



Article Common Fixed Point Theorems for Two Mappings in Complete *b*-Metric Spaces

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Abstract: Our paper is devoted to the issue of the existence and uniqueness of common fixed points for two mappings in complete *b*-metric spaces by virtue of the new functions *F* and θ , respectively. Moreover, two specific examples to indicate the validity of our results are also given. Eventually, the generalized forms of Jungck fixed point theorem in the above spaces is investigated. Different from related literature, the conditions that the function *F* needs to satisfy are weakened, and *F* only needs to be non-decreasing in this paper. To some extent, our conclusions and methods improve the results of previous literature.

Keywords: existence and uniqueness; common fixed point theorems; *b*-metric spaces

MSC: 47H09; 47H10



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1. Introduction

The conceptual framework of *b*-metric spaces, as a meaningful generalization of metric spaces, was first formally proposed by Czerwik [1] who discussed the convergence of measurable functions and also established the Banach contraction principle in *b*-metric spaces. Subsequently, the Banach contraction principle plays an important role in *b*-metric spaces, and it is one of the most valid tools in the research fields of nonlinear analysis and its applications. In fact, it is extensively regarded as the beginnings of metric fixed point theory.

Thereafter, many scholars have focused on fixed point problems in the tendency of the generalization of *b*-metric spaces. To be specific, Samet [2] fully certified that the class of (α, ψ) -type contractions contains a good deal of contraction-type operators, and the fixed points of the operators can be obtained in virtue of the Picard iteration. Mohanta [3] discussed the existence and uniqueness of common fixed points for mappings defined on a *b*-metric space endowed with a graph. In [4], the authors gave some common fixed point results for a pair of self-mappings that satisfy *g*-generalized weakly contractive conditions in a *b*-metric space endowed with an amorphous binary relation.

In [5], an interesting generalization of the Banach contraction principle was shown by introducing the notion of *F*-contractions, which as a new type of contraction, have been applied to obtain fixed point results for single-valued mappings and multi-valued mappings in *b*-metric spaces. In [6], Cosentinoet et al. introduced the notion of Hardy– Rogers-type *F*-contractions as a generalization of *F*-contractions in complete metric spaces. Moreover, a number of consequences related to *F*-contractions and their extensions have been obtained, for details please see [7–17]. Suzuki [18] investigated fixed point theorems for set-valued *F*-contractions in complete *b*-metric spaces and also proposed a fixed point theorem for single-valued *F*-contractions in complete *b*-metric spaces. Moreover, Mirmostafaee et al. [19] established a set-valued version of Suzuki's fixed point theorem in complete *b*-metric spaces. Jang [20] presented Hardy–Rogers-type and Reich-type common fixed point theorems in complete metric spaces which generalize and unify previously known results. Recently, the existence and uniqueness of fixed points for *F*-contractions in complete Branciari *b*-metric spaces were considered in [21]. The theory of set-valued mappings has been established to extend the framework of fixed point theorems in b-metric spaces and has applications in control theory, convex optimization, differential inclusions, and economics.

Motivated by the above-mentioned discussions, we mainly study the existence and uniqueness of common fixed points for two mappings in complete *b*-metric spaces by virtue of the new functions F and θ , respectively. Furthermore, we present two specific instances to show the availability of our results. In the specific proof process, we discuss and deal with various cases in detail. Compared with the previous results, we weaken the conditions of the function F, which only needs to be non-decreasing. Hence, to some extent, our conclusions and methods improve the results of previous literature.

2. Preliminaries

We begin with some auxiliary lemmas and basic definitions in this section. Let \mathcal{N} be the set of all positive integers and \mathcal{R} be the set of all real numbers respectively.

Definition 1 ([22]). Let V be a non-empty set and consider $s \ge 1$ be a given real number. A function $d_b: V \times V \to [0,\infty)$ is a b-metric if the following conditions are satisfied for every $c_1, c_2, c_3 \in V$:

 $(a_1) d_b(c_1, c_2) = 0$ if and only if $c_1 = c_2$; $(a_2) d_b(c_1, c_2) = d_b(c_2, c_1)$;

$$(a_2) d_b(c_1, c_2) = d_b(c_2, c_1)$$

 $(a_3) d_b(c_1, c_2) \leq s[d_b(c_1, c_3) + d_b(c_3, c_2)].$

In this case, the pair (V, d_b) is called a b-metric space with constant $s \ge 1$.

Definition 2 ([5]). A function \mathbb{F} : $(0, +\infty) \to \mathbb{R}$ belongs to \mathcal{F} if the following conditions are satisfied:

- $(F_1) \mathbb{F}$ is strictly decreasing;
- (F₂) for each sequence $\{\alpha_n\}$ ($\alpha_n > 0$), $\lim_{n \to +\infty} \alpha_n = 0$ if and only if $\lim_{n \to +\infty} \mathbb{F}(\alpha_n) = -\infty$;
- (*F*₃) there exists $k \in (0, 1)$ such that $\lim_{k \to \infty} x^k \mathbb{F}(x) = 0$.

Definition 3 ([5]). Let (V, d) be a metric space and $P : V \to V$ be a mapping. Assume that there exist $\mathbb{F} \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in V$, the inequality d(Px, Py) > 0 implies $\tau + \mathbb{F}(sd(Px, Py)) \leq \mathbb{F}(d(x, y))$, then P is called an F-contraction.

Example 1 ([5]). Let (V, d) be a b-metric space and $F : (0, \infty) \to \mathbb{R}$ be defined by $F(\kappa) = \ln \kappa$. Then F satisfies (F1)-(F3). Each mapping $P: V \to V$ satisfying the inequality of Definition 3 is an *F*-contraction such that

$$d_b(Px, Py) \le e^{-\tau} d_b(x, y)$$
, for all $x, y \in V, Px \neq Py$.

In the case of Px = Py, the inequality also holds, we obtain that every Banach contraction is an F-contraction.

Example 2 ([5]). Consider $F(\kappa) = \ln \kappa + \kappa$, $\kappa > 0$. F satisfies (F1)-(F3) and the inequality of *Definition* **3** *implies*

$$\frac{d_b(Px, Py)}{d_b(x, y)}e^{d_b(Px, Py)-d_b(x, y)} \le e^{-\tau}, \text{for all } x, y \in V, Px \neq Py.$$

Let $L, J: V \to V$ be two mappings defined on a metric space (V, d). If there exists $u \in V$ such that u = L(u), then u is said to be a fixed point of L. Moreover, if u = L(u) = UJ(u), then u is said to be a common fixed point of the mappings L and J.

Theorem 1 ([23]). Let *L* be a continuous self-map on a complete metric space (*V*, *d*). Then *L* has a fixed point if and only if there exist a constant $\varphi \in [0, 1)$ and a map $J : V \to V$ which commutes with *L* and the following conditions hold

$$J(V) \subseteq L(V)$$
 and $d(J(u), J(t)) \leq \varphi d(L(u), L(t))$ for all $u, t \in V$.

Indeed, L and J have a unique common fixed point if the above conditions hold.

Theorem 2 ([24]). Let $L: V \to V$ be a self-map on a complete metric space (V, d). If there exist the constants $\varphi, \omega, \mu, \eta, \lambda \in [0, 1)$ with $\varphi + \omega + \mu + \eta + \lambda < 1$ such that for all $u, t \in V$,

 $d(L(u), L(t)) \le \varphi d(u, L(u)) + \omega d(t, L(t)) + \mu d(u, L(t)) + \eta d(t, L(u)) + \lambda d(u, t),$

then L has a unique fixed point.

Before giving the main results, we first show a useful lemma.

Lemma 1. Let *L* and *J* be self-mappings on a b-metric space (V, d_b) with $s \ge 1$. Suppose that there exist $\varphi, \omega, \mu, \eta, \lambda \in [0, 1)$ with $\varphi + \omega + s\mu + s\eta + \lambda < 1$ such that

$$d_b(L(u_1), J(u_2)) \le \varphi d_b(u_1, L(u_1)) + \omega d_b(u_2, J(u_2)) + \mu d_b(u_1, J(u_2)) + \eta d_b(u_2, L(u_1)) + \lambda d_b(u_1, u_2),$$

for all $u_1, u_2 \in V$. Then

$$d_b(L(u), J(L(u))) \le \frac{\varphi + s\mu + \lambda}{1 - s\mu - \omega} d_b(u, L(u)),$$

$$d_b(J(t), L(J(t))) \le \frac{\omega + s\eta + \lambda}{1 - s\eta - \varphi} d_b(t, J(t)),$$

for all $u \in V$ and $t \in L(V)$.

Proof. Let $u \in V$. It is not difficult to see that

$$d_b(L(u), J(L(u))) \le \varphi d_b(u, L(u)) + \omega d_b(L(u), J(L(u))) + \mu d_b(u, J(L(u))) + \lambda d_b(u, L(u)).$$
(1)

By using condition (a_3) of Definition 1 and adding $-s\mu d_b(u, L(u))$ on both sides of (1), we obtain

$$\begin{aligned} &d_b(L(u), J(L(u))) - s\mu d_b(u, L(u)) \\ &\leq \varphi d_b(u, L(u)) + \omega d_b(L(u), J(L(u))) + \mu d_b(u, J(L(u))) \\ &- s\mu d_b(u, L(u)) + \lambda d_b(u, L(u)) \\ &\leq \varphi d_b(u, L(u)) + \omega d_b(L(u), J(L(u))) + s\mu d_b(L(u), J(L(u))) + \lambda d_b(u, L(u)). \end{aligned}$$

Thus

$$d_b(L(u), J(L(u))) \le \frac{\varphi + s\mu + \lambda}{1 - s\mu - \omega} d_b(u, L(u)).$$

Let $t \in L(V)$. Notice that

$$d_{b}(J(t), L(J(t))) = d_{b}(L(J(t)), J(t)) \leq \varphi d_{b}(J(t), L(J(t))) + \omega d_{b}(t, J(t)) + \eta d_{b}(t, L(J(t))) + \lambda d_{b}(J(t), t).$$
(2)

By adding $-s\eta d_b(J(t), L(J(t)))$ on both sides of (2), we have

$$\begin{aligned} d_b(J(t), L(J(t))) &- s\eta d_b(J(t), L(J(t))) \le \varphi d_b(J(t), L(J(t))) + \omega d_b(t, J(t)) \\ &+ s\eta d_b(t, J(t)) + \lambda d_b(t, J(t)). \end{aligned}$$

Thus

$$d_b(J(t), L(J(t))) \leq \frac{\omega + s\eta + \lambda}{1 - s\eta - \varphi} d_b(t, J(t)).$$

3. Main Results

In this section, we consider the problem of the existence and uniqueness of common fixed points of two mappings on a complete *b*-metric space.

3.1. Existence and Uniqueness of Common Fixed Points for Two Mappings

Theorem 3. Let $L, J : V \to V$ be two mappings on a complete b-metric space (V, d_b) with $s \ge 1$. Suppose that $F: [0, +\infty) \to \mathcal{R}$ is a non-decreasing function, and there exist $\tau > 0, 0 < \varphi, \omega < 1$ and $0 \le \mu, \eta, \lambda < 1$ satisfying the following properties:

(1) $\varphi + \omega + 2s\mu + \lambda < 1$ and $\varphi + \omega + 2s\eta + \lambda < 1$, (2) $s^2 < \frac{1}{pq}$ and $\frac{s\varphi}{1-\eta} < 1$, and (3) for $u, t \in V$, the inequality $d_b(L(u), J(t)) > 0$ implies

 $\tau + F(d_b(L(u), J(t))) \le F(\varphi d_b(u, L(u)) + \omega d_b(t, J(t)) + \mu d_b(u, J(t)) + \eta d_b(t, L(u)) + \lambda d_b(u, t)).$

Then L and J have a unique common fixed point.

Proof. Let *u* be an element of *V*. Put $u_0 = u$. For each $n \in \mathcal{N}$, we define

$$u_{2n-1} = L(u_{2n-2})$$
 and $u_{2n} = J(u_{2n-1})$.

We consider the following four cases:

Case I. If $u_0 = u_1$, that is $u_0 = L(u_0)$, then $u_0 = J(u_0)$. Indeed, if $u_0 \neq J(u_0) = J(L(u_0)) = J(u_1)$, then $d_b(L(u_0), J(u_1)) > 0$. From condition (3), it follows that

$$F(d_b(L(u_0), J(u_1))) < \tau + F(d_b(L(u_0), J(u_1)))$$

$$\leq F(\varphi d_b(u_0, L(u_0)) + \omega d_b(u_1, J(u_1)) + \mu d_b(u_0, J(u_1)))$$

$$+ \eta d_b(u_1, L(u_0)) + \lambda d_b(u_0, u_1))$$

$$= F(\omega d_b(u_1, J(u_1)) + \mu d_b(u_0, J(u_1))).$$

Since *F* is non-decreasing, we obtain

$$(1 - \omega - s\mu)d_b(L(u_0), J(u_1)) \le 0,$$

which yields $1 - \omega - s\mu \le 0$, a contradiction. Hence, u_0 is a common fixed point of the mappings *L* and *J*.

Case II. If $u_1 = u_2$, that is $u_1 = J(u_1)$, then $u_1 = L(u_1)$. If $u_1 \neq L(u_1) = L(J(u_1)) = L(u_2)$, then $d_b(J(u_1), L(u_2)) > 0$. From condition (3) and the fact that *F* is non-decreasing, we have

$$d_b(L(u_2), J(u_1)) \le \varphi d_b(u_2, L(u_2) + \omega d_b(u_1, J(u_1)) + \mu d_b(u_2, J(u_1)) + \eta d_b(u_1, L(u_2)) + \lambda d_b(u_2, u_1)).$$

By Lemma 1, we get

$$(1 - \varphi - s\eta)d_b(L(u_2), J(u_1)) \le 0,$$

which implies $d_b(L(u_2), J(u_1)) = 0$, a contradiction. Hence, u_1 is a common fixed point of *L* and *J*.

Case III. Similarly, if $u_n = u_{n+1}$ for some *n*, we can also obtain that u_n is a common fixed point of *L* and *J*.

Case IV. If $u_0 \neq u_1$ and $u_1 \neq u_2$, then from condition (1), we get

$$\varphi + \omega + s\mu + s\eta + \lambda < 1.$$

Combining with Lemma 1, we deduce that

$$d_b(u_1, u_2) \le pd_b(u_0, f(u_0)),$$

$$d_b(u_2, u_3) \le qd_b(u_1, u_2) \le pqd_b(u_0, f(u_0)),$$

where $p = \frac{\varphi + s\mu + \lambda}{1 - s\mu - \omega}$, $q = \frac{\omega + s\eta + \lambda}{1 - s\eta - \varphi}$ and $0 \le p$, q < 1. Repeating this process, it is not difficult to see that

$$d_b(u_n, u_{n+1}) \leq \begin{cases} p^{\frac{n}{2}} q^{\frac{n}{2}} d_b(u_0, f(u_0)), & \text{if } n \text{ is even,} \\ p^{\frac{n+1}{2}} q^{\frac{n-1}{2}} d_b(u_0, f(u_0)), & \text{if } n \text{ is odd.} \end{cases}$$

Now, we will verify that $\{u_n\}_{n \in N}$ is a Cauchy sequence. To this end, we discuss the following two cases.

 (b_1) Let m = n + i, if *i* is odd and i > 2, we have

$$\begin{aligned} d_b(u_n, u_m) &\leq s(d_b(u_n, u_{n+1}) + d_b(u_{n+1}, u_m)) \\ &\leq sd_b(u_n, u_{n+1}) + s^2 d_b(u_{n+1}, u_{n+2}) + s^2 d_b(u_{n+2}, u_m) \\ &\leq sd_b(u_n, u_{n+1}) + s^2 d_b(u_{n+1}, u_{n+2}) + s^3 d_b(u_{n+2}, u_{n+3}) + s^3 d_b(u_{n+3}, u_m) \\ &\leq sd_b(u_n, u_{n+1}) + s^2 d_b(u_{n+1}, u_{n+2}) + s^3 d_b(u_{n+2}, u_{n+3}) \\ &+ s^4 d_b(u_{n+3}, u_{n+4}) + \dots + s^{m-n-1} d_b(u_{m-2}, u_{m-1}) + s^{m-n-1} d_b(u_{m-1}, u_m). \end{aligned}$$

In this case, when *n* is even, we obtain

$$\begin{split} d_{b}(u_{n},u_{m}) &\leq sp^{\frac{n}{2}}q^{\frac{n}{2}}d_{b}(u_{0},u_{1}) + s^{2}p^{\frac{n+2}{2}}q^{\frac{n}{2}}d_{b}(u_{0},u_{1}) \\ &+ s^{3}p^{\frac{n+2}{2}}q^{\frac{n+2}{2}}d_{b}(u_{0},u_{1}) + s^{4}p^{\frac{n+4}{2}}q^{\frac{n+2}{2}}d_{b}(u_{0},u_{1}) + \dots \\ &+ s^{m-n-1}(p^{\frac{m-1}{2}}q^{\frac{m-3}{2}} + p^{\frac{m-1}{2}}q^{\frac{m-1}{2}})d_{b}(u_{0},u_{1}) \\ &= sp^{\frac{n}{2}}q^{\frac{n}{2}}(1 + sp + s^{2}pq + s^{3}p^{2}q + s^{4}p^{2}q^{2} + \dots \\ &+ s^{m-n-2}p^{\frac{m-n-1}{2}}q^{\frac{m-n-3}{2}} + s^{m-n-2}p^{\frac{m-n-1}{2}}q^{\frac{m-n-1}{2}})d_{b}(u_{0},u_{1}) \\ &= sp^{\frac{n}{2}}q^{\frac{n}{2}}[(1 + s^{2}pq + s^{4}p^{2}q^{2} + \dots + s^{m-n-1}p^{\frac{m-n-1}{2}}q^{\frac{m-n-1}{2}}) \\ &+ (sp + s^{3}p^{2}q + s^{5}p^{3}q^{2} + \dots + s^{m-n-2}p^{\frac{m-n-1}{2}}q^{\frac{m-n-3}{2}})]d_{b}(u_{0},u_{1}) \\ &= [\frac{1 - s^{m-n+1}p^{\frac{m-n+1}{2}}q^{\frac{m-n+1}{2}}}{1 - s^{2}pq} + \frac{sp(1 - s^{m-n-1}p^{\frac{m-n-1}{2}}q^{\frac{m-n-1}{2}})}{1 - s^{2}pq}]sp^{\frac{n}{2}}q^{\frac{n}{2}}d_{b}(u_{0},u_{1}) \\ &= [\frac{1 - s^{i+1}p^{\frac{i+1}{2}}q^{\frac{i+1}{2}}}{1 - s^{2}pq} + \frac{sp(1 - s^{i-1}p^{\frac{i-1}{2}}q^{\frac{i-1}{2}})}{1 - s^{2}pq}]sp^{\frac{n}{2}}q^{\frac{n}{2}}d_{b}(u_{0},u_{1}) \end{split}$$

$$= \left[\frac{1+sp-(1+sq)s^{i}p^{\frac{i-1}{2}}q^{\frac{i-1}{2}}}{1-s^{2}pq}\right]sp^{\frac{n}{2}}q^{\frac{n}{2}}d_{b}(u_{0},u_{1}).$$

When n is odd, we get

$$\begin{split} d_b(u_n, u_m) &\leq sp^{\frac{n+1}{2}} q^{\frac{n-1}{2}} d_b(u_0, u_1) + s^2 p^{\frac{n+1}{2}} q^{\frac{n+1}{2}} d_b(u_0, u_1) \\ &+ s^3 p^{\frac{n+3}{2}} q^{\frac{n+1}{2}} d_b(u_0, u_1) + s^4 p^{\frac{n+3}{2}} q^{\frac{n+3}{2}} d_b(u_0, u_1) + \dots \\ &+ s^{m-n-1} (p^{\frac{m-2}{2}} q^{\frac{m-2}{2}} + p^{\frac{m}{2}} q^{\frac{m-2}{2}}) d_b(u_0, u_1) \\ &= sp^{\frac{n+1}{2}} q^{\frac{n-1}{2}} (1 + sq + s^2 pq + s^3 pq^2 + s^4 p^2 q^2 + \dots \\ &+ s^{m-n-2} p^{\frac{m-n-3}{2}} q^{\frac{m-n-1}{2}} + s^{m-n-2} p^{\frac{m-n-1}{2}} q^{\frac{m-n-1}{2}}) d_b(u_0, u_1) \\ &= sp^{\frac{n+1}{2}} q^{\frac{n-1}{2}} [(1 + s^2 pq + s^4 p^2 q^2 + \dots + s^{m-n-1} p^{\frac{m-n-1}{2}} q^{\frac{m-n-1}{2}}) \\ &+ (sq + s^3 pq^2 + s^5 p^2 q^3 + \dots + s^{m-n-2} p^{\frac{m-n-3}{2}} q^{\frac{m-n-1}{2}})] d_b(u_0, u_1) \\ &= [\frac{1 - s^{m-n+1} p^{\frac{m-n+1}{2}} q^{\frac{m-n+1}{2}}}{1 - s^2 pq} + \frac{sq(1 - s^{m-n-1} p^{\frac{m-n-1}{2}} q^{\frac{m-n-1}{2}})}{1 - s^2 pq}]] d_b(u_0, u_1) \\ &= [\frac{1 - s^{i+1} p^{\frac{i+1}{2}} q^{\frac{i+1}{2}}}{1 - s^2 pq} + \frac{sq(1 - s^{i-1} p^{\frac{i-1}{2}} q^{\frac{i-1}{2}})}{1 - s^2 pq}] sp^{\frac{n+1}{2}} q^{\frac{n-1}{2}} d_b(u_0, u_1) \\ &= [\frac{1 + sq - (1 + sp)s^i p^{\frac{i-1}{2}} q^{\frac{i+1}{2}}}{1 - s^2 pq}] sp^{\frac{n+1}{2}} q^{\frac{n-1}{2}} d_b(u_0, u_1). \end{split}$$

 (b_2) If *i* is even and i > 2, by the similar argument, we deduce that

$$\begin{split} d_{b}(u_{n},u_{m}) &\leq sp^{\frac{n}{2}}q^{\frac{n}{2}}d_{b}(u_{0},u_{1}) + s^{2}p^{\frac{n+2}{2}}q^{\frac{n}{2}}d_{b}(u_{0},u_{1}) \\ &+ s^{3}p^{\frac{n+2}{2}}q^{\frac{n+2}{2}}d_{b}(u_{0},u_{1}) + s^{4}p^{\frac{n+4}{2}}q^{\frac{n+2}{2}}d_{b}(u_{0},u_{1}) + \dots \\ &+ s^{m-n-1}(p^{\frac{m-2}{2}}q^{\frac{m-2}{2}} + p^{\frac{m}{2}}q^{\frac{m-2}{2}})d_{b}(u_{0},u_{1}) \\ &\leq sp^{\frac{n}{2}}q^{\frac{n}{2}}(1 + sp + s^{2}pq + s^{3}p^{2}q + s^{4}p^{2}q^{2} + \dots \\ &+ s^{m-n-2}p^{\frac{m-n-2}{2}}q^{\frac{m-n-2}{2}} + s^{m-n-2}p^{\frac{m-n}{2}}q^{\frac{m-n-2}{2}})d_{b}(u_{0},u_{1}) \\ &\leq sp^{\frac{n}{2}}q^{\frac{n}{2}}[(1 + s^{2}pq + s^{4}p^{2}q^{2} + \dots + s^{m-n-2}p^{\frac{m-n-2}{2}}q^{\frac{m-n-2}{2}}) \\ &+ (sp + s^{3}p^{2}q + s^{5}p^{3}q^{2} + \dots + s^{m-n-2}p^{\frac{m-n}{2}}q^{\frac{m-n-2}{2}})]d_{b}(u_{0},u_{1}) \\ &\leq sp^{\frac{n}{2}}q^{\frac{n}{2}}(1 + sp)\frac{1 - s^{m-n}p^{\frac{m-n}{2}}q^{\frac{m-n}{2}}}{1 - s^{2}pq}d_{b}(u_{0},u_{1}), \end{split}$$

where *n* and *m* are even. Moreover,

$$\begin{split} d_b(u_n, u_m) &\leq sp^{\frac{n+1}{2}} q^{\frac{n-1}{2}} d_b(u_0, u_1) + s^2 p^{\frac{n+1}{2}} q^{\frac{n+1}{2}} d_b(u_0, u_1) \\ &+ s^3 p^{\frac{n+3}{2}} q^{\frac{n+1}{2}} d_b(u_0, u_1) + s^4 p^{\frac{n+3}{2}} q^{\frac{n+3}{2}} d_b(u_0, u_1) + \dots \\ &+ s^{m-n-1} (p^{\frac{m-1}{2}} q^{\frac{m-3}{2}} + p^{\frac{m-1}{2}} q^{\frac{m-1}{2}}) d_b(u_0, u_1) \\ &\leq sp^{\frac{n+1}{2}} q^{\frac{n-1}{2}} (1 + sq) \frac{1 - s^{m-n} p^{\frac{m-n}{2}} q^{\frac{m-n}{2}}}{1 - s^2 pq} d_b(u_0, u_1) \end{split}$$

$$\leq sp^{\frac{n+1}{2}}q^{\frac{n-1}{2}}(1+sq)\frac{1-s^{i}p^{\frac{i}{2}}q^{\frac{i}{2}}}{1-s^{2}pq}d_{b}(u_{0},u_{1}),$$

where *n* is odd and *m* is even. Thus,

$$\begin{split} d_b(u_n,u_m) &\leq \\ & \left\{ \begin{array}{ll} sp^{\frac{n}{2}}q^{\frac{n}{2}}[\frac{1+sp-(1+sq)s^ip^{\frac{i+1}{2}}q^{\frac{i-1}{2}}}{1-s^2pq}]d_b(u_0,u_1), & \text{if i is odd and n is even,} \\ & sp^{\frac{n+1}{2}}q^{\frac{n-1}{2}}[\frac{1+sq-(1+sp)s^ip^{\frac{i-1}{2}}q^{\frac{i+1}{2}}}{1-s^2pq}]d_b(u_0,u_1), & \text{if i is odd and n is odd,} \\ & sp^{\frac{n}{2}}q^{\frac{n}{2}}(1+sp)\frac{1-s^ip^{\frac{i}{2}}q^{\frac{i}{2}}}{1-s^2pq}d_b(u_0,u_1), & \text{if i is even and n is even,} \\ & sp^{\frac{n+1}{2}}q^{\frac{n-1}{2}}(1+sq)\frac{1-s^ip^{\frac{i}{2}}q^{\frac{i}{2}}}{1-s^2pq}d_b(u_0,u_1), & \text{if i is even and n is odd.} \\ \end{split}$$

Letting $n, m \to \infty$, we obtain that $d_b(u_n, u_m) \to 0$, since $0 \le p, q < 1$. Hence, we draw the conclusion that $\{u_n\}$ is a Cauchy sequence. Noticing that the completeness of the space, there exists $r \in V$ such that

$$\lim_{n\to\infty}d_b(u_n,r)=0$$

Next, we verify that *r* is a common fixed point of *L* and *J*. Indeed, if $d_b(L(r), r) > 0$, condition (3) yields

$$F(d_b(L(r), u_{2n})) < \tau + F(d_b(L(r), u_{2n})) = \tau + F(d_b(L(r), J(u_{2n-1})))$$

$$\leq F(\varphi d_b(r, L(r)) + \omega d_b(u_{2n-1}, u_{2n}) + \mu d_b(r, u_{2n}) + \eta d_b(u_{2n-1}, L(r)) + \lambda d_b(r, u_{2n-1})).$$

Since *F* is non-decreasing, we get

$$d_b(L(r), u_{2n}) \le \varphi d_b(r, L(r)) + \omega d_b(u_{2n-1}, u_{2n}) + \mu d_b(r, u_{2n}) + \eta d_b(u_{2n-1}, L(r)) + \lambda d_b(r, u_{2n-1}).$$

Moreover, from condition (a_3) of Definition 1, it follows that

$$d_b(L(r), r) \le s[d_b(L(r), u_{2n}) + d_b(u_{2n}, r)].$$

Then

.

$$\frac{1}{s}d_b(L(r),r) \leq \lim_{n \to \infty} \sup d_b(L(r),u_{2n}) \leq \frac{\varphi}{1-s\eta}d_b(L(r),r).$$

Hence, we deduce that $\frac{1}{s} \leq \frac{\varphi}{1-s\eta}$ which contradicts the fact $\frac{s\varphi}{1-s\eta} < 1$. Therefore, we obtain $d_b(L(r), r) = 0$. Similarly, we can get J(r) = r. Therefore, we can write

$$L(r) = J(r) = r.$$

For the uniqueness, we assume that r and r^* are two distinct common fixed points of f and g. Then

$$\begin{aligned} \tau + F(d_b(r, r^*)) &= \tau + F(d_b(L(r), J(r^*))) \\ &\leq F(\varphi d_b(r, L(r)) + \omega d_b(r^*, J(r^*)) + \mu d_b(r, J(r^*)) \\ &+ \eta d_b(r^*, L(r)) + \lambda d_b(r, r^*)) \\ &= F(\mu d_b(r, J(r^*)) + \eta d_b(r^*, L(r)) + \lambda d_b(r, r^*)). \end{aligned}$$

Since *F* is non-decreasing, we deduce that

$$d_b(r, r^*) \le \mu d_b(r, J(r^*)) + \eta d_b(r^*, L(r)) + \lambda d_b(r, r^*),$$

which implies

$$(1-\lambda-\mu-\eta)d_b(r,r^*)\leq 0.$$

Then

$$1 - \lambda - \mu - \eta \le 0,$$

which is a contradiction. Hence, $r = r^*$. \Box

The following example shows the validity of Theorem 3.

Example 3. Let V = [0, 8], and L, $J : V \to V$ be two mappings defined by

$$L(u) = \begin{cases} 7, & if \ u \in (0,8], \\ 8, & if \ u = 0, \end{cases}$$

and

$$J(u) = \begin{cases} 7, & if \ u \in (0,8], \\ 6, & if \ u = 0. \end{cases}$$

We define a b-metric $d_b : V \times V \rightarrow [0, \infty)$ *by*

$$d_b(u,t) = (u-t)^2$$
, for all $u, t \in V$.

Clearly, (V, d_b) is a complete b-metric space with constant s = 2 (see [25] for details). We observe that $d_b(L(u), J(t)) > 0$, when $(u, t) \in D = \{(u, t) : u \in (0, 8], t = 0\} \cup \{(u, t) : u = 0, t \in (0, 8]\} \cup \{(u, t) : u = 0, t = 0\}$.

Define $E: V \times V \rightarrow [0, \infty)$ by

$$E(u,t) = \frac{1}{4}d_b(u,L(u)) + \frac{1}{4}d_b(t,J(u)) + \frac{1}{16}d_b(u,t), \ u,t \in V.$$

Next, we discuss the following three cases: (*i*) If $u \in (0, 8]$ and t = 0, then we obtain

$$\frac{1}{6} - \frac{1}{d_b(L(u), J(0)) + 1} \leq \frac{1}{6} - \frac{1}{2} = -\frac{1}{3} \\
< -\frac{4}{36} = -\frac{4}{d_b(0, J(0))} \\
\leq -\frac{1}{E(u, 0)} \\
< -\frac{1}{E(u, 0) + 1}.$$
(3)

(*ii*) If $t \in (0, 8]$ and u = 0, then we have

$$\frac{1}{6} - \frac{1}{d_b(L(0), J(t)) + 1} \leq \frac{1}{6} - \frac{1}{2} = -\frac{1}{3}
< -\frac{4}{64} = -\frac{4}{d_b(0, f(0))}
\leq -\frac{1}{E(0, t)}
< -\frac{1}{E(0, t) + 1}.$$
(4)

(*iii*) If t = 0 and u = 0, then we get

$$\frac{1}{6} - \frac{1}{d_b(L(0), J(0)) + 1} \leq \frac{1}{6} - \frac{1}{3} \\
< -\frac{1}{25} \\
\leq -\frac{1}{E(0, 0)} \\
< -\frac{1}{E(0, 0) + 1}.$$
(5)

Combining with (3), (4), and (5), we deduce that

$$\frac{1}{6} - \frac{1}{d_b(L(u), J(t)) + 1} < -\frac{1}{E(u, t) + 1},$$

for all $(u, t) \in D$.

Therefore, if we consider $F(h) = -\frac{1}{h+1}$, where $h \ge 0$, $t \in (0, \infty)$, and $\tau = \frac{1}{6}$, then all the conditions of Theorem 3 are satisfied with $\varphi = \frac{1}{4}$, $\omega = \frac{1}{4}$, $\mu = \eta = 0$, $\lambda = \frac{1}{16}$. Meanwhile, we notice that L(z) = J(z) = z if and only if z = 7. Hence, L and J have a unique common fixed point.

Next, we give the following result which shows the uniqueness and existence of the common fixed point for two mappings in a complete *b*-metric space by virtue of the function θ .

Theorem 4. Let *L* and *J* be two self-maps on a complete b-metric space (V, d_b) with constant $s \ge 1$. Let $\varphi, \omega \in (0,1)$, $\mu, \eta, \lambda \in [0,1)$ be the constants with $\varphi + \omega + s\mu + s\eta + \lambda < 1$, $s^2\varphi + s^2\eta + (s^3 + s^4)(\mu + \lambda) < 1$, and $s^2\omega + s^2\mu + (s^3 + s^4)(\eta + \lambda) < 1$. Define $\theta : [0,1] \times [0,1] \times [0,1] \times [0,1] \to (0,1]$ by

$$\begin{split} \theta(\varphi, \omega, \mu, \eta, \lambda) &= \min\{\frac{1 - s^2 \varphi - s^2 \eta - (s^3 + s^4)(\mu + \lambda)}{1 - s^3 \mu - s^2 \eta - s^3 \lambda}, \\ &\frac{1 - s^2 \omega - s^2 \mu - (s^3 + s^4)(\eta + \lambda)}{1 - s^2 \mu - s^3 \eta - s^3 \lambda}\}. \end{split}$$

Suppose that each of the conditions

$$\begin{aligned} \theta(\varphi, \omega, \mu, \eta, \lambda) d_b(u, L(u)) &\leq s^2 d_b(u, t) \qquad or \\ \theta(\varphi, \omega, \mu, \eta, \lambda) d_b(t, J(t)) &\leq s^2 d_b(u, t), \end{aligned}$$

implies

$$d_b(L(u), J(t)) \le \varphi d_b(u, L(u)) + \omega d_b(t, J(t)) + \mu d_b(u, J(t)) + \eta d_b(t, L(u)) + \lambda d_b(u, t),$$

for $u, t \in V$. Then L and J have a unique common fixed point r.

Proof. It is not difficult to see that $\theta(\varphi, \omega, \mu, \eta, \lambda) \in (0, 1)$, since

$$1 - s^2 \varphi - s^2 \eta - (s^3 + s^4)(\mu + \lambda) < 1 - s^3 \mu - s^2 \eta - s^3 \lambda_{\lambda}$$

and

$$1-s^2\omega-s^2\mu-(s^3+s^4)(\eta+\lambda)<1-s^2\mu-s^3\eta-s^3\lambda.$$

$$\theta(\varphi, \omega, \mu, \eta, \lambda)d_b(e_0, L(e_0)) \le s^2 d_b(e_0, L(e_0)).$$

Hence, we have

$$\begin{aligned} d_b(e_1, e_2) &= d_b(L(e_0), J(e_1)) \\ &\leq \varphi d_b(e_0, e_1) + \omega d_b(e_1, e_2) + s \mu [d_b(e_0, e_1) + d_b(e_1, e_2)] + \lambda d_b(e_0, e_1). \end{aligned}$$

It yields

$$(1-\omega-s\mu)d_b(e_1,e_2)\leq 0,$$

which shows that $d_b(e_0, J(e_1)) = 0$, a contradiction. Hence, e_0 is a common fixed point of the mappings *L* and *J*.

Similarly, as in Case II and Case III of Theorem 3, we conclude that if $e_n = e_{n+1}$ for some *n*, then e_n is also the common fixed point of the mappings *L* and *J*.

If $e_0 \neq e_1$ and $e_1 \neq e_2$, we notice that

$$\theta(\varphi, \omega, \mu, \eta, \lambda)d_b(e_0, L(e_0)) \le s^2 d_b(e_0, L(e_0)),$$

since $\theta(\varphi, \omega, \mu, \eta, \lambda) < 1 \le s^2$. Then, we have

$$d_b(e_1, e_2) = d_b(L(e_0), J(e_1))$$

$$\leq \varphi d_b(e_0, e_1) + \omega d_b(e_1, e_2) + \mu d_b(e_0, e_2) + \lambda d_b(e_0, e_1).$$

From Lemma 1, it follows that

$$d_b(e_1, e_2) \le p d_b(e_0, e_1), \quad where \quad p = \frac{\varphi + s\mu + \lambda}{1 - s\mu - \omega} \in (0, 1).$$

Similarly, we can also deduce that

$$\theta(\varphi, \omega, \mu, \eta, \lambda)d_b(e_1, e_2) \leq s^2 d_b(e_1, e_2).$$

From Lemma 1, we obtain

$$d_b(e_2, e_3) \leq qd_b(d_b(e_1, e_2)) \leq pqd_b(e_0, e_1), \quad where \quad q = \frac{\omega + s\eta + \lambda}{1 - s\eta - \varphi} \in (0, 1).$$

By repeating the above process, we can establish using a similar argument as in Theorem 3 that $\{e_n\}$ is a Cauchy sequence and $\lim_{n\to\infty} e_n = r$ for some $r \in V$. Now we will prove that r is a common fixed point of the mappings L and J. Let $u \neq r$ be an arbitrary element of V. Then there exists $N \in \mathcal{N}$ such that $d_b(e_n, r) \leq \frac{1}{2+s}d_b(u, r)$ for all $n \geq \mathcal{N}$. Hence, we have

$$d_b(u,r) \le s[d_b(u,e_{2n-1}) + d_b(e_{2n-1},r)] \le s[d_b(u,e_{2n-1}) + \frac{1}{2+s}d_b(u,r)],$$

for large enough *n*. Thus

$$\frac{2}{2+s}d_b(u,r) \le sd_b(u,e_{2n-1}).$$

What is more,

$$\begin{aligned} \theta(\varphi, \omega, \mu, \eta, \lambda) d_b(e_{2n-1}, J(e_{2n-1})) &\leq d_b(e_{2n-1}, J(e_{2n-1})) \\ &\leq s[(d_b(e_{2n-1}, r) + d_b(r, e_{2n}))] \\ &\leq \frac{2s}{2+s} d_b(u, r) \\ &\leq s^2 d_b(u, e_{2n-1}). \end{aligned}$$

Hence, we deduce that

 $\begin{aligned} d_b(L(u), J(e_{2n-1})) &\leq \varphi d_b(u, L(u)) + \omega d_b(e_{2n-1}, J(e_{2n-1})) \\ &+ \mu d_b(u, J(e_{2n-1})) + \eta d_b(e_{2n-1}, L(u)) + \lambda d_b(u, e_{2n-1}). \end{aligned}$

Then

$$\begin{aligned} d_{b}(L(u),r) &\leq s[d_{b}(L(u),e_{2n}) + d_{b}(e_{2n},r)] \\ &\leq s\varphi d_{b}(u,L(u)) + s\omega d_{b}(e_{2n-1},J(e_{2n-1})) \\ &+ s^{2}\mu[d_{b}(u,r) + d_{b}(r,e_{2n})] + s^{2}\eta[d_{b}(e_{2n-1},r) + d_{b}(L(u),r)] \\ &+ s^{2}\lambda[d_{b}(u,r) + d_{b}(r,e_{2n-1})] + sd_{b}(e_{2n},r) \\ &\leq \lim_{n \to \infty} \sup\{s\varphi d_{b}(u,L(u)) + s\omega d_{b}(e_{2n-1},J(e_{2n-1})) \\ &+ s^{2}\mu[d_{b}(u,r) + d_{b}(r,e_{2n})] + s^{2}\eta[d_{b}(e_{2n-1},r) + d_{b}(L(u),r)] \\ &+ s^{2}\lambda[d_{b}(u,r) + d_{b}(r,e_{2n-1})] + sd_{b}(e_{2n},r) \\ &= s\varphi d_{b}(u,L(u)) + s^{2}(\mu + \lambda)d_{b}(u,r) + s^{2}\eta d_{b}(L(u),r). \end{aligned}$$
(6)

It follows that

$$d_b(L(r),r) \le s\varphi d_b(L(r),r) + s^2 \eta d_b(L(r),r),$$

which implies $(1 - s\varphi - s^2\eta)d_b(L(r), r) \le 0$. Notice that

$$1-s\varphi-s^2\eta>(s^2-s)\varphi+s^2\omega+(s^3+s^4)(\mu+\lambda)>0.$$

Hence, we deduce $d_b(L(r), r) = 0$. By adding $-s^3(\mu + \lambda)d_b(L(u), r)$ on both sides of (6), we have

$$\begin{aligned} &d_b(L(u), r) - s^3(\mu + \lambda)d_b(L(u), r) \\ &\leq s\varphi d_b(u, L(u)) + s^2(\mu + \lambda)d_b(u, r) - s^3(\mu + \lambda)d_b(L(u), r) + s^2\eta d_b(L(u), r) \\ &\leq s\varphi d_b(u, L(u)) + s^3(\mu + \lambda)d_b(u, L(u)) + s^2\eta d_b(L(u), r). \end{aligned}$$

Thus

$$d_b(L(u),r) \le \frac{s\varphi + s^3\mu + s^3\lambda}{1 - s^3\mu - s^2\eta - s^3\lambda} d_b(u,L(u)),\tag{7}$$

and

$$d_b(u, L(u)) \le s[d_b(u, r) + d_b(L(u), r)]$$

$$\le sd_b(u, r) + \frac{s^2\varphi + s^4\mu + s^4\lambda}{1 - s^3\mu - s^2\eta - s^3\lambda}d_b(u, L(u)).$$

Therefore, we obtain

$$\frac{1 - s^2 \varphi - s^2 \eta - (s^3 + s^4)(\mu + \lambda)}{1 - s^2 \eta s^3 - (\mu + \lambda)} d_b(u, L(u)) \le s^2 d_b(u, r).$$
(8)

Similarly, we can also deduce $d_b(J(r), r) = 0$ and

$$\frac{1 - s^2\omega - s^2\mu - (s^3 + s^4)(\eta + \lambda)}{1 - s^2\mu - s^3(\eta + \lambda)}d_b(u, g(u)) \le s^2d_b(u, r).$$
(9)

Hence, *r* is a common fixed point of the mappings *L* and *J*. Furthermore, we also have

$$\theta(\varphi,\omega,\mu,\eta,\lambda)d_b(u,L(u)) \leq s^2 d_b(u,r), \ \theta(\varphi,\omega,\mu,\eta,\lambda)d_b(u,J(u)) \leq s^2 d_b(u,r).$$

Next, we show the uniqueness of the common fixed point of the mappings *L* and *J*. Assume that there exist $r, r' \in V$ such that L(r) = J(r) = r and L(r') = r' = J(r'). Note that

$$\theta(\varphi, \omega, \mu, \eta, \lambda) d_b(r, L(r)) \le s^2 d_b(r, r')$$

Then we have

$$d_b(r, r') = d_b(L(r), J(r')) \le \varphi d_b(r, L(r)) + \omega d_b(r', J(r')) + \mu d_b(r, J(r')) + \eta d_b(r', L(r)) + \lambda d_b(r, r'),$$

which yields

 $(1-\mu-\eta-\lambda)d_b(r,r')\leq 0.$

Since

$$1-\mu-\eta-\lambda > \varphi+\omega > 0,$$

we get $d_b(r, r') = 0$. This completes the proof. \Box

Example 4. Let V = [0,1] be equipped with b-metric $d_b(u,t) = (u-t)^2$ and $u, t \in V$, where s = 2. Define L and J by

$$L(u) = \frac{u}{2}, \quad J(u) = \frac{u}{3}, \quad u \in V,$$

and set

$$E(u,t) = \frac{1}{32}d_b(u,L(u)) + \frac{1}{32}d_b(t,J(t)) + \frac{1}{64}d_b(u,t),$$

for all $u, t \in V$. Let $\varphi = \frac{1}{32}$, $\omega = \frac{1}{32}$, $\mu = \eta = 0$, $\lambda = \frac{1}{64}$, it is easy to verify that

$$\begin{split} \varphi + \omega + s\mu + s\eta + \lambda &< 1, \\ s^2 \varphi + s^2 \eta + (s^3 + s^4)(\mu + \lambda) &< 1, \\ s^2 \omega + s^2 \mu + (s^3 + s^4)(\eta + \lambda) &< 1 \end{split}$$

and

$$\theta(\varphi, \omega, \mu, \eta, \lambda) = \min\{\frac{1 - s^2\omega - s^2\mu - (s^3 + s^4)\eta - (s^3 + s^4)\lambda}{1 - s^2\mu - s^3\eta - s^3\lambda}, \frac{1 - s^2\varphi - (s^3 + s^4)\mu - s^2\eta - (s^3 + s^4)\lambda}{1 - s^3\mu - s^2\eta - s^3\lambda}\} = \frac{4}{7}.$$

Next, we claim that the condition

$$\begin{aligned} \theta(\varphi, \omega, \mu, \eta, \lambda) d_b(u, L(u)) &\leq s^2 d_b(u, t) \quad \text{or} \\ \theta(\varphi, \omega, \mu, \eta, \lambda) d_b(t, J(t)) &\leq s^2 d_b(u, t), \end{aligned}$$

implies

$$d_b(L(u), J(t)) \le \varphi d_b(u, L(u)) + \omega d_b(t, J(t)) + \mu d_b(u, J(t)) + \eta d_b(t, L(u)) + \lambda d_b(u, t),$$

for $u, t \in V$. Indeed,

$$d_b(L(u), J(t)) = \left(\frac{u}{2} - \frac{t}{3}\right)^2 \le \frac{1}{128}u^2 + \frac{1}{72}t^2 + \frac{1}{64}(u-t)^2, \ u, t \in V.$$
(10)

~

Moreover, if

$$\theta(\varphi, \omega, \mu, \eta, \lambda)d_b(u, L(u)) \le s^2 d_b(u, t),$$

then we deduce

$$\frac{u^2}{28} \le (u-t)^2. \tag{11}$$

Now, we study the following two cases: (i) If u > t, we have

$$\leq (1 - \frac{1}{\sqrt{28}})u. \tag{12}$$

Combining (10) and (12) yields

$$Au^2 \leq 0,$$

where

$$A = (\frac{1}{4} - \frac{1}{128} - \frac{1}{64}) + (\frac{1}{9} - \frac{1}{72} - \frac{1}{64})(1 - \frac{1}{\sqrt{28}})^2 + (\frac{1}{32} - \frac{1}{3})(1 - \frac{1}{\sqrt{28}}) > 0,$$

which means that u = t = 0. Therefore, when u = t = 0, the condition

t

$$\theta(\varphi, \omega, \mu, \eta, \lambda)d_b(u, L(u)) \leq s^2 d_b(u, t),$$

implies

$$d_b(L(u), J(t)) \le \varphi d_b(u, L(u)) + \omega d_b(t, J(t)) + \mu d_b(u, J(t)) + \eta d_b(t, L(u)) + \lambda d_b(u, t).$$

(*ii*) If
$$u < t$$
, we get

$$u \le \frac{\sqrt{28}}{\sqrt{28}+1}t.\tag{13}$$

Then by (10) and (13), we obtain

$$Bt^2 \leq 0$$
,

where

$$B = \left(\frac{1}{4} - \frac{1}{128} - \frac{1}{64}\right)\left(\frac{\sqrt{28}}{\sqrt{28} + 1}\right)^2 + \left(\frac{1}{9} - \frac{1}{72} - \frac{1}{64}\right) + \left(\frac{1}{32} - \frac{1}{3}\right)\frac{\sqrt{28}}{\sqrt{28} + 1} < 0.$$

Therefore, for $u, t \in [0, 1]$ *, when* $u \leq \frac{\sqrt{28}}{\sqrt{28}+1}t$ *, the condition*

$$\theta(\varphi, \omega, \mu, \eta, \lambda) d_b(u, L(u)) \le s^2 d_b(u, t),$$

implies

$$d_b(L(u), J(t)) \le \varphi d_b(u, L(u)) + \omega d_b(t, J(t)) + \mu d_b(u, J(t)) + \eta d_b(t, L(u)) + \lambda d_b(u, t).$$

As a consequence, we claim that all the conditions of Theorem 4 are satisfied with $\varphi = \frac{1}{32}, \omega = \frac{1}{32}, \mu = \eta = 0, \lambda = \frac{1}{64}$. Meanwhile, it is not difficult to see that 0 is the unique common fixed point of the mappings L and J.

3.2. A Generalized Form of Jungck Fixed Point Theorem

Next, we give a generalized form of Jungck fixed point theorem in complete *b*-metric spaces.

Theorem 5. Let *L* be a continuous self-map on a complete b-metric space (V, d_b) . Then *L* has a fixed point if and only if there exist the constants $\omega \in (0,1)$, $\varphi, \mu, \eta, \lambda \in [0,1)$ with $\varphi + \omega + \mu + 2s\eta + \lambda < 1$ and a continuous self-map *J* on *V* satisfying the following properties:

(1) $J(V) \subseteq L(V)$,

(2) *L* and *J* commute under composition (i.e., J(L(u)) = L(J(u)) for all $u \in V$),

(3) $d_b(J(u), J(t)) \leq \varphi d_b(L(u), J(u)) + \omega d_b(L(t), J(t)) + \mu d_b(L(u), J(t)) + \eta d_b(L(t), J(u)) + \lambda d_b(L(u), L(t))$ for all $u, t \in V$.

If the above conditions hold, then L and J have a unique common fixed point.

Proof. If *L* has a fixed point $r \in V$, then we define $J : V \to V$ by J(u) = r for all $u \in V$. It follows immediately that $J(V) \subseteq L(V)$. Moreover, for any $u \in V$, J(L(u)) = r and L(J(u)) = L(r) = r, which implies that J(L(u)) = L(J(u)) for all $u \in V$. In addition, for any $\varphi, \omega, \mu, \eta, \lambda \in [0, 1)$ and $u, t \in V$, we obtain

$$d_b(J(u), J(t)) = d_b(r, r) = 0$$

$$\leq \varphi d_b(L(u), r) + \omega d_b(L(t), r) + \mu d_b(L(u), r)$$

$$+ \eta d_b(L(t), r) + \lambda d_b(L(u), L(t)).$$

On the other hand, if conditions (1)-(3) are satisfied, we claim that *L* has a fixed point, and *L* and *J* have a unique common fixed point. To see this, let $u_0 \in V$. It follows from condition (1) that there exists $u_1 \in V$ such that $L(u_1) = J(u_0)$. Repeating this process, we can find $\{u_n\} \subset V$ such that $L(u_n) = J(u_{n-1})$ for $n \in \mathcal{N}$, which together with condition (3) yields

$$\begin{aligned} d_b(L(u_{n+1}), L(u_n)) &= d_b(J(u_n), J(u_{n-1})) \\ &\leq \varphi d_b(L(u_n), L(u_{n+1})) + \omega d_b(L(u_{n-1}), L(u_n)) \\ &+ \eta d_b(L(u_{n-1}), L(u_{n+1})) + \lambda d_b(L(u_n), L(u_{n-1})) \\ &\leq \varphi d_b(L(u_n), L(u_{n+1})) + \omega d_b(L(u_{n-1}), L(u_n)) \\ &+ s\eta [d_b(L(u_{n-1}), L(u_n)) + d_b(L(u_n), L(u_{n+1}))] \\ &+ \lambda d_b(L(u_n), L(u_{n-1})). \end{aligned}$$

Then

$$d_b(L(u_{n+1}), L(u_n)) \le \frac{\omega + s\eta + \lambda}{1 - \varphi - s\eta} d_b(L(u_n), L(u_{n-1})).$$

Since $\frac{\omega+s\eta+\lambda}{1-\varphi-s\eta} < 1$, we see that $\{L(u_n)\}$ is a Cauchy sequence. Moreover, by the completeness of the space, we deduce that $\lim_{n\to\infty} L(u_n) = r$ for some $r \in V$. From $L(u_n) = J(u_{n-1})$, it follows that $\lim_{n\to\infty} J(u_n) = r$. By condition (2), we get

$$L(r) = \lim_{n \to \infty} L(J(u_n)) = \lim_{n \to \infty} J(L(u_n)) = J(r),$$

which yields L(L(r)) = L(J(r)) = J(L(r)) = J(J(r)). By condition (3), we deduce that

$$d_b(J(r), J(J(r))) \le \varphi d_b(L(r), J(r)) + \omega d_b(L(J(r)), J(J(r))) + \mu d_b(L(r), J(J(r))) + \eta d_b(L(J(r)), J(r)) + \lambda d_b(L(r), L(J(r))) = \mu d_b(L(r), L(L(r))) + \eta d_b(L(r), L(L(r))) + \lambda d_b(L(r), L(L(r))),$$

which implies $(1 - \mu - \eta - \lambda)d_b(L(r), L(L(r))) \le 0$. Since $1 - \mu - \eta - \lambda > 0$, we deduce that L(r) is a common fixed point of *L* and *J*.

For the uniqueness, if there exist r, $k \in V$ such that L(r) = r = J(r) and L(k) = k = J(k), then by condition (3), it follows that

$$d_b(r,k) = d_b(J(r), J(k))$$

$$\leq \varphi d_b(L(r), J(r)) + \omega d_b(L(k), J(k))$$

$$+ \mu d_b(L(r), J(k)) + \eta d_b(L(k), J(r)) + \lambda d_b(L(r), L(k))$$

$$= \mu d_b(r,k) + \eta d_b(r,k) + \lambda d_b(r,k).$$

Clearly, we can get $(1 - \mu - \eta - \lambda)d_b(r, k) \le 0$. Since $1 - \mu - \eta - \lambda > 0$, we have $d_b(r, k) = 0$ which implies r = k. \Box

4. Conclusions

In this paper, we mainly study the existence and uniqueness of common fixed points for two mappings in complete *b*-metric spaces by virtue of the new functions *F* and θ , respectively. Moreover, we present two specific instances to show the availability of our results. Compared with the previous results, we weaken the conditions of the function *F*, which only needs to be non-decreasing. Hence, to some extent, our conclusions and methods improve the results of previous literature. However, for the application of the results, we need to continue to study, especially the existence of solutions of some integral equations and differential equations.

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