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Stability of a Nonlinear Langevin System of ML-Type Fractional Derivative Affected by Time-Varying Delays and Differential Feedback Control

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Abstract: The Langevin system is an important mathematical model to describe Brownian motion. The research shows that fractional differential equations have more advantages in viscoelasticity. The exploration of fractional Langevin system dynamics is novel and valuable. Compared with the fractional system of Caputo or Riemann–Liouville (RL) derivatives, the system with Mittag–Leffler (ML)-type fractional derivatives can eliminate singularity such that the solution of the system has better analytical properties. Therefore, we concentrate on a nonlinear Langevin system of ML-type fractional derivatives affected by time-varying delays and differential feedback control in the manuscript. We first utilize two fixed-point theorems proposed by Krasnoselskii and Schauder to investigate the existence of a solution. Next, we employ the contraction mapping principle and nonlinear analysis to establish the stability of types such as Ulam–Hyers (UH) and Ulam–Hyers–Rassias (UHR) as well as generalized UH and UHR. Lastly, the theoretical analysis and numerical simulation of some interesting examples are carried out by using our main results and the DDESD toolbox of MATLAB.

Keywords: Langevin system; ML-fractional derivative; existence and stability; time-varying delay; differential feedback control

MSC: 34K37; 34K20; 37C25

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1. Introduction

This work mostly takes into account a nonlinear Langevin system of ML-type fractional derivatives affected by time-varying delays and differential feedback control as follows:

$$\begin{cases} {}^{ML}\mathcal{D}^{\beta}_{0+}\left[{}^{ML}\mathcal{D}^{\alpha}_{0+}-\lambda\right]u(t)=f(t,u(t-\sigma_1(t)),w(t-\sigma_2(t))),\ t\in(0,b], \\ w'(t)=g(t,u(t-\sigma_3(t)),w(t-\sigma_4(t))),\ t\in J=[0,b], \\ u(t)=\omega_1(t),\ w(t)=\omega_2(t),\ {}^{ML}\mathcal{D}^{\alpha}_{0+}u(t)=\omega_3(t),\ t\in[-\sigma,0], \end{cases} \tag{1}$$

where $0<\alpha,\beta\leq 1$ and $b,\ \lambda>0$, ${}^{ML}\mathcal{D}^{\kappa}_{0+}$ is the ML-type fractional derivative with κ-order, for all $0<\kappa\leq 1$. $f,g\in C(J\times\mathbb{R}^2,\mathbb{R}), \sigma_l\in C(J,\mathbb{R}^+)(l=1,2,3,4), \sigma=\max_{1\leq l\leq 4}\{\max_{t\in J}\sigma_l(t)\}$, $\omega_m\in C([-\sigma,0],\mathbb{R})(m=1,2,3)$.

The main motivation comes from the important theory and wide application of the Langevin system. In 1908, the French physicist Paul Langevin put forward the famous Langevin equation to expound the random motion of particles annihilated in the fluid due to the collision between particles and fluid molecules. In [1,2], the authors provided a large number of random examples of the Langevin system as mathematical models. The fractional-order Langevin system makes up for the deficiency of the integer-order Langevin system and has been widely used and studied. The integer-order Langevin system can not meet the accuracy requirements in describing complex viscoelasticity. Therefore, the classical Langevin system has been extended and modified. For example, Kubo [3,4] imitated a complex viscoelastic abnormal diffusion process by applying a general Langevin system. It is worth noting that the derivatives in these generalized equations are of integer order. Because fractional derivatives have advantages in describing the process of memory and viscoelasticity, another modification is to replace the integral derivative with a fractional derivative in the Langevin system. The early research on the fractional Langevin system as a model was conducted by Eab and Lim [5] as well as Sandev and Tomovski [6]. They used the fractional Langevin system to study the one-dimensional diffusion process and the motion of free particles driven by power law noise, respectively. Recently, some new fields have emerged in the application of fractional differential systems (see [7–10]). Additionally, some important achievements have been made in the research of the fractional Langevin system (see [11–17]).

Furthermore, for a system with practical application background, its stability is very important. According to practical needs, scientists have put forward many concepts of system stability. UH-stability is one of the most important stabilities, which was posed by Ulam and Hyers [18,19] in the 1940s. In the last ten years, the research on the UH-stability of fractional systems has been highly praised by many scholars. There have many papers dealing with the UH-stability [20–26] and generalized UH-stability [27–29] of fractional system. However, only a few papers [30–34] have discussed that the fractional Langevin system is UH-stable.

It is worth noting that the previous works on the fractional Langevin system basically involve Caputo- or RL-fractional derivatives. However, under certain circumstances, the singularity of Caputo or Riemann–Liouville fractional derivatives is unavoidable. This leads to huge errors and even distortion in the application of the Caputo or Riemann–Liouville fractional system in some physical fields. Hence, some scholars began to modify the previous definition of a fractional derivative to eliminate singularity. For instance, Caputo and Febrizio raised a nonsingular fractional derivative (see [35]), which is called the Caputo–Febrizio fractional derivative. In [36], Atangana and Baleanu proposed another novel nonsingular fractional derivative with an ML-kernel. In both theory [37–40] and application [41–47], many scholars have focused on the study of these nonsingular fractional systems since they were put forward. However, only rare published works [48,49] have shown that some ML-type fractional Langevin systems are UH-stable.

Inspired by the aforementioned, the main purpose of this manuscript is to find some sufficient conditions for the existence and stability of the solution of system (1). The main contributions of our work are reflected in the following aspects. (i) Comparison with previous works on fractional Langevin equation such as [30–32], our system (1) involves nonsingular fractional derivatives with an ML-kernel. (ii) In system (1), we consider the influence of delayed differential feedback control, which is not found in previous papers. (iii) We obtain some new criteria of existence and UH-type stability of solutions. (iv) By applying the DDESD toolbox of MATLAB, the simulations and numerical solutions are given.

In Section 2, we will review some concepts and lemmas used later. Based on some fixed-point theorems, we prove that system (1) has at least a solution in Section 3. The UH-, UHR-, GUH-, and GUHR-stability of (1) are built in Section 4. As applications, we conduct theoretical analysis and numerical simulation on some examples to verify the correctness and effectiveness of our main results in Section 5. Finally, a brief summary is made in Section 6. In addition, we put lengthy and complex theoretical proofs in the appendices to improve the readability and conciseness of the manuscript.

2. Preliminaries

Definition 1 ([50])**.** *Let $0 < \alpha \leq 1$, $b > 0$, $u : [0, b] \to \mathbb{R}$. Define an integral*

$$ {}^{ML}\mathcal{I}^{\alpha}_{0^+} u(t) = \frac{1-\alpha}{\mathcal{N}(\alpha)} u(t) + \frac{\alpha}{\mathcal{N}(\alpha)\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} u(x) dx, $$

where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$, *the normalized constant* $\mathcal{N}(\alpha)$ *satisfies* $\mathcal{N}(0) = \mathcal{N}(1) = 1$. *If the above integral exists, then it is called the* α*-order ML-fractional integral along the left of* u.

Definition 2 ([36]). *Let* $0 < \alpha \leq 1$, $b > 0$ *and* $u \in C^1(0, b)$. *The* α*-order ML-fractional derivative along the left of* u *in sense of Caputo is defined by*

$$^{ML}\mathcal{D}^{\alpha}_{0+}u(t) = \frac{\mathcal{N}(\alpha)}{(1-\alpha)} \int_0^t \mathbb{E}_\alpha\Big[-\frac{\alpha}{1-\alpha}(t-x)^\alpha\Big]u'(x)dx,$$

where $\mathbb{E}_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}$ *is is called the one-parameter ML-function.*

Lemma 1 ([37]). *Given* $0 < \nu \leq 1$ *and* $h \in C[0, b]$, *then the following ML-fractional system*

$$\begin{cases} ^{ML}\mathcal{D}^{\nu}_{0+}w(t) = h(t), \ t \in (0, b), \\ w(0) = w_0, \end{cases}$$

is uniquely solved by

$$w(t) = w_0 + \frac{1-\nu}{\mathcal{N}(\nu)}[h(t) - h(0)] + \frac{\nu}{\mathcal{N}(\nu)\Gamma(\nu)} \int_0^t (t-x)^{\nu-1}h(x)dx.$$

Remark 1. *Definition 2 and Lemma 1 imply that* $^{ML}\mathcal{D}^{\alpha}_{0+}u(t) \equiv 0$ *iff* $u(t)$ *is identical to a constant.*

The whole paper needs the following fundamental assumptions.

(A_1) $\lambda, b > 0$ and $0 < \alpha, \beta \leq 1$ are given constants such that $\Delta \triangleq 1 - \frac{\lambda(1-\alpha)}{\mathcal{N}(\alpha)} \neq 0$.

(A_2) $f, g \in C(J \times \mathbb{R}^2, \mathbb{R})$, $\sigma_l \in C(J, \mathbb{R}^+)(l = 1, 2, 3, 4)$, $\sigma = \max_{1\leq l\leq 4}\{\max_{t\in J} \sigma_l(t)\}$, $\omega_m \in C([-\sigma, 0], \mathbb{R})(m = 1, 2, 3)$.

Consider the below nonlinear integral system

$$\begin{cases} u(t) = \omega_1(0) + \frac{1}{\Delta}\big\{\frac{\omega_3(0)-\lambda\omega_1(0)}{\mathcal{N}(\alpha)\Gamma(\alpha)}t^\alpha - \frac{(1-\alpha)(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)}f(0, \omega_1(-\sigma_1(0)), \omega(-\sigma_2(0))) \\ \quad + \frac{(1-\alpha)(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)}f(t, u(t-\sigma_1(t)), w(t-\sigma_2(t))) + \frac{\lambda\alpha}{\mathcal{N}(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds \\ \quad + \frac{(1-\alpha)\beta}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}f(s, u(s-\sigma_1(s)), w(s-\sigma_2(s)))ds \\ \quad + \frac{\alpha(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s, u(s-\sigma_1(s)), w(s-\sigma_2(s)))ds \\ \quad + \frac{\alpha\beta}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}f(s, u(s-\sigma_1(s)), w(s-\sigma_2(s)))ds\big\}, \ t \in J, \\ w(t) = \omega_2(0) + \int_0^t g(s, u(s-\sigma_3(s)), w(s-\sigma_4(s)))ds, \ t \in J, \\ u(t) = \omega_1(t), \ w(t) = \omega_2(t), \ t \in [-\sigma, 0]. \end{cases} \tag{2}$$

Lemma 2. *If conditions* (A_1) *and* (A_2) *are true, then, solving the nonlinear ML-fractional system* (1) *and solving the nonlinear integral system* (2) *are equivalent.*

Remark 2. *Lemma 2 is the foundation of studying the existence and stability of solutions to system* (1). *Its proof is shown in Appendix A.*

3. Existence of Solutions

In this section, by applying the following important fixed-point theorems, we emphasize studying the existence of solutions for system (1).

Lemma 3 (Krasnoselskii's fixed-point theorem [51]). *Let* $\mathbb{X}$ *be a Banach space, and* $\phi \neq \mathbb{Y} \subset \mathbb{X}$ *be closed convex. If the mappings* $\mathscr{P}, \mathscr{Q} : \mathbb{Y} \to \mathbb{Y}$ *fulfill the following conditions:*

(i) $\mathscr{P}u + \mathscr{Q}v \in \mathbb{Y}, \forall\, u, v \in \mathbb{Y}$.
(ii) $\mathscr{P}$ *is contraction and* $\mathscr{Q}$ *is continuous and compact.*
Then, there has at least a solution $u^* \in \mathbb{Y}$ *such that* $u^* = \mathscr{P}u^* + \mathscr{Q}u^*$.

Lemma 4 (Schauder's fixed point theorem [52])**.** *Given a Banach space* $\mathbb{X}$*, let* $\Omega \subset \mathbb{X}$ *be closed convex, and define a mapping* $\mathscr{T} : \overline{\Omega} \to \overline{\Omega}$*. Then, there is a* $u^* \in \overline{\Omega}$ *such that* $\mathscr{T}u^* = u^*$*, provided that* $\mathscr{T}$ *is completely continuous.*

It is easy to see from (2) that the solution of (1) can only meet the continuity. So we define some Banach spaces $\mathbb{Y} = C(I, \mathbb{R})$ with the norm $\|z\|_C = \max_{t\in I} |z(t)|$, and $\mathbb{X} = \mathbb{Y} \times \mathbb{Y}$ equipped with the norm $\|U\| = \max\{\|u\|_C, \|w\|_C\}$, where $I = [-\sigma, T]$ and $U = (u, v) \in \mathbb{X}$. We shall study the existence and stability of the solution of (1) in $(\mathbb{X}, \|\cdot\|)$.

Under the basic assumptions (A_1) and (A_2), we obtain the following two solvability results of system (1).

Theorem 1. *Assume that* (A_1) *and* (A_2) *hold. If conditions* (A_3) *and* (A_4) *are satisfied:*
(A_3) *There exist some functions* $m_1(t), m_2(t) \in C(J, \mathbb{R}^+)$ *such that*

$$|f(t, u, w)| \leq m_1(t), \;\; |g(t, u, w)| \leq m_2(t), \;\; \forall\; t \in J, u, w \in \mathbb{R};$$

(A_4) $0 < \kappa \triangleq \frac{\lambda b^\alpha}{|\Delta|\mathcal{N}(\alpha)\Gamma(\alpha)} < 1$;
then system (1) *has at least one solution* $(u^*(t), w^*(t)) \in \mathbb{X}$.

Theorem 2. *Assume that* (A_1) *and* (A_2) *hold. If conditions* (A_5) *and* (A_6) *are satisfied:*
(A_5) *There exist some functions* $M_k(t), N_k(t) \in C(J, \mathbb{R}^+)(k = 1, 2, 3)$ *such that,* $\forall\; t \in J, u, w \in \mathbb{R}$,

$$|f(t, u, w)| \leq M_1(t) + M_2(t)|u| + M_3(t)|w|, \;\; |g(t, u, w)| \leq N_1(t) + N_2(t)|u| + N_3(t)|w|;$$

(A_6) $0 < \Theta_1, \Theta_2 < 1$, *where* $\Theta_1 = \frac{\lambda b^\alpha}{|\Delta|\mathcal{N}(\alpha)\Gamma(\alpha)} + \frac{\|M_2\|_b + \|M_3\|_b}{|\Delta|\mathcal{N}(\alpha)\mathcal{N}(\beta)}\left[2(1-\alpha)(1-\beta) + \frac{(1-\alpha)b^\beta}{\Gamma(\beta)} + \frac{(1-\beta)b^\alpha}{\Gamma(\alpha)} + \frac{\alpha\beta b^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right]$, $\Theta_2 = b(\|N_2\|_b + \|N_3\|_b)$, $\|\cdot\|_b = \max_{t\in[0,b]} |\cdot|$ *and* $\|\cdot\|_\sigma = \max_{t\in[-\sigma,0]} |\cdot|$;
then system (1) *has at least one solution* $(u^*(t), w^*(t)) \in \mathbb{X}$.

Remark 3. *We apply Lemma 3 to prove Theorem 1 in three steps. Step 1 is to define two appropriate mappings* $\mathscr{P}, \mathscr{Q} : \mathbb{X} \to \mathbb{X}$ *as*

$$(\mathscr{P}U)(t) = ((\mathscr{P}_1U)(t), (\mathscr{P}_2U)(t)), \;\; (\mathscr{Q}U)(t) = ((\mathscr{Q}_1U)(t), (\mathscr{Q}_2U)(t)).$$

Step 2 is to prove that $\mathscr{P}$ *is contraction. Step 3 is to show that* $\mathscr{Q}$ *is continuous and compact based on the Arzelá–Ascoli theorem. The detailed proof of Theorem 1 can be seen in Appendix B. Similarly, an appropriate mapping* $\mathscr{T} : \mathbb{X} \to \mathbb{X}$ *is defined by*

$$(\mathscr{T}U)(t) = ((\mathscr{T}_1U)(t), (\mathscr{T}_2U)(t)), \;\; \forall\, t \in I = [-\sigma, T], \, U = (u, w) \in \mathbb{X},$$

we also employ Schauder's fixed-point theorem (i.e., Lemma 4) to give the concrete proof of Theorem 2 in the Appendix C.

Remark 4. *Theorem 1 and Theorem 2 do not imply each other. In fact, condition* (A_3) *is true* $\Rightarrow$ *condition* (A_5) *is also true, otherwise, it may not be true. However, condition* (A_6) *is true* $\Rightarrow$ *condition* (A_4) *is also true, otherwise, it may not be true.*

4. Stability of Ulam–Hyers type Systems

This section mainly discusses the stability of types such as Ulam–Hyers (UH) and Ulam–Hyers–Rassias (UHR) as well as generalized UH and UHR for system (1).

Lemma 5. *[52] Given a Banach space $\mathbb{X}$, let $\phi \neq \mathbb{E} \subset \mathbb{X}$ be closed, and define a mapping $\mathscr{T} : \mathbb{E} \to \mathbb{E}$. Assume that $\mathscr{T}$ is contractive, then, there has a unique $u^* \in \mathbb{E}$ such that $\mathscr{T}u^* = u^*$.*

Consider two inequalities as follows:

$$\begin{cases} |{}^{ML}\mathcal{D}_{0^+}^{\beta}\left[{}^{ML}\mathcal{D}_{0^+}^{\alpha} - \lambda\right]x(t) - f(t, x(t-\sigma_1(t)), y(t-\sigma_2(t)))| \leq \epsilon, \ 0 < t \leq b, \\ |y'(t) - g(t, x(t-\sigma_3(t)), y(t-\sigma_4(t)))| \leq \epsilon, \ 0 < t \leq b, \\ x(t) = \omega_1(t), \ y(t) = \omega_2(t), \ {}^{ML}\mathcal{D}_{0^+}^{\alpha}x(t) = \omega_3(t), \ t \in [-\sigma, 0], \end{cases} \tag{3}$$

and

$$\begin{cases} |{}^{ML}\mathcal{D}_{0^+}^{\beta}\left[{}^{ML}\mathcal{D}_{0^+}^{\alpha} - \lambda\right]x(t) - f(t, x(t-\sigma_1(t)), y(t-\sigma_2(t)))| \leq \varphi(t)\epsilon, \ 0 < t \leq b, \\ |y'(t) - g(t, x(t-\sigma_3(t)), y(t-\sigma_4(t)))| \leq \varphi(t)\epsilon, \ 0 < t \leq b, \\ x(t) = \omega_1(t), \ y(t) = \omega_2(t), \ {}^{ML}\mathcal{D}_{0^+}^{\alpha}x(t) = \omega_3(t), \ t \in [-\sigma, 0], \end{cases} \tag{4}$$

where $\epsilon > 0$, $0 < \alpha, \beta \leq 1$, and $\varphi : [-\sigma, b] \to \mathbb{R}^+$ is continuously non-decreasing.

Similar to [18,19], we define the UH-type stability corresponding to (1) as follows.

Definition 3. *Let $X = (x, y)$ and $U = (u, w)$ in $\mathbb{X}$ be any solution of inequality (3) and the unique solution of system (1). For each $\epsilon > 0$,*

(1) *if there exists a constant $C_1 > 0$ such that*

$$\|X(t) - U(t)\| \leq C_1\epsilon,$$

then system (1) is called UH-stable;

(2) *if there exists a continuous function $\theta : \mathbb{R} \to \mathbb{R}^+$ with $\theta(0) = 0$ such that*

$$\|X(t) - U(t)\| \leq \theta(\epsilon),$$

then system (1) is called generalized UH-stable.

Definition 4. *Let $X = (x, y)$ and $U = (u, w)$ in $\mathbb{X}$ be any solution of inequality (4) and the unique solution of system (1). For each $\epsilon > 0$,*

(1) *if there exists a constant $C_2 > 0$ such that*

$$\|X(t) - U(t)\| \leq C_2\varphi(t)\epsilon, \quad t \in [-\sigma, b],$$

then system (1) is called UHR-stable;

(2) *if there exists a constant $C_3 > 0$ such that*

$$\|X(t) - U(t)\| \leq C_3\varphi(t), \quad t \in [-\sigma, b],$$

then system (1) is called generalized UHR-stable.

Obviously, UH-stable $\Rightarrow$ generalized UH-stable, and UHR-stable $\Rightarrow$ generalized UHR-stable.

Remark 5. *A function $X = (x, y) \in \mathbb{X}$ solves the inequality (3) iff there has a continuous vector function $\phi = (\phi_1, \phi_2)$ in $(0, b]$ such that*

(1) $|\phi_1(t)| \leq \epsilon$, $|\phi_2(t)| \leq \epsilon$, $0 < t \leq b$.

(2) $^{ML}\mathcal{D}^{\beta}_{0^+}\left[^{ML}\mathcal{D}^{\alpha}_{0^+} - \lambda\right]x(t) = f(t, x(t - \sigma_1(t)), y(t - \sigma_2(t))) + \phi_1(t), 0 < t \le b.$
(3) $y'(t) = g(t, x(t - \sigma_3(t)), y(t - \sigma_4(t))) + \phi_2(t), 0 < t \le b.$
(4) $x(t) = \omega_1(t),\ y(t) = \omega_2(t),\ ^{ML}\mathcal{D}^{\alpha}_{0^+}x(t) = \omega_3(t),\ t \in [-\sigma, 0].$

Remark 6. *A function $X = (x, y) \in \mathbb{X}$ solves the inequality (4) iff there has a continuous vector function $\psi = (\psi_1, \psi_2)$ in $(0, b]$ such that*
(1) $|\psi_1(t)| \le \varphi(t)\epsilon, |\psi_2(t)| \le \varphi(t)\epsilon, 0 < t \le b.$
(2) $^{ML}\mathcal{D}^{\beta}_{0^+}\left[^{ML}\mathcal{D}^{\alpha}_{0^+} - \lambda\right]x(t) = f(t, x(t - \sigma_1(t)), y(t - \sigma_2(t))) + \psi_1(t), 0 < t \le b.$
(3) $y'(t) = g(t, x(t - \sigma_3(t)), y(t - \sigma_4(t))) + \psi_2(t), 0 < t \le b.$
(4) $x(t) = \omega_1(t),\ y(t) = \omega_2(t),\ ^{ML}\mathcal{D}^{\alpha}_{0^+}x(t) = \omega_3(t),\ t \in [-\sigma, 0].$

Theorem 3. *Assume that* (A_1) *and* (A_2) *hold. If conditions* (A_7) *and* (A_8) *are satisfied:*
(A_7) *There exist $a_1(t), a_2(t), b_1(t), b_2(t) \in C(J, \mathbb{R}^+)$ such that, $\forall\ t \in J, u, \overline{u}, w, \overline{w} \in \mathbb{R}$,*

$$|f(t, u, w) - |f(t, \overline{u}, \overline{w})| \le a_1(t)|u - \overline{u}| + a_2(t)|w - \overline{w}|,$$

$$|g(t, u, w) - g(t, \overline{u}, \overline{w})| \le b_1(t)|u - \overline{u}| + b_2(t)|w - \overline{w}|;$$

(A_8) $0 < Y_1, Y_2 < 1$, *where* $Y_1 = \frac{\lambda b^{\alpha}}{|\Delta|\mathcal{N}(\alpha)\Gamma(\alpha)} + \frac{\|a_1\|_b + \|a_2\|_b}{|\Delta|\mathcal{N}(\alpha)\mathcal{N}(\beta)}\left[2(1-\alpha)(1-\beta) + \frac{(1-\alpha)b^{\beta}}{\Gamma(\beta)} + \frac{(1-\beta)b^{\alpha}}{\Gamma(\alpha)} + \frac{\alpha\beta b^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right]$, $Y_2 = b(\|b_1\|_b + \|b_2\|_b)$ *and* $\|\cdot\|_b = \max_{t\in[0,b]}|\cdot|$;
then there are the following claims:
(a) *System (1) has a unique solution $U^*(t) = (u^*(t), w^*(t)) \in \mathbb{X}$.*
(b) *System (1) is UH-stable and also generalized UH-stable.*
(c) *System (1) is UHR-stable and also generalized UHR-stable.*

Remark 7. *Similar to the proof of Theorem 2, a mapping $\mathscr{T} : \mathbb{X} \to \mathbb{X}$ is defined by*

$$(\mathscr{T}U)(t) = ((\mathscr{T}_1 U)(t), (\mathscr{T}_2 U)(t)),\ \ \forall t \in [-\sigma, b],\ U = (u, w) \in \mathbb{X},$$

By the contraction mapping principle, namely, Lemma 5, we prove Theorem 3 in Appendix D. In addition, compared with Theorems 1–2, without the restriction of complete continuity of operator $\mathscr{T}$, Theorem 3 ensures that the unique solution of system (1) is UH-type stable.

5. Applications

In this section, we intend to make a theoretical analysis of several examples by applying the main theorems obtained in this paper. Simultaneously, we conduct some numerical simulations by means of MATLAB.

5.1. Theoretical Analysis

Example 1. *In (1), take $b = 1$, $\alpha = 0.9$, $\beta = 0.3$, $\lambda = \frac{2}{3}$, $f(t, u, w) = \sin(t)[\sin(u) + \cos(w)]$, $g(t, u, w) = \cos(t)\arctan(u + w)$, $\sigma_1(t) = 0.1$, $\sigma_2(t) = \frac{1+\sin(2t)}{6}$, $\sigma_3(t) = \frac{2+\cos(2t)}{8}$, $\sigma_4(t) = 0.6$, $\omega_1(t) = 2\cos(t)$, $\omega_2(t) = t$, $\omega_3(t) = \sin(t)$, $\mathcal{N}(x) = 1 - x + \frac{x}{\Gamma(x)}, 0 < x \le 1$. A simple calculation gives $\sigma = 0.6$, $m_1(t) = 2|\sin(t)|$, $m_2(t) = \frac{\pi}{2}|\cos(t)|$, $\mathcal{N}(0) = \mathcal{N}(1) = 1$ and*

$$\Delta = 1 - \frac{\lambda(1-\alpha)}{\mathcal{N}(\alpha)} \approx 0.9292 > 0, \quad \kappa = \frac{\lambda\alpha}{|\Delta|\mathcal{N}(\alpha)\Gamma(\alpha)} \approx 0.7125 < 1.$$

Thus conditions (A_1)-(A_4) *are all fulfilled. It follows from Theorem 1 that the system of Example 1 has at least a solution $U^*(t) = (u^*(t), w^*(t)) \in C([-0.6, 1], \mathbb{R}) \times C([-0.6, 1], \mathbb{R})$.*

Example 2. *In (1), take $b = 1$, $\alpha = 0.4$, $\beta = 0.5$, $\lambda = \frac{1}{4}$, $f(t, u, w) = \log(1 + t) + \frac{1}{20}e^t u + \frac{t}{10}w$, $g(t, u, w) = t + \frac{1}{10}e^{-t}(u + w)$, $\sigma_1(t) = 0.1$, $\sigma_2(t) = \frac{1+\sin(2t)}{6}$, $\sigma_3(t) = \frac{2+\cos(2t)}{8}$, $\sigma_3(t) = 0.6$,*

$\omega_1(t) = 2\cos(t)$, $\omega_2(t) = t$, $\omega_3(t) = \sin(t)$, $\mathcal{N}(x) = 1 - x + \frac{x}{\Gamma(x)}, 0 < x \leq 1$. *By calculation, we get* $\sigma = 0.6$, $M_1(t) = |\log(1+t)|$, $M_2(t) = \frac{e^t}{20}$, $M_3(t) = \frac{|t|}{10}$, $N_1(t) = |t|$, $N_2(t) = N_3(t) = \frac{1}{10}e^{-t}$, $\mathcal{N}(0) = \mathcal{N}(1) = 1$, $\|M_2\|_b = \frac{e}{20}$, $\|M_3\|_b = \frac{1}{10}$, $\|N_2\|_b = \|N_3\|_b = \frac{1}{10}$ *and*

$$\Delta = 1 - \frac{\lambda(1-\alpha)}{\mathcal{N}(\alpha)} \approx 0.8078 > 0, \quad \Theta_2 = b\big(\|N_2\|_b + \|N_3\|_b\big) = 0.2 < 1,$$

$$\begin{aligned}\Theta_1 =& \frac{\lambda b^\alpha}{|\Delta|\mathcal{N}(\alpha)\Gamma(\alpha)} + \frac{\|M_2\|_b + \|M_3\|_b}{|\Delta|\mathcal{N}(\alpha)\mathcal{N}(\beta)}\Big[2(1-\alpha)(1-\beta) + \frac{(1-\alpha)b^\beta}{\Gamma(\beta)} \\ &+ \frac{(1-\beta)b^\alpha}{\Gamma(\alpha)} + \frac{\alpha\beta b^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big] \approx 0.8353 < 1.\end{aligned}$$

Thus, conditions (A_1), (A_2), (A_5), *and* (A_6) *are all true. One concludes from Theorem 2 that the system of Example 2 has at least a solution* $U^*(t) = (u^*(t), w^*(t)) \in C([-0.6,1],\mathbb{R}) \times C([-0.6,1],\mathbb{R})$.

Example 3. *In (1), take* $b = 1$, $\alpha = 0.6$, $\beta = 0.8$, $\lambda = \frac{1}{5}$, $f(t,u,w) = \cos(t) + \frac{t^2}{20}\log(1+u^2+w^2)$, $g(t,u,w) = \sin(t) + \frac{2+\sin(3t)}{30}\arctan(u+w)$, $\sigma_1(t) = 0.1$, $\sigma_2(t) = \frac{1+\sin(2t)}{6}$, $\sigma_3(t) = \frac{2+\cos(2t)}{8}$, $\sigma_4(t) = 0.6$, $\omega_1(t) = 2\cos(t)$, $\omega_2(t) = t$, $\omega_3(t) = \sin(t)$, $\mathcal{N}(x) = 1 - x + \frac{x}{\Gamma(x)}, 0 < x \leq 1$. *By a direct calculation, one has* $\mathcal{N}(0) = \mathcal{N}(1) = 1$, $a_1(t) = a_2(t) = \frac{t^2}{20}$, $b_1(t) = b_2(t) = \frac{2+\sin(3t)}{30}$, $\|a_1\|_b = \|a_2\|_b = \frac{1}{20}$, $\|b_1\|_b = \|b_2\|_b = \frac{1}{10}$ *and*

$$\Delta = 1 - \frac{\lambda(1-\alpha)}{\mathcal{N}(\alpha)} \approx 0.9004 > 0, \quad \Upsilon_2 = b\big(\|b_1\|_b + \|b_2\|_b\big) = 0.2 < 1,$$

$$\begin{aligned}\Upsilon_1 =& \frac{\lambda b^\alpha}{|\Delta|\mathcal{N}(\alpha)\Gamma(\alpha)} + \frac{\|a_1\|_b + \|a_2\|_b}{|\Delta|\mathcal{N}(\alpha)\mathcal{N}(\beta)}[2(1-\alpha)(1-\beta) + \frac{(1-\alpha)b^\beta}{\Gamma(\beta)} \\ &+ \frac{(1-\beta)b^\alpha}{\Gamma(\alpha)} + \frac{\alpha\beta b^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}] \approx 0.3455 < 1.\end{aligned}$$

Thus, we verify that conditions (A_1), (A_2), (A_7), *and* (A_8) *hold. From Theorem 3, we know that the system of Example 3 has a unique solution* $U^* = (u^*(t), w^*(t)) \in C([-0.6,1],\mathbb{R}) \times C([-0.6,1],\mathbb{R})$. *Meanwhile, this system is UH-, UHR-, GUH-, and GUHR-stable.*

5.2. Numerical Simulation

In this subsection, we first give the numerical simulation algorithm of system (1) as follows:

Step 1: Transform the system into a system of equations. Let $v(t) = ({}^{ML}\mathcal{D}^\alpha_{0^+} - \lambda)u(t)$, then system (1) becomes the following equations

$$\begin{cases} {}^{ML}\mathcal{D}^\alpha_{0^+}u(t) = \lambda u(t) + v(t), \ t \in (0,b], \\ {}^{ML}\mathcal{D}^\beta_{0^+}v(t) = f(t, u(t-\sigma_1(t)), w(t-\sigma_2(t))), \ t \in (0,b], \\ w'(t) = g(t, u(t-\sigma_3(t)), w(t-\sigma_4(t))), \ t \in J = [0,b], \\ u(t) = \omega_1(t), \ w(t) = \omega_2(t), \ v(t) = \omega_3(t) - \lambda\omega_1(t), \ t \in [-\sigma, 0]. \end{cases} \tag{5}$$

Step 2: Transform Equation (5) into the integral equations. According to the proof of Lemma 2, we have

$$\begin{cases} u(t) = \omega_1(0) + \frac{1-\alpha}{\mathcal{N}(\alpha)}\left[\lambda[u(t)-\omega_1(0)] + [v(t) - (\omega_3(0) - \lambda\omega_1(0))]\right] \\ \quad + \frac{\alpha}{\mathcal{N}(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[\lambda u(s) + v(s)]ds,\ t \in (0,b], \\ v(t) = \omega_3(0) - \lambda\omega_1(0) + \frac{1-\beta}{\mathcal{N}(\beta)}\left[f(t, u(t-\sigma_1(t)), w(t-\sigma_2(t))) \right. \\ \quad \left. - f(0, u(-\sigma_1(0)), w(-\sigma_2(0)))\right],\ t \in (0,b], \\ w(t) = \omega_2(0) + \int_0^t g(s, u(s-\sigma_3(s)), w(s-\sigma_4(s)))ds,\ t \in J = [0,b], \\ u(t) = \omega_1(t),\ w(t) = \omega_2(t),\ v(t) = \omega_3(t) - \lambda\omega_1(t),\ t \in [-\sigma, 0]. \end{cases} \tag{6}$$

Step 3: Transform the integral equations (Equations (6)) into a differential equations of integer order. Find the first derivative at both ends of Equation (6), then Equation (6) becomes a first-order differential equation. Next, we can get a system of delayed ordinary differential equations with some simplifications and arrangements. Finally, the simulation can be carried out with the help of the DDESD toolbox in MATLAB R2019b. In addition, the error estimation and convergence order of the algorithm in the DDESD toolbox are given by Shampine [53].

To simulate and compare with the integer order equation corresponding to system (1), the integer order differential Langevin equation corresponding to system (1) is formulated by

$$\begin{cases} [u'(t) - \lambda u(t)]' = f(t, u(t-\sigma_1(t)), w(t-\sigma_2(t))),\ t \in (0,b], \\ w'(t) = g(t, u(t-\sigma_3(t)), w(t-\sigma_4(t))),\ t \in J = [0,b], \\ u(t) = \omega_1(t),\ w(t) = \omega_2(t),\ u'(t) = \omega_3(t),\ t \in [-\sigma, 0]. \end{cases} \tag{7}$$

Let $v(t) = (\frac{d}{dt} - \lambda)u(t)$, then the system of equations equivalent to Equation (7) is formed as

$$\begin{cases} u'(t) = \lambda u(t) + v(t),\ t \in (0,b], \\ v'(t) = f(t, u(t-\sigma_1(t)), w(t-\sigma_2(t))),\ t \in (0,b], \\ w'(t) = g(t, u(t-\sigma_3(t)), w(t-\sigma_4(t))),\ t \in J = [0,b], \\ u(t) = \omega_1(t),\ w(t) = \omega_2(t),\ v(t) = \omega_3(t) - \lambda\omega_1(t),\ t \in [-\sigma, 0]. \end{cases} \tag{8}$$

Based on the above preparations, we will now discuss and simulate the solutions of Examples 1–3, respectively.

(a) When all parameter values remain invariant, the simulations of solutions of Example 1 and its corresponding integer-order differential equation are shown as Figures 1 and 2, respectively. $u(t)$ is the solution of the Langevin equation in the figures. The comparison of numerical solutions of $u(t)$ is given in Table 1. It can be seen from the simulation figures and table that the solution of the integer-order equation changes sharply, while the fractional-order equation changes gently.

Table 1. Comparison of numerical solutions of $u(t)$ in Example 1.

t	0	0.2	0.4	0.6	0.8	1
Fractional order	2.0000	1.9971	1.9858	1.9582	1.9033	1.8187
Integer order	2.0000	1.8101	1.6193	1.4444	1.3058	1.2184

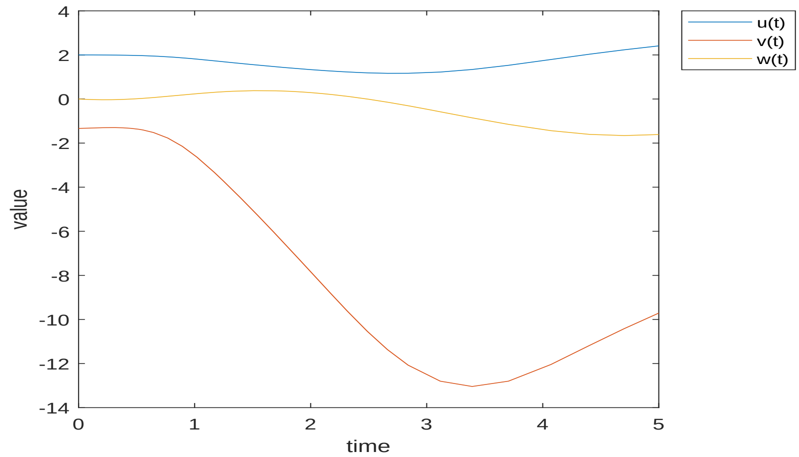

Figure 1. Simulation of solutions in Example 1.

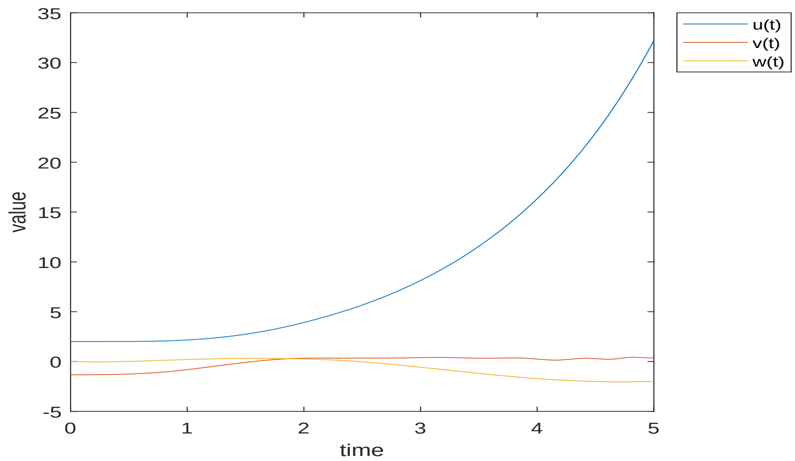

Figure 2. Simulation of solutions of the integer-order equation corresponding to Example 1.

(b) When all parameter values remain invariant, the simulations of solutions of Example 2 and its corresponding integer-order differential equation are shown as Figures 3 and 4, respectively. $u(t)$ is the solution of the Langevin equation in the figures. The comparison of numerical solutions of $u(t)$ is given in Table 2. It can be seen from the simulation figures and table that the solution of the integer-order equation changes sharply, while the fractional-order equation changes gently.

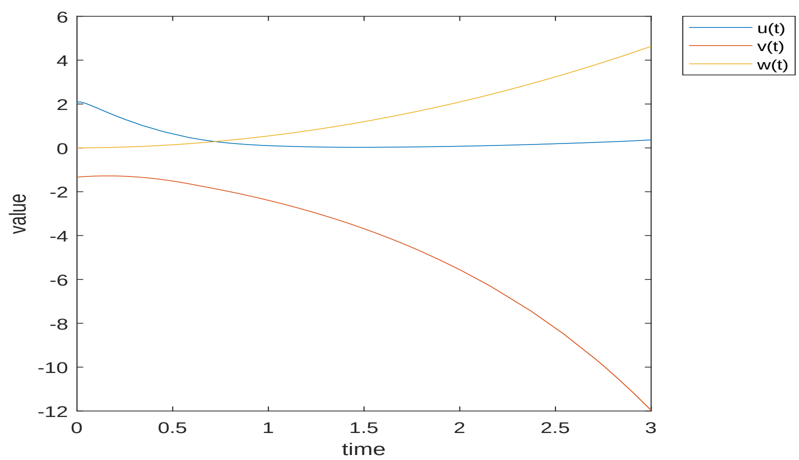

Figure 3. Simulation of solutions in Example 2.

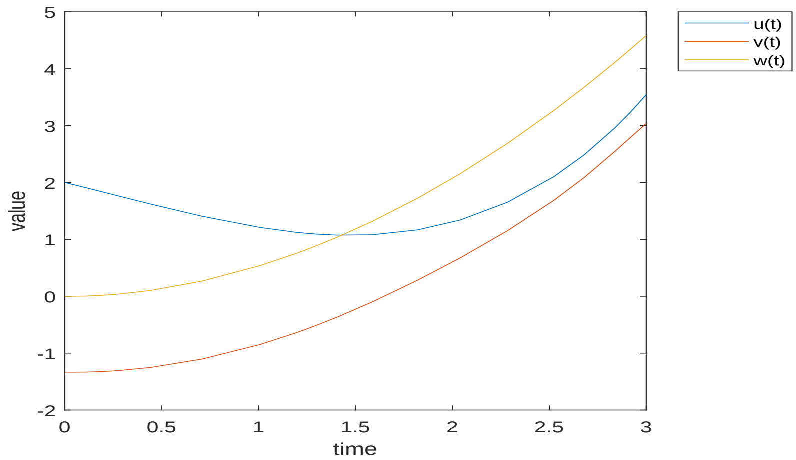

Figure 4. Simulation of solutions of the integer-order equation corresponding to Example 2.

Table 2. Comparison of numerical solutions of $u(t)$ in Example 2.

t	0	0.2	0.4	0.6	0.8	1
Fractional order	2.0000	1.4708	0.8650	0.4482	0.2147	0.1024
Integer order	2.0000	1.8100	1.6190	1.4355	1.2686	1.1276

(c) When all parameter values remain invariant, the simulations of solutions of Example 3 and its corresponding integer-order differential equation are shown as Figures 5 and 6, respectively. The simulations of Ulam–Hyers stability of Example 3 is shown as Figure 7. $u(t)$ is the solution of the Langevin equation in the figures. The comparison of numerical solutions of $u(t)$ is given in Tables 3 and 4. It can be seen from the simulation figures and tables that the solution of the integer-order equation changes sharply, while the fractional-order equation changes gently. When $\epsilon \to 0^+$, the solution curves of Example 3 and inequality (3) almost coincide. This makes clear that Example 3 is UH-stable.

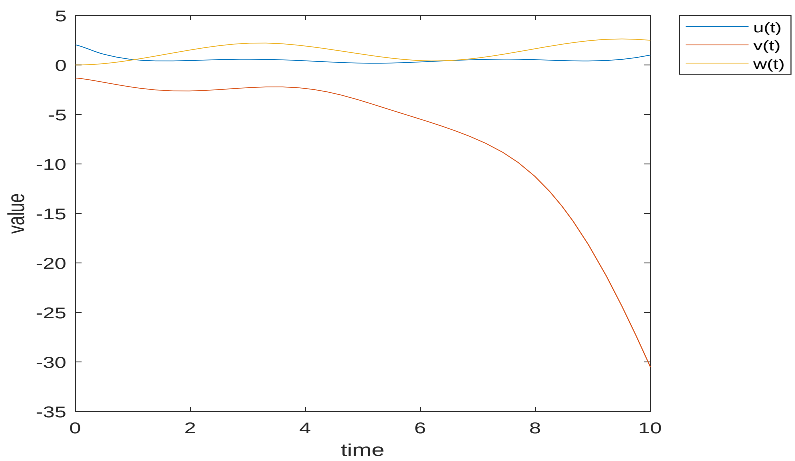

Figure 5. Simulation of solutions in Example 3.

Table 3. Comparison of numerical solutions of $u(t)$ in Example 3.

t	0	0.2	0.4	0.6	0.8	1
Fractional order	2.0000	1.6614	1.2576	0.9385	0.7052	0.5470
Integer order	2.0000	1.8298	1.6924	1.5871	1.5115	1.4627

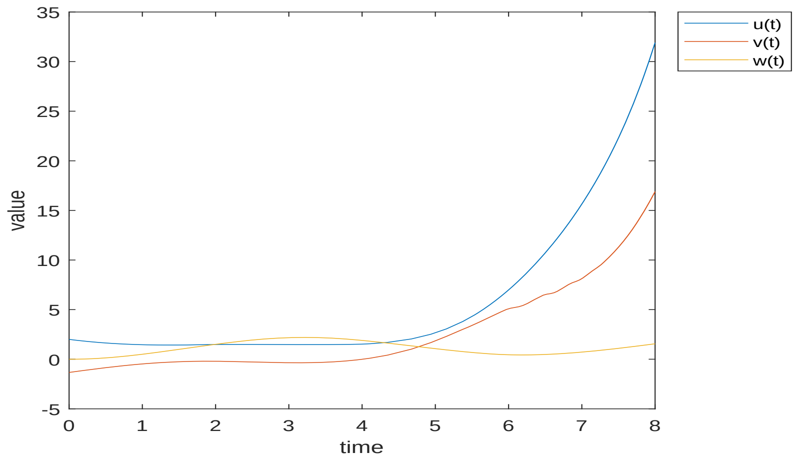

Figure 6. Simulation of solutions of the integer-order equation corresponding to Example 3.

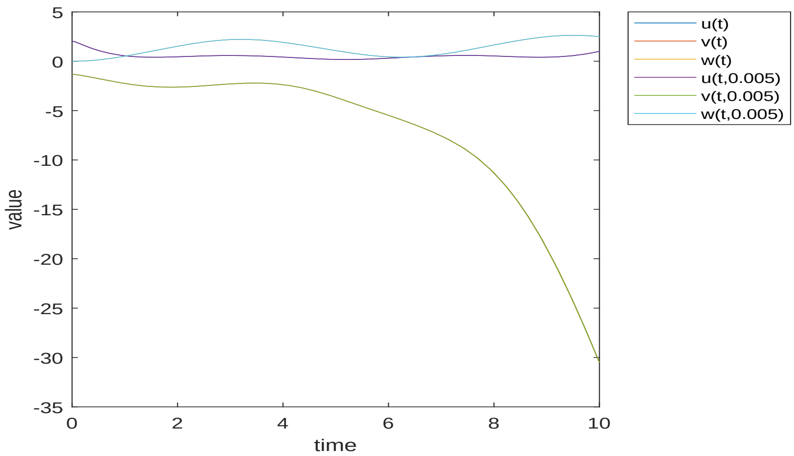

Figure 7. Simulation of UH-stability in Example 3 with $\epsilon = 0, 0.005$.

Table 4. Comparison of numerical solutions for the UH-stability of $u(t)$ in Example 3.

t	0	0.2	0.4	0.6	0.8	1
$\varepsilon = 0$	2.0000	1.6614	1.2576	0.9385	0.7052	0.5470
$\varepsilon = 0.005$	2.0000	1.6611	1.2569	0.9376	0.7042	0.5459

6. Conclusions

It is well known that the Langevin equation is a powerful tool for describing the random motion of particles in fluids. In a particularly complex viscous liquid, the integer-order Langevin equation describing the motion of particles is no longer accurate. Some scholars began to use the fractional Langevin equation as a model to study this problem and achieved good results. Most previous research on the theory and application of fractional differential systems mainly involve the Caputo- or RL-derivative. However, under certain circumstances, the Caputo- or RL-derivative is singular. To overcome this singularity, the ML-type fractional derivative is introduced. In this manuscript, we investigate the existence, uniqueness, and UH-type stability of solutions for the nonlinear ML-type fractional Langevin equation (Equation (1)). Theoretical analysis and numerical simulation of some examples verify the correctness and effectiveness of our main results. Compared with the numerical simulation of the integer-order differential equation corresponding to these examples, we find that the fractional-order differential equation is more detailed and accurate than the integer-order differential equation in describing the random motion of free particles. In practical applications, the fractional Langevin system is often affected by impulsive and random effects. Sometimes it is necessary to consider the asymptotic stability of the fractional Langevin system in the sense of Lyapunov. These are not considered in this work and need to be studied in the future.

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Appendix A

Proof of Lemma 2. Let $v(t) = \left[{}^{ML}\mathcal{D}^{\alpha}_{0^+} - \lambda\right]u(t)$, it follows from (1) that

$$\begin{cases} {}^{ML}\mathcal{D}^{\beta}_{0^+}v(t) = f(t, u(t-\sigma_1(t)), w(t-\sigma_2(t))),\ t \in (0,b], \\ {}^{ML}\mathcal{D}^{\alpha}_{0^+}u(t) = \lambda u(t) + v(t),\ t \in (0,b]. \end{cases} \tag{A1}$$

Adequacy.Let $u, w \in C^1[0,b]$, if $(u(t), w(t))$ solves system (1), then, from Lemma 1 and (A1), one has, for $t \in J = [0,b]$,

$$\begin{aligned} v(t) =\ & v(0) + \tfrac{1-\beta}{\mathcal{N}(\beta)}\left[f(t, u(t-\sigma_1(t)), w(t-\sigma_2(t))) - f(0, u(-\sigma_1(0)), w(-\sigma_2(0)))\right] \\ & + \tfrac{\beta}{\mathcal{N}(\beta)\Gamma(\beta)} \textstyle\int_0^t (t-\tau)^{\beta-1} f(\tau, u(\tau-\sigma_1(\tau)), w(\tau-\sigma_2(\tau)))d\tau, \end{aligned} \tag{A2}$$

and

$$\begin{aligned} u(t) =\ & u(0) + \tfrac{1-\alpha}{\mathcal{N}(\alpha)}\left[\lambda[u(t) - u(0)] + [v(t) - v(0)]\right] \\ & + \tfrac{\alpha}{\mathcal{N}(\alpha)\Gamma(\alpha)} \textstyle\int_0^t (t-s)^{\alpha-1}[\lambda u(s) + v(s)]ds. \end{aligned} \tag{A3}$$

Together with (A2) and (A3), $u(0) = \omega_1(0)$ and $v(0) = {}^{ML}\mathcal{D}^{\alpha}_{0^+}u(0) - \lambda u(0) = \omega_3(0) - \lambda\omega_1(0)$, one yields

$$\begin{aligned} u(t) &= \omega_1(0) + \frac{1-\alpha}{\mathcal{N}(\alpha)}\Bigg[\lambda[u(t) - \omega_1(0)] + \frac{1-\beta}{\mathcal{N}(\beta)}[f(t, u(t-\sigma_1(t)), w(t-\sigma_2(t))) \\ &\quad - f(0, \omega_1(-\sigma_1(0)), \omega_2(-\sigma_2(0)))] + \frac{\beta}{\mathcal{N}(\beta)\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1} f(\tau, u(\tau-\sigma_1(\tau)), \\ &\quad w(\tau-\sigma_2(\tau)))d\tau\Bigg] + \frac{\alpha}{\mathcal{N}(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Bigg[(\omega_3(0) - \lambda\omega_1(0)) + \lambda u(s) \\ &\quad + \frac{1-\beta}{\mathcal{N}(\beta)} f(s, u(s-\sigma_1(s)), w(t-\sigma_2(s))) + \frac{\beta}{\mathcal{N}(\beta)\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} \\ &\quad \times f(\tau, u(\tau-\sigma_1(\tau)), w(\tau-\sigma_2(\tau)))d\tau\Bigg]ds \\ &= \left[1 - \frac{\lambda(1-\alpha)}{\mathcal{N}(\alpha)}\right]\omega_1(0) + \frac{\lambda(1-\alpha)}{\mathcal{N}(\alpha)}u(t) + \frac{(1-\alpha)(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)}[f(t, u(t-\sigma_1(t)), w(t-\sigma_2(t))) \\ &\quad - f(0, \omega_1(-\sigma_1(0)), \omega_2(-\sigma_2(0)))] + \frac{(1-\alpha)\beta}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1} f(\tau, u(\tau-\sigma_1(\tau)), \\ &\quad w(\tau-\sigma_2(\tau)))d\tau + \frac{\omega_3(0) - \lambda\omega_1(0)}{\mathcal{N}(\alpha)\Gamma(\alpha)}t^{\alpha} + \frac{\lambda\alpha}{\mathcal{N}(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds \\ &\quad + \frac{\alpha(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, u(s-\sigma_1(s)), w(s-\sigma_2(s)))ds + \frac{\alpha\beta}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\alpha)\Gamma(\beta)} \\ &\quad \times \int_0^t (t-s)^{\alpha-1}\left[\int_0^s (s-\tau)^{\beta-1} f(\tau, u(\tau-\sigma_1(\tau)), w(\tau-\sigma_2(\tau)))d\tau\right]ds. \end{aligned} \tag{A4}$$

To calculate the last quadratic integral of (A4), we exchange the order of integrals to get

$$
\begin{aligned}
&\int_0^t (t-s)^{\alpha-1}\left[\int_0^s (s-\tau)^{\beta-1} f(\tau, u(\tau-\sigma_1(\tau)), w(\tau-\sigma_2(\tau)))d\tau\right]ds \\
=&\int_0^t f(\tau, u(\tau-\sigma_1(\tau)), w(\tau-\sigma_2(\tau)))\left[\int_\tau^t (t-s)^{\alpha-1}(s-\tau)^{\beta-1}ds\right]d\tau \\
=&\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\int_0^t (t-\tau)^{\alpha+\beta-1} f(\tau, u(\tau))d\tau.
\end{aligned}
\tag{A5}
$$

Combining (A4) and (A5), one has

$$
\begin{aligned}
u(t) =&\omega_1(0) + \frac{1}{\Delta}\left\{\frac{\omega_3(0)-\lambda\omega_1(0)}{\mathcal{N}(\alpha)\Gamma(\alpha)}t^\alpha - \frac{(1-\alpha)(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)} f(0, \omega_1(-\sigma_1(0)), \omega(-\sigma_2(0)))\right. \\
&+ \frac{(1-\alpha)(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)} f(t, u(t-\sigma_1(t)), w(t-\sigma_2(t))) + \frac{\lambda\alpha}{\mathcal{N}(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds \\
&+ \frac{(1-\alpha)\beta}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1} f(\tau, u(\tau-\sigma_1(\tau)), w(\tau-\sigma_2(\tau)))d\tau \\
&+ \frac{\alpha(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, u(s-\sigma_1(s)), w(s-\sigma_2(s)))ds \\
&\left.+ \frac{\alpha\beta}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\alpha+\beta)}\int_0^t (t-\tau)^{\alpha+\beta-1} f(\tau, u(\tau-\sigma_1(\tau)), w(\tau-\sigma_2(\tau)))d\tau\right\}.
\end{aligned}
\tag{A6}
$$

Obviously, Equation (A6) is the first equation of (2).

Now, integrating both sides of the second equation of (1) from 0 to t, we have

$$
\begin{aligned}
w(t) =&w(0) + \int_0^t g(s, u(s-\sigma_3(s)), w(s-\sigma_4(s)))ds \\
=&\omega_2(0) + \int_0^t g(s, u(s-\sigma_3(s)), w(s-\sigma_4(s)))ds, \ t \in J.
\end{aligned}
\tag{A7}
$$

When $t \in [-\sigma, 0]$, it is evident that $u(t) = \omega_1(t)$ and $w(t) = \omega_2(t)$ hold. Thus, one has completely derived the nonlinear integral system (2). In other words, $(u(t), w(t))$ solves the nonlinear integral system (2).

Necessity.For $t \in [-\sigma, 0]$, let ${}^{ML}\mathcal{D}^\alpha_{0+}u(t) = \omega_3(t)$, then $u(t) = \omega_1(t)$, $w(t) = \omega_2(t)$ and ${}^{ML}\mathcal{D}^\alpha_{0+}u(t) = \omega_3(t)$ meet with system (2) $\Rightarrow u(t) = \omega_1(t)$, $w(t) = \sigma_2(t)$ and ${}^{ML}\mathcal{D}^\alpha_{0+}u(t) = \omega_3(t)$ meet with system (1). For $t \in J$, $u, w \in C^1[0, b]$, if $(u(t), w(t))$ solves the integral system (2), then, one deduces (A1) and the second equation of (1) by finding the one-order derivative in (A7), β-order ML-derivative in (A2), and α-order ML-derivative of (A3). Thus, for $u, w \in C^1[0, b]$, $(u(t), w(t))$ also solves system (1). This completes the proof. □

Appendix B

Proof of Theorem 1. Based on Lemma 2, for all $U = (u, w) \in \mathbb{X}$, we define two mappings $\mathscr{P}, \mathscr{Q} : \mathbb{X} \to \mathbb{X}$ as follows:

$$
(\mathscr{P}U)(t) = ((\mathscr{P}_1U)(t), (\mathscr{P}_2U)(t)), \ (\mathscr{Q}U)(t) = ((\mathscr{Q}_1U)(t), (\mathscr{Q}_2U)(t)),
\tag{A8}
$$

where

$$
(\mathscr{P}_1U)(t) = \begin{cases} \omega_1(0) + \frac{1}{\Delta}\left\{\frac{\omega_3(0)-\lambda\omega_1(0)}{\mathcal{N}(\alpha)\Gamma(\alpha)}t^\alpha + \frac{\lambda\alpha}{\mathcal{N}(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds\right\}, \ t \in J, \\ \omega_1(t), \ t \in [-\sigma, 0], \end{cases}
\tag{A9}
$$

$$(\mathscr{P}_2 U)(t) = \begin{cases} \omega_2(0), \ t \in J, \\ \omega_2(t), \ t \in [-\sigma, 0], \end{cases} \tag{A10}$$

$$(\mathscr{Q}_1 U)(t) = \begin{cases} \frac{1}{\Delta}\{ -\frac{(1-\alpha)(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)} f(0, \omega_1(-\sigma_1(0)), \omega(-\sigma_2(0))) + \frac{(1-\alpha)(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)} \\ \quad \times f(t, u(t-\sigma_1(t)), w(t-\sigma_2(t))) + \frac{(1-\alpha)\beta}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \\ \quad \times f(s, u(s-\sigma_1(s)), w(s-\sigma_2(s)))ds + \frac{\alpha(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\ \quad \times f(s, u(s-\sigma_1(s)), w(s-\sigma_2(s)))ds + \frac{\alpha\beta}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\alpha+\beta)} \\ \quad \times \int_0^t (t-s)^{\alpha+\beta-1} f(s, u(s-\sigma_1(s)), w(s-\sigma_2(s)))ds\}, \ t \in J, \\ 0, \ t \in [-\sigma, 0], \end{cases} \tag{A11}$$

and

$$(\mathscr{Q}_2 U)(t) = \begin{cases} \int_0^t g(s, u(s-\sigma_3(s)), w(s-\sigma_4(s)))ds, \ t \in J, \\ 0, \ t \in [-\sigma, 0]. \end{cases} \tag{A12}$$

It is easy to see from (A8)–(A12) that $\mathscr{P}U + \mathscr{Q}V \in \mathbb{X}$, $\forall\ U, V \in \mathbb{X}$, that is, condition (i) in Lemma 3 holds. Next, we need to verify that condition (ii) in Lemma 3 also holds. In fact, $\forall\ t \in I, U = (u_1, w_1), V = (u_2, w_2) \in \mathbb{X}$, when $t \in [0, b]$, we have

$$|(\mathscr{P}_1 U)(t) - (\mathscr{P}_1 V)(t)| = \left| \frac{\lambda\alpha}{\Delta\mathcal{N}(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[u_1(s) - u_2(s)]ds \right|$$

$$\begin{aligned} &\leq \frac{\lambda\alpha}{|\Delta|\mathcal{N}(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|u_1(s) - u_2(s)|ds \\ &\leq \frac{\lambda\alpha}{|\Delta|\mathcal{N}(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}ds \cdot \|u_1 - u_2\|_C \\ &= \frac{\lambda t^\alpha}{|\Delta|\mathcal{N}(\alpha)\Gamma(\alpha)} \|u_1 - u_2\|_C \leq \frac{\lambda b^\alpha}{|\Delta|\mathcal{N}(\alpha)\Gamma(\alpha)} \|U - V\| = \kappa\|U - V\|, \end{aligned} \tag{A13}$$

and

$$|(\mathscr{P}_2 U)(t) - (\mathscr{P}_3 V)(t)| = |\omega_2(0) - \omega_2(0)| = 0. \tag{A14}$$

When $t \in [-\sigma, 0]$, we yield

$$|(\mathscr{P}_1 U)(t) - (\mathscr{P}_1 V)(t)| = |\omega_1(t) - \omega_1(t)| = 0. \tag{A15}$$

and

$$|(\mathscr{P}_2 U)(t) - (\mathscr{P}_3 V)(t)| = |\omega_2(t) - \omega_2(t)| = 0. \tag{A16}$$

It follows from (A13)–(A16) that

$$\|\mathscr{P}U - \mathscr{P}V\| \leq \frac{\lambda b^\alpha}{|\Delta|\mathcal{N}(\alpha)\Gamma(\alpha)} \|U - V\| = \kappa\|U - V\|. \tag{A17}$$

In light of (A_4) and (A17), one concludes that $\mathscr{P} : \mathbb{X} \to \mathbb{X}$ is contractive.

Next, we shall apply the Arzelá–Ascoli theorem to prove that $\mathscr{Q} : \mathbb{X} \to \mathbb{X}$ is completely continuous. For all $t \in I = [-\sigma, b], U = (u, v) \in \mathbb{X}$, when $t \in J = [0, b]$, we derive from (A_3) that

$$
\begin{aligned}
|(\mathcal{Q}_1 U)(t)| \le{}& \frac{1}{|\Delta|}\bigg\{\frac{(1-\alpha)(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)}|f(0,\omega_1(-\sigma_1(0)),\omega(-\sigma_2(0)))| + \frac{(1-\alpha)(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)} \\
&\times |f(t,u(t-\sigma_1(t)),w(t-\sigma_2(t)))| + \frac{(1-\alpha)\beta}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} \\
&\times |f(s,u(s-\sigma_1(s)),w(s-\sigma_2(s)))|ds + \frac{\alpha(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \\
&\times |f(s,u(s-\sigma_1(s)),w(s-\sigma_2(s)))|ds + \frac{\alpha\beta}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\alpha+\beta)} \\
&\times \int_0^t (t-s)^{\alpha+\beta-1}|f(s,u(s-\sigma_1(s)),w(s-\sigma_2(s)))|ds\bigg\} \\
\le{}& \frac{1}{|\Delta|}\bigg\{\frac{(1-\alpha)(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)}m_1(t) + \frac{(1-\alpha)(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)}m_1(t) + \frac{(1-\alpha)\beta}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\beta)} \\
&\times \int_0^t (t-s)^{\beta-1}m_1(s)ds + \frac{\alpha(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}m_1(s)ds \\
&+ \frac{\alpha\beta}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}m_1(s)ds\bigg\} \\
\le{}& \frac{\max_{t\in J}\{m_1(t)\}}{|\Delta|}\bigg\{\frac{2(1-\alpha)(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)} + \frac{(1-\alpha)t^\beta}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\beta)} \\
&+ \frac{(1-\beta)t^\alpha}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\alpha)} + \frac{\alpha\beta t^{\alpha+\beta}}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\alpha+\beta+1)}\bigg\} \\
\le{}& \frac{\max_{t\in J}\{m_1(t)\}}{|\Delta|}\bigg\{\frac{2(1-\alpha)(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)} + \frac{(1-\alpha)b^\beta}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\beta)} \\
&+ \frac{(1-\beta)b^\alpha}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\alpha)} + \frac{\alpha\beta b^{\alpha+\beta}}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\alpha+\beta+1)}\bigg\},
\end{aligned} \tag{A18}
$$

and

$$
|(\mathcal{Q}_2 U)(t)| \le \int_0^t |g(s,u(s-\sigma_3(s)),w(s-\sigma_4(s)))|ds \le b\max_{t\in J}\{m_2(t)\}. \tag{A19}
$$

When $t \in [-\sigma, 0]$, we obtain

$$
|(\mathcal{Q}_1 U)(t)| = |(\mathcal{Q}_2 U)(t)| = 0. \tag{A20}
$$

From (A18)–(A20), one knows that $\mathcal{Q} : \mathbb{X} \to \mathbb{X}$ is uniformly bounded.

Furthermore, for all $U = (u, w) \in \mathbb{X}$, $t_1, t_2 \in I = [-\sigma, b]$ with $t_1 < t_2$, we will argue the equicontinuity of operator $\mathcal{Q}$ in three cases. For the convenience of later writing, we denote

$$
F_{uw}(t) = f(t,u(t-\sigma_1(t)),w(t-\sigma_2(t))),\ G_{uw}(t) = g(t,u(t-\sigma_3(t)),w(t-\sigma_4(t))). \tag{A21}
$$

Case 1: When $0 \le t_1 < t_2 \le b$, it follows from (A21), (A_1), (A_2), and (A_3) that

$$
\begin{aligned}
&|(\mathscr{Q}_1 U)(t_2) - (\mathscr{Q}_1 U)(t_1)| = \frac{1}{|\Delta|}\bigg|\frac{(1-\alpha)(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)}[F_{uw}(t_2) - F_{uw}(t_1)] \\
&+ \frac{(1-\alpha)\beta}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\beta)}\bigg[\int_0^{t_2}(t_2-s)^{\beta-1}F_{uw}(s)ds - \int_0^{t_1}(t_1-s)^{\beta-1}F_{uw}(s)ds\bigg] \\
&+ \frac{\alpha(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\alpha)}\bigg[\int_0^{t_2}(t_2-s)^{\alpha-1}F_{uw}(s)ds - \int_0^{t_1}(t_1-s)^{\alpha-1}F_{uw}(s)ds\bigg] \\
&+ \frac{\alpha\beta}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\alpha+\beta)}\bigg[\int_0^{t_2}(t_2-s)^{\alpha+\beta-1}F_{uw}(s)ds - \int_0^{t_1}(t_1-s)^{\alpha+\beta-1}F_{uw}(s)ds\bigg]\bigg| \\
=&\frac{1}{|\Delta|}\bigg|\frac{(1-\alpha)(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)}[F_{uw}(t_2) - F_{uw}(t_1)] + \frac{(1-\alpha)\beta}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\beta)} \\
&\times\bigg[\int_{t_1}^{t_2}(t_2-s)^{\beta-1}F_{uw}(s)ds + \int_0^{t_1}\big[(t_2-s)^{\beta-1} - (t_1-s)^{\beta-1}\big]f(s,u(s))ds\bigg] \\
&+ \frac{\alpha(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\alpha)}\bigg[\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}F_{uw}(s)ds + \int_0^{t_1}\big[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\big] \\
&\times F_{uw}(s)ds\bigg] + \frac{\alpha\beta}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\alpha+\beta)}\bigg[\int_{t_1}^{t_2}(t_2-s)^{\alpha+\beta-1}F_{uw}(s)ds \\
&+ \int_0^{t_1}\big[(t_2-s)^{\alpha+\beta-1} - (t_1-s)^{\alpha+\beta-1}\big]F_{uw}(s)ds\bigg]\bigg| \\
\le&\frac{1}{|\Delta|}\bigg\{\frac{(1-\alpha)(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)}|F_{uw}(t_2) - F_{uw}(t_1)| + \frac{(1-\alpha)\beta}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\beta)} \\
&\times\bigg[\int_{t_1}^{t_2}(t_2-s)^{\beta-1}|F_{uw}(s)|ds + \int_0^{t_1}\big|(t_2-s)^{\beta-1} - (t_1-s)^{\beta-1}\big|\cdot|F_{uw}(s)|ds\bigg] \\
&+ \frac{\alpha(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\alpha)}\bigg[\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}|F_{uw}(s)|ds + \int_0^{t_1}\big|(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\big| \\
&\times|F_{uw}(s)|ds\bigg] + \frac{\alpha\beta}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\alpha+\beta)}\bigg[\int_{t_1}^{t_2}(t_2-s)^{\alpha+\beta-1}|F_{uw}(s)|ds \\
&+ \int_0^{t_1}\big|(t_2-s)^{\alpha+\beta-1} - (t_1-s)^{\alpha+\beta-1}\big||F_{uw}(s)|ds\bigg]\bigg\} \\
\le&\frac{1}{|\Delta|}\bigg\{\frac{(1-\alpha)(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)}|F_{uw}(t_2) - F_{uw}(t_1)| + \frac{\max_{t\in J}\{m_1(t)\}(1-\alpha)\beta}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\beta)}\bigg[\int_{t_1}^{t_2}(t_2-s)^{\beta-1}ds \\
&+ \int_0^b\big|(t_2-s)^{\beta-1} - (t_1-s)^{\beta-1}\big|ds\bigg] + \frac{\max_{t\in J}\{m_1(t)\}\alpha(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\alpha)}\bigg[\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds \\
&+ \int_0^b\big|(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\big|ds\bigg] + \frac{\max_{t\in J}\{m_1(t)\}\alpha\beta}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\alpha+\beta)}\bigg[\int_{t_1}^{t_2}(t_2-s)^{\alpha+\beta-1}ds \\
&+ \int_0^b\big|(t_2-s)^{\alpha+\beta-1} - (t_1-s)^{\alpha+\beta-1}\big|ds\bigg]\bigg\} \\
=&\frac{1}{|\Delta|}\bigg\{\frac{(1-\alpha)(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)}|F_{uw}(t_2) - F_{uw}(t_1)| + \frac{\max_{t\in J}\{m_1(t)\}(1-\alpha)\beta}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\beta)}\bigg[\frac{1}{\beta}(t_2-t_1)^{\beta} \\
&+ \int_0^b\big|(t_2-s)^{\beta-1} - (t_1-s)^{\beta-1}\big|ds\bigg] + \frac{\max_{t\in J}\{m_1(t)\}\alpha(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\alpha)}\bigg[\frac{1}{\alpha}(t_2-t_1)^{\alpha} \\
&+ \int_0^b\big|(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\big|ds\bigg] + \frac{\max_{t\in J}\{m_1(t)\}\alpha\beta}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\alpha+\beta)}\bigg[\frac{1}{\alpha+\beta}(t_2-t_1)^{\alpha+\beta-1} \\
&+ \int_0^b\big|(t_2-s)^{\alpha+\beta-1} - (t_1-s)^{\alpha+\beta-1}\big|ds\bigg]\bigg\} \to 0, \quad \text{as} \quad t_2 \to t_1,
\end{aligned}
\tag{A22}
$$

and

$$|(\mathscr{Q}_2U)(t_2)-(\mathscr{Q}_2U)(t_1)| = \left|\int_0^{t_2} G_{uw}(s)ds - \int_0^{t_1} G_{uw}(s)ds\right| = \left|\int_{t_1}^{t_2} G_{uw}(s)ds\right|$$
$$\leq \int_{t_1}^{t_2} |G_{uw}(s)|ds \leq \max_{t\in J}\{m_2(t)\}(t_2-t_1) \to 0, \quad \text{as} \quad t_2 \to t_1. \tag{A23}$$

Case 2: When $-\sigma \leq t_1 < 0 < t_2 \leq b$, then $t_2 \to t_1$ means that $t_1 \to 0^-$ and $t_2 \to 0^+$. From (A11) and (A12), we obtain

$$(\mathscr{Q}U)(t_2)-(\mathscr{Q}U)(t_1) \to (\mathscr{Q}U)(0^+)-(\mathscr{Q}U)(0^-) = (0,0), \quad \text{as} \quad t_2 \to t_1. \tag{A24}$$

Case 3:When $-\sigma \leq t_1 < t_2 \leq 0$, then

$$(\mathscr{Q}U)(t_2)-(\mathscr{Q}U)(t_1) \equiv (0,0)-(0,0) = (0,0) \to (0,0), \quad \text{as} \quad t_2 \to t_1. \tag{A25}$$

From (A22)–(A25), we conclude that, $\forall\, \epsilon > 0$ (small enough), $\exists\, \zeta = \zeta(\epsilon) > 0$, for all $t_1, t_2 \in I$, $u \in \mathbb{X}$, when $|t_2 - t_1| < \zeta$, there has $\|(\mathscr{Q}u)(t_2)-(\mathscr{Q}u)(t_1)\| < \epsilon$, namely, $\mathscr{Q}: \mathbb{X} \to \mathbb{X}$ is equicontinuous. Thereby, according to Lemmas 2 and 3, one asserts that there has at least a fixed point $U^*(t) = (u^*(t), w^*(t)) \in \mathbb{X}$ such that $U^*(t) = (\mathscr{P}U^*)(t) + (\mathscr{Q}U^*)(t)$, which meets with system (1). This completes the proof. □

Appendix C

Proof of Throrem 2. According to Lemma 2, a self-mapping $\mathscr{T}$ on $\mathbb{X}$ is defined as

$$(\mathscr{T}U)(t) = ((\mathscr{T}_1U)(t), (\mathscr{T}_2U)(t)), \quad \forall\, t \in I = [-\sigma, b],\ U = (u,w) \in \mathbb{X}, \tag{A26}$$

$$(\mathscr{T}_1U)(t) = \begin{cases} \omega_1(0) + \frac{1}{\Delta}\Big\{\frac{\omega_3(0)-\lambda\omega_1(0)}{\mathcal{N}(\alpha)\Gamma(\alpha)}t^\alpha - \frac{(1-\alpha)(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)}F_{uw}(0) \\ \quad + \frac{(1-\alpha)(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)}F_{uw}(t) + \frac{\lambda\alpha}{\mathcal{N}(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds \\ \quad + \frac{(1-\alpha)\beta}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}F_{uw}(s)ds \\ \quad + \frac{\alpha(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F_{uw}(s)ds \\ \quad + \frac{\alpha\beta}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}F_{uw}(s)ds\Big\},\ t \in J, \\ \omega_1(t),\ t \in [-\sigma, 0], \end{cases} \tag{A27}$$

$$(\mathscr{T}_2U)(t) = \begin{cases} \omega_2(0) + \int_0^t G_{uw}(s)ds,\ t \in J, \\ \omega_2(t),\ t \in [-\sigma, 0], \end{cases} \tag{A28}$$

where $F_{uw}(t)$ and $G_{uw}(t)$ are denoted as (A21).

Take $\Omega = \{U = (u,w) \in \mathbb{X} : \|U\| < R\}$, where $R = \max\{\|\omega_1\|_\sigma, \|\omega_2\|_\sigma, \frac{\Sigma_1}{1-\Theta_1}, \frac{\Sigma_2}{1-\Theta_2}\}$, $\Sigma_1 = \|\omega_1\|_\sigma + \frac{\|\omega_3\|_\sigma+\lambda\|\omega_1\|_\sigma}{|\Delta|\mathcal{N}(\alpha)\Gamma(\alpha)}b^\alpha + \frac{\|M_1\|_b}{|\Delta|\mathcal{N}(\alpha)\mathcal{N}(\beta)}\Big[2(1-\alpha)(1-\beta) + \frac{(1-\alpha)b^\beta}{\Gamma(\beta)} + \frac{(1-\beta)b^\alpha}{\Gamma(\alpha)} + \frac{\alpha\beta b^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big]$ and $\Sigma_2 = \|\omega_2\|_\sigma + b\|N_1\|_b$. Now let us prove that $\mathscr{T}(\overline{\Omega}) \subset \overline{\Omega}$. In fact, for all $U = (u,w) \in \overline{\Omega}$, namely, $\|U\| \leq R$, when $t \in [-\sigma, 0]$, we have

$$|(\mathscr{T}_1U)(t)| = |\omega_1(t)| \leq \|\omega_1\|_\sigma \leq R,\ |(\mathscr{T}_2U)(t)| = |\omega_2(t)| \leq \|\omega_2\|_\sigma \leq R. \tag{A29}$$

When $t \in [0, b]$, it follows from (A_5) that

$$|F_{uw}(t)| \leq M_1(t) + M_2(t)|u(t-\sigma_1(t))| + M_3(t)|w(t-\sigma_2(t))|$$
$$\leq \|M_1\|_b + R\|M_2\|_b + R\|M_3\|_b, \tag{A30}$$

and

$$\begin{aligned}|G_{uw}(t)| \le& N_1(t)+N_2(t)|u(t-\sigma_3(t))|+N_3(t)|w(t-\sigma_4(t))|\\ \le& \|N_1\|_b+R\|N_2\|_b+R\|N_3\|_b.\end{aligned} \tag{A31}$$

By (A30), (A31), (A_1), (A_2), and (A_6), we derive

$$\begin{aligned}|(\mathscr{T}_1U)(t)| \le& |\omega_1(0)|+\frac{1}{|\Delta|}\Big\{\frac{|\omega_3(0)|+\lambda|\omega_1(0)|}{\mathcal{N}(\alpha)\Gamma(\alpha)}b^\alpha+\frac{(1-\alpha)(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)}|F_{uw}(0)|\\
&+\frac{(1-\alpha)(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)}|F_{uw}(t)|+\frac{\lambda\alpha}{\mathcal{N}(\alpha)\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|u(s)|ds\\
&+\frac{(1-\alpha)\beta}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}|F_{uw}(s)|ds+\frac{\alpha(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|F_{uw}(s)|ds\\
&+\frac{\alpha\beta}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}|F_{uw}(s)|ds\Big\}\\
\le& \|\omega_1\|_\sigma+\frac{1}{|\Delta|}\Big\{\frac{\|\omega_3\|_\sigma+\lambda\|\omega_1\|_\sigma}{\mathcal{N}(\alpha)\Gamma(\alpha)}b^\alpha+\frac{(1-\alpha)(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)}\Big[\|M_1\|_b+R\|M_2\|_b+R\|M_3\|_b\Big]\\
&+\frac{(1-\alpha)(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)}\Big[\|M_1\|_b+R\|M_2\|_b+R\|M_3\|_b\Big]+\frac{\lambda\alpha}{\mathcal{N}(\alpha)\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}Rds\\
&+\frac{(1-\alpha)\beta}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}\Big[\|M_1\|_b+R\|M_2\|_b+R\|M_3\|_b\Big]ds\\
&+\frac{\alpha(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\Big[\|M_1\|_b+R\|M_2\|_b+R\|M_3\|_b\Big]ds\\
&+\frac{\alpha\beta}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}\Big[\|M_1\|_b+R\|M_2\|_b+R\|M_3\|_b\Big]ds\Big\}\\
\le& \|\omega_1\|_\sigma+\frac{1}{|\Delta|}\Big\{\frac{\|\omega_3\|_\sigma+\lambda\|\omega_1\|_\sigma}{\mathcal{N}(\alpha)\Gamma(\alpha)}b^\alpha+\frac{2(1-\alpha)(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)}\Big[\|M_1\|_b+R\|M_2\|_b+R\|M_3\|_b\Big]\\
&+\frac{\lambda b^\alpha R}{\mathcal{N}(\alpha)\Gamma(\alpha)}+\frac{(1-\alpha)b^\beta}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\beta)}\Big[\|M_1\|_b+R\|M_2\|_b+R\|M_3\|_b\Big]\\
&+\frac{(1-\beta)b^\alpha}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\alpha)}\Big[\|M_1\|_b+R\|M_2\|_b+R\|M_3\|_b\Big]\\
&+\frac{\alpha\beta b^{\alpha+\beta}}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\alpha+\beta+1)}\Big[\|M_1\|_b+R\|M_2\|_b+R\|M_3\|_b\Big]\Big\}\\
=& \frac{\|\omega_3\|_\sigma+\lambda\|\omega_1\|_\sigma}{|\Delta|\mathcal{N}(\alpha)\Gamma(\alpha)}b^\alpha+\frac{\|M_1\|_b}{|\Delta|\mathcal{N}(\alpha)\mathcal{N}(\beta)}\Big[2(1-\alpha)(1-\beta)+\frac{(1-\alpha)b^\beta}{\Gamma(\beta)}+\frac{(1-\beta)b^\alpha}{\Gamma(\alpha)}\\
&+\frac{\alpha\beta b^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big]+\Big\{\frac{\lambda b^\alpha}{|\Delta|\mathcal{N}(\alpha)\Gamma(\alpha)}+\frac{\|M_2\|_b+\|M_3\|_b}{|\Delta|\mathcal{N}(\alpha)\mathcal{N}(\beta)}\Big[2(1-\alpha)(1-\beta)\\
&+\frac{(1-\alpha)b^\beta}{\Gamma(\beta)}+\frac{(1-\beta)b^\alpha}{\Gamma(\alpha)}+\frac{\alpha\beta b^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big]\Big\}R=\Sigma_1+\Theta_1R\le R,\end{aligned} \tag{A32}$$

and

$$\begin{aligned}|(\mathscr{T}_2U)(t)|=&\left|\omega_2(0)+\int_0^t G_{uw}(s)ds\right|\le|\omega_2(0)|+\int_0^t|G_{uw}(s)|ds\\ \le&\|\omega_2\|_\sigma+b\big[\|N_1\|_b+R\|N_2\|_b+R\|N_3\|_b\big]=\Sigma_2+\Theta_2R\le R.\end{aligned} \tag{A33}$$

From (A29), (A32) and (A33), one has $\|(\mathscr{T}U)(t)\|\le R$, which implies that $(\mathscr{T}U)(t)\in\overline{\Omega}$, and $\mathscr{T}$ is uniformly bounded in $\overline{\Omega}$. Similar to (A22)–(A25) in the proof of Theorem 1, it is easy to prove that $\mathscr{T}:\overline{\Omega}\to\overline{\Omega}$ is equicontinuous, so we omit it. Therefore, the complete continuity of $\mathscr{T} : \overline{\Omega} \to \overline{\Omega}$ is established by the Arzelá–Ascoli theorem. Thus, it follows from Lemmas 2 and 4 that the mapping $\mathscr{T}$ has a fixed point $U^* = (u^*, w^*) \in \overline{\Omega}$, which meets with system (1). This completes the proof. □

Appendix D

Proof of Theorem 3. Firstly, we utilize Lemma 5 to prove claim (a). Define a mapping $\mathscr{T} : \mathbb{X} \to \mathbb{X}$ as (A26)–(A28), then, for all $U_1 = (u_1, w_1), U_2 = (u_2, w_2) \in \mathbb{X}$, when $t \in J = [-\sigma, 0]$, one has

$$|(\mathscr{T}_1 U_1)(t) - (\mathscr{T}_1 U_2)(t)| = |\omega_1(t) - \omega_1(t)| = 0, \tag{A34}$$

$$|(\mathscr{T}_2 U_1)(t) - (\mathscr{T}_2 U_2)(t)| = |\omega_2(t) - \omega_2(t)| = 0. \tag{A35}$$

When $t \in J = [0, b]$, similar to (A30) and (A31), one derives from (A_1), (A_2), and (A_7) that

$$|F_{u_1 w_1}(t) - F_{u_2 w_2}(t)| \le \left[\|a_1\|_b + \|a_2\|_b\right] \|U_1 - U_2\|, \tag{A36}$$

$$|G_{u_1 w_1}(t) - G_{u_2 w_2}(t)| \le \left[\|b_1\|_b + \|b_2\|_b\right] \|U_1 - U_2\|. \tag{A37}$$

From (A36) and (A37), one gets

$$\begin{aligned}
&|(\mathscr{T}_1 U_1)(t) - (\mathscr{T}_1 U_2)(t)| \le \frac{1}{|\Delta|}\left\{ \frac{(1-\alpha)(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)} |F_{u_1 w_1}(0) - F_{u_2 w_2}(0)| \right. \\
&+ \frac{(1-\alpha)(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)} |F_{u_1 w_1}(t) - F_{u_2 w_2}(t)| \\
&+ \frac{\lambda\alpha}{\mathcal{N}(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |u_1(s) - u_2(s)| ds \\
&+ \frac{(1-\alpha)\beta}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} |F_{u_1 w_1}(s) - F_{u_2 w_2}(s)| ds \\
&+ \frac{\alpha(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |F_{u_1 w_1}(s) - F_{u_2 w_2}(s)| ds \\
&\left. + \frac{\alpha\beta}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} |F_{u_1 w_1}(s) - F_{u_2 w_2}(s)| ds \right\} \\
\le & \frac{1}{|\Delta|}\left\{ \frac{(1-\alpha)(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)} \left[\|a_1\|_b + \|a_2\|_b\right] \|U_1 - U_2\| \right. \\
&+ \frac{(1-\alpha)(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)} \left[\|a_1\|_b + \|a_2\|_b\right] \|U_1 - U_2\| \\
&+ \frac{\lambda\alpha}{\mathcal{N}(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \cdot \|U_1 - U_2\| \\
&+ \frac{(1-\alpha)\beta}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} ds \cdot \left[\|a_1\|_b + \|a_2\|_b\right] \|U_1 - U_2\| \\
&+ \frac{\alpha(1-\beta)}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \cdot \left[\|a_1\|_b + \|a_2\|_b\right] \|U_1 - U_2\| \\
&\left. + \frac{\alpha\beta}{\mathcal{N}(\alpha)\mathcal{N}(\beta)\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} ds \cdot \left[\|a_1\|_b + \|a_2\|_b\right] \|U_1 - U_2\| \right\} \\
\le & \left\{ \frac{\lambda b^\alpha}{|\Delta|\mathcal{N}(\alpha)\Gamma(\alpha)} + \frac{\|a_1\|_b + \|a_2\|_b}{|\Delta|\mathcal{N}(\alpha)\mathcal{N}(\beta)} \left[2(1-\alpha)(1-\beta) + \frac{(1-\alpha)b^\beta}{\Gamma(\beta)} \right.\right. \\
&\left.\left. + \frac{(1-\beta)b^\alpha}{\Gamma(\alpha)} + \frac{\alpha\beta b^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \right] \right\} \|U_1 - U_2\| = \Upsilon \|U_1 - U_2\|,
\end{aligned} \tag{A38}$$

and

$$|(\mathscr{T}_2U_1)(t)-(\mathscr{T}_2U_2)(t)|\le\int_0^t|G_{u_1w_1}(s)-G_{u_2w_2}(s)|ds\le b\big(\|b_1\|_b+\|b_2\|_b\big)\|U_1-U_2\|. \tag{A39}$$

Taking $k=\max\{\Upsilon_1,\Upsilon_2\}$, we derive from (A34), (A35), (A38) and (A39) that

$$\|\mathscr{T}U_1-\mathscr{T}U_2\|\le k\|U_1-U_2\|. \tag{A40}$$

By (A_8) and (A40), one knows that $\mathscr{T}:\mathbb{X}\to\mathbb{X}$ is contractive. Thus it follows from Lemmas 2 and 5 that the operator $\mathscr{T}$ has a unique fixed point $U^*(t)=(u^*(t),w^*(t))\in\mathbb{X}$, which is a unique solution of system (1).

Next, we shall prove claim (b). Define the operator $\mathscr{T}:\mathbb{X}\to\mathbb{X}$ as (A26)–(A28). From claim (a), system (1) exists a unique solution $U^*(t)=(u^*(t),w^*(t)\in\mathbb{X}$ satisfying $\mathscr{T}U^*=U^*$. Let $\theta(t)\equiv(0,0)$ and $X(t)=(x(t),y(t))\in\mathbb{X}$ be a solution of inequality (3). Based upon Remark 5 and Lemma 2, we easily verify that $X(t)=(x(t),y(t))$ satisfies the operator equation as below:

$$X(t)=(\mathscr{T}X)(t)+(\mathscr{T}\phi)(t)-(\mathscr{T}\theta)(t). \tag{A41}$$

As an analogue to (A40), one derives from (A41) and (A_8) that

$$\begin{aligned}\|X-U^*\|&=\|(\mathscr{T}X)(t)-(\mathscr{T}U^*)(t)+(\mathscr{T}\phi)(t)-(\mathscr{T}\theta)(t)\|\\&\le\|(\mathscr{T}X)(t)-(\mathscr{T}U^*)(t)\|+\|(\mathscr{T}\phi)(t)-(\mathscr{T}\theta)(t)\|\\&\le k\|X-U^*\|+k\|\phi-\theta\|\le k\|X-U^*\|+k\epsilon,\end{aligned} \tag{A42}$$

which implies that

$$\|X-U^*\|\le\frac{k}{1-k}\epsilon. \tag{A43}$$

Thus, (A43) shows that system (1) is UH-stable and also generalized UH-stable. The proof of claim (c) resembles claim (b), so we omit it. The proof is completed. □

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