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New Results for Weakly Compatible (WC) and R-Weakly Commuting (RWC) Mappings with an Application in Dynamic Programming

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Abstract: The aim of this paper is to obtain some new results about common fixed points. Our results use weaker conditions than those previously used. We have relaxed the conditions for commutating pair mappings and compatible mappings of the type (A), which were introduced in 1976. The theorems are enriched by using the concept of WC and various types of weakly commuting pairs of maps in metric spaces. To discuss the existence and uniqueness of the common solutions, we have obtained an application to the functional equations in dynamic programming.

Keywords: weak contraction; weakly compatible (WC) mappings; R-weakly commuting (RWC) mappings; dynamic programming (DP)

MSC: 47H10; 54H25

1. Introduction and Preliminaries

In the literature of metric fixed point theory, the Poincaré concept was introduced by Banach [1] in a metric space called contraction mapping, and this was the first result after Brouwer [2] in which a fixed point of the contraction map was unique. The main beauty of the Banach fixed point theorem is the richness of the hypothesis and the elegant proof of the theorem. One can observe that the metric fixed point theory has immense applications in the fields of financial economics, medical sciences (for most approximate diagnoses of medicine), defense (missile technology to penetrate the target accurately), and various branches of mathematical and computational sciences.

Jungck [3] was perhaps the first to utilize the concept of commutative pairs of mappings for obtaining a unique fixed point by generalizing the contraction condition introduced by Banach.

Later, it was quite natural to ask a question among the researchers: Does there exist any condition weaker than commuting pairs of maps? The answer given by Sessa [4] in the year 1982 was in the affirmative by generalizing the commuting pair of maps and introduced the weakly commuting pair of maps in a metric space. A pair of self-mappings (Ω, Υ) on a metric space $(\mho, \rho)$ is said to be weakly commuting [4] if $\rho(\Omega\Upsilon\kappa, \Upsilon\Omega\kappa) \leq \rho(\Upsilon\kappa, \Omega\kappa)$ for all $\kappa \in \mho$. Furthermore, in 1986, Jungck [5] defined more generalized commutativity, known as compatibility. A pair of self-mappings (Ω, Υ) on a metric space $(\mho, \rho)$ is said to be compatible [5] if $\lim_{n\to+\infty} \rho(\Omega\Upsilon\kappa_n, \Upsilon\Omega\kappa_n) = 0$ whenever $\{\kappa_n\}$ is a sequence in $\mho$ such that $\lim_{n\to+\infty} \Omega\kappa_n = \lim_{n\to+\infty} \Upsilon\kappa_n = \kappa$ for some κ in $\mho$.

Most of the common fixed point results of compatible mappings and its variants require the following:

(1) Continuity of one of the maps under consideration;
(2) Containment of the range spaces;
(3) Completeness of the spaces or range spaces.

In 1996, Jungck [6] extended the notion of compatible mappings to a larger class of mappings known as *WC*. Let Ω and Υ be two mappings from a metric space $(\mho, \rho)$ into itself. If Ω and Υ commute at their coincidence point (i.e., if $\Omega\kappa = \Upsilon\kappa$ for some $\kappa \in \mho$ implies $\Omega\Upsilon\kappa = \Upsilon\Omega\kappa$), then Ω and Υ are called *WC* [6]. In 1994, Pant [7] introduced the notion of *RWC* mappings in metric spaces first to widen the scope of the study of common fixed point theorems from the class of compatibility to the wider class of *RWC* mappings. Secondly, the maps are not necessarily continuous at the fixed point. A pair of self-mappings (Ω, Υ) on a metric space $(\mho, \rho)$ is said to be *RWC* [7] if there exists some $R \geq 0$ such that $\rho(\Omega\Upsilon\kappa, \Upsilon\Omega\kappa) \leq R\rho(\Omega\kappa, \Upsilon\kappa)$ for all $\kappa \in \mho$.

In 1997, Pathak et al. [8] introduced the improved notions of *RWC* mappings and called these maps *RWC* mappings of the type (A_Ω) and *RWC* mappings of the type (A_Υ):

Definition 1 ([8])**.** *A pair of self-mappings (Ω, Υ) on a metric space $(\mho, \rho)$ is said to be the following:*

(1) RWC mappings of the type (A_Ω) if there exists some $R > 0$ such that $\rho(\Omega\Upsilon\kappa, \Upsilon\Upsilon\kappa) \leq R\rho(\Omega\kappa, \Upsilon\kappa)$ for all $\kappa \in \mho$;
(2) RWC mappings of the type (A_Υ) if there exists some $R > 0$ such that $\rho(\Upsilon\Omega\kappa, \Omega\Omega\kappa) \leq R\rho(\Omega\kappa, \Upsilon\kappa)$ for all $\kappa \in \mho$;

In 2009, Kumar et al. [9] introduced the notion of *RWC* mappings of the type (P) as follows:

Definition 2 ([9])**.** *A pair of self-mappings (Ω, Υ) on a metric space $(\mho, \rho)$ is said to be RWC mappings of the type (P) if there exists some $R > 0$ such that $\rho(\Omega\Omega\kappa, \Upsilon\Upsilon\kappa) \leq R\rho(\Omega\kappa, \Upsilon\kappa)$ for all $\kappa \in \mho$.*

Example 1. *Let $\mho = [-2, 2]$ and ρ be an usual metric on $\mho$. We define the self-mappings Ω and Υ on a metric space $(\mho, \rho)$ as*

$$\Omega(\kappa) = |\kappa| \text{ and } \Upsilon(\kappa) = |\kappa| - 2.$$

Then, $\rho(\Omega\kappa, \Upsilon\kappa) = 2$, $\rho(\Omega\Upsilon\kappa, \Upsilon\Omega\kappa) = 2(2 - |\kappa|)$, $\rho(\Omega\Upsilon\kappa, \Upsilon\Upsilon\kappa) = 2$, $\rho(\Upsilon\Omega\kappa, \Omega\Omega\kappa) = 2$, and $\rho(\Omega\Omega\kappa, \Upsilon\Upsilon\kappa) = 2|\kappa|$ for all κ in $\mho$.

From Example 1, we have the following:

(1) The pair (Ω, Υ) is not weakly commuting;
(2) For $R = 2$, the pair (Ω, Υ) is *RWC*, *RWC* of the type (A_Ω), *RWC* of the type (A_Υ) and *RWC* of the type (P);
(3) For $R = \frac{3}{2}$, the pair (Ω, Υ) is *RWC* of the type (A_Ω) but not *RWC* of the types (P) or *RWC*.

For the results for common fixed points, see [10–17]. Now, we are ready to establish some common fixed point theorems in metric spaces by using *WC* and *RWC* pairs of maps which are weaker than the variants of weak commuting pairs of maps in metric spaces and other abstract spaces. The results in this paper are new, and other published papers do not cover them.

2. Main Results

In 2021, Kumar et al. [18] introduced a new weak contraction that involves the cubic terms of a distance function and proved the common fixed point theorems for compatible mappings and their variants:

Theorem 1 ([18]). *Let ξ, ζ, Ω, and Υ be four mappings of a complete metric space $(\mho, \rho)$ in itself satisfying the following conditions:*
(C1) $\xi(\mho) \subset \Upsilon(\mho), \zeta(\mho) \subset \Omega(\mho)$;

$$\begin{aligned}(C2)\ \ \rho^3(\xi\kappa, \zeta\omega) \leq & p\max\{\frac{1}{2}[\rho^2(\Omega\kappa, \xi\kappa)\rho(\Upsilon\omega, \zeta\omega) + \rho(\Omega\kappa, \xi\kappa)\rho^2(\Upsilon\omega, \zeta\omega)], \\ & \rho(\Omega\kappa, \xi\kappa)\rho(\Omega\kappa, \zeta\omega)\rho(\Upsilon\omega, \xi\kappa), \rho(\Omega\kappa, \zeta\omega)\rho(\Upsilon\omega, \xi\kappa)\rho(\Upsilon\omega, \zeta\omega)\} \\ & - \phi(m(\Omega\kappa, \Upsilon\omega)),\end{aligned}$$

for all $\kappa, \omega \in \mho$, where

$$\begin{aligned}m(\Omega\kappa, \Upsilon\omega) = \max\{ & \rho^2(\Omega\kappa, \Upsilon\omega), \rho(\Omega\kappa, \xi\kappa)\rho(\Upsilon\omega, \zeta\omega), \\ & \rho(\Omega\kappa, \zeta\omega)\rho(\Upsilon\omega, \xi\kappa), \\ & \frac{1}{2}[\rho(\Omega\kappa, \xi\kappa)\rho(\Omega\kappa, \zeta\omega) + \rho(\Upsilon\omega, \xi\kappa)\rho(\Upsilon\omega, \zeta\omega)]\}\end{aligned}$$

In addition, p is a real number satisfying $0 < p < 1$ and a continuous function $\phi : [0, +\infty) \to [0, +\infty)$ with $\phi(0) = 0$ and $\phi(s) > 0$ for $s > 0$;
(C3) One of ξ, ζ, Ω, or Υ is continuous.

Suppose that the pairs (ξ, Ω) and (ζ, Υ) are type-(A) compatible, type-(B) compatible, type-(C) compatible, or type-(P) compatible. Then, ξ, ζ, Ω and Υ have a unique common fixed point in $\mho$.

Now, we extend Theorem 1 from the class of compatible mappings to a larger class of mappings having weak compatibility without appealing to the continuity:

Theorem 2. *Let ξ, ζ, Ω, and Υ be four self-mappings on a metric space $(\mho, \rho)$ satisfying $(C1)$, $(C2)$, and the following condition:*
$(C4)$ One of the subspaces $\xi\mho$, $\zeta\mho$, $\Omega\mho$, or $\Upsilon\mho$ is complete.

Then ξ, ζ, Ω, and Υ have a unique common fixed point, provided that the pairs (ξ, Ω) and (ζ, Υ) are WC.

Proof. Let $\kappa_0 \in \mho$ be an arbitrary point. From $(C1)$, we can find κ_1 such that $\xi(\kappa_0) = \Upsilon(\kappa_1) = \omega_0$. For this κ_1, one can find $\kappa_2 \in \mho$ such that $\zeta(\kappa_1) = \Omega(\kappa_2) = \omega_1$. By continuing in this way, one can construct a sequence $\{\omega_n\}$ such that

$$\omega_{2n} = \xi(\kappa_{2n}) = \Upsilon(\kappa_{2n+1}), \quad \omega_{2n+1} = \zeta(\kappa_{2n+1}) = \Omega(\kappa_{2n+2}) \text{ for each } n \geq 0.$$

From the proof of Theorem 1 [18], $\{\omega_n\}$ is a Cauchy sequence.

Let $\Omega(\mho)$ be complete subspace of $\mho$. Then, there exist $\eta \in \mho$ such that

$$\omega_{2n+1} = \zeta(\kappa_{2n+1}) = \Omega(\kappa_{2n+2}) \to \eta \text{ as } n \to +\infty.$$

Accordingly, we can find $\vartheta \in \mho$ such that $\Omega\vartheta = \eta$. A Cauchy sequence $\{\omega_n\}$ has a convergent subsequence $\{\omega_{2n}\}$, and therefore we have

$$\omega_{2n} = \xi(\kappa_{2n}) = \Upsilon(\kappa_{2n+1}) \to \eta \text{ as } n \to +\infty.$$

We show that $\xi\vartheta = \eta$. By putting $\kappa = \vartheta$ and $\omega = \kappa_{2n+1}$ into $(C2)$, we have

$$\begin{aligned}\rho^3(\xi\vartheta, \zeta\kappa_{2n+1}) \leq & p \max\Big\{\frac{1}{2}[\rho^2(\Omega\vartheta, \xi\vartheta)\rho(\Upsilon\kappa_{2n+1}, \zeta\kappa_{2n+1}) \\ & + \rho(\Omega\vartheta, \xi\vartheta)\rho^2(\Upsilon\kappa_{2n+1}, \zeta\kappa_{2n+1})], \\ & \rho(\Omega\vartheta, \xi\vartheta)\rho(\Omega\vartheta, \zeta\kappa_{2n+1})\rho(\Upsilon\kappa_{2n+1}, \xi\vartheta), \\ & \rho(\Omega\vartheta, \zeta\kappa_{2n+1})\rho(\Upsilon\kappa_{2n+1}, \xi\vartheta)\rho(\Upsilon\kappa_{2n+1}, \zeta\kappa_{2n+1})\Big\} \\ & - \phi(m(\Omega\vartheta, \Upsilon\kappa_{2n+1})),\end{aligned}$$

where

$$\begin{aligned}m(\Omega\vartheta, \Upsilon\kappa_{2n+1}) = & \max\Big\{\rho^2(\Omega\vartheta, \Upsilon\kappa_{2n+1}), \rho(\Omega\vartheta, \xi\vartheta)\rho(\Upsilon\kappa_{2n+1}, \zeta\kappa_{2n+1}), \\ & \rho(\Omega\vartheta, \zeta\kappa_{2n+1})\rho(\Upsilon\kappa_{2n+1}, \xi\vartheta), \frac{1}{2}[\rho(\Omega\vartheta, \xi\vartheta)\rho(\Omega\vartheta, \zeta\kappa_{2n+1}) \\ & + \rho(\Upsilon\kappa_{2n+1}, \xi\vartheta)\rho(\Upsilon\kappa_{2n+1}, \zeta\kappa_{2n+1})]\Big\}.\end{aligned}$$

By letting $n \to +\infty$, we have

$$\begin{aligned}\rho^3(\xi\vartheta, \eta) \leq & p \max\{\frac{1}{2}[\rho^2(\Omega\vartheta, \xi\vartheta)\rho(\eta, \eta) + \rho(\Omega\vartheta, \xi\vartheta)\rho^2(\eta, \eta)], \\ & \rho(\Omega\vartheta, \xi\vartheta)\rho(\Omega\vartheta, \eta)\rho(\eta, \xi\vartheta), \rho(\Omega\vartheta, \eta)\rho(\eta, \xi\vartheta)\rho(\eta, \eta)\} \\ & - \phi(m(\Omega\vartheta, \eta)),\end{aligned}$$

where

$$\begin{aligned}m(\Omega\vartheta, \eta) = & \max\Big\{\rho^2(\Omega\vartheta, \eta), \rho(\Omega\vartheta, \xi\vartheta)\rho(\eta, \eta), \rho(\Omega\vartheta, \eta)\rho(\eta, \xi\vartheta), \\ & \frac{1}{2}[\rho(\Omega\vartheta, \xi\vartheta)\rho(\Omega\vartheta, \eta) + \rho(\eta, \xi\vartheta)\rho(\eta, \eta)]\Big\} = 0.\end{aligned}$$

Upon simplification, we have

$$\rho^3(\xi\vartheta, \eta) \leq p \max\Big\{\frac{1}{2}[0 + 0], 0, 0\Big\} - \phi(0).$$

This implies that $\xi\vartheta = \eta$ and hence $\xi\vartheta = \Omega\vartheta = \eta$. Therefore, ϑ is a point of coincidence of ξ and Ω. Since $\eta = \xi\vartheta \in \xi\mho \subset \Upsilon\mho$, there exist $\nu \in \mho$ such that $\eta = \Upsilon\nu$.

Next, we show that $\zeta\nu = \eta$. Upon putting $\kappa = \kappa_{2n}$ and $\omega = \nu$ into $(C2)$, we have

$$\begin{aligned}\rho^3(\xi\kappa_{2n}, \zeta\nu) \leq & p \max\Big\{\frac{1}{2}[\rho^2(\Omega\kappa_{2n}, \xi\kappa_{2n})\rho(\Upsilon\nu, \zeta\nu) \\ & + \rho(\Omega\kappa_{2n}, \xi\kappa_{2n})\rho^2(\Upsilon\nu, \zeta\nu)], \\ & \rho(\Omega\kappa_{2n}, \xi\kappa_{2n})\rho(\Omega\kappa_{2n}, \zeta\nu)\rho(\Upsilon\nu, \xi\kappa_{2n}), \\ & \rho(\Omega\kappa_{2n}, \zeta\nu)\rho(\Upsilon\nu, \xi\kappa_{2n})\rho(\Upsilon\nu, \zeta\nu)\Big\} \\ & - \phi(m(\Omega\kappa_{2n}, \Upsilon\nu)),\end{aligned}$$

where

$$\begin{aligned}m(\Omega\kappa_{2n}, \Upsilon\nu) = & \max\Big\{\rho^2(\Omega\kappa_{2n}, \Upsilon\nu), \rho(\Omega\kappa_{2n}, \xi\kappa_{2n})\rho(\Upsilon\nu, \zeta\nu), \\ & \rho(\Omega\kappa_{2n}, \zeta\nu)\rho(\Upsilon\nu, \xi\kappa_{2n}), \frac{1}{2}[\rho(\Omega\kappa_{2n}, \xi\kappa_{2n})\rho(\Omega\kappa_{2n}, \zeta\nu) \\ & + \rho(\Upsilon\nu, \xi\kappa_{2n})\rho(\Upsilon\nu, \zeta\nu)]\Big\}.\end{aligned}$$

By letting $n \to +\infty$, we obtain

$$\begin{aligned}\rho^3(\eta, \zeta\nu) \leq & p \max\left\{\frac{1}{2}[\rho^2(\eta, \eta)\rho(\Upsilon\nu, \zeta\nu) + \rho(\eta, \eta)\rho^2(\Upsilon\nu, \zeta\nu)],\right. \\ & \left.\rho(\eta, \eta)\rho(\eta, \zeta\nu)\rho(\Upsilon\nu, \eta), \rho(\eta, \zeta\nu)\rho(\Upsilon\nu, \eta)\rho(\Upsilon\nu, \zeta\nu)\right\} \\ & - \phi(m(\eta, \Upsilon\nu)),\end{aligned}$$

where

$$\begin{aligned}m(\eta, \Upsilon\nu) = & \max\left\{\rho^2(\eta, \Upsilon\nu), \rho(\eta, \eta)\rho(\Upsilon\nu, \zeta\nu), \rho(\eta, \zeta\nu)\rho(\Upsilon\nu, \eta),\right. \\ & \left.\frac{1}{2}[\rho(\eta, \eta)\rho(\eta, \zeta\nu) + \rho(\Upsilon\nu, \eta)\rho(\Upsilon\nu, \zeta\nu)]\right\} = 0.\end{aligned}$$

Upon simplification, we have

$$\rho^3(\eta, \zeta\nu) \leq p \max\left\{\frac{1}{2}[0+0], 0, 0\right\} - \phi(0).$$

This implies that $\zeta\nu = \eta$ and hence $\eta = \zeta\nu = \Upsilon\nu$. Thus, ν is a coincidence point of ζ and Υ. Since the pairs (ξ, Ω) and (ζ, Υ) are WC, therefore

$$\xi\eta = \xi(\Omega\vartheta) = \Omega(\xi\vartheta) = \Omega\eta, \quad \zeta\eta = \zeta(\Upsilon\nu) = \Upsilon(\zeta\nu) = \Upsilon\eta.$$

Next, we show that $\xi\eta = \eta$. Suppose that $\xi\eta \neq \eta$. Upon putting $\kappa = \eta$ and $\omega = \kappa_{2n+1}$ into $(C2)$, we obtain

$$\begin{aligned}\rho^3(\xi\eta, \zeta\kappa_{2n+1}) \leq & p \max\left\{\frac{1}{2}[\rho^2(\Omega\eta, \xi\eta)\rho(\Upsilon\kappa_{2n+1}, \zeta\kappa_{2n+1})\right. \\ & + \rho(\Omega\eta, \xi\eta)\rho^2(\Upsilon\kappa_{2n+1}, \zeta\kappa_{2n+1})], \\ & \rho(\Omega\eta, \xi\eta)\rho(\Omega\eta, \zeta\kappa_{2n+1})\rho(\Upsilon\kappa_{2n+1}, \xi\eta), \\ & \left.\rho(\Omega\eta, \zeta\kappa_{2n+1})\rho(\Upsilon\kappa_{2n+1}, \xi\eta)\rho(\Upsilon\kappa_{2n+1}, \zeta\kappa_{2n+1})\right\} \\ & - \phi(m(\Omega\eta, \Upsilon\kappa_{2n+1})),\end{aligned}$$

where

$$\begin{aligned}m(\Omega\eta, \Upsilon\kappa_{2n+1}) = & \max\left\{\rho^2(\Omega\eta, \Upsilon\kappa_{2n+1}), \rho(\Omega\eta, \xi\eta)\rho(\Upsilon\kappa_{2n+1}, \zeta\kappa_{2n+1}),\right. \\ & \rho(\Omega\eta, \zeta\kappa_{2n+1})\rho(\Upsilon\kappa_{2n+1}, \xi\eta), \frac{1}{2}[\rho(\eta, \xi\eta)\rho(\Omega\eta, \zeta\kappa_{2n+1}) \\ & \left.+ \rho(\Upsilon\kappa_{2n+1}, \xi\eta)\rho(\Upsilon\kappa_{2n+1}, \zeta\kappa_{2n+1})]\right\}.\end{aligned}$$

By letting $n \to +\infty$, and upon simplification, we obtain

$$\rho^3(\xi\eta, \eta) \leq p \max\left\{\frac{1}{2}[0+0], 0, 0\right\} - \phi(d^2(\xi\eta, \eta)).$$

This implies that $\rho^3(\xi\eta, \eta) \leq -\phi(\rho^2(\xi\eta, \eta))$, which is a contradiction, and hence $\xi\eta = \eta$.

Thus, we have $\xi\eta = \Omega\eta = \eta$.

Now, we show that $\zeta\eta = \eta$. Suppose that $\zeta\eta \neq \eta$. We put $\kappa = \kappa_{2n}$ and $\omega = \eta$ into $(C2)$, and we have

$$\begin{aligned}\rho^3(\xi\kappa_{2n}, \zeta\eta) \leq & p \max\Big\{\frac{1}{2}[\rho^2(\Omega\kappa_{2n}, \xi\kappa_{2n})\rho(\Upsilon\eta, \zeta\eta) \\ & + \rho(\Omega\kappa_{2n}, \xi\kappa_{2n})\rho^2(\Upsilon\eta, \zeta\eta)], \\ & \rho(\Omega\kappa_{2n}, \xi\kappa_{2n})\rho(\Omega\kappa_{2n}, \zeta\eta)\rho(\Upsilon\eta, \xi\kappa_{2n}), \\ & \rho(\Omega\kappa_{2n}, \zeta\eta)\rho(\Upsilon\eta, \xi\kappa_{2n})\rho(\Upsilon\eta, \zeta\eta)\Big\} \\ & - \phi(m(\Omega\kappa_{2n}, \Upsilon\eta)),\end{aligned}$$

where

$$\begin{aligned}m(\Omega\kappa_{2n}, \Upsilon\eta) = & \max\Big\{\rho^2(\Omega\kappa_{2n}, \Upsilon\eta), \rho(\Omega\kappa_{2n}, \xi\kappa_{2n})\rho(\Upsilon\eta, \zeta\eta), \\ & \rho(\Omega\kappa_{2n}, \zeta\eta)\rho(\Upsilon\eta, \xi\kappa_{2n}), \frac{1}{2}[\rho(\Omega\kappa_{2n}, \xi\kappa_{2n})\rho(\Omega\kappa_{2n}, \zeta\eta) \\ & + \rho(\Upsilon\eta, \xi\kappa_{2n})\rho(\Upsilon\eta, \zeta\eta)]\Big\}.\end{aligned}$$

By letting $n \to +\infty$, and upon simplification, we obtain

$$\rho^3(\eta, \zeta\eta) \leq \phi(\rho^2(\eta, \zeta\eta)), \text{ a contradiction }.$$

Thus, we have $\eta = \zeta\eta = \Upsilon\eta$.

Therefore, η is a common fixed point of ξ, ζ, Ω, and Υ.

Similarly, one can complete the proof by taking $\zeta\mho$, $\xi\mho$, or $\Upsilon\mho$ as a complete subspace of $\mho$. The uniqueness of a common fixed point follows easily from the condition $(C2)$. This completes the proof. □

If we put $\xi = \zeta$ into Theorem 2, then we obtain the following result:

Corollary 1. *Let ξ, Ω, and Υ be self-mappings on a complete metric space $(\mho, \rho)$ satisfying the following conditions:*

(C5) $\xi(\mho) \subset \Upsilon(\mho)$, $\xi(\mho) \subset \Omega(\mho)$;

$(C6)$

$$\begin{aligned}\rho^3(\xi\kappa, \xi\omega) \leq & p \max\Big\{\frac{1}{2}[\rho^2(\Omega\kappa, \xi\kappa)\rho(\Upsilon\omega, \xi\omega) + \rho(\Omega\kappa, \xi\kappa)\rho^2(\Upsilon\omega, \xi\omega)], \\ & \rho(\Omega\kappa, \xi\kappa)\rho(\Omega\kappa, \xi\omega)\rho(\Upsilon\omega, \xi\kappa), \rho(\Omega\kappa, \xi\omega)\rho(\Upsilon\omega, \xi\kappa)\rho(\Upsilon\omega, \xi\omega)\Big\} \\ & -\phi(m(\Omega\kappa, \Upsilon\omega)),\end{aligned}$$

for all $\kappa, \omega \in \mho$, where

$$\begin{aligned}m(\Omega\kappa, \Upsilon\omega) = & \max\Big\{\rho^2(\Omega\kappa, \Upsilon\omega), \rho(\Omega\kappa, \xi\kappa)\rho(\Upsilon\omega, \zeta\omega), \rho(\Omega\kappa, \xi\omega)\rho(\Upsilon\omega, \xi\kappa), \\ & \frac{1}{2}[\rho(\Omega\kappa, \xi\kappa)\rho(\Omega\kappa, \xi\omega) + \rho(\Upsilon\omega, \xi\kappa)\rho(\Upsilon\omega, \xi\omega)]\Big\}.\end{aligned}$$

Furthermore, p is a real number such that $0 < p < 1$ and $\phi : [0, +\infty) \to [0, +\infty)$ is a continuous function with $\phi(0) = 0$ and $\phi(s) > 0$ for each $s > 0$;

$(C7)$ *One of the subspaces $\xi\mho$, $\Omega\mho$, or $\Upsilon\mho$ is complete.*

Then ξ, Ω, and Υ have a unique common fixed point, provided that the pairs (ξ, Ω) and (ξ, Υ) are WC.

If we put $\Omega = \Upsilon = I$ (identity map) in Theorem 2, then we obtain the following result:

Corollary 2. *Let ξ and ζ be two self-mappings on a metric space $(\mho, \rho)$ satisfying the following conditions:*

$$\begin{aligned}\rho^3(\xi\kappa, \xi\omega) \leq & p \max\Big\{\frac{1}{2}[\rho^2(\kappa, \xi\kappa)\rho(\omega, \xi\omega) + \rho(\kappa, \xi\kappa)\rho^2(\omega, \xi\omega)], \\ & \rho(\kappa, \xi\kappa)\rho(\kappa, \xi\omega)\rho(\omega, \xi\kappa), \rho(\kappa, \xi\omega)\rho(\omega, \xi\kappa)\rho(\omega, \xi\omega)\Big\} \\ & -\phi(m(\kappa, \omega)),\end{aligned}$$

for all $\kappa, \omega \in \mho$, where

$$\begin{aligned}m(\kappa, \omega) = \max & \Big\{\rho^2(\kappa, \omega), \rho(\kappa, \xi\kappa)\rho(\omega, \zeta\omega), \rho(\kappa, \xi\omega)\rho(\omega, \xi\kappa), \\ & \frac{1}{2}[\rho(\kappa, \xi\kappa)\rho(\kappa, \xi\omega) + \rho(\omega, \xi\kappa)\rho(\omega, \xi\omega)]\Big\}.\end{aligned}$$

Furthermore, p is a real number such that $0 < p < 1$ and $\phi : [0, +\infty) \to [0, +\infty)$ is a continuous function with $\phi(0) = 0$ and $\phi(s) > 0$ for each $s > 0$.
Assume that one subspace $\xi\mho$ or $\zeta\mho$ is complete. Then, ξ and ζ have a unique common fixed point.

Now, we prove a theorem for WC mappings by avoiding the condition of completeness of the subspaces.

Theorem 3. *Let ξ, ζ, Ω, and Υ be four self-mappings on a complete metric space $(\mho, \rho)$ satisfying $(C1)$, $(C2)$, and the following condition:*
$(C8)$ One of subspace $\xi\mho$, $\zeta\mho$, $\Omega\mho$, or $\Upsilon\mho$ is closed;
Then ξ, ζ, Ω, and Υ have a unique common fixed point provided that the pairs (ξ, Ω) and (ζ, Υ) are WC.

Proof. Since the subspace of a complete metric space is complete if and only if it is closed, the conclusion easily follows from Theorem 2.
This completes the proof. □

In the next theorem, we are going to replace the concept of WC pairs of maps in previously established Theorems 2 and 3 by variants of weakly commuting pair of maps. In addition, we can realize that the conclusions of said theorems still hold well without changing the rest of the hypothesis:

Theorem 4. *The Theorems 2 and 3 remain true if the WC property of the pairs (ξ, Ω) and (ζ, Υ) is replaced by any one (retaining the rest of hypothesis) of the following:*
(1) Pairs (ξ, Ω) and (ζ, Υ) satisfy the RWC property;
(2) Pairs (ξ, Ω) and (ζ, Υ) satisfy the RWC property of types (A_ξ) and (A_ζ), respectively;
(3) Pairs (ξ, Ω) and (ζ, Υ) satisfy the RWC property of types (A_Ω) and (A_Υ), respectively;
(4) Pairs (ξ, Ω) and (ζ, Υ) satisfy the RWC property of type (P);
(5) Pairs (ξ, Ω) and (ζ, Υ) satisfy the weakly commuting property.

Proof. Since all the conditions of Theorems 2 and 3 are satisfied, then the existence of coincidence points for both the pairs is ensured. Let μ and ν be arbitrary points of coincidence for the pairs (ζ, Ω) and (ζ, Υ), respectively. Then, using the RWC property, we obtain

$$\rho(\xi\Omega\kappa, \Omega\xi\kappa) \leq R\rho(\xi\kappa, \Omega\kappa) = 0$$

and

$$\rho(\zeta\Upsilon\nu, \Upsilon\zeta\nu) \leq R\rho(\zeta\nu, \Upsilon\nu) = 0,$$

which implies that $\xi\Omega\kappa = \Omega\xi\kappa$ and $\zeta\Upsilon\nu = \Upsilon\zeta\nu$. Thus, the pairs (ζ, Ω) and (ζ, Υ) are WC. Now, using Theorems 2 and 3, we obtain ξ, ζ, Ω, and Υ have a unique common fixed point.

In case the pair (ξ, Ω) satisfies the RWC property of type (A_ξ), then

$$\rho(\xi\Omega\kappa, \Omega\Omega\kappa) \leq R\rho(\xi\kappa, \Omega\kappa) = 0,$$

which implies that $\xi\Omega\kappa = \Omega\Omega\kappa$.

Additionally, $\rho(\xi\Omega\kappa, \Omega\xi\kappa) \leq \rho(\xi\Omega\kappa, \Omega\Omega\kappa) + \rho(\Omega\Omega\kappa, \Omega\xi\kappa) = 0$, which provides $\xi\Omega\kappa = \Omega\xi\kappa$. Similarly, for the pair (ζ, Υ), we have $\zeta\Upsilon\nu = \Upsilon\zeta\nu$.

Similarly, if the pairs (ξ, Ω) and (ζ, Υ) are RWCs of types (A_Ω) and (A_Υ), respectively, RWC of type (P) or weakly commuting, then (ξ, Ω) and (ζ, Υ) also commute at their points of coincidence. Now, in view of Theorems 2 and 3, in all four cases, ξ, ζ, Ω, and Υ have a unique common fixed point. □

Example 2. *Let $\mho = [2, 20]$ and d be a usual metric. Let $\xi, \zeta, \Omega,$ and Υ be four self-mappings on $\mho$ defined by*

$$\xi(\kappa) = \begin{cases} 2, & \kappa \in [2, 4) \\ \frac{27}{10}, & \kappa \in [4, 20], \end{cases} \quad \zeta(\kappa) = \begin{cases} \kappa, & \kappa = 2 \\ \frac{12}{5}, & \kappa \in (2, 20], \end{cases}$$

$$\Omega(\kappa) = \begin{cases} 2, & \kappa \in [2, 4) \\ \frac{12}{5}, & \kappa = 4 \\ \kappa - \frac{1}{2}, & \kappa \in (4, 20], \end{cases} \quad \Upsilon(\kappa) = \begin{cases} \kappa, & \kappa = 2, \frac{27}{10} \\ 6, & \kappa \in (2, 20] - \{\frac{27}{10}\}. \end{cases}$$

Let $\phi : [0, +\infty) \to [0, +\infty)$ be a function defined by $\phi(s) = \frac{s}{16}$ for $s \geq 0$. Then, one can easily verify that all the conditions of Theorems 2 and 3 are satisfied for $p = \frac{9}{10}$, and two is the unique common fixed point of $\xi, \zeta, \Omega,$ and Υ.

3. Applications

Assume that $\Omega \subset \mho$ is the state space and $D \subset Y$ is the decision space, where $\mho$ and Y are Banach spaces. Let $\mathbb{R} = (-\infty, +\infty)$ and $B(\Omega)$ denote the set of all bounded real-valued functions on Ω. Following Bellman and Lee [19], the basic form of the functional equation of dynamic programming is defined as follows:

$$\xi(\kappa) = opt_\omega H(\kappa, \omega, \xi(\Upsilon(\kappa, \omega))),$$

where κ and ω represent the state and decision vectors, respectively, Υ is the transformation of the process, and $\xi(\kappa)$ is the optimal return with the initial state κ, where the opt denotes the maximum or minimum.

In this section, we shall discuss the existence and uniqueness of a common solution to the following functional equations arising in dynamic programming:

$$\xi_i(\kappa) = \sup_{\omega \in D} H_i(\kappa, \omega, \xi_i(\Upsilon(\kappa, \omega))), \kappa \in \Omega, \tag{1}$$

$$\zeta_i(\kappa) = \sup_{\omega \in D} F_i(\kappa, \omega, \zeta_i(\Upsilon(\kappa, \omega))), \kappa \in \Omega, \tag{2}$$

where $\Upsilon : \Omega \times D \to \Omega$ and $H_i, F_i : \Omega \times D \times R \to R, i = 1, 2$:

Theorem 5. *Assume that the following conditions are satisfied:*

(1) *For $i = 1, 2$, H_i and F_i are bounded;*

(2)

$$|H_1(\kappa, \omega, \hbar(s)) - H_2(\kappa, \omega, \aleph(s))|^3 \leq p \max\{\frac{1}{2}[|\Upsilon_1\hbar(s) - Q_1\hbar(s)|^2 \cdot |\Upsilon_2\aleph(s) - Q_2\aleph(s)| + |\Upsilon_1\hbar(s) - Q_1\hbar(s)| \cdot |\Upsilon_2\aleph(s) - Q_2\aleph(s)|^2], |\Upsilon_1\hbar(s) - Q_1\hbar(s)| \cdot |\Upsilon_1\hbar(s) - Q_2\aleph(s)| \cdot |\Upsilon_2\aleph(s) - Q_1\hbar(s)|, |\Upsilon_1\hbar(s) - Q_2\aleph(s)| \cdot |\Upsilon_2\aleph(s) - Q_1\hbar(s)| \cdot |\Upsilon_2\aleph(s) - Q_2\aleph(s)|\} - \phi(m(\Upsilon_1\hbar(s), \Upsilon_2\aleph(s))),$$

for all $(\kappa, \omega) \in \Omega \times D$, $\hbar, \aleph \in B(\Omega)$ and $s \in \Omega$, where

$$m(\Upsilon_1\hbar(s), \Upsilon_2\aleph(s)) = \max\{|\Upsilon_1\hbar(s) - \Upsilon_2\aleph(s)|^2, |\Upsilon_1\hbar(s) - Q_1\hbar(s)| \cdot |\Upsilon_2\aleph(s) - Q_2\aleph(s)|, |\Upsilon_1\hbar(s) - Q_2\aleph(s)| \cdot |\Upsilon_2\aleph(s) - Q_1\hbar(s)|, \frac{1}{2}[|\Upsilon_1\hbar(s) - Q_1\hbar(s)| \cdot |\Upsilon_1\hbar(s) - Q_2\aleph(s)| + |\Upsilon_2\aleph(s) - Q_1\hbar(s)| \cdot |\Upsilon_2\aleph(s) - Q_2\aleph(s)|]\}$$

and p and ϕ are the same as in Theorem 1. Additionally, the mappings Q_i and Υ_i are defined as follows:

$$Q_i\hbar(\kappa) = \sup_{\omega \in D} H_i(\kappa, \omega, \hbar(\Upsilon(\kappa, \omega))), \kappa \in \Omega, \hbar \in B(\Omega), i = 1, 2,$$

$$\Upsilon_i\aleph(\kappa) = \sup_{\omega \in D} F_i(\kappa, \omega, k(\Upsilon(\kappa, \omega))), \kappa \in \Omega, \aleph \in B(\Omega), i = 1, 2,$$

(3) *For any $\hbar, \aleph \in B(\Omega)$, there exist $\aleph_1, \aleph_2 \in B(\Omega)$ such that*

$$Q_1\hbar(\kappa) = \Upsilon_2\aleph_1(\kappa), \qquad Q_2\hbar(\kappa) = \Upsilon_1\aleph_2(\kappa), \kappa \in \Omega,$$

(4) *For any $\hbar \in B(\Omega)$, if $Q_i\hbar = \Upsilon_i\hbar$, then $Q_i\Upsilon_i\hbar = \Upsilon_iQ_i\hbar$ and $i = 1, 2$.*
Then, the system of functional Equations (1) and (2) has a unique common solution in $B(\Omega)$.

Proof. Let $\rho(\hbar, \aleph) = \sup\{|\hbar(\kappa) - \aleph(\kappa)| : \kappa \in \Omega\}$ for any $\hbar, \aleph \in B(\Omega)$. Then, $(B(\Omega), \rho)$ is a complete metric space. From conditions (1–4), Q_i and Υ_i are self-mappings of $B(\Omega)$, $i = 1, 2$, $Q_1(B(\Omega)) \subset \Upsilon_2(B(\Omega))$, and $Q_2(B(\Omega)) \subset \Upsilon_1(B(\Omega))$, and the pairs of mappings Q_i, Υ_i, and $i = 1, 2$ are *WC*. Let $\hbar_i (i = 1, 2)$ be any two pints of $B(\Omega)$, $\kappa \in \Omega$ and α be any positive number. Suppose that there exists $\omega_i (i = 1, 2)$ in D such that

$$Q_i\hbar_i(\kappa) < H_i(\kappa, \omega_i, \hbar_i(\kappa_i)) + \alpha, \tag{3}$$

where $\kappa_i = \Upsilon(\kappa, \omega_i)$, $i = 1, 2$. In addition, we have

$$Q_1\hbar_1(\kappa) \geq H_1(\kappa, \omega_2, \hbar_1(\kappa_2)), \tag{4}$$

$$Q_2\hbar_2(\kappa) \geq H_2(\kappa, \omega_1, \hbar_2(\kappa_1)). \tag{5}$$

Since α is any positive number, from Equations (2), (3) and (5), we have

$$\begin{aligned}
&(Q_1\hbar_1(\kappa) - Q_2\hbar_2(\kappa))^3 \\
&< (H_1(\kappa,\omega_1,\hbar_1(\kappa_1)) - H_2(\kappa,\omega_1,\hbar_2(\kappa_1)))^3 + \alpha \\
&\leq (|H_1(\kappa,\omega_1,\hbar_1(\kappa_1)) - H_2(\kappa,\omega_1,\hbar_2(\kappa_1))|)^3 + \alpha \\
&\leq p\max\{\frac{1}{2}[|\Upsilon_1\hbar_1(\kappa_1) - Q_1\hbar_1(\kappa_1)|^2.|\Upsilon_2\hbar_2(\kappa_1) - Q_2\hbar_2(\kappa_1)| \\
&+ |\Upsilon_1\hbar_1(\kappa_1) - Q_1\hbar_1(\kappa_1)| \cdot |\Upsilon_2\hbar_2(\kappa_1) - Q_2\hbar_2(\kappa_1)|^2], |\Upsilon_1\hbar_1(\kappa_1) - Q_1\hbar_1(\kappa_1)| \\
&\cdot |\Upsilon_1\hbar_1(\kappa_1) - Q_2\hbar_2(\kappa_1)| \cdot |\Upsilon_2\hbar_2(\kappa_1) - Q_1\hbar_1(\kappa_1)|, |\Upsilon_1\hbar_1(\kappa_1) - Q_2\hbar_2(\kappa_1)| \\
&\cdot |\Upsilon_2\hbar_2(\kappa_1) - Q_1\hbar_1(\kappa_1)| \cdot |\Upsilon_2\hbar_2(\kappa_1) - Q_2\hbar_2(\kappa_1)|\} - \phi(m(\Upsilon_1\hbar_1(\kappa_1), \Upsilon_2\hbar_2(\kappa_1))) + \alpha,
\end{aligned} \tag{6}$$

where

$$\begin{aligned}
&m(\Upsilon_1\hbar_1(\kappa_1), \Upsilon_2\hbar_2(\kappa_1)) \\
&\quad = \max\{|\Upsilon_1\hbar_1(\kappa_1) - \Upsilon_2\hbar_2(\kappa_1)|^2, |\Upsilon_1\hbar_1(\kappa_1) - Q_1\hbar_1(\kappa_1)| \cdot |\Upsilon_2\hbar_2(\kappa_1) - Q_2\hbar_2(\kappa_1)|, \\
&\quad |\Upsilon_1\hbar_1(\kappa_1) - Q_2\hbar_2(\kappa_1)| \cdot |\Upsilon_2\hbar_2(\kappa_1) - Q_1\hbar_1(\kappa_1)|, \frac{1}{2}[|\Upsilon_1\hbar_1(\kappa_1) - Q_1\hbar_1(\kappa_1)| \\
&\quad \cdot |\Upsilon_1\hbar_1(\kappa_1) - Q_2\hbar_2(\kappa_1)| + |\Upsilon_2\hbar_2(\kappa_1) - Q_1\hbar_1(\kappa_1)| \cdot |\Upsilon_2\hbar_2(\kappa_1) - Q_2\hbar_2(\kappa_1)|]\}.
\end{aligned}$$

From Equation (6), we have

$$\begin{aligned}
&(Q_1\hbar_1(\kappa) - Q_2\hbar_2(\kappa))^3 \\
&\leq p\max\{\frac{1}{2}[\rho^2(\Upsilon_1\hbar_1, Q_1\hbar_1)\rho(\Upsilon_2\hbar_2, Q_2\hbar_2) + \rho(\Upsilon_1\hbar_1, Q_1\hbar_1)\rho^2(\Upsilon_2\hbar_2, Q_2\hbar_2)], \\
&\rho(\Upsilon_1\hbar_1, Q_1\hbar_1)\rho(\Upsilon_1\hbar_1, Q_2\hbar_2)\rho(\Upsilon_2\hbar_2, Q_1\hbar_1), \rho(\Upsilon_1\hbar_1, Q_2\hbar_2)\rho(\Upsilon_2\hbar_2, Q_1\hbar_1) \\
&\rho(\Upsilon_2\hbar_2, Q_2\hbar_2)\} - \phi(m(\Upsilon_1\hbar_1(\kappa_1), \Upsilon_2\hbar_2(\kappa_1))) + \alpha,
\end{aligned} \tag{7}$$

where

$$\begin{aligned}
m(\Upsilon_1\hbar_1(\kappa_1), \Upsilon_2\hbar_2(\kappa_1)) = \max\{&\rho^2(\Upsilon_1\hbar_1, \Upsilon_2\hbar_2), \rho(\Upsilon_1\hbar_1, Q_1\hbar_1)\rho(\Upsilon_2\hbar_2, Q_2\hbar_2), \\
&\rho(\Upsilon_1\hbar_1, Q_2\hbar_2)\rho(\Upsilon_2\hbar_2, Q_1\hbar_1), \frac{1}{2}[\rho(\Upsilon_1\hbar_1, Q_1\hbar_1) \\
&\rho(\Upsilon_1\hbar_1, Q_2\hbar_2) + \rho(\Upsilon_2\hbar_2, Q_1\hbar_1)\rho(\Upsilon_2\hbar_2, Q_2\hbar_2)]\}.
\end{aligned}$$

Since α is any positive number, from Equations (3) and (4) and condition (2), we have

$$\begin{aligned}
&(Q_1\hbar_1(\kappa) - Q_2\hbar_2(\kappa))^3 \\
&\geq -p\max\{\frac{1}{2}[\rho^2(\Upsilon_1\hbar_1, Q_1\hbar_1)\rho(\Upsilon_2\hbar_2, Q_2\hbar_2) + \rho(\Upsilon_1\hbar_1, Q_1\hbar_1)\rho^2(\Upsilon_2\hbar_2, Q_2\hbar_2)], \\
&\rho(\Upsilon_1\hbar_1, Q_1\hbar_1)\rho(\Upsilon_1\hbar_1, Q_2\hbar_2)\rho(\Upsilon_2\hbar_2, Q_1\hbar_1), \rho(\Upsilon_1\hbar_1, Q_2\hbar_2)\rho(\Upsilon_2\hbar_2, Q_1\hbar_1) \\
&\rho(\Upsilon_2\hbar_2, Q_2\hbar_2)\} + \phi(m(\Upsilon_1\hbar_1(\kappa_1), \Upsilon_2\hbar_2(\kappa_1))) - \alpha,
\end{aligned} \tag{8}$$

where $m(\Upsilon_1\hbar_1(\kappa_1), \Upsilon_2\hbar_2(\kappa_1))$ is same as in Equation (7). The combination of Equations (7) and (8) gives

$$\begin{aligned}
&|Q_1\hbar_1(\kappa) - Q_2\hbar_2(\kappa)|^3 \\
&\leq p\max\{\frac{1}{2}[\rho^2(\Upsilon_1\hbar_1, Q_1\hbar_1)\rho(\Upsilon_2\hbar_2, Q_2\hbar_2) + \rho(\Upsilon_1\hbar_1, Q_1\hbar_1)\rho^2(\Upsilon_2\hbar_2, Q_2\hbar_2)], \\
&\rho(\Upsilon_1\hbar_1, Q_1\hbar_1)\rho(\Upsilon_1\hbar_1, Q_2\hbar_2)\rho(\Upsilon_2\hbar_2, Q_1\hbar_1), \rho(\Upsilon_1\hbar_1, Q_2\hbar_2)\rho(\Upsilon_2\hbar_2, Q_1\hbar_1) \\
&\rho(\Upsilon_2\hbar_2, Q_2\hbar_2)\} - \phi(m(\Upsilon_1\hbar_1(\kappa_1), \Upsilon_2\hbar_2(\kappa_1))) + \alpha.
\end{aligned} \tag{9}$$

Since Equation (9) holds for any $\kappa \in \Omega$, and α is any positive number, upon taking supremum over all $\kappa \in \Omega$, we have

$$\begin{aligned}
&\rho^3(Q_1\hbar_1, Q_2\hbar_2) \\
&\le p \max\{\frac{1}{2}[\rho^2(\Upsilon_1\hbar_1, Q_1\hbar_1)\rho(\Upsilon_2\hbar_2, Q_2\hbar_2) + \rho(\Upsilon_1\hbar_1, Q_1\hbar_1)\rho^2(\Upsilon_2\hbar_2, Q_2\hbar_2)], \\
&\rho(\Upsilon_1\hbar_1, Q_1\hbar_1)\rho(\Upsilon_1\hbar_1, Q_2\hbar_2)\rho(\Upsilon_2\hbar_2, Q_1\hbar_1), \rho(\Upsilon_1\hbar_1, Q_2\hbar_2)\rho(\Upsilon_2\hbar_2, Q_1\hbar_1) \\
&.\rho(\Upsilon_2\hbar_2, Q_2\hbar_2)\} - \phi(m(\Upsilon_1\hbar_1(\kappa_1), \Upsilon_2\hbar_2(\kappa_1))).
\end{aligned}$$

Therefore, by Theorem 2, Q_1, Q_2, Υ_1, and Υ_2 have a unique common fixed point $\hbar' \in B(\Omega)$ (i.e., $\hbar'(\kappa)$ is a unique solution of the functional Equations (1) and (2)). This completes the proof. □

Remark 1. *On replacing condition (4) of Theorem 5, by any one of the following conditions (a–e), then we obtain applications for Theorem 4.*

(a) *For all $\hbar(\kappa) \in B(\Omega)$, there exists some $R, R' \ge 0$ such that*

$$\sup_{\kappa\in\Omega}|Q_1\Upsilon_1\hbar(\kappa) - \Upsilon_1Q_1\hbar(\kappa)| \le R \sup_{\kappa\in\Omega}|Q_1\hbar(\kappa) - \Upsilon_1\hbar(\kappa)|$$

and

$$\sup_{\kappa\in\Omega}|Q_2\Upsilon_2\hbar(\kappa) - \Upsilon_2Q_2\hbar(\kappa)| \le R' \sup_{\kappa\in\Omega}|Q_2\hbar(\kappa) - \Upsilon_2\hbar(\kappa)|.$$

(b) *For all $\hbar(\kappa) \in B(\Omega)$, there exists some $R, R' > 0$ such that*

$$\sup_{\kappa\in\Omega}|Q_1\Upsilon_1\hbar(\kappa) - \Upsilon_1\Upsilon_1\hbar(\kappa)| \le R \sup_{\kappa\in\Omega}|Q_1\hbar(\kappa) - \Upsilon_1\hbar(\kappa)|$$

and

$$\sup_{\kappa\in\Omega}|Q_2\Upsilon_2\hbar(\kappa) - \Upsilon_2\Upsilon_2\hbar(\kappa)| \le R' \sup_{\kappa\in\Omega}|Q_2\hbar(\kappa) - \Upsilon_2\hbar(\kappa)|.$$

(c) *For all $\hbar(\kappa) \in B(\Omega)$, there exists some $R, R' > 0$ such that*

$$\sup_{\kappa\in\Omega}|\Upsilon_1Q_1\hbar(\kappa) - Q_1Q_1\hbar(\kappa)| \le R \sup_{\kappa\in\Omega}|Q_1\hbar(\kappa) - \Upsilon_1\hbar(\kappa)|$$

and

$$\sup_{\kappa\in\Omega}|\Upsilon_2Q_2\hbar(\kappa) - Q_2Q_2\hbar(\kappa)| \le R' \sup_{\kappa\in\Omega}|Q_2\hbar(\kappa) - \Upsilon_2\hbar(\kappa)|.$$

(d) *For all $\hbar(\kappa) \in B(\Omega)$, there exists some $R, R' > 0$ such that*

$$\sup_{\kappa\in\Omega}|Q_1Q_1\hbar(\kappa) - \Upsilon_1\Upsilon_1\hbar(\kappa)| \le R \sup_{\kappa\in\Omega}|Q_1\hbar(\kappa) - \Upsilon_1\hbar(\kappa)|$$

and

$$\sup_{\kappa\in\Omega}|Q_2Q_2\hbar(\kappa) - \Upsilon_2\Upsilon_2\hbar(\kappa)| \le R' \sup_{\kappa\in\Omega}|Q_2\hbar(\kappa) - \Upsilon_2\hbar(\kappa)|.$$

(e) *For all $\hbar(\kappa) \in B(\Omega)$, we have*

$$\sup_{\kappa\in\Omega}|Q_1\Upsilon_1\hbar(\kappa) - \Upsilon_1Q_1\hbar(\kappa)| \le \sup_{\kappa\in\Omega}|Q_1\hbar(\kappa) - \Upsilon_1\hbar(\kappa)|$$

and

$$\sup_{\kappa\in\Omega}|Q_2\Upsilon_2\hbar(\kappa) - \Upsilon_2Q_2\hbar(\kappa)| \le \sup_{\kappa\in\Omega}|Q_2\hbar(\kappa) - \Upsilon_2\hbar(\kappa)|.$$

4. Conclusions

We have demonstrated the power of the very essential tools in this paper, such as WC mappings and variants of RWC pairs of maps. We made use of satisfying the weak contraction condition in which cubic terms exist in the metric function. The results provided here are the extension of the results from the class of compatible mappings to a larger class of mappings having weak compatibility without appealing to continuity in the context of metric fixed point theory and applications. Our results were also obtained using the condition of WC to avoid the condition of completeness of the subspaces. Finally, as an application of our results, we have discussed the existence and uniqueness of common solutions to the functional equations arising in dynamic programming.

Retrospect:

- The present study under the given title sounds as though a lot of research can also be performed in the area of contraction and weak contraction conditions.
- On the applications side, a lot of work is in progress for applying the concept of the variants of weak commutativity and weak compatibility to the nonlinear integral equations.
- We are also exploring the possibility of obtaining applications of fixed point theory to day-to-day life, such as the recently faced COVID-19 pandemic, for the most appropriate diagnosis.

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