fractal and fractional

Article

On the Solvability of Mixed-Type Fractional-Order Non-Linear Functional Integral Equations in the Banach Space $C(I)$

Vijai Kumar Pathak [1], Lakshmi Narayan Mishra [1,*], Vishnu Narayan Mishra [2] and Dumitru Baleanu [3,4,5]

1 Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore 632 014, Tamil Nadu, India
2 Department of Mathematics, Indira Gandhi National Tribal University, Lalpur, Amarkantak, Anuppur 484 887, Madhya Pradesh, India
3 Department of Mathematics, Faculty of Arts and Sciences, Cankaya University, Ankara 09790, Turkey
4 Institute of Space Sciences, 077125 Magurele, Ilfov, Romania
5 Department of Natural Sciences, School of Arts and Sciences, Lebanese American University, Beirut 11022801, Lebanon
* Correspondence: lakshminarayanmishra04@gmail.com; Tel.: +91-98383-75431

Abstract: This paper is concerned with the existence of the solution to mixed-type non-linear fractional functional integral equations involving generalized proportional (κ, ϕ)-Riemann–Liouville along with Erdélyi–Kober fractional operators on a Banach space $C([1, \mathtt{T}])$ arising in biological population dynamics. The key findings of the article are based on theoretical concepts pertaining to the fractional calculus and the Hausdorff measure of non-compactness (MNC). To obtain this goal, we employ Darbo's fixed-point theorem (DFPT) in the Banach space. In addition, we provide two numerical examples to demonstrate the applicability of our findings to the theory of fractional integral equations.

Keywords: functional integral equations; measure of non-compactness; Darbo's fixed-point theorem; fractional operators; Banach space

MSC: 26A33; 47H08; 47H10; 26D15

Citation: Pathak, V.K.; Mishra, L.N.; Mishra, V.N.; Baleanu, D. On the Solvability of Mixed-Type Fractional-Order Non-Linear Functional Integral Equations in the Banach Space $C(I)$. *Fractal Fract.* **2022**, *6*, 744. https://doi.org/10.3390/fractalfract6120744

Academic Editors: Ivanka Stamova, Zoran D. Mitrovic, Reny Kunnel Chacko George and Liliana Guran

Received: 14 November 2022
Accepted: 11 December 2022
Published: 16 December 2022

1. Introduction

Fractional calculus is a well-known mathematical tool for the description of anomalous and non-local diffusion together with physical investigation and has also found applications in various fields from physics and engineering to the investigation of natural phenomena and financial analysis. The field of fractional calculus plays a central role in mathematical analysis which analyses the derivatives and integrals of any real or complex order by employing the Euler gamma function. Fractional calculus enables us to illustrate different occurrences and impacts in various science disciplines as well as in engineering, including frequency dispersion of power types, long-range interactions of power-law types, spatial dispersion of power types, intrinsic dissipation, fractional diffusion waves, fractional viscoelasticity, fractional electrochemistry, fractional relaxation-oscillation, fractional electromagnetics, fading memory (forgetting), fractional biological population models, the openness of systems, optics, signals processing, the vibration of earthquake motion, and several others. In the 16th century, the idea of fractional calculus was introduced. The first application of fractional calculus to engineering problems is considered to be Abel's study of the tautochrone problem. During the 19th and early 20th centuries, the ideas and multiple practical invocations of fractional calculus were substantially developed.

Functional integral equations play a vital role in distinct disciplines, as well as in the analysis of many real-life problems, and can be modeled by utilizing fractional operators very efficiently to describe a range of phenomena, including media with non-integer mass dimensions, seepage flow in porous media, the fractal structure of matter and non-linear oscillations of earthquakes. Non-linear fractional integral equations are of practical importance in distinct areas of modeling, including fluid dynamic traffic models, the theory of radioactive transmission, the theory of statistical mechanics, cytotoxic activity, and acoustic scattering [1–5].

Different real-life situations, which are modeled by the application of fractional integral equations, can be studied by employing fixed-point theory (FPT) and MNC [6–13]. In recent years, FPT, first proposed by Stephen Banach, has been widely used in different scientific fields. FPT has been applied in relation to recrystalization theory, phase-transition theory, object-oriented analysis, and programming language analysis, together with having several potential applications in immunology, aerospace, neural networks, and healthcare, among others. Dhage [14] discussed global attractivity results for non-linear functional integral equations via a Krasnoselskii-type fixed-point theorem, Aghajani et al. [15] studied fixed-point theorems for Meir–Keeler condensing operators via a measure of non-compactness, and Javahernia et al. [16] studied common fixed points in generalized Mizoguchi–Takahashi-type contractions. Mohammadi et al. [17] also investigated the existence of solutions for a system of integral equations using a generalization of Darbo's fixed-point theorem. Jleli et al. [18] proved some generalizations of Darbo's theorem and studied applications to fractional integral equations. FPT can also been used to seek solutions for fractional functional integral equations. Fractional functional integral equations of various types have made essential contributions to a wide range of real-world problems. Many problems in mathematics, science, engineering and astronomy can be explained by utilizing particular types of fractional integral equations. For examples, please see [19–25].

Recently, several research articles have been published in connection with applications of FPT.

In 2020, Arab et al. [26] discussed the solvability of functional-integral equations (fractional order) using a measure of non-compactness

$$u(t) = f(t, u(t)) + \frac{Hu(t)}{\Gamma(\gamma)} \int_0^t \frac{g'(s)}{(g(t) - g(s))^{1-\gamma}} k(t, s, u(s)) ds,$$

where $t \in I = [0, 1]$, $\gamma \in (0, 1)$. Existence results were obtained through the techniques of MNC and a generalized version of Darbo's fixed-point theorem by introducing a new μ-set contraction operator using control functions in Banach spaces.

In 2022, Das et al. [27] investigated the generalization of a Darbo-type theorem and its application to the existence of implicit fractional integral equations in tempered sequence spaces

$$z_n(\varsigma) = K_n\left(\varsigma, z(\varsigma), \int_a^\varsigma \frac{g'(w) H_n(\varsigma, w, z(w))}{(g(\varsigma) - g(w))^{1-\alpha}} dw\right),$$

where $\alpha \in (0, 1)$, $\varsigma \in I = [a, T]$, $T > 0$, $a \geq 0$, $z(\varsigma) = (z_n(\varsigma))_{n=1}^{\infty} \in \mathbb{E}$ and $\mathbb{E}$ is a Banach sequence space. Existence results were obtained through the techniques of MNC and Darbo's fixedpoint theorem in tempered sequence space ℓ_p^α.

In 2022, Mohiuddine et al. [28] established the existence of solutions for non-linear integral equations in tempered sequence spaces via a generalized Darbo-type theorem

$$\Omega_n(\xi) = \mathbb{F}_n\left(\xi, \Omega(\xi), \int_0^\xi \mathbb{G}_n(\xi, s, \Omega(s)) ds\right),$$

for $n \in \mathbb{N}$, where $\Omega(\xi) = (\Omega_n(\xi))_{n=1}^{\infty}$, $\xi \in I = [0, a]$, $a > 0$. To realize the existence of the solutions of the integral equations, the authors used the concept of MNC and Darbo-type fixed point in tempered sequence space $C([I, \ell_p^\alpha])$.

In 2022, Das et al. [29] investigated the iterative algorithm and theoretical treatment of the existence of a solution for (k, z)-Riemann—Liouville fractional integral equations

$$\Psi(h) = \Theta\big(h, \mathcal{G}(h, \Psi(h)), {}_k^z\mathrm{J}^\alpha f(h)\big),$$

for $z \in \mathbb{R}^+ \setminus \{-1\}$, $1 > k > 0$, $\alpha > 0$, $h \in I = [1, T]$. To realize the existence of the solution of those integral equations, the authors used the concept of MNC and Darbo's fixed-point theorem in the Banach space $C([1, T])$. They also discussed an iterative algorithm which was constructed by a homotopy perturbation method to find the approximate solution.

In the present paper, we present the (κ, ϕ)-type generalized proportional Riemann–Liouville fractional integral operator ${}^{\phi}_{\kappa}\mathtt{I}^{\wp,\upsilon}_{a}$, where $\upsilon \in (0, 1]$, $\phi \in \mathbb{R}^+ \setminus \{-1\}$ and a, $\wp$, $\kappa > 0$, for a continuous function $\Psi(\varrho)$ is given by

$$({}^{\phi}_{\kappa}\mathtt{I}^{\wp,\upsilon}_{a}\Psi)(\varrho) = \frac{(\phi+1)^{1-\frac{\wp}{\kappa}}}{\upsilon^{\frac{\wp}{\kappa}}\kappa\Gamma\kappa(\wp)} \int_a^{\varrho} \exp\left[\frac{(\upsilon-1)(\varrho^{\phi+1} - \varsigma^{\phi+1})}{\upsilon}\right] (\varrho^{\phi+1} - \varsigma^{\phi+1})^{\frac{\wp}{\kappa}-1} \varsigma^{\phi} \Psi(\varsigma) d\varsigma.$$

In addition, we present the Erdélyi–Kober fractional integral operator $\hat{\mathtt{I}}^{\alpha}_{\zeta,a}$, where $\zeta > 0, a > 0$, and $0 < \alpha < 1$, for a continuous function $\Psi(\varrho)$ is given by

$$(\hat{\mathtt{I}}^{\alpha}_{\zeta,a}\Psi)(\varrho) = \frac{\zeta}{\Gamma(\alpha)} \int_a^{\varrho} \frac{\varsigma^{\zeta-1}\Psi(\varsigma)}{(\varrho^{\zeta} - \varsigma^{\zeta})^{1-\alpha}} d\varsigma.$$

The study of biological population dynamics can be analyzed using different types of fractional operators which have been defined and can be categorized into broad classes according to their properties and behaviors. In our study, we establish an important connection between (κ, ϕ)-type generalized proportional Riemann–Liouville and Erdélyi–Kober fractional operators, writing one in terms of the other by making use of the theory of fractional calculus with respect to the same function on the Banach space $C([1, T])$.

Moreover, in terms of application, the main goal of this paper is to study the non-linear fractional order biological population model including the determination of the surge in the birthrate $\Psi(\varrho)$ at any time ϱ to allow for future necessary planning. The dependence of the birthrate $\Psi(\varrho)$ on previous birthrates $\Psi(\varrho^{\eta} - \varsigma^{\eta})$, for women in the child-bearing age range $1 < \varsigma < \mathtt{T}$, $\eta > 1$, is given by the mixed type integral equation associated with generalized proportional (κ, ϕ)-Riemann–Liouville along with Erdélyi–Kober fractional operators as follows:

$$\Psi(\varrho) = \mathtt{g}(\varrho) + \mathtt{f}(\varrho, \mathtt{q}(\varrho, \Psi(\varrho)), ({}^{\phi}_{\kappa}\mathtt{I}^{\wp,\upsilon}_{1}\Psi)(\varrho)) + \mathtt{F}(\varrho, \mathtt{h}(\varrho, \Psi(\varrho)), (\hat{\mathtt{I}}^{\alpha}_{\zeta,1}\Psi)(\varrho)), \tag{1}$$

where $\Psi(\varsigma)$ is the probability that the female lives to age ς. $\mathtt{g}(\varrho)$, $\mathtt{q}(\varrho, \Psi(\varrho))$, and $\mathtt{h}(\varrho, \Psi(\varrho))$ are the terms added to allow for girls already born before the oldest child-bearing women of age ($\varsigma = \mathtt{T}$) were born. $\mathtt{F}$ and $\mathtt{f}$ are the survival functions, which are the fraction of the number of people that survive to age ϱ. Further, $\upsilon \in (0, 1]$, $\wp > 1$, $\kappa > 0$, $\phi \in \mathbb{R}^+ \setminus \{-1\}$, $\zeta > 0$, $0 < \alpha < 1$ and $\varrho \in I = [1, \mathtt{T}]$.

This model was studied by Gurtin and MacCamy in [30] and numerous authors have conducted in-depth research on it. In [31], Metz and Diekmann gave a detailed account of the use of mathematical models for physiologically structured populations. In [32], Cushing provides a broad survey of the literature in the area of delay in population dynamics. For a deep evaluation of age-dependent population dynamics, one can refer to [33–42].

We discuss below the motivation for studying Equation (1) as well as the nature of our findings. In this paper, we sought to extend the theory of fractional calculus methods by considering fractional integral equations in relation to the modeling of biological population dynamics. Secondly, we sought to review relevant work in this area. Thirdly, we consider that the proposed fixed-point theorem has the advantage of relaxing the constraint of the domain of compactness, which is necessary for several fixed-point theorems. Our findings generalize, extend, and complement previously published findings.

The paper is organized as follows: Section 2 presents the preliminary concepts concerning fractional calculus, MNC, and FPT that are pertinent to our study. In Section 3, we focus on the solvability of Equation (1). In Section 4, we present two examples to illustrate the applicability of our findings. in Section 5, our conclusions are presented.

2. Preliminaries

In this section, we provide notations, definitions, and additional information to support discussion of our principal findings.

Suppose that $\mathtt{E}$ is a Banach space with the norm $\|.\|_{\mathtt{E}}$. The symbol $\mathtt{B}[\theta, \mathtt{v}_0]$ represents the closed ball centered at θ together with radius $\mathtt{v}_0$ in $\mathtt{E}$. The symbols $\bar{\Omega}$, $Conv\, \Omega$ represent the closure and convex hull of a subset Ω of $\mathtt{E}$, respectively. Denote by $\mathbb{R}$ the set of all real numbers and $\mathbb{R}^+ = [0, \infty)$. Denote by $\mathbb{N}^*$ the set of all natural numbers without zero and $\varnothing$ represents the empty set. Further, assume that $\mathtt{M}_{\mathtt{E}}$ indicates the family of all non-empty and bounded subsets of $\mathtt{E}$, and $\mathtt{N}_{\mathtt{E}}$ indicates its subfamily of all relatively compact subsets.

Suppose that $\mathtt{E} = C(I)$ is the space of real-valued continuous maps defined on I, wherein $I = [1, \mathtt{T}]$. Then, $\mathtt{E}$ is a Banach space together with the norm:

$$\|\mathtt{z}\| = \sup\{|\mathtt{z}(\varrho)| : \varrho \in I\}, \text{ for some } \mathtt{z} \in \mathtt{E}.$$

Definition 1 ([43]). *A function $\chi : \mathtt{M}_{\mathtt{E}} \to \mathbb{R}^+$ is called an MNC in $\mathtt{E}$ if it fulfils the following conditions:*

(i) $\Omega \in \mathtt{M}_{\mathtt{E}}$ *and* $\chi(\Omega) = 0$ *provides* Ω *is precompact;*
(ii) *ker* $\emptyset = \{\Omega \in \mathtt{M}_{\mathtt{E}} : \chi(\Omega) = 0\}$ *is non-void and ker* $\emptyset \subset \mathtt{N}_{\mathtt{E}}$*;*
(iii) $\Omega \subseteq \Omega_1 \Rightarrow \chi(\Omega) \leq \chi(\Omega_1)$*;*
(iv) $\chi(\bar{\Omega}) = \chi(\Omega)$*;*
(v) $\chi(Conv\, \Omega) = \chi(\Omega)$*;*
(vi) $\chi(\rho\Omega + (1-\rho)\Omega_1) \leq \rho\chi(\Omega) + (1-\rho)\chi(\Omega_1)$, $\forall\, 0 \leq \rho \leq 1$*;*
(vii) *if* $\Omega_m \in \mathtt{M}_{\mathtt{E}}$, $\chi(\Omega) = \chi(\bar{\Omega})$, $\Omega_{m+1} \subset \Omega_m$, *where* $m = 1, 2, ...$ *and that* $\lim_{m \to +\infty} \chi(\Omega_n) = 0$. *Then, we can write* $\Omega_\infty = \bigcap_{m=1}^{+\infty} \mathtt{X}_m \neq \varnothing$.

Remark 1. *The family ker* $\emptyset$ *is called the kernel of MNC* χ. *Further,* $\Omega_\infty \in ker\chi$ *and* $\chi(\Omega_\infty) \leq \chi(\Omega_m)$ *for* $m = 1, 2, 3, ...$, *we can find* $\chi(\Omega_\infty) = 0$. *This implies that* $\Omega_\infty \in$ *ker* $\emptyset$.

Theorem 1 ([44], DFPT). *Suppose that χ is an MNC, $\mathtt{E}$ is a Banach space, and $\mathtt{Q} \subseteq \mathtt{E}$ is non-empty, bounded, closed, and convex. In addition, consider $\mathtt{U} : \mathtt{Q} \to \mathtt{Q}$ be a continuous map. If there is*

$$\chi(\mathtt{US}) \leq k\chi(\mathtt{S}), \ \mathtt{S} \subseteq \mathtt{Q},$$

for a constant $k \in [0, 1)$. Then, $\mathtt{U}$ has a fixed point in the set $\mathtt{Q}$.

Definition 2 ([45–47]). *The Riemann–Liouville fractional integral of order $\alpha > 0$, for a continuous map f on $[a, b]$, is defined by*

$$I_a^\alpha f(r) = \frac{1}{\Gamma(\alpha)} \int_a^r f(s)(r-s)^{\alpha-1} ds, \ a < r \leq b,$$

wherein $\Gamma(.)$ is the Euler gamma function. The Riemann–Liouville integral is motivated by the well-known Cauchy formula:

$$\int_a^r ds_1 \int_a^{s_1} ds_2 ... \int_a^{s_{n-1}} f(s_n) ds_n = \frac{1}{(n-1)!} \int_a^r f(s)(r-s)^{n-1} ds, \ n \in \mathbb{N}^*.$$

Definition 3. *The Erdélyi–Kober operator is a fractional integral [48] operator proposed by Arthur Erdélyi (1940) and Hermann Kober (1940). The Erdélyi–Kober fractional integral operator $I_{\zeta,a}^{\nu,\alpha}$, where $\zeta > 0, \alpha > 0, a > 0$, and $\nu \in \mathbb{R}$, for a sufficiently well-behaved continuous function $f(\omega)$ is defined by*

$$I_{\zeta,a}^{\nu,\alpha} f(\omega) = \frac{\zeta}{\Gamma(\alpha)} \omega^{-\zeta(\alpha+\nu)} \int_a^\omega \frac{s^{\zeta(\nu+1)-1} f(s)}{(\omega^\zeta - s^\zeta)^{1-\alpha}} ds.$$

Definition 4. *The κ-gamma function is a generalization of the classical gamma function introduced by Diaz and Pariguan [49], denoted and defined as:*

$$\Gamma\kappa(\wp) = \lim_{n\to\infty} \frac{n!\kappa^n (n\kappa)^{\frac{\wp}{\kappa}-1}}{(\wp)_{n,\kappa}},\ \kappa > 0,\ \wp > 0,$$

where the notation $(\wp)_{n,\kappa}$ is the Pochhammer's κ-symbol [50] for factorial function. The integral form of the κ-gamma function is denoted and defined as [51]

$$\Gamma\kappa(\wp) = \int_0^\infty e^{-\frac{s^\kappa}{\kappa}} s^{\wp-1} ds,\ \kappa > 0,\ \wp > 0.$$

Further, the Riemann–Liouville κ-fractional integral of the function f of order $\alpha > 0$, as introduced by Mubeen and Habibullah [52], is denoted and defined as

$${}_\kappa I_0^\alpha f(r) = \frac{1}{\kappa\Gamma\kappa(\wp)} \int_0^r f(s)(r-s)^{\frac{\alpha}{\kappa}-1} ds,\ \kappa > 0,\ r > 0.$$

Definition 5. *Suppose that $\mathtt{W}(\neq \emptyset) \subseteq C(I)$ is bounded. Then, the modulus of continuity of $\mathtt{z}$, where $\mathtt{z} \in \mathtt{W}$, and $\epsilon > 0$ is stated as follows:*

$$\mathtt{w}(\mathtt{z}, \epsilon) = \sup\{|\mathtt{z}(\varrho_2) - \mathtt{z}(\varrho_1)| : \varrho_1, \varrho_2 \in I;\ |\varrho_2 - \varrho_1| \leq \epsilon\},$$

together with

$$\mathtt{w}(\mathtt{W}, \epsilon) = \sup\{\mathtt{w}(\mathtt{z}, \epsilon) : \mathtt{z} \in \mathtt{W}\},$$
$$\mathtt{w}_0(\mathtt{W}) = \lim_{\epsilon\to 0} \mathtt{w}(\mathtt{W}, \epsilon),$$

where the map $\mathtt{w}_0(\mathtt{W})$ is a regular MNC in $C(I)$. There also exists a Hausdorff MNC χ, which is governed by $\chi(\mathtt{W}) = \frac{1}{2}\mathtt{w}_0(\mathtt{W})$ (see [43]).

3. New Results

This section mainly concentrates on the solvability of the Equation (1) in the Banach space $C(I)$.

Let $\mathtt{B}_{\mathtt{v}_0} = \{\Psi \in \mathtt{E} : \|\Psi\| \leq \mathtt{v}_0\}$. We consider the following essential hypotheses for proving our main theorem as follows:

$\mathbf{H_1}$. *The function $\mathtt{g} : I \to \mathbb{R}$ is continuous and bounded with $\mathtt{a}_1 = \sup_{\varrho\in I} |\mathtt{g}(\varrho)|$.*

$\mathbf{H_2}$. *The functions $\mathtt{f} : I \times \mathbb{R}^2 \to \mathbb{R}$, $\mathtt{q} : I \times \mathbb{R} \to \mathbb{R}$ are continuous, such that there exist constants $\mathtt{a}_2, \mathtt{a}_3, \mathtt{a}_4 \geq 0$ such that*

$$|\mathtt{f}(\varrho, \mathtt{q}, \mathtt{I}_1) - \mathtt{f}(\varrho, \bar{\mathtt{q}}, \bar{\mathtt{I}}_1)| \leq \mathtt{a}_2|\mathtt{q} - \bar{\mathtt{q}}| + \mathtt{a}_3|\mathtt{I}_1 - \bar{\mathtt{I}}_1|, \text{ for } \mathtt{q}, \bar{\mathtt{q}}, \mathtt{I}_1, \bar{\mathtt{I}}_1 \in \mathbb{R} \text{ and } \varrho \in I.$$

Further, $|\mathtt{q}(\varrho, \mathtt{P}_1) - \mathtt{q}(\varrho, \mathtt{P}_2)| \leq \mathtt{a}_4|\mathtt{P}_1 - \mathtt{P}_2|$, $\mathtt{P}_1, \mathtt{P}_2 \in \mathbb{R}$.

$\mathbf{H_3}$. *The functions $\mathtt{F} : I \times \mathbb{R}^2 \to \mathbb{R}$, $\mathtt{h} : I \times \mathbb{R} \to \mathbb{R}$ are continuous, such that there exist constants $\mathtt{a}_5, \mathtt{a}_6, \mathtt{a}_7 \geq 0$ satisfying*

$$|\mathtt{F}(\varrho, \mathtt{h}, \hat{\mathtt{I}}_1) - \mathtt{F}(\varrho, \bar{\mathtt{h}}, \bar{\hat{\mathtt{I}}}_1)| \leq \mathtt{a}_5(|\mathtt{h} - \bar{\mathtt{h}}|) + \mathtt{a}_6|\hat{\mathtt{I}}_1 - \bar{\hat{\mathtt{I}}}_1|,$$

for $\varrho \in I$ and $\mathtt{h}, \hat{\mathtt{I}}_1, \bar{\mathtt{h}}, \bar{\hat{\mathtt{I}}}_1 \in \mathbb{R}$.
Further, $|\mathtt{h}(\varrho, \mathtt{Q}_1) - \mathtt{h}(\varrho, \mathtt{Q}_2)| \leq \mathtt{a}_7|\mathtt{Q}_1 - \mathtt{Q}_2|$, $\mathtt{Q}_1, \mathtt{Q}_2 \in \mathbb{R}$.

$\mathbf{H_4}$. *There exists $\mathtt{v}_0 \in \mathbb{R}^+$ satisfying*

$$\sup\{|\mathtt{g}(\varrho) + \mathtt{f}(\varrho, \mathtt{q}, \mathtt{I}_1) + \mathtt{F}(\varrho, \mathtt{h}, \hat{\mathtt{I}}_1)| : \varrho \in I,\ \mathtt{q} \in [-\mathtt{q}', \mathtt{q}'],\ \mathtt{I}_1 \in [-\mathtt{I}_1{}', \mathtt{I}_1{}'],\ \mathtt{h} \in [-\mathtt{h}', \mathtt{h}'],$$
$$\hat{\mathtt{I}}_1 \in [-\hat{\mathtt{I}}_1{}', \hat{\mathtt{I}}_1{}']\} \leq \mathtt{v}_0,$$

where

$$\begin{gathered}\mathtt{q}' = \sup\{|\mathtt{q}(\varrho,\Psi(\varrho))| : \varrho \in I,\ \Psi(\varrho) \in [-\mathtt{v}_0,\mathtt{v}_0]\},\\ \mathtt{I_1}' = \sup\{({}^{\phi}_{\kappa}\mathtt{I}_1^{\wp,v}\Psi)(\varrho)| : \varrho \in I,\ \Psi(\varrho) \in [-\mathtt{v}_0,\mathtt{v}_0]\},\\ \mathtt{h}' = \sup\{|\mathtt{h}(\varrho,\Psi(\varrho))| : \varrho \in I,\ \Psi(\varrho) \in [-\mathtt{v}_0,\mathtt{v}_0]\},\\ \text{and } \hat{\mathtt{I}}_1' = \sup\{(\hat{\mathtt{I}}^{\alpha}_{\zeta,1}\Psi)(\varrho)| : \varrho \in I,\ \Psi(\varrho) \in [-\mathtt{v}_0,\mathtt{v}_0]\}.\end{gathered}$$

Further, $\mathtt{a}_2\mathtt{a}_4 + \mathtt{a}_5\mathtt{a}_7 < 1$.

H$_5$. *There exists a positive solution* $\mathtt{v}_0 \in \mathbb{R}^+$ *such that*

$$\mathtt{a}_1 + (\mathtt{a}_2\mathtt{a}_4 + \mathtt{a}_5\mathtt{a}_7)\mathtt{v}_0 + \mathtt{a}_3\frac{\mathtt{v}_0 \exp\left[\frac{(v-1)(\mathtt{T}^{\phi+1})}{v}\right](\phi+1)^{-\frac{\varrho}{\kappa}}}{\wp v^{\frac{\varrho}{\kappa}}\kappa^{\frac{\varrho}{\kappa}-1}\Gamma(\frac{\varrho}{\kappa})}(\mathtt{T}^{\phi+1}-1)^{\frac{\varrho}{\kappa}} + \mathtt{a}_6\frac{\mathtt{v}_0}{\Gamma(\alpha+1)}\mathtt{T}^{\zeta\alpha}$$
$$\leq \mathtt{v}_0.$$

Remark 2. *As a consequence of the hypotheses (H$_2$) and (H$_3$), we find*

$$\begin{gathered}|\mathtt{q}(\varrho,0)| = 0,\\ |\mathtt{f}(\varrho,0,0)| = 0,\\ |\mathtt{h}(\varrho,0)| = 0,\\ \textit{and } |\mathtt{F}(\varrho,0,0)| = 0.\end{gathered}$$

Theorem 2. *Under the assumptions (H$_1$)–(H$_5$) with* `Remark 2`*, we are enabled to assert that Equation (1) possesses a solution in* $C(I)$.

Proof. Let $\mathtt{U} : \mathtt{B}_{\mathtt{v}_0} \to \mathtt{E}$ be an operator stated as follows:

$$(\mathtt{U}\Psi)(\varrho) = \mathtt{g}(\varrho) + \mathtt{f}(\varrho,\mathtt{q}(\varrho,\Psi(\varrho)),({}^{\phi}_{\kappa}\mathtt{I}_1^{\wp,v}\Psi)(\varrho)) + \mathtt{F}(\varrho,\mathtt{h}(\varrho,\Psi(\varrho)),(\hat{\mathtt{I}}^{\alpha}_{\zeta,1}\Psi)(\varrho)).$$

Step 1: We show that $\mathtt{U}$ maps $\mathtt{B}_{\mathtt{v}_0}$ into $\mathtt{B}_{\mathtt{v}_0}$. Let us assert that $\mathtt{U} \in \mathtt{B}_{\mathtt{v}_0}$, we estimate

$$\begin{aligned}|(\mathtt{U}\Psi)(\varrho)| &= |\mathtt{g}(\varrho) + \mathtt{f}(\varrho,\mathtt{q}(\varrho,\Psi(\varrho)),({}^{\phi}_{\kappa}\mathtt{I}_1^{\wp,v}\Psi)(\varrho)) + \mathtt{F}(\varrho,\mathtt{h}(\varrho,\Psi(\varrho)),(\hat{\mathtt{I}}^{\alpha}_{\zeta,1}\Psi)(\varrho))|\\ &\leq |\mathtt{g}(\varrho)| + |\mathtt{f}(\varrho,\mathtt{q}(\varrho,\Psi(\varrho)),({}^{\phi}_{\kappa}\mathtt{I}_1^{\wp,v}\Psi)(\varrho)) - \mathtt{f}(\varrho,0,0)| + |\mathtt{f}(\varrho,0,0)|\\ &\quad + |\mathtt{F}(\varrho,\mathtt{h}(\varrho,\Psi(\varrho)),(\hat{\mathtt{I}}^{\alpha}_{\zeta,1}\Psi)(\varrho)) - \mathtt{F}(\varrho,0,0)| + |\mathtt{F}(\varrho,0,0)|\\ &\leq \mathtt{a}_1 + \mathtt{a}_2|\mathtt{q}(\varrho,\Psi(\varrho))| + \mathtt{a}_3|({}^{\phi}_{\kappa}\mathtt{I}_1^{\wp,v}\Psi)(\varrho)| + |\mathtt{f}(\varrho,0,0)| + \mathtt{a}_5|\mathtt{h}(\varrho,\Psi(\varrho))|\\ &\quad + \mathtt{a}_6|(\hat{\mathtt{I}}^{\alpha}_{\zeta,1}\Psi)(\varrho)| + |\mathtt{F}(\varrho,0,0)|\\ &\leq \mathtt{a}_1 + \mathtt{a}_2|\mathtt{q}(\varrho,\Psi(\varrho)) - \mathtt{q}(\varrho,0)| + |\mathtt{q}(\varrho,0)| + \mathtt{a}_3|({}^{\phi}_{\kappa}\mathtt{I}_1^{\wp,v}\Psi)(\varrho)|\\ &\quad + \mathtt{a}_5|\mathtt{h}(\varrho,\Psi(\varrho)) - \mathtt{h}(\varrho,0)| + |\mathtt{h}(\varrho,0)| + \mathtt{a}_6|(\hat{\mathtt{I}}^{\alpha}_{\zeta,1}\Psi)(\varrho)|\\ &\leq \mathtt{a}_1 + \mathtt{a}_2\mathtt{a}_4|\Psi(\varrho)| + \mathtt{a}_3|({}^{\phi}_{\kappa}\mathtt{I}_1^{\wp,v}\Psi)(\varrho)| + \mathtt{a}_5\mathtt{a}_7|\Psi(\varrho)|\\ &\quad + \mathtt{a}_6|(\hat{\mathtt{I}}^{\alpha}_{\zeta,1}\Psi)(\varrho)|,\end{aligned}$$

wherein

$$\begin{aligned}&|({}^{\phi}_{\kappa}\mathtt{I}_1^{\wp,v}\Psi)(\varrho)|\\ &= \left|\frac{(\phi+1)^{1-\frac{\varrho}{\kappa}}}{v^{\frac{\varrho}{\kappa}}\kappa^{\frac{\varrho}{\kappa}}\Gamma(\frac{\varrho}{\kappa})}\int_1^{\varrho}\exp\left[\frac{(v-1)(\varrho^{\phi+1}-\varsigma^{\phi+1})}{v}\right](\varrho^{\phi+1}-\varsigma^{\phi+1})^{\frac{\varrho}{\kappa}-1}\varsigma^{\phi}\Psi(\varsigma)d\varsigma\right|\\ &\leq \frac{(\phi+1)^{1-\frac{\varrho}{\kappa}}}{v^{\frac{\varrho}{\kappa}}\kappa^{\frac{\varrho}{\kappa}}\Gamma(\frac{\varrho}{\kappa})}\left|\int_1^{\varrho}\exp\left[\frac{(v-1)(\varrho^{\phi+1}-\varsigma^{\phi+1})}{v}\right](\varrho^{\phi+1}-\varsigma^{\phi+1})^{\frac{\varrho}{\kappa}-1}\varsigma^{\phi}\Psi(\varsigma)d\varsigma\right|\\ &\leq \frac{\mathtt{v}_0\exp\left[\frac{(v-1)(T^{\phi+1})}{v}\right](\phi+1)^{1-\frac{\varrho}{\kappa}}}{v^{\frac{\varrho}{\kappa}}\kappa^{\frac{\varrho}{\kappa}}\Gamma(\frac{\varrho}{\kappa})}\int_1^{\varrho}(\varrho^{\phi+1}-\varsigma^{\phi+1})^{\frac{\varrho}{\kappa}-1}\varsigma^{\phi}d\varsigma\\ &\leq \frac{\mathtt{v}_0\exp\left[\frac{(v-1)(\mathtt{T}^{\phi+1})}{v}\right](\phi+1)^{-\frac{\varrho}{\kappa}}}{\wp v^{\frac{\varrho}{\kappa}}\kappa^{\frac{\varrho}{\kappa}-1}\Gamma(\frac{\varrho}{\kappa})}(\mathtt{T}^{\phi+1}-1)^{\frac{\varrho}{\kappa}},\end{aligned}$$

and

$$
\begin{aligned}
|(\hat{\mathtt{I}}^{\alpha}_{\zeta,1}\Psi)(\varrho)| &= \left|\frac{\zeta}{\Gamma(\alpha)}\int_1^{\varrho}\frac{\varsigma^{\zeta-1}\Psi(\varsigma)}{(\varrho^{\zeta}-\varsigma^{\zeta})^{1-\alpha}}d\varsigma\right| \\
&\leq \frac{\zeta}{\Gamma(\alpha)}\int_1^{\varrho}\frac{\varsigma^{\zeta-1}|\Psi(\varsigma)|}{(\varrho^{\zeta}-\varsigma^{\zeta})^{1-\alpha}}d\varsigma \\
&< \frac{\mathtt{v}_0\zeta}{\Gamma(\alpha)}\int_1^{\varrho}\frac{\varsigma^{\zeta-1}}{(\varrho^{\zeta}-\varsigma^{\zeta})^{1-\alpha}}d\varsigma \\
&< \frac{\mathtt{v}_0}{\Gamma(\alpha+1)}\mathtt{T}^{\zeta\alpha}.
\end{aligned}
$$

Thus, if $\|\Psi\| < \mathtt{v}_0$, then

$$
\begin{aligned}
&\|(\mathtt{U}\Psi)\| \\
&< \mathtt{a}_1 + (\mathtt{a}_2\mathtt{a}_4 + \mathtt{a}_5\mathtt{a}_7)\mathtt{v}_0 + \mathtt{a}_3\frac{\mathtt{v}_0\exp\left[\frac{(\upsilon-1)(\mathtt{T}^{\phi+1})}{\upsilon}\right](\phi+1)^{-\frac{\wp}{\kappa}}}{\wp\upsilon^{\frac{\wp}{\kappa}}\kappa^{\frac{\wp}{\kappa}-1}\Gamma(\frac{\wp}{\kappa})}(\mathtt{T}^{\phi+1}-1)^{\frac{\wp}{\kappa}} \\
&\quad +\mathtt{a}_6\frac{\mathtt{v}_0}{\Gamma(\alpha+1)}\mathtt{T}^{\zeta\alpha}.
\end{aligned}
$$

Finally, from the hypothesis H_5, we infer that $\|(\mathtt{U}\Psi)\| < \mathtt{v}_0$, i.e., $\mathtt{U}$ maps $\mathtt{B}_{\mathtt{v}_0}$ into itself.

Step 2: We show that $\mathtt{U}$ is continuous in $\mathtt{B}_{\mathtt{v}_0}$. To do this, suppose that $\epsilon > 0$, together with $\Psi, \bar{\Psi} \in \mathtt{B}_{\mathtt{v}_0}$ such that $\|\Psi - \bar{\Psi}\| < \epsilon$, we estimate

$$
\begin{aligned}
&|(\mathtt{U}\Psi)(\varrho) - (\mathtt{U}\bar{\Psi})(\varrho)| \\
&\leq |\mathtt{g}(\varrho) + \mathtt{f}(\varrho,\mathtt{q}(\varrho,\Psi(\varrho)),({}^{\phi}_{\kappa}\mathtt{I}^{\wp,\upsilon}_1\Psi)(\varrho)) + \mathtt{F}(\varrho,\mathtt{h}(\varrho,\Psi(\varrho)),(\hat{\mathtt{I}}^{\alpha}_{\zeta,1}\Psi)(\varrho)) \\
&\quad -\mathtt{g}(\varrho) + \mathtt{f}(\varrho,\mathtt{q}(\varrho,\bar{\Psi}(\varrho)),({}^{\phi}_{\kappa}\mathtt{I}^{\wp,\upsilon}_1\bar{\Psi})(\varrho)) + \mathtt{F}(\varrho,\mathtt{h}(\varrho,\bar{\Psi}(\varrho)),(\hat{\mathtt{I}}^{\alpha}_{\zeta,1}\bar{\Psi})(\varrho))|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \mathtt{a}_2|\mathtt{q}(\varrho,\Psi(\varrho)) - \mathtt{q}(\varrho,,\bar{\Psi}(\varrho))| + \mathtt{a}_5|\mathtt{h}(\varrho,\Psi(\varrho)) - \mathtt{h}(\varrho,\bar{\Psi}(\varrho))| \\
&\quad +\mathtt{a}_3|({}^{\phi}_{\kappa}\mathtt{I}^{\wp,\upsilon}_1\Psi)(\varrho) - ({}^{\phi}_{\kappa}\mathtt{I}^{\wp,\upsilon}_1\bar{\Psi})(\varrho)| + \mathtt{a}_6|(\hat{\mathtt{I}}^{\alpha}_{\zeta,1}\Psi)(\varrho)) - (\hat{\mathtt{I}}^{\alpha}_{\zeta,1}\bar{\Psi}(\varrho))| \\
&\leq \mathtt{a}_2\mathtt{a}_4|\Psi(\varrho) - \bar{\Psi}(\varrho)| + \mathtt{a}_5\mathtt{a}_7|\Psi(\varrho) - \bar{\Psi}(\varrho)| + \mathtt{a}_3|({}^{\phi}_{\kappa}\mathtt{I}^{\wp,\upsilon}_1\Psi)(\varrho) - ({}^{\phi}_{\kappa}\mathtt{I}^{\wp,\upsilon}_1\bar{\Psi})(\varrho)| \\
&\quad +\mathtt{a}_6|(\hat{\mathtt{I}}^{\alpha}_{\zeta,1}\Psi)(\varrho)) - (\hat{\mathtt{I}}^{\alpha}_{\zeta,1}\bar{\Psi}(\varrho))|,
\end{aligned}
$$

wherein

$$
\begin{aligned}
&|({}^{\phi}_{\kappa}\mathtt{I}^{\wp,\upsilon}_1\Psi)(\varrho) - ({}^{\phi}_{\kappa}\mathtt{I}^{\wp,\upsilon}_1\bar{\Psi})(\varrho)| \\
&= \Bigg|\frac{(\phi+1)^{1-\frac{\wp}{\kappa}}}{\upsilon^{\frac{\wp}{\kappa}}\kappa^{\frac{\wp}{\kappa}}\Gamma(\frac{\wp}{\kappa})}\int_1^{\varrho}\exp\left[\frac{(\upsilon-1)(\varrho^{\phi+1}-\varsigma^{\phi+1})}{\upsilon}\right](\varrho^{\phi+1}-\varsigma^{\phi+1})^{\frac{\wp}{\kappa}-1}\varsigma^{\phi} \\
&\qquad (\Psi(\varsigma)-\bar{\Psi}(\varsigma))d\varsigma\Bigg| \\
&\leq \frac{(\phi+1)^{1-\frac{\wp}{\kappa}}}{\upsilon^{\frac{\wp}{\kappa}}\kappa^{\frac{\wp}{\kappa}}\Gamma(\frac{\wp}{\kappa})}\Bigg|\int_1^{\varrho}\exp\left[\frac{(\upsilon-1)(\varrho^{\phi+1}-\varsigma^{\phi+1})}{\upsilon}\right](\varrho^{\phi+1}-\varsigma^{\phi+1})^{\frac{\wp}{\kappa}-1}\varsigma^{\phi} \\
&\qquad (\Psi(\varsigma)-\bar{\Psi}(\varsigma))d\varsigma\Bigg| \\
&< \frac{\epsilon\exp\left[\frac{(\upsilon-1)(\mathtt{T}^{\phi+1})}{\upsilon}\right](\phi+1)^{-\frac{\wp}{\kappa}}}{\wp\upsilon^{\frac{\wp}{\kappa}}\kappa^{\frac{\wp}{\kappa}-1}\Gamma(\frac{\wp}{\kappa})}(\mathtt{T}^{\phi+1}-1)^{\frac{\wp}{\kappa}},
\end{aligned}
$$

and

$$
\begin{aligned}
|(\hat{\mathtt{I}}^{\alpha}_{\zeta,1}\Psi)(\varrho)) - (\hat{\mathtt{I}}^{\alpha}_{\zeta,1}\bar{\Psi}(\varrho))| &= \left|\frac{\zeta}{\Gamma(\alpha)}\int_1^{\varrho}\frac{\varsigma^{\zeta-1}(\Psi(\varsigma)-\bar{\Psi}(\varsigma))}{(\varrho^{\zeta}-\varsigma^{\zeta})^{1-\alpha}}d\varsigma\right| \\
&\leq \frac{\zeta}{\Gamma(\alpha)}\int_1^{\varrho}\frac{\varsigma^{\zeta-1}|\Psi(\varsigma)-\bar{\Psi}(\varsigma)|}{(\varrho^{\zeta}-\varsigma^{\zeta})^{1-\alpha}}d\varsigma \\
&< \frac{\zeta\epsilon}{\Gamma(\alpha)}\int_1^{\varrho}\frac{\varsigma^{\zeta-1}}{(\varrho^{\zeta}-\varsigma^{\zeta})^{1-\alpha}}d\varsigma \\
&< \frac{\epsilon}{\Gamma(\alpha+1)}\mathtt{T}^{\zeta\alpha}.
\end{aligned}
$$

Thus, if $\|\Psi - \bar{\Psi}\| < \epsilon$, then

$$\begin{aligned}
&\|(\mathtt{U}\Psi)(\varrho) - (\mathtt{U}\bar{\Psi})(\varrho)\| \\
&< \mathtt{a}_2\mathtt{a}_4\epsilon + \mathtt{a}_5\mathtt{a}_7\epsilon + \mathtt{a}_3 \frac{\epsilon \exp\left[\frac{(v-1)(\mathtt{T}^{\phi+1})}{v}\right](\phi+1)^{-\frac{\wp}{\kappa}}}{\wp v^{\frac{\wp}{\kappa}} \kappa^{\frac{\wp}{\kappa}-1} \Gamma(\frac{\wp}{\kappa})} (\mathtt{T}^{\phi+1} - 1)^{\frac{\wp}{\kappa}} + \mathtt{a}_6 \frac{\epsilon}{\Gamma(\alpha+1)} \mathtt{T}^{\zeta\alpha}.
\end{aligned}$$

Now, we obtain

$$\|(\mathtt{U}\Psi)(\varrho) - (\mathtt{U}\bar{\Psi})(\varrho)\| \to 0, \text{ as } \epsilon \to 0.$$

This implies that $\mathtt{U} : \mathtt{B}_{\mathtt{v}_0} \to \mathtt{B}_{\mathtt{v}_0}$ is continuous.

Step 3: We prove that an estimate of $\mathtt{U}$ with respect to w_0.
To do this, suppose a fixed, arbitrary, $\epsilon > 0$, and $\mathtt{W}$ are a non-empty subset of $\mathtt{B}_{\mathtt{v}_0}$. Further, we take $\Psi \in \mathtt{W}$ and $\varrho_1, \varrho_2 \in I = [1, \mathtt{T}]$ together with $\varrho_1 \leq \varrho_2$ so that $|\varrho_2 - \varrho_1| \leq \epsilon$. Then, we estimate

$$\begin{aligned}
&|(\mathtt{U}\Psi)(\varrho_2) - (\mathtt{U}\Psi)(\varrho_1)| \\
&= |\mathtt{g}(\varrho_2) + \mathtt{f}(\varrho_2, \mathtt{q}(\varrho_2, \Psi(\varrho_2)), ({}_{\kappa}^{\phi}\mathtt{I}_1^{\wp,v}\Psi)(\varrho_2)) + \mathtt{F}(\varrho_2, \mathtt{h}(\varrho_2, \Psi(\varrho_2)), (\hat{\mathtt{I}}_{\zeta,1}^{\alpha}\Psi)(\varrho_2)) \\
&\quad - \mathtt{g}(\varrho_1) + \mathtt{f}(\varrho_1, \mathtt{q}(\varrho_1, \Psi(\varrho_1)), ({}_{\kappa}^{\phi}\mathtt{I}_1^{\wp,v}\Psi)(\varrho_1)) + \mathtt{F}(\varrho_1, \mathtt{h}(\varrho_1, \Psi(\varrho_1)), (\hat{\mathtt{I}}_{\zeta,1}^{\alpha}\Psi)(\varrho_1))| \\
&\leq |\mathtt{g}(\varrho_2) - \mathtt{g}(\varrho_1)| \\
&\quad + |\mathtt{f}(\varrho_2, \mathtt{q}(\varrho_2, \Psi(\varrho_2)), ({}_{\kappa}^{\phi}\mathtt{I}_1^{\wp,v}\Psi)(\varrho_2)) - \mathtt{f}(\varrho_1, \mathtt{q}(\varrho_1, \Psi(\varrho_1)), ({}_{\kappa}^{\phi}\mathtt{I}_1^{\wp,v}\Psi)(\varrho_1))| \\
&\quad + |\mathtt{F}(\varrho_2, \mathtt{h}(\varrho_2, \Psi(\varrho_2)), (\hat{\mathtt{I}}_{\zeta,1}^{\alpha}\Psi)(\varrho_2)) - \mathtt{F}(\varrho_1, \mathtt{h}(\varrho_1, \Psi(\varrho_1)), (\hat{\mathtt{I}}_{\zeta,1}^{\alpha}\Psi)(\varrho_1))| \\
&\leq |\mathtt{g}(\varrho_2) - \mathtt{g}(\varrho_1)| \\
&\quad + |\mathtt{f}(\varrho_2, \mathtt{q}(\varrho_2, \Psi(\varrho_2)), ({}_{\kappa}^{\phi}\mathtt{I}_1^{\wp,v}\Psi)(\varrho_2)) - \mathtt{f}(\varrho_2, \mathtt{q}(\varrho_2, \Psi(\varrho_2)), ({}_{\kappa}^{\phi}\mathtt{I}_1^{\wp,v}\Psi)(\varrho_1))| \\
&\quad + |\mathtt{f}(\varrho_2, \mathtt{q}(\varrho_2, \Psi(\varrho_2)), ({}_{\kappa}^{\phi}\mathtt{I}_1^{\wp,v}\Psi)(\varrho_1)) - \mathtt{f}(\varrho_2, \mathtt{q}(\varrho_1, \Psi(\varrho_1)), ({}_{\kappa}^{\phi}\mathtt{I}_1^{\wp,v}\Psi)(\varrho_1))| \\
&\quad + |\mathtt{f}(\varrho_2, \mathtt{q}(\varrho_1, \Psi(\varrho_1)), ({}_{\kappa}^{\phi}\mathtt{I}_1^{\wp,v}\Psi)(\varrho_1)) - \mathtt{f}(\varrho_1, \mathtt{q}(\varrho_1, \Psi(\varrho_1)), ({}_{\kappa}^{\phi}\mathtt{I}_1^{\wp,v}\Psi)(\varrho_1))| \\
&\quad + |\mathtt{F}(\varrho_2, \mathtt{h}(\varrho_2, \Psi(\varrho_2)), (\hat{\mathtt{I}}_{\zeta,1}^{\alpha}\Psi)(\varrho_2)) - \mathtt{F}(\varrho_2, \mathtt{h}(\varrho_2, \Psi(\varrho_2)), (\hat{\mathtt{I}}_{\zeta,1}^{\alpha}\Psi)(\varrho_1))| \\
&\quad + |\mathtt{F}(\varrho_2, \mathtt{h}(\varrho_2, \Psi(\varrho_2)), (\hat{\mathtt{I}}_{\zeta,1}^{\alpha}\Psi)(\varrho_1)) - \mathtt{F}(\varrho_2, \mathtt{h}(\varrho_1, \Psi(\varrho_1)), (\hat{\mathtt{I}}_{\zeta,1}^{\alpha}\Psi)(\varrho_1))| \\
&\quad + |\mathtt{F}(\varrho_2, \mathtt{h}(\varrho_1, \Psi(\varrho_1)), (\hat{\mathtt{I}}_{\zeta,1}^{\alpha}\Psi)(\varrho_1)) - \mathtt{F}(\varrho_1, \mathtt{h}(\varrho_1, \Psi(\varrho_1)), (\hat{\mathtt{I}}_{\zeta,1}^{\alpha}\Psi)(\varrho_1))| \\
&\leq \mathtt{w}(\mathtt{g}, \epsilon) + \mathtt{a}_3|({}_{\kappa}^{\phi}\mathtt{I}_1^{\wp,v}\Psi)(\varrho_2) - ({}_{\kappa}^{\phi}\mathtt{I}_1^{\wp,v}\Psi)(\varrho_1)| + \mathtt{a}_2|\mathtt{q}(\varrho_2, \Psi(\varrho_2)) - \mathtt{q}(\varrho_1, \Psi(\varrho_1))| \\
&\quad + \mathtt{w}_{\mathtt{f}}(I, \epsilon) + \mathtt{a}_6|(\hat{\mathtt{I}}_{\zeta,1}^{\alpha}\Psi)(\varrho_2)) - (\hat{\mathtt{I}}_{\zeta,1}^{\alpha}\Psi)(\varrho_1))| + \mathtt{a}_5|\mathtt{h}(\varrho_2, \Psi(\varrho_2)) - \mathtt{h}(\varrho_1, \Psi(\varrho_1))| \\
&\quad + w_{\mathtt{F}}(I, \epsilon) \\
&\leq \mathtt{w}(\mathtt{g}, \epsilon) + \mathtt{a}_3|({}_{\kappa}^{\phi}\mathtt{I}_1^{\wp,v}\Psi)(\varrho_2) - ({}_{\kappa}^{\phi}\mathtt{I}_1^{\wp,v}\Psi)(\varrho_1)| + \mathtt{a}_2|\mathtt{q}(\varrho_2, \Psi(\varrho_2)) - \mathtt{q}(\varrho_2, \Psi(\varrho_1))| \\
&\quad + |\mathtt{q}(\varrho_2, \Psi(\varrho_1)) - \mathtt{q}(\varrho_1, \Psi(\varrho_1))| + \mathtt{w}_{\mathtt{f}}(I, \epsilon) \\
&\quad + \mathtt{a}_6|(\hat{\mathtt{I}}_{\zeta,1}^{\alpha}\Psi)(\varrho_2)) - (\hat{\mathtt{I}}_{\zeta,1}^{\alpha}\Psi)(\varrho_1))| + \mathtt{a}_5|\mathtt{h}(\varrho_2, \Psi(\varrho_2)) - \mathtt{h}(\varrho_2, \Psi(\varrho_1))| \\
&\quad + |\mathtt{h}(\varrho_2, \Psi(\varrho_1)) - \mathtt{h}(\varrho_1, \Psi(\varrho_1))| + w_{\mathtt{F}}(I, \epsilon) \\
&\leq \mathtt{w}(\mathtt{g}, \epsilon) + \mathtt{a}_2\mathtt{a}_4|(\Psi(\varrho_2) - \Psi(\varrho_1))| + \mathtt{a}_2\mathtt{w}_{\mathtt{q}}(I, \epsilon) + \mathtt{w}_{\mathtt{f}}(I, \epsilon) \\
&\quad + \mathtt{a}_3|({}_{\kappa}^{\phi}\mathtt{I}_1^{\wp,v}\Psi)(\varrho_2) - ({}_{\kappa}^{\phi}\mathtt{I}_1^{\wp,v}\Psi)(\varrho_1)| + \mathtt{a}_6|(\hat{\mathtt{I}}_{\zeta,1}^{\alpha}\Psi)(\varrho_2)) - (\hat{\mathtt{I}}_{\zeta,1}^{\alpha}\Psi)(\varrho_1))| \\
&\quad + \mathtt{a}_5\mathtt{a}_7|\Psi(\varrho_2)) - \Psi(\varrho_1))| + \mathtt{a}_5\mathtt{w}_{\mathtt{h}}(I, \epsilon) + \mathtt{w}_{\mathtt{F}}(I, \epsilon),
\end{aligned}$$

wherein

$$\begin{aligned}
\mathtt{w}_{\mathtt{F}}(I, \epsilon) &= \sup\{|\mathtt{F}(\varrho_2, \mathtt{h}, \mathtt{J}_1) - \mathtt{F}(\varrho_1, \mathtt{h}, \mathtt{J}_1)| : \varrho_1, \varrho_2 \in I; |\varrho_2 - \varrho_1| \leq \epsilon\}, \\
\mathtt{w}_{\mathtt{q}}(I, \epsilon) &= \sup\{|\mathtt{q}(\varrho_2, \Psi) - \mathtt{q}(\varrho_1, \Psi)| : \varrho_1, \varrho_2 \in I; |\varrho_2 - \varrho_1| \leq \epsilon\}, \\
\mathtt{w}_{\mathtt{h}}(I, \epsilon) &= \sup\{|\mathtt{h}(\varrho_2, \Psi) - \mathtt{h}(\varrho_1, \Psi)| : \varrho_1, \varrho_2 \in I; |\varrho_2 - \varrho_1| \leq \epsilon\}, \\
\mathtt{w}_{\mathtt{f}}(I, \epsilon) &= \sup\{|\mathtt{f}(\varrho_2, \mathtt{q}, \mathtt{I}_1) - \mathtt{f}(\varrho_1, \mathtt{q}, \mathtt{I}_1)| : \varrho_1, \varrho_2 \in I; |\varrho_2 - \varrho_1| \leq \epsilon\}, \\
\mathtt{w}(\mathtt{g}, \epsilon) &= \sup\{|\mathtt{g}(\varrho_2) - \mathtt{g}(\varrho_1)| : \varrho_1, \varrho_2 \in I; |\varrho_2 - \varrho_1| \leq \epsilon\}.
\end{aligned}$$

Also

$$
\begin{aligned}
&|({}^{\phi}_{\kappa}\mathtt{I}_1^{\wp,v}\Psi)(\varrho_2) - ({}^{\phi}_{\kappa}\mathtt{I}_1^{\wp,v}\Psi)(\varrho_1)| \\
&= \Bigg| \frac{(\phi+1)^{1-\frac{\wp}{\kappa}}}{v^{\frac{\wp}{\kappa}}\kappa^{\frac{\wp}{\kappa}}\Gamma(\frac{\wp}{\kappa})} \int_1^{\varrho_2} \exp\left[\frac{(v-1)(\varrho_2^{\phi+1}-\varsigma^{\phi+1})}{v}\right](\varrho_2^{\phi+1}-\varsigma^{\phi+1})^{\frac{\wp}{\kappa}-1}\varsigma^{\phi}\Psi(\varsigma)d\varsigma \\
&\quad - \frac{(\phi+1)^{1-\frac{\wp}{\kappa}}}{v^{\frac{\wp}{\kappa}}\kappa^{\frac{\wp}{\kappa}}\Gamma(\frac{\wp}{\kappa})} \int_1^{\varrho_1} \exp\left[\frac{(v-1)(\varrho_1^{\phi+1}-\varsigma^{\phi+1})}{v}\right](\varrho_1^{\phi+1}-\varsigma^{\phi+1})^{\frac{\wp}{\kappa}-1}\varsigma^{\phi}\Psi(\varsigma)d\varsigma \Bigg| \\
&= \frac{(\phi+1)^{1-\frac{\wp}{\kappa}}}{v^{\frac{\wp}{\kappa}}\kappa^{\frac{\wp}{\kappa}}\Gamma(\frac{\wp}{\kappa})} \Bigg| \int_1^{\varrho_2} \exp\left[\frac{(v-1)(\varrho_2^{\phi+1}-\varsigma^{\phi+1})}{v}\right](\varrho_2^{\phi+1}-\varsigma^{\phi+1})^{\frac{\wp}{\kappa}-1}\varsigma^{\phi}\Psi(\varsigma)d\varsigma \\
&\quad - \int_1^{\varrho_1} \exp\left[\frac{(v-1)(\varrho_2^{\phi+1}-\varsigma^{\phi+1})}{v}\right](\varrho_2^{\phi+1}-\varsigma^{\phi+1})^{\frac{\wp}{\kappa}-1}\varsigma^{\phi}\Psi(\varsigma)d\varsigma \Bigg| \\
&\quad + \frac{(\phi+1)^{1-\frac{\wp}{\kappa}}}{v^{\frac{\wp}{\kappa}}\kappa^{\frac{\wp}{\kappa}}\Gamma(\frac{\wp}{\kappa})} \Bigg| \int_1^{\varrho_1} \exp\left[\frac{(v-1)(\varrho_2^{\phi+1}-\varsigma^{\phi+1})}{v}\right](\varrho_2^{\phi+1}-\varsigma^{\phi+1})^{\frac{\wp}{\kappa}-1}\varsigma^{\phi}\Psi(\varsigma)d\varsigma \\
&\quad - \int_1^{\varrho_1} \exp\left[\frac{(v-1)(\varrho_1^{\phi+1}-\varsigma^{\phi+1})}{v}\right](\varrho_1^{\phi+1}-\varsigma^{\phi+1})^{\frac{\wp}{\kappa}-1}\varsigma^{\phi}\Psi(\varsigma)d\varsigma \Bigg| \\
&\leq \frac{(\phi+1)^{1-\frac{\wp}{\kappa}}}{v^{\frac{\wp}{\kappa}}\kappa^{\frac{\wp}{\kappa}}\Gamma(\frac{\wp}{\kappa})} \int_{\varrho_1}^{\varrho_2} \exp\left[\frac{(v-1)(\varrho_2^{\phi+1}-\varsigma^{\phi+1})}{v}\right](\varrho_2^{\phi+1}-\varsigma^{\phi+1})^{\frac{\wp}{\kappa}-1}\varsigma^{\phi}|\Psi(\varsigma)|d\varsigma \\
&\quad + \frac{(\phi+1)^{1-\frac{\wp}{\kappa}}}{v^{\frac{\wp}{\kappa}}\kappa^{\frac{\wp}{\kappa}}\Gamma(\frac{\wp}{\kappa})} \int_1^{\varrho_1} \Bigg| \Bigg(\exp\left[\frac{(v-1)(\varrho_2^{\phi+1}-\varsigma^{\phi+1})}{v}\right](\varrho_2^{\phi+1}-\varsigma^{\phi+1})^{\frac{\wp}{\kappa}-1} \\
&\quad - \exp\left[\frac{(v-1)(\varrho_1^{\phi+1}-\varsigma^{\phi+1})}{v}\right](\varrho_1^{\phi+1}-\varsigma^{\phi+1})^{\frac{\wp}{\kappa}-1}\Bigg)\varsigma^{\phi}\Psi(\varsigma)\Bigg| d\varsigma \\
&\leq \frac{\exp\left[\frac{(v-1)(T^{\phi+1})}{v}\right](\phi+1)^{-\frac{\wp}{\kappa}}}{\wp v^{\frac{\wp}{\kappa}}\kappa^{\frac{\wp}{\kappa}-1}\Gamma(\frac{\wp}{\kappa})}(T^{\phi+1}-1)^{\frac{\wp}{\kappa}}\|\Psi\| \\
&\quad + \|\Psi\| \frac{(\phi+1)^{1-\frac{\wp}{\kappa}}}{v^{\frac{\wp}{\kappa}}\kappa^{\frac{\wp}{\kappa}}\Gamma(\frac{\wp}{\kappa})} \int_1^{\varrho_1} \Bigg| \Bigg(\exp\left[\frac{(v-1)(\varrho_2^{\phi+1}-\varsigma^{\phi+1})}{v}\right](\varrho_2^{\phi+1}-\varsigma^{\phi+1})^{\frac{\wp}{\kappa}-1} \\
&\quad - \exp\left[\frac{(v-1)(\varrho_1^{\phi+1}-\varsigma^{\phi+1})}{v}\right](\varrho_1^{\phi+1}-\varsigma^{\phi+1})^{\frac{\wp}{\kappa}-1}\Bigg)\varsigma^{\phi}\Bigg| d\varsigma,
\end{aligned}
$$

and

$$
\begin{aligned}
&|(\hat{\mathtt{I}}^{\alpha}_{\zeta,1}\Psi)(\varrho_2) - (\hat{\mathtt{I}}^{\alpha}_{\zeta,1}\Psi)(\varrho_1)| \\
&= \left| \frac{\zeta}{\Gamma(\alpha)} \int_1^{\varrho_2} \frac{\varsigma^{\zeta-1}\Psi(\varsigma)}{(\varrho_2^{\zeta}-\varsigma^{\zeta})^{1-\alpha}}d\varsigma - \frac{\zeta}{\Gamma(\alpha)} \int_1^{\varrho_1} \frac{\varsigma^{\zeta-1}\Psi(\varsigma)}{(\varrho_1^{\zeta}-\varsigma^{\zeta})^{1-\alpha}}d\varsigma \right| \\
&\leq \frac{\zeta}{\Gamma(\alpha)} \left| \int_1^{\varrho_2} \frac{\varsigma^{\zeta-1}\Psi(\varsigma)}{(\varrho_2^{\zeta}-\varsigma^{\zeta})^{1-\alpha}}d\varsigma - \int_1^{\varrho_1} \frac{\varsigma^{\zeta-1}\Psi(\varsigma)}{(\varrho_2^{\zeta}-\varsigma^{\zeta})^{1-\alpha}}d\varsigma \right| \\
&\quad + \frac{\zeta}{\Gamma(\alpha)} \left| \int_1^{\varrho_1} \frac{\varsigma^{\zeta-1}\Psi(\varsigma)}{(\varrho_2^{\zeta}-\varsigma^{\zeta})^{1-\alpha}}d\varsigma - \int_1^{\varrho_1} \frac{\varsigma^{\zeta-1}\Psi(\varsigma)}{(\varrho_1^{\zeta}-\varsigma^{\zeta})^{1-\alpha}}d\varsigma \right| \\
&\leq \frac{\zeta}{\Gamma(\alpha)} \int_{\varrho_1}^{\varrho_2} \frac{\varsigma^{\zeta-1}|\Psi(\varsigma)|}{(\varrho_2^{\zeta}-\varsigma^{\zeta})^{1-\alpha}}d\varsigma + \frac{\zeta}{\Gamma(\alpha)} \int_1^{\varrho_1} \left(\frac{\varsigma^{\zeta-1}}{(\varrho_2^{\zeta}-\varsigma^{\zeta})^{1-\alpha}} - \frac{\varsigma^{\zeta-1}}{(\varrho_1^{\zeta}-\varsigma^{\zeta})^{1-\alpha}} \right)|\Psi(\varsigma)|d\varsigma \\
&\leq \frac{\|\Psi\|}{\Gamma(\alpha+1)} \left[2(\varrho_2^{\zeta}-\varrho_1^{\zeta})^{\alpha} - (\varrho_2^{\zeta}-1)^{\alpha} + (\varrho_1^{\zeta}-1)^{\alpha} \right].
\end{aligned}
$$

Thus, if $|\varrho_2 - \varrho_1| \leq \epsilon$, and $\epsilon \to 0$, we get

$$\varrho_2 \to \varrho_1,$$

$$|({}^{\phi}_{\kappa}\mathtt{I}_1^{\wp,v}\Psi)(\varrho_2) - ({}^{\phi}_{\kappa}\mathtt{I}_1^{\wp,v}\Psi)(\varrho_1)| \to 0,$$

and

$$|(\hat{\mathtt{I}}^{\alpha}_{\zeta,1}\Psi)(\varrho_2) - (\hat{\mathtt{I}}^{\alpha}_{\zeta,1}\Psi)(\varrho_1)| \to 0.$$

Hence

$$
\begin{aligned}
& |(\mathtt{U}\Psi)(\varrho_2) - (\mathtt{U}\Psi)(\varrho_1)| \\
& \leq \mathtt{w}(\mathtt{g},\epsilon) + \mathtt{a}_2\mathtt{a}_4\mathtt{w}(\Psi,\epsilon) + \mathtt{a}_2 w_{\mathtt{q}}(I,\epsilon) + \mathtt{w}_{\mathtt{f}}(I,\epsilon) \\
& \quad + \mathtt{a}_3|({}^{\phi}_{\kappa}\mathtt{I}_1^{\wp,\upsilon}\Psi)(\varrho_2) - ({}^{\phi}_{\kappa}\mathtt{I}_1^{\wp,\upsilon}\Psi)(\varrho_1)| + \mathtt{a}_6|(\hat{\mathtt{I}}^{\alpha}_{\zeta,1}\Psi)(\varrho_2)) - (\hat{\mathtt{I}}^{\alpha}_{\zeta,1}\Psi)(\varrho_1))| \\
& \quad + \mathtt{a}_5\mathtt{a}_7 w(\Psi,\epsilon) + \mathtt{a}_5 w_{\mathtt{h}}(I,\epsilon) + w_{\mathtt{F}}(I,\epsilon), \\
\text{i.e.,} \quad & \mathtt{w}(\mathtt{U}\Psi,\epsilon) \\
& \leq \mathtt{w}(\mathtt{g},\epsilon) + (\mathtt{a}_5\mathtt{a}_7 + \mathtt{a}_2\mathtt{a}_4)\mathtt{w}(\Psi,\epsilon) + \mathtt{a}_2 w_{\mathtt{q}}(I,\epsilon) + \mathtt{w}_{\mathtt{f}}(I,\epsilon) \\
& \quad + \mathtt{a}_3|({}^{\phi}_{\kappa}\mathtt{I}_1^{\wp,\upsilon}\Psi)(\varrho_2) - ({}^{\phi}_{\kappa}\mathtt{I}_1^{\wp,\upsilon}\Psi)(\varrho_1)| + \mathtt{a}_6|(\hat{\mathtt{I}}^{\alpha}_{\zeta,1}\Psi)(\varrho_2)) - (\hat{\mathtt{I}}^{\alpha}_{\zeta,1}\Psi)(\varrho_1))| \\
& \quad + \mathtt{a}_5\mathtt{w}_{\mathtt{h}}(I,\epsilon) + \mathtt{w}_{\mathtt{F}}(I,\epsilon).
\end{aligned}
$$

By utilizing the uniform continuity of the functions $\mathtt{g}$, $\mathtt{q}$, $\mathtt{f}$, $\mathtt{h}$, and $\mathtt{F}$ on I, $I \times [-\mathtt{v}_0, \mathtt{v}_0]$, $I \times [-\mathtt{q}', \mathtt{q}'] \times [-\mathtt{I}_1{}', \mathtt{I}_1{}']$, $I \times [-\mathtt{v}_0, \mathtt{v}_0]$, and $I \times [-\mathtt{h}', \mathtt{h}'] \times [-\hat{\mathtt{I}}_1{}', \hat{\mathtt{I}}_1{}']$, respectively, we obtain $\mathtt{w}(\mathtt{g},\epsilon) \to 0$, $\mathtt{w}_{\mathtt{f}}(I,\epsilon) \to 0$, $\mathtt{w}_{\mathtt{q}}(I,\epsilon) \to 0$, $\mathtt{w}_{\mathtt{h}}(I,\epsilon) \to 0$, and $\mathtt{w}_{\mathtt{F}}(I,\epsilon) \to 0$, when $\epsilon \to 0$.
Thus, taking $\sup_{\Psi \in \mathtt{W}}$ and $\epsilon \to 0$, we find$\mathtt{w}_0(\mathtt{UW}) \leq (\mathtt{a}_2\mathtt{a}_4 + \mathtt{a}_5\mathtt{a}_7)\mathtt{w}_0(\mathtt{W})$.
Hence, by utilizing DFPT, we can say that $\mathtt{U}$ possesses a fixed point in $\mathtt{W} \subseteq \mathtt{B}_{\mathtt{v}_0}$.
Consequently, the functional integral Equation (1) possesses a solution in $C(I)$.

□

Now, we will study some applications to verify the efficiency of our findings that arise in modeling biological populations.

4. Applications

Example 1. *Let us consider that the fractional order model emerges in the form of mixed-type non-linear functional integral equations given as follows:*

$$
\Psi(\varrho) = \frac{\varrho e^{-\frac{\varrho^2}{2}}}{6} + \frac{\varrho^2 \arctan \Psi(\varrho)}{4 + 5\varrho^2} + \frac{\sin \Psi(\varrho)}{1 + \varrho^2} + \frac{({}^{\frac{1}{3}}_{\frac{1}{3}}\mathtt{I}_1^{3,\frac{1}{3}}\Psi)(\varrho)}{243} + \frac{(\hat{\mathtt{I}}^{\frac{1}{7}}_{\frac{1}{5},1}\Psi)(\varrho)}{35}, \tag{2}
$$

wherein $\Psi(\varrho)$ denotes the surge in the birthrate at any time ϱ, $\varrho \in [1,2] = I$.

Comparing Equation (2) with Equation(1), we get

$$
\mathtt{g}(\varrho) = \frac{\varrho e^{-\frac{\varrho^2}{2}}}{6},
$$

$$
\mathtt{f}(\varrho, \mathtt{q}, \mathtt{I}_1) = \mathtt{q}(\varrho, \Psi) + \frac{\mathtt{I}_1}{243},
$$

$$
\mathtt{q}(\varrho, \Psi) = \frac{\varrho^2 \arctan \Psi(\varrho)}{4+5\varrho^2},
$$

$$
({}^{\frac{1}{3}}_{\frac{1}{3}}\mathtt{I}_1^{3,\frac{1}{3}}\Psi)(\varrho)) = \frac{3^{20}}{4^8\Gamma(9)} \int_1^{\varrho} \exp[-2(\varrho^{\frac{4}{3}} - \varsigma^{\frac{4}{3}})](\varrho^{\frac{4}{3}} - \varsigma^{\frac{4}{3}})^8 \varsigma^{\frac{1}{3}} \Psi(\varsigma) d\varsigma,
$$

$$
(\hat{\mathtt{I}}^{\frac{1}{7}}_{\frac{1}{5},1}\Psi)(\varrho) = \frac{1}{5\Gamma(\frac{1}{7})} \int_1^{\varrho} \frac{\varsigma^{\frac{-4}{5}}}{(\varrho^{\frac{1}{7}} - \varsigma^{\frac{1}{7}})^{\frac{6}{7}}} \Psi(\varsigma) d\varsigma,
$$

$$
\mathtt{F}(\varrho, \mathtt{h}, \hat{\mathtt{I}}_1) = \mathtt{h}(\varrho, \Psi) + \frac{\hat{\mathtt{I}}_1}{35},
$$

$$
\textit{and} \quad \mathtt{h}(\varrho, \Psi) = \frac{\sin \Psi(\varrho)}{1+\varrho^2},
$$

wherein $\Psi(\varsigma)$ denotes the probability that the female lives to age ς. $\mathtt{g}(\varrho)$, $\mathtt{q}(\varrho,\Psi)$, and $\mathtt{h}(\varrho,\Psi)$ denote the terms added to allow for girls already born before the oldest child-bearing women of age ($\varsigma = 2$) were born. $\mathtt{F}$ and $\mathtt{f}$ denote the survival functions, which are the fraction of the number of people that survive to age ϱ.

It is clear that the functions $\mathtt{g}$, $\mathtt{f}$, $\mathtt{q}$, $\mathtt{F}$, and $\mathtt{h}$ are continuous satisfying

$$\begin{aligned}
&|\mathtt{f}(\varrho,\mathtt{q},\mathtt{I}_1) - \mathtt{f}(\varrho,\bar{\mathtt{h}},\bar{\mathtt{I}}_1)| \leq |\mathtt{q} - \bar{\mathtt{q}}| + \tfrac{1}{243}|\mathtt{I}_1 - \bar{\mathtt{I}}_1|,\\
&|\mathtt{q}(\varrho,\mathtt{P}_1) - \mathtt{q}(\varrho,\mathtt{P}_2)| \leq \tfrac{|\mathtt{P}_1-\mathtt{P}_2|}{6},\\
&|\mathtt{F}(\varrho,\mathtt{h},\hat{\mathtt{I}}_1) - \mathtt{F}(\varrho,\bar{\mathtt{h}},\bar{\hat{\mathtt{I}}}_1)| \leq |\mathtt{h} - \bar{\mathtt{h}}| + \tfrac{1}{35}|\hat{\mathtt{I}}_1 - \bar{\hat{\mathtt{I}}}_1|,\\
&\textit{and } |\mathtt{h}(\varrho,\mathtt{Q}_1) - \mathtt{h}(\varrho,\mathtt{Q}_2)| \leq \tfrac{|\mathtt{Q}_1-\mathtt{Q}_2|}{2},
\end{aligned}$$

Hence, $\mathtt{a}_1 = 0.1010,\ \mathtt{a}_2 = 1,\ \mathtt{a}_3 = \frac{1}{243},\ \mathtt{a}_4 = \frac{1}{6},\ \mathtt{a}_5 = 1,\ \mathtt{a}_6 = \frac{1}{35},\ \mathtt{a}_7 = \frac{1}{2}$, *and* $\mathtt{a}_2\mathtt{a}_4 + \mathtt{a}_5\mathtt{a}_7 = \frac{2}{3} < 1$.

If $\|\Psi\| \leq \mathtt{v}_0$, *then*

$$\mathtt{q}' = \frac{\mathtt{v}_0}{6},\ \mathtt{h}' = \frac{\mathtt{v}_0}{2},\ \mathtt{I}_1{}' = \frac{\mathtt{v}_0 \exp(-2(2^{\frac{4}{3}}))3^{25}(2^{\frac{4}{3}}-1)^9}{4^9\Gamma(9)},\ \hat{\mathtt{I}}_1 = \frac{7\mathtt{v}_0(2^{\frac{1}{5}}-1)^{\frac{1}{7}}}{\Gamma(\frac{1}{7})}.$$

Further, the inequality arising in assumption (H_4) becomes

$$0.1010 + \frac{2}{3}\mathtt{v}_0 + \frac{\mathtt{v}_0 \exp(-2(2^{\frac{4}{3}}))3^{20}(2^{\frac{4}{3}}-1)^9}{4^9\Gamma(9)} + \frac{\mathtt{v}_0(2^{\frac{1}{5}}-1)^{\frac{1}{7}}}{5\Gamma(\frac{1}{7})} \leq \mathtt{v}_0.$$

If we choose $\mathtt{v}_0 = 3$, we get

$$\mathtt{q}' = \frac{1}{2},\ \mathtt{h}' = \frac{3}{2},\ \mathtt{I}_1{}' = \frac{\exp(-2(2^{\frac{4}{3}}))3^{26}(2^{\frac{4}{3}}-1)^9}{4^9\Gamma(9)},\ \hat{\mathtt{I}}_1' = \frac{21(2^{\frac{1}{5}}-1)^{\frac{1}{7}}}{\Gamma(\frac{1}{7})}.$$

Furthermore, the inequality arising in assumption H_5 becomes

$$0.1010 + 2 + \frac{\exp(-2(2^{\frac{4}{3}}))3^{21}(2^{\frac{4}{3}}-1)^9}{4^9\Gamma(9)} + \frac{3(2^{\frac{1}{5}}-1)^{\frac{1}{7}}}{5\Gamma(\frac{1}{7})} < 3.$$

Thus, all the assumptions from (H_1)–(H_5) with `Remark 2` *are satisfied. Hence, based on* `Theorem 2`, *we may conclude that Equation (1) has a solution in $C(I)$.*

Example 2. *In the second example, we consider the following fractional order model emerges in the form of mixed-type non-linear functional integral equations:*

$$\Psi(\varrho) = \frac{e^{-\varrho}}{1+\varrho} + \frac{e^{-(\varrho-1)^2}\Psi(\varrho)}{5} + \frac{\Psi(\varrho)}{3+\varrho^2} + \frac{({}^{\frac{1}{9}}_{\frac{2}{9}}\mathtt{I}_1^{\frac{3}{2},\frac{2}{3}}\Psi)(\varrho)}{3^6} + \frac{(\hat{\mathtt{I}}^{\frac{2}{9}}_{\frac{1}{9},1}\Psi)(\varrho)}{3^4}, \tag{3}$$

wherein $\Psi(\varrho)$ denotes the surge in the birthrate at any time ϱ, $\varrho \in [1,2] = I$.

Comparing Equation (3) with Equation (1), we get

$$\mathtt{g}(\varrho) = \tfrac{e^{-\varrho}}{1+\varrho},$$

$$\mathtt{f}(\varrho, \mathtt{q}, \mathtt{I}_1) = \mathtt{q}(\varrho, \Psi) + \tfrac{\mathtt{I}_1}{3^6},$$

$$\mathtt{q}(\varrho, \Psi) = \frac{e^{-(\varrho-1)^2}\Psi(\varrho)}{5},$$

$$({}^{\frac{1}{9}}_{\frac{2}{9}}\mathtt{I}_1^{\frac{3}{2},\frac{2}{3}}\Psi)(\varrho) = \tfrac{3^{31.75}}{5^{5.75}2^{19.25}\Gamma(6.75)}\int_1^{\varrho}\exp[-0.5(\varrho^{\frac{10}{9}} - \varsigma^{\frac{10}{9}})](\varrho^{\frac{10}{9}} - \varsigma^{\frac{10}{9}})^{5.75}\varsigma^{\frac{1}{9}}\Psi(\varsigma)d\varsigma,$$

$$(\hat{\mathtt{I}}^{\frac{2}{9}}_{\frac{1}{9},1}\Psi)(\varrho) = \tfrac{1}{9\Gamma(\frac{2}{9})}\int_1^{\varrho}\tfrac{\varsigma^{\frac{-8}{9}}}{(\varrho^{\frac{1}{9}}-\varsigma^{\frac{1}{9}})^{\frac{7}{9}}}\Psi(\varsigma)d\varsigma,$$

$$\mathtt{F}(\varrho, \mathtt{h}, \hat{\mathtt{I}}_1) = \mathtt{h}(\varrho, \Psi) + \tfrac{\hat{\mathtt{I}}_1}{3^4},$$

$$\textit{and} \quad \mathtt{h}(\varrho, \Psi) = \tfrac{\Psi(\varrho)}{3+\varrho^2},$$

wherein $\Psi(\varsigma)$ *denotes the probability that the female lives to age* ς*.* $\mathtt{g}(\varrho)$*,* $\mathtt{q}(\varrho, \Psi)$*, and* $\mathtt{h}(\varrho, \Psi)$ *denote the terms added to allow for girls already born before the oldest child-bearing women of age* $(\varsigma = 2)$ *were born.* $\mathtt{F}$ *and* $\mathtt{f}$ *denote the survival functions, which are the fraction of the number of people that survive to age* ϱ*.*

Herein, it is clear that the functions $\mathtt{g}$, $\mathtt{f}$, $\mathtt{q}$, $\mathtt{F}$, *and* $\mathtt{h}$ *are continuous satisfying*

$$|\mathtt{f}(\varrho, \mathtt{q}, \mathtt{I}_1) - \mathtt{f}(\varrho, \bar{\mathtt{h}}, \bar{\mathtt{I}}_1)| \leq |\mathtt{q} - \bar{\mathtt{q}}| + \tfrac{1}{3^6}|\mathtt{I}_1 - \bar{\mathtt{I}}_1|,$$

$$|\mathtt{q}(\varrho, \mathtt{P}_1) - \mathtt{q}(\varrho, \mathtt{P}_2)| \leq \tfrac{|\mathtt{P}_1 - \mathtt{P}_2|}{5},$$

$$|\mathtt{F}(\varrho, \mathtt{h}, \hat{\mathtt{I}}_1) - \mathtt{F}(\varrho, \bar{\mathtt{h}}, \bar{\hat{\mathtt{I}}}_1)| \leq |\mathtt{h} - \bar{\mathtt{h}}| + \tfrac{1}{3^4}|\hat{\mathtt{I}}_1 - \bar{\hat{\mathtt{I}}}_1|,$$

$$\textit{and } |\mathtt{h}(\varrho, \mathtt{Q}_1) - \mathtt{h}(\varrho, \mathtt{Q}_2)| \leq \tfrac{|\mathtt{Q}_1 - \mathtt{Q}_2|}{4}.$$

Hence, $\mathtt{a}_1 = 0.1839$, $\mathtt{a}_2 = 1$, $\mathtt{a}_3 = \frac{1}{3^6}$, $\mathtt{a}_4 = \frac{1}{5}$, $\mathtt{a}_5 = 1$, $\mathtt{a}_6 = \frac{1}{3^4}$, $\mathtt{a}_7 = \frac{1}{4}$, *and* $\mathtt{a}_2\mathtt{a}_4 + \mathtt{a}_5\mathtt{a}_7 = 0.45 < 1$.

If $\|\Psi\| \leq \mathtt{v}_0$*, then*

$$\mathtt{q}' = \tfrac{\mathtt{v}_0}{5},\ \mathtt{h}' = \tfrac{\mathtt{v}_0}{4},\ \mathtt{I}_1{}' = \tfrac{\mathtt{v}_0\exp(-0.5(2^{\frac{10}{9}}))3^{8.75}(2^{\frac{10}{9}}-1)^{6.75}}{5^{6.75}2^{7.75}\Gamma(6.75)},\ \hat{\mathtt{I}}_1' = \tfrac{4.5\mathtt{v}_0(2^{\frac{1}{9}}-1)^{\frac{2}{9}}}{\Gamma(\frac{2}{9})}.$$

Further, the inequality arising in assumption (H$_4$) becomes

$$0.1839 + 0.45\mathtt{v}_0 + \frac{\mathtt{v}_0\exp(-0.5(2^{\frac{10}{9}}))3^{2.75}(2^{\frac{10}{9}} - 1)^{6.75}}{5^{6.75}2^{7.75}\Gamma(6.75)} + \frac{\mathtt{v}_0(2^{\frac{1}{9}} - 1)^{\frac{2}{9}}}{18\Gamma(\frac{2}{9})} \leq \mathtt{v}_0.$$

If we choose $\mathtt{v}_0 = 5.5$*, we get*

$$\mathtt{q}' = \frac{5.5}{5},\ \mathtt{h}' = \frac{5.5}{4},\ \mathtt{I}_1{}' = \frac{5.5\exp(-0.5(2^{\frac{10}{9}}))3^{8.75}(2^{\frac{10}{9}} - 1)^{6.75}}{5^{6.75}2^{7.75}\Gamma(6.75)},\ \hat{\mathtt{I}}_1' = \frac{24.75(2^{\frac{1}{9}} - 1)^{\frac{2}{9}}}{\Gamma(\frac{2}{9})}.$$

Furthermore, the inequality arising in assumption (H$_5$) becomes

$$0.1839 + 2.475 + \frac{5.5\exp(-0.5(2^{\frac{10}{9}}))3^{2.75}(2^{\frac{10}{9}} - 1)^{6.75}}{5^{6.75}2^{7.75}\Gamma(6.75)} + \frac{1.375(2^{\frac{1}{9}} - 1)^{\frac{2}{9}}}{6\Gamma(\frac{2}{9})} < 5.5.$$

Thus, all the assumptions from (H_1)*–*(H_5) *with* `Remark 2` *are satisfied. Hence, based on* `Theorem 2`*, we may conclude that Equation* (1) *has a solution in* $C(I)$*.*

5. Conclusions

In this paper, we investigated the solvability of mixed-type non-linear functional integral equations involving the (κ, ϕ)-type generalized proportional Riemann–Liouville fractional together with Erdélyi–Kober fractional operators arising in biological populations. To do this, we employed fractional calculus, DFPT, and Hausdorff MNC in the Banach space $C(I)$. We also demonstrated the efficiency of our findings with the aid of two relevant numerical examples. This technique can be utilized for various functional integral equations involving distinct fractional operators.

Author Contributions: Conceptualization, V.K.P., L.N.M. and V.N.M.; Methodology, V.K.P., L.N.M. and V.N.M.; Validation, L.N.M., V.N.M. and D.B.; Formal analysis, L.N.M., V.N.M. and D.B.; Investigation, V.K.P., L.N.M. and V.N.M.; Resources, L.N.M., V.N.M. and D.B.; Writing—original draft, V.K.P. and L.N.M.; Writing—review & editing, L.N.M.; Visualization, L.N.M., V.N.M. and D.B.; Supervision, L.N.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare that there are no conflict of interest regarding the publication of this paper.

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