). The next terms are

$$\psi_1 = L_2^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}\left(L_2L_t\left[x_2\frac{\partial}{\partial x_1}\left(x_1\frac{\partial\psi_0}{\partial x_1}\right)+x_1\frac{\partial}{\partial x_2}\left(x_2\frac{\partial\psi_0}{\partial x_2}\right)\right]\right)dp_1dp_2\right]$$
$$+L_2^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}\left(L_2L_t\left[x_2\frac{\partial^2}{\partial x_1\partial t}\left(x_1\frac{\partial\psi_0}{\partial x_1}\right)+x_1\frac{\partial^2}{\partial x_2\partial t}\left(x_2\frac{\partial\psi_0}{\partial x_2}\right)\right]\right)dp_1dp_2\right]$$
$$+L_2^{-1}L_s^{-1}\left[\frac{1}{s^{\alpha}}\int_0^{p_1}\int_0^{p_2}\left(L_2L_t\left[x_1x_2^2A_0+x_1^2x_2\,B_0\right]\right)dp_1dp_2\right]$$
$$\psi_1 = 0.$$

Following in a similar manner, we have

$$\psi_2 = 0,\quad \psi_3 = 0, \psi_4 = 0, \dots$$

Hence, according to Equation (24) we have

$$\psi(x_1,x_2,t) = \left(x_1^2-x_2^2\right)-\left(x_1^2-x_2^2\right)t+\left(x_1^2-x_2^2\right)t^{2\alpha}\Xi_{1,2\alpha+1}(-t),$$

if we set $\alpha = 1$ then, the exact solutions of Equation (38) is presented by

$$\psi(x_1,x_2,t) = \left(x^2-y^2\right)e^{-t}.$$

4. Singular m-D Coupled Pseudo-Hyperbolic Equation and 3-DLADM

The aim of this section is to establish the solution of the coupled singular m-D pseudo-hyperbolic equation by using (3-DLADM).

Now, consider the coupled singular 2-D pseudo-hyperbolic equations as follows:

$$\begin{aligned} D_{tt}^{\alpha}\varphi &= \sum_{i=1}^{m}\frac{1}{x_i}\frac{\partial}{\partial x_i}\left(x_i\frac{\partial\varphi}{\partial x_i}\right)+\sum_{i=1}^{m}\frac{1}{x_i}\frac{\partial^2}{\partial x_i\partial t}\left(x_i\frac{\partial\varphi}{\partial x_i}\right)+\omega+f(x_1,x_2,\dots,x_m,t),\\ D_{tt}^{\alpha}\omega &= \sum_{i=1}^{m}\frac{1}{x_i}\frac{\partial}{\partial x_i}\left(x_i\frac{\partial\omega}{\partial x_i}\right)+\sum_{i=1}^{m}\frac{1}{x_i}\frac{\partial^2}{\partial x_i\partial t}\left(x_i\frac{\partial\omega}{\partial x_i}\right)+\varphi+g(x_1,x_2,\dots,x_m,t), \end{aligned} \tag{42}$$

subject to

$$\begin{aligned} \varphi(x_1,x_2,\dots,x_m,0) &= f_1(x_1,x_2,\dots,x_m),\quad \varphi_t(x_1,x_2,\dots,x_m,0)=f_2(x_1,x_2,\dots,x_m)\\ \omega(x_1,x_2,\dots,x_m,0) &= g_1(x_1,x_2,\dots,x_m),\quad \omega_t(x_1,x_2,\dots,x_m,0)=g_2(x_1,x_2,\dots,x_m), \end{aligned} \tag{43}$$

where $f(x_1,x_2,\dots,x_m,t)$, $g(x_1,x_2,\dots,x_m,t)$, $f_1(x_1,x_2,\dots,x_m)$, $f_2(x_1,x_2,\dots,x_m)$, $g_1(x_1,x_2,\dots,x_m)$ and $g_2(x_1,x_2,\dots,x_m)$, are given functions, by using ($m+1$-DLADM), this method contains the following steps.

Step (1): Multiplying both sides of Equation (42) by xy leads to the following equation

$$\begin{aligned} \prod_{j=1}^{m}x_jD_{tt}^{\alpha}\varphi &= \sum_{i=1}^{m}\prod_{\substack{j=1\\j\neq i}}^{m}x_j\frac{\partial}{\partial x_i}\left(x_i\frac{\partial\varphi}{\partial x_i}\right)+\sum_{i=1}^{m}\prod_{\substack{j=1\\j\neq i}}^{m}x_j\frac{\partial^2}{\partial x_i\partial t}\left(x_i\frac{\partial\varphi}{\partial x_i}\right)\\ &\quad+\prod_{j=1}^{m}x_j\omega+\prod_{j=1}^{m}x_j(f(x_1,x_2,\dots,x_m,t))\\ \prod_{j=1}^{m}x_jD_{tt}^{\alpha}\omega &= \sum_{i=1}^{m}\prod_{\substack{j=1\\j\neq i}}^{m}x_j\frac{\partial}{\partial x_i}\left(x_i\frac{\partial\omega}{\partial x_i}\right)+\sum_{i=1}^{m}\prod_{\substack{j=1\\j\neq i}}^{m}x_j\frac{\partial^2}{\partial x_i\partial t}\left(x_i\frac{\partial\omega}{\partial x_i}\right)\\ &\quad+\prod_{j=1}^{m}x_j\varphi+\prod_{j=1}^{m}x_j(g(x_1,x_2,\dots,x_m,t)). \end{aligned} \tag{44}$$

Step (2): Apply 3-DLT to Equation (43) and 2-DLT to Equation (44), then we obtain

$$
\begin{aligned}
\frac{\partial^m}{\partial p_1 \partial p_2 \ldots \partial p_m}\Psi(p_1, p_2, \ldots, p_m, s) = {} & \frac{1}{s}\frac{\partial^m}{\partial p_1 \partial p_2 \ldots \partial p_m} L_m[f_1(x_1, x_2, \ldots, x_m)] \\
& + \frac{1}{s^2}\frac{\partial^m}{\partial p_1 \partial p_2 \ldots \partial p_m} L_m[f_2(x_1, x_2, \ldots, x_m)] \\
& + \frac{1}{s^{2\alpha}}\frac{\partial^m}{\partial p_1 \partial p_2 \ldots \partial p_m}(L_m L_t[f(x_1, x_2, \ldots, x_m, t)]) \\
& + \frac{1}{s^{2\alpha}} L_m L_t\left[\sum_{i=1}^{m}\prod_{\substack{j=1 \\ j\neq i}}^{m} x_j \frac{\partial}{\partial x_i}\left(x_i \frac{\partial \varphi}{\partial x_i}\right)\right], \\
& + \frac{1}{s^{2\alpha}} L_m L_t\left[\sum_{i=1}^{m}\prod_{\substack{j=1 \\ j\neq i}}^{m} x_j \frac{\partial^2}{\partial x_i \partial t}\left(x_i \frac{\partial \varphi}{\partial x_i}\right)\right] \\
& + \frac{1}{s^{2\alpha}} L_m L_t\left[\prod_{j=1}^{m} x_j \omega\right],
\end{aligned} \tag{45}
$$

and

$$
\begin{aligned}
\frac{\partial^m}{\partial p_1 \partial p_2 \ldots \partial p_m}\Phi(p_1, p_2, \ldots, p_m, s) = {} & \frac{1}{s}\frac{\partial^m}{\partial p_1 \partial p_2 \ldots \partial p_m} L_m[f_1(x_1, x_2, \ldots, x_m)] \\
& + \frac{1}{s^2}\frac{\partial^m}{\partial p_1 \partial p_2 \ldots \partial p_m} L_m[f_2(x_1, x_2, \ldots, x_m)] \\
& + \frac{1}{s^{2\alpha}}\frac{\partial^m}{\partial p_1 \partial p_2 \ldots \partial p_m}(L_m L_t[f(x_1, x_2, \ldots, x_m, t)]) \\
& + \frac{1}{s^{2\alpha}} L_m L_t\left[\sum_{i=1}^{m}\prod_{\substack{j=1 \\ j\neq i}}^{m} x_j \frac{\partial}{\partial x_i}\left(x_i \frac{\partial \omega}{\partial x_i}\right)\right], \\
& + \frac{1}{s^{2\alpha}} L_m L_t\left[\sum_{i=1}^{m}\prod_{\substack{j=1 \\ j\neq i}}^{m} x_j \frac{\partial^2}{\partial x_i \partial t}\left(x_i \frac{\partial \omega}{\partial x_i}\right)\right] \\
& + \frac{1}{s^{2\alpha}} L_m L_t\left[\prod_{j=1}^{m} x_j \varphi\right].
\end{aligned} \tag{46}
$$

and **Step (3):** Operating the integral of Equation (46), from 0 to p_1, 0 to p_2,. . ., 0 to p_m with respect to $p_1, p_2, \ldots, p_m$, respectively, we obtain

$$\begin{aligned}
\Psi(p_1, p_2, \ldots, p_m, s) = {} & \frac{1}{s}\int_0^{p_1}\cdots\int_0^{p_m}\left(\frac{\partial^m}{\partial p_1 \partial p_2 \ldots \partial p_m} L_m[f_1(x_1, x_2, \ldots, x_m)]\right) dp_1 \ldots dp_m \\
& + \frac{1}{s^2}\int_0^{p_1}\cdots\int_0^{p_m}\left(\frac{\partial^m}{\partial p_1 \partial p_2 \ldots \partial p_m} L_m[f_2(x_1, x_2, \ldots, x_m)]\right) dp_1 \ldots dp_m \\
& + \frac{1}{s^{2\alpha}}\int_0^{p_1}\cdots\int_0^{p_m}\left(\frac{\partial^m}{\partial p_1 \partial p_2 \ldots \partial p_m} L_m L_t[f(x_1, x_2, \ldots, x_m, t)]\right) dp_1 \ldots dp_m \\
& + \frac{1}{s^{2\alpha}}\int_0^{p_1}\cdots\int_0^{p_m}\left(L_m L_t\left[\sum_{i=1}^{m}\prod_{\substack{j=1 \\ j\neq i}}^{m} x_j \frac{\partial}{\partial x_i}\left(x_i \frac{\partial \varphi}{\partial x_i}\right)\right]\right) dp_1 \ldots dp_m \\
& + \frac{1}{s^{2\alpha}}\int_0^{p_1}\cdots\int_0^{p_m}\left(L_m L_t\left[\sum_{i=1}^{m}\prod_{\substack{j=1 \\ j\neq i}}^{m} x_j \frac{\partial^2}{\partial x_i \partial t}\left(x_i \frac{\partial \varphi}{\partial x_i}\right)\right]\right) dp_1 \ldots dp_m, \\
& \frac{1}{s^{2\alpha}}\int_0^{p_1}\cdots\int_0^{p_m}\left(L_m L_t\left[\prod_{j=1}^{m} x_j \omega\right]\right) dp_1 \ldots dp_m,
\end{aligned} \tag{47}$$

and

$$\begin{aligned}
\Psi(p_1, p_2, \ldots, p_m, s) = {} & \frac{1}{s}\int_0^{p_1}\cdots\int_0^{p_m}\left(\frac{\partial^m}{\partial p_1 \partial p_2 \ldots \partial p_m} L_m[f_1(x_1, x_2, \ldots, x_m)]\right) dp_1 \ldots dp_m \\
& + \frac{1}{s^2}\int_0^{p_1}\cdots\int_0^{p_m}\left(\frac{\partial^m}{\partial p_1 \partial p_2 \ldots \partial p_m} L_m[f_2(x_1, x_2, \ldots, x_m)]\right) dp_1 \ldots dp_m \\
& + \frac{1}{s^{2\alpha}}\int_0^{p_1}\cdots\int_0^{p_m}\left(\frac{\partial^m}{\partial p_1 \partial p_2 \ldots \partial p_m} L_m L_t[f(x_1, x_2, \ldots, x_m, t)]\right) dp_1 \ldots dp_m \\
& + \frac{1}{s^{2\alpha}}\int_0^{p_1}\cdots\int_0^{p_m}\left(L_m L_t\left[\sum_{i=1}^{m}\prod_{\substack{j=1 \\ j\neq i}}^{m} x_j \frac{\partial}{\partial x_i}\left(x_i \frac{\partial \omega}{\partial x_i}\right)\right]\right) dp_1 \ldots dp_m \\
& + \frac{1}{s^{2\alpha}}\int_0^{p_1}\cdots\int_0^{p_m}\left(L_m L_t\left[\sum_{i=1}^{m}\prod_{\substack{j=1 \\ j\neq i}}^{m} x_j \frac{\partial^2}{\partial x_i \partial t}\left(x_i \frac{\partial \omega}{\partial x_i}\right)\right]\right) dp_1 \ldots dp_m, \\
& \frac{1}{s^{2\alpha}}\int_0^{p_1}\cdots\int_0^{p_m}\left(L_m L_t\left[\prod_{j=1}^{m} x_j \varphi\right]\right) dp_1 \ldots dp_m.
\end{aligned} \tag{48}$$

Now the series solution is entirely determined by:

$$\begin{aligned}
\varphi(x_1, x_2, \ldots, x_m, t) &= \sum_{n=0}^{\infty} \varphi_n(x_1, x_2, \ldots, x_m, t), \\
\omega(x_1, x_2, \ldots, x_m, t) &= \sum_{n=0}^{\infty} \omega_n(x_1, x_2, \ldots, x_m, t),
\end{aligned} \tag{49}$$

now applying the m-DLT both sides of Equation (47) and apply Equation (49), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\varphi_n(x_1,x_2,\dots,x_m,t) &= f_1(x_1,x_2,\dots,x_m)+f_2(x_1,x_2,\dots,x_m)t\\ &+L_m^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\dots\int_0^{p_m}\left(\frac{\partial^m}{\partial p_1\partial p_2\dots\partial p_m}L_mL_t[f(x_1,x_2,\dots,x_m,t)]\right)dp_1\dots dp_m\right]\\ &+L_m^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\dots\int_0^{p_m}\left(L_mL_t\left[\sum_{i=1}^{m}\prod_{\substack{j=1\\ j\neq i}}^{m}x_j\frac{\partial}{\partial x_i}\left(x_i\frac{\partial}{\partial x_i}\sum_{n=0}^{\infty}\varphi_n\right)\right]\right)dp_1\dots dp_m\right]\\ &+L_m^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\dots\int_0^{p_m}\left(L_mL_t\left[\sum_{i=1}^{m}\prod_{\substack{j=1\\ j\neq i}}^{m}x_j\frac{\partial^2}{\partial x_i\partial t}\left(x_i\frac{\partial}{\partial x_i}\sum_{n=0}^{\infty}\varphi_n\right)\right]\right)dp_1\dots dp_m\right]\\ &+L_m^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\dots\int_0^{p_m}\left(L_mL_t\left[\prod_{j=1}^{m}x_j\sum_{n=0}^{\infty}\omega_n\right]\right)dp_1\dots dp_m\right],\end{aligned}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}\omega_n(x_1,x_2,\dots,x_m,t) &= f_1(x_1,x_2,\dots,x_m)+f_2(x_1,x_2,\dots,x_m)t\\ &+L_m^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\dots\int_0^{p_m}\left(\frac{\partial^m}{\partial p_1\partial p_2\dots\partial p_m}L_mL_t[f(x_1,x_2,\dots,x_m,t)]\right)dp_1\dots dp_m\right]\\ &+L_m^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\dots\int_0^{p_m}\left(L_mL_t\left[\sum_{i=1}^{m}\prod_{\substack{j=1\\ j\neq i}}^{m}x_j\frac{\partial}{\partial x_i}\left(x_i\frac{\partial}{\partial x_i}\sum_{n=0}^{\infty}\omega_n\right)\right]\right)dp_1\dots dp_m\right]\\ &+L_m^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\dots\int_0^{p_m}\left(L_mL_t\left[\sum_{i=1}^{m}\prod_{\substack{j=1\\ j\neq i}}^{m}x_j\frac{\partial^2}{\partial x_i\partial t}\left(x_i\frac{\partial}{\partial x_i}\sum_{n=0}^{\infty}\omega_n\right)\right]\right)dp_1\dots dp_m\right]\\ &+L_m^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\dots\int_0^{p_m}\left(L_mL_t\left[\prod_{j=1}^{m}x_j\sum_{n=0}^{\infty}\varphi_n\right]\right)dp_1\dots dp_m\right],\end{aligned}$$

the approximation,

$$\begin{aligned}\varphi_0 &= f_1(x_1,x_2,\dots,x_m)+f_2(x_1,x_2,\dots,x_m)t\\ &+L_m^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\dots\int_0^{p_m}\left(\frac{\partial^m}{\partial p_1\partial p_2\dots\partial p_m}L_mL_t[f(x_1,x_2,\dots,x_m,t)]\right)dp_1\dots dp_m\right], \end{aligned}\tag{50}$$

and the remaining components φ_{n+1}, $n\geq 0$, are denoted by

$$\begin{aligned}\varphi_{n+1} &= L_m^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\dots\int_0^{p_m}\left(L_mL_t\left[\sum_{i=1}^{m}\prod_{\substack{j=1\\ j\neq i}}^{m}x_j\frac{\partial}{\partial x_i}\left(x_i\frac{\partial\varphi}{\partial x_i}\right)\right]\right)dp_1\dots dp_m\right]\\ &+L_m^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\dots\int_0^{p_m}\left(L_mL_t\left[\sum_{i=1}^{m}\prod_{\substack{j=1\\ j\neq i}}^{m}x_j\frac{\partial^2}{\partial x_i\partial t}\left(x_i\frac{\partial\varphi}{\partial x_i}\right)\right]\right)dp_1\dots dp_m\right]\\ &+L_m^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\dots\int_0^{p_m}\left(L_mL_t\left[\prod_{j=1}^{m}x_j\omega\right]\right)dp_1\dots dp_m\right],\end{aligned}\tag{51}$$

and

$$\begin{aligned}\omega_0 &= g_1(x_1,x_2,\ldots,x_m)+g_2(x_1,x_2,\ldots,x_m)t\\&+L_m^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\cdots\int_0^{p_m}\left(\frac{\partial^m}{\partial p_1\partial p_2\ldots\partial p_m}L_mL_t[g(x_1,x_2,\ldots,x_m,t)]\right)dp_1\ldots dp_m\right],\end{aligned}\tag{52}$$

and the remaining components ω_{n+1}, $n\geq 0$, are denoted by

$$\begin{aligned}\omega_{n+1}=L_m^{-1}L_s^{-1}&\left[\tfrac{1}{s^{2\alpha}}\int_0^{p_1}\cdots\int_0^{p_m}\left(L_mL_t\left[\sum_{i=1}^{m}\prod_{\substack{j=1\\j\neq i}}^{m}x_j\tfrac{\partial}{\partial x_i}\left(x_i\tfrac{\partial\omega}{\partial x_i}\right)\right]\right)dp_1\ldots dp_m\right]\\+L_m^{-1}L_s^{-1}&\left[\tfrac{1}{s^{2\alpha}}\int_0^{p_1}\cdots\int_0^{p_m}\left(L_mL_t\left[\sum_{i=1}^{m}\prod_{\substack{j=1\\j\neq i}}^{m}x_j\tfrac{\partial^2}{\partial x_i\partial t}\left(x_i\tfrac{\partial\omega}{\partial x_i}\right)\right]\right)dp_1\ldots dp_m\right]\\+L_m^{-1}L_s^{-1}&\left[\tfrac{1}{s^{2\alpha}}\int_0^{p_1}\cdots\int_0^{p_m}\left(L_mL_t\left[\prod_{j=1}^{m}x_j\varphi\right]\right)dp_1\ldots dp_m\right].\end{aligned}\tag{53}$$

To check the applicability of the present method, we consider $m=2$.

Example 3. *Time fractional coupled pseudo-hyperbolic equations are given by*

$$\begin{aligned}D_{tt}^{\alpha}\varphi &= \sum_{i=1}^{m}\frac{1}{x_i}\frac{\partial}{\partial x_i}\left(x_i\frac{\partial\varphi}{\partial x_i}\right)+\sum_{i=1}^{m}\frac{1}{x_i}\frac{\partial^2}{\partial x_i\partial t}\left(x_i\frac{\partial\varphi}{\partial x_i}\right)+\omega\\D_{tt}^{\alpha}\omega &= \sum_{i=1}^{m}\frac{1}{x_i}\frac{\partial}{\partial x_i}\left(x_i\frac{\partial\omega}{\partial x_i}\right)+\sum_{i=1}^{m}\frac{1}{x_i}\frac{\partial^2}{\partial x_i\partial t}\left(x_i\frac{\partial\omega}{\partial x_i}\right)+\varphi,\end{aligned}\tag{54}$$

where

$$0\leq x,y,t<\infty,\ \ 1,\ 0<\alpha\leq 1$$

with initial condition

$$\begin{aligned}\varphi(x_1,x_2,\ldots,x_m,0) &= x_1^2-x_2^2,\ \ \varphi_t(x_1,x_2,\ldots,x_m,0)=x_1^2-x_2^2\\\omega(x_1,x_2,\ldots,x_m,0) &= x_1^2-x_2^2,\ \ \omega_t(x_1,x_2,\ldots,x_m,0)=x_1^2-x_2^2.\end{aligned}\tag{55}$$

Using the (2-DLADM) procedure Equations (51)–(53), we obtain following components:

$$\varphi_0=x_1^2-x_2^2+\left(x_1^2-x_2^2\right)t,\ \ \omega_0=x_1^2-x_2^2+\left(x_1^2-x_2^2\right)t,$$

at $n=0$,

$$\begin{aligned}\varphi_1=L_2^{-1}L_s^{-1}&\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}\left(L_2L_t\left[x_2\frac{\partial}{\partial x_1}\left(x_1\frac{\partial\varphi_0}{\partial x_1}\right)+x_1\frac{\partial}{\partial x_2}\left(x_2\frac{\partial\varphi_0}{\partial x_2}\right)\right]\right)dp_1dp_2\right]\\+L_2^{-1}L_s^{-1}&\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}\left(L_2L_t\left[x_2\frac{\partial^2}{\partial x_1\partial t}\left(x_1\frac{\partial\varphi_0}{\partial x_1}\right)+x_1\frac{\partial^2}{\partial x_2\partial t}\left(x_2\frac{\partial\varphi_0}{\partial x_2}\right)\right]\right)dp_1dp_2\right]\\+L_2^{-1}L_s^{-1}&\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}(L_2L_t[x_1x_2\omega_0])dp_1dp_2\right],\end{aligned}$$

and

$$
\begin{aligned}
\omega_1 &= L_2^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}\left(L_2L_t\left[x_2\frac{\partial}{\partial x_1}\left(x_1\frac{\partial \omega_0}{\partial x_1}\right)+x_1\frac{\partial}{\partial x_2}\left(x_2\frac{\partial \omega_0}{\partial x_2}\right)\right]\right)dp_1dp_2\right] \\
&+ L_2^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}\left(L_2L_t\left[x_2\frac{\partial^2}{\partial x_1\partial t}\left(x_1\frac{\partial \omega_0}{\partial x_1}\right)+x_1\frac{\partial^2}{\partial x_2\partial t}\left(x_2\frac{\partial \omega_0}{\partial x_2}\right)\right]\right)dp_1dp_2\right] \\
&+ L_2^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}(L_2L_t[x_1x_2\varphi_0])dp_1dp_2\right],
\end{aligned}
$$

therefore

$$
\begin{aligned}
\varphi_1 &= L_2^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}\left(x_1^3x_2-x_1x_2^3+x_1^3x_2t-x_1x_2^3t\right)dp_1dp_2\right] \\
&= L_2^{-1}L_s^{-1}\left[\frac{2}{p_1^3p_2s^{2\alpha+1}}-\frac{2}{p_2^3p_1s^{2\alpha+1}}+\frac{2}{p_1^3p_2s^{2\alpha+2}}-\frac{2}{p_2^3p_1s^{2\alpha+2}}\right] \\
\varphi_1 &= \frac{x_1^2t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{x_2^2t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{x_1^2t^{2\alpha+1}}{\Gamma(2\alpha+2)}-\frac{x_2^2t^{2\alpha+1}}{\Gamma(2\alpha+2)},
\end{aligned}
$$

$$
\begin{aligned}
\omega_1 &= L_2^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}\left(x_1^3x_2-x_1x_2^3+x_1^3x_2t-x_1x_2^3t\right)dp_1dp_2\right] \\
&= L_2^{-1}L_s^{-1}\left[\frac{2}{p_1^3p_2s^{2\alpha+1}}-\frac{2}{p_2^3p_1s^{2\alpha+1}}+\frac{2}{p_1^3p_2s^{2\alpha+2}}-\frac{2}{p_2^3p_1s^{2\alpha+2}}\right] \\
\omega_1 &= \frac{x_1^2t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{x_2^2t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{x_1^2t^{2\alpha+1}}{\Gamma(2\alpha+2)}-\frac{x_2^2t^{2\alpha+1}}{\Gamma(2\alpha+2)},
\end{aligned}
$$

at $n = 1$

$$
\begin{aligned}
\varphi_2 &= L_2^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}\left(L_2L_t\left[x_2\frac{\partial}{\partial x_1}\left(x_1\frac{\partial \varphi_1}{\partial x_1}\right)+x_1\frac{\partial}{\partial x_2}\left(x_2\frac{\partial \varphi_1}{\partial x_2}\right)\right]\right)dp_1dp_2\right] \\
&+ L_2^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}\left(L_2L_t\left[x_2\frac{\partial^2}{\partial x_1\partial t}\left(x_1\frac{\partial \varphi_1}{\partial x_1}\right)+x_1\frac{\partial^2}{\partial x_2\partial t}\left(x_2\frac{\partial \varphi_1}{\partial x_2}\right)\right]\right)dp_1dp_2\right] \\
&+ L_2^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}(L_2L_t[x_1x_2\omega_1])dp_1dp_2\right],
\end{aligned}
$$

and

$$
\begin{aligned}
\omega_1 &= L_2^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}\left(L_2L_t\left[x_2\frac{\partial}{\partial x_1}\left(x_1\frac{\partial \omega_1}{\partial x_1}\right)+x_1\frac{\partial}{\partial x_2}\left(x_2\frac{\partial \omega_1}{\partial x_2}\right)\right]\right)dp_1dp_2\right] \\
&+ L_2^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}\left(L_2L_t\left[x_2\frac{\partial^2}{\partial x_1\partial t}\left(x_1\frac{\partial \omega_1}{\partial x_1}\right)+x_1\frac{\partial^2}{\partial x_2\partial t}\left(x_2\frac{\partial \omega_1}{\partial x_2}\right)\right]\right)dp_1dp_2\right] \\
&+ L_2^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}(L_2L_t[x_1x_2\varphi_1])dp_1dp_2\right],
\end{aligned}
$$

$$
\begin{aligned}
\varphi_2 &= L_2^{-1}L_s^{-1}\left[\frac{1}{s^{2\alpha}}\int_0^{p_1}\int_0^{p_2}\left(\frac{x_1^3x_2t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{x_1x_2^3t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{x_1^3x_2t^{2\alpha+1}}{\Gamma(2\alpha+2)}-\frac{x_1x_2^3t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right)dp_1dp_2\right] \\
&= L_2^{-1}L_s^{-1}\left[\frac{2}{p_1^3p_2s^{4\alpha+1}}-\frac{2}{p_2^3p_1s^{4\alpha+1}}+\frac{2}{p_1^3p_2s^{4\alpha+2}}-\frac{2}{p_2^3p_1s^{4\alpha+2}}\right] \\
\varphi_2 &= \frac{x_1^2t^{4\alpha}}{\Gamma(4\alpha+1)}-\frac{x_2^2t^{4\alpha}}{\Gamma(4\alpha+1)}+\frac{x_1^2t^{4\alpha+1}}{\Gamma(4\alpha+2)}-\frac{x_2^2t^{4\alpha+1}}{\Gamma(4\alpha+2)},
\end{aligned}
$$

similarly

$$\varphi_2 = \frac{x_1^2 t^{4\alpha}}{\Gamma(4\alpha+1)} - \frac{x_2^2 t^{4\alpha}}{\Gamma(4\alpha+1)} + \frac{x_1^2 t^{4\alpha+1}}{\Gamma(4\alpha+2)} - \frac{x_2^2 t^{4\alpha+1}}{\Gamma(4\alpha+2)},$$

and

$$\omega_2 = \frac{x_1^2 t^{4\alpha}}{\Gamma(4\alpha+1)} - \frac{x_2^2 t^{4\alpha}}{\Gamma(4\alpha+1)} + \frac{x_1^2 t^{4\alpha+1}}{\Gamma(4\alpha+2)} - \frac{x_2^2 t^{4\alpha+1}}{\Gamma(4\alpha+2)},$$

by the same way, at $n = 2$, *we have*

$$\varphi_3 = \frac{x_1^2 t^{6\alpha}}{\Gamma(6\alpha+1)} - \frac{x_2^2 t^{6\alpha}}{\Gamma(6\alpha+1)} + \frac{x_1^2 t^{6\alpha+1}}{\Gamma(6\alpha+2)} - \frac{x_2^2 t^{6\alpha+1}}{\Gamma(6\alpha+2)},$$

$$\omega_3 = \frac{x_1^2 t^{6\alpha}}{\Gamma(6\alpha+1)} - \frac{x_2^2 t^{6\alpha}}{\Gamma(6\alpha+1)} + \frac{x_1^2 t^{6\alpha+1}}{\Gamma(6\alpha+2)} - \frac{x_2^2 t^{6\alpha+1}}{\Gamma(6\alpha+2)}.$$

On using Equation (35), the approximate solutions follow

$$\begin{aligned}\varphi(x_1, x_2, t) \quad = \quad & \left(x_1^2 - x_2^2\right) + \left(x_1^2 - x_2^2\right)t + \left(x_1^2 - x_2^2\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \left(x_1^2 - x_2^2\right)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \\ & + \left(x_1^2 - x_2^2\right)\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \left(x_1^2 - x_2^2\right)\frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)} \\ & + \left(x_1^2 - x_2^2\right)\frac{t^{6\alpha}}{\Gamma(6\alpha+1)} + \left(x_1^2 - x_2^2\right)\frac{t^{6\alpha+1}}{\Gamma(6\alpha+2)},\end{aligned}$$

and

$$\begin{aligned}\omega(x_1, x_2, t) \quad = \quad & \left(x_1^2 - x_2^2\right) + \left(x_1^2 - x_2^2\right)t + \left(x_1^2 - x_2^2\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \left(x_1^2 - x_2^2\right)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \\ & + \left(x_1^2 - x_2^2\right)\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \left(x_1^2 - x_2^2\right)\frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)} \\ & + \left(x_1^2 - x_2^2\right)\frac{t^{6\alpha}}{\Gamma(6\alpha+1)} + \left(x_1^2 - x_2^2\right)\frac{t^{6\alpha+1}}{\Gamma(6\alpha+2)}.\end{aligned}$$

If we set $\alpha = 1$, the fractional solution becomes

$$\begin{aligned}\varphi(x_1, x_2, t) &= \varphi_0 + \varphi_1 + \varphi_2 + \varphi_3 + \ldots = \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \ldots\right)\left(x_1^2 - x_2^2\right) \\ \omega(x_1, x_2, t) &= \omega_0 + \omega_1 + \omega_2 + \omega_3 + \ldots = \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \ldots\right)\left(x_1^2 - x_2^2\right),\end{aligned}$$

and hence, the exact solution becomes

$$\varphi(x_1, x_2, t) = \left(x_1^2 - x_2^2\right)e^{-t}, \quad \omega(x_1, x_2, t) = \left(x_1^2 - x_2^2\right)e^{-t}.$$

5. Conclusions

In this study, we have presented the multi-dimensional Laplace transform (m+1-DLADM) in order to find the approximate and series solutions of the generalized singular time-fractional M-D pseudo-hyperbolic equation. We studied three different examples related to the singular time-fractional 2-D pseudo-hyperbolic equations. By examining the examples, we note that (m+1-DLADM) is a strong tool for the solution of generalized linear, nonlinear, and coupled systems of fractional singular M-dimensional pseudo-hyperbolic equations, and compared with the Adomian decomposition method, homotopy analysis method (HAM) and variational iteration method(VAM). However, there is still an open problem to examine the rate of convergence to the exact solution for these types of problems. It is also possible to study (M+1- DLADM) by applying an analytical solution and to the other fractional singular M-dimensional partial differential equations, which appear very often in applied science as well as engineering which may provide a better understanding of the real-world problems that represent the singular M-dimensional fractional partial differential equations.

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