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Fractional $p(\cdot)$-Kirchhoff Type Problems Involving Variable Exponent Logarithmic Nonlinearity

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Abstract: In this paper, we investigate a fractional $p(\cdot)$-Kirchhoff type problem involving variable exponent logarithmic nonlinearity. With the help of the Nehari manifold approach, the existence and multiplicity of nontrivial weak solutions for the above problem are obtained. The main aspect and challenges of this paper are the presence of double non-local terms and logarithmic nonlinearity.

Keywords: fractional $p(\cdot)$-Kirchhoff type; logarithmic nonlinearity; variable growth; Nehari manifold

MSC: 47G20; 35J60

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1. Introduction

The study of differential equations and variational issues involving $p(x)$-growth conditions has received a majority of attention in recent years. The development of numerous significant models in electrorheological and thermorheological fluids, image processing, and other fields inspired a systematic study of partial differential equations with variable exponents; see [1–3]. The literature on the study of such operators is very large and rich, but we only list some newly published articles for interested readers, see, e.g., [4–8].

The study of elliptic equations with fractional operators is one of the most fascinating areas of nonlinear analysis. These issues have received much attention in both pure mathematics study and practical applications. In reality, this sort of operator often appears in a variety of settings. Few authors have also studied elliptic problems involving inequalities [9,10]. As far as we know, the fractional Sobolev spaces with variable exponents and the fractional $p(\cdot)$-Laplacian were introduced firstly by U.Kaufmann, J.D.Rossi and R.Vidal in [11]. Here, the authors obtained the embedding result of fractional Sobolev spaces with variable exponents to variable-exponent Lebesgue spaces. In addition, they also discussed the existence result of a fractional $p(\cdot)$-Laplacian problem.

After that, many mathematicians were concerned with equations involving the operator and studied it extensively, see [12–17]. In particular, this combination of fractional $p(x)$-Laplace operators and Kirchhoff functions is very interesting. For example, E. Azroul et al. [13] investigated a class of fractional $p(\cdot)$-Kirchhoff type problems using the mountain pass lemma, direct variational method, Ekeland's variational principle and concluded the existence of nontrivial weak solutions for the above problem in various cases of the competition between the growth rates of functions. In addition, we recommend that interested readers read the literature [18]. The basic Kirchhoff problem was first introduced by Kirchhoff [19] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic string. Kirchhoff's model takes into account the changes in the length of the string produced by transverse vibrations. A detailed advancement in the Kirchhoff elliptic problem and its physical interpretation can be seen in [20].

On the other hand, elliptic, parabolic and hyperbolic equations with logarithmic nonlinearity have received extensive attention from many scholars, and many mathematicians have conducted extensive research; see [21–27]. In particular, we point out that Xiang et al. [26] investigated the existence of two local least energy solutions for fractional p-Kirchhoff problems involving logarithmic nonlinearity by means of the Nehari manifold approach. This method is used essentially because the functional corresponding to the equation is not bounded below in the whole workspace, so it is difficult to find the critical points in the whole workspace, and thus, we need to find the critical points on a smaller set. For more details on this approach, we recommend some very good papers for interested readers [28–31].

To our best knowledge, there are no results concerned with the Kirchhoff type problem driven by a $p(\cdot)$-fractional Laplace operator with logarithmic nonlinearity. Motivated by the works discussed above, in this paper, we are interested in the existence of two nontrivial weak solutions for the following fractional $p(\cdot)$-Kirchhoff type problems.

$$\begin{cases} K\left(\int\int_{\mathbb{R}^{2N}} \frac{1}{p(x,y)} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dxdy\right)(-\Delta)^s_{p(\cdot)} u = h(x)|u|^{\overline{p}(x)\theta-2} u \ln|u| + \beta |u|^{q(x)-2} u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (1)$$

where $\Omega \subset \mathbb{R}^n$ is a smooth and bounded domain with $N > p(x,y)s$ for any $(x,y) \in \overline{\Omega} \times \overline{\Omega}$, $\overline{p}(x) = p(x,x)$ for $x \in \overline{\Omega}$, β is a positive parameter, $2 < q(x) < \theta\overline{p}(x) < \overline{p_s^*}(x)$ for any $x \in \overline{\Omega}$ and $h(x) \in C(\overline{\Omega})$ is a positive function, M is a Kirchhoff function model, $(-\Delta)^s_{p(\cdot)}$ is a $p(\cdot)$-fractional Laplace operator, with $s \in (0,1)$, defined as follows: for each $x \in \Omega$,

$$(-\Delta)^s_{p(x)} \varphi(x) = \text{p.v.} \int_{\mathbb{R}^N} \frac{|'(x) - '(y)|^{p(x,y)-2}('(x) - '(y))}{|x-y|^{N+sp(x,y)}} dy,$$

along any $\varphi \in C_0^\infty(\Omega)$, where p.v. is considered in the principal value sense.

Let

$$p^- := \inf_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,y) \le \sup_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,y) =: p^+.$$

$$q^- := \min_{x\in\overline{\Omega}} q(x) \le q(x) \le q^+ := \max_{x\in\overline{\Omega}} q(x).$$

A model of K proposed by Kirchhoff is of the form $K(t) = a + bt^{\alpha-1}, a, b \ge 0, a + b > 0, t \ge 0$ and $\alpha \in (1, +\infty)$ if $b > 0, \alpha = 1$ if $b = 0$. When $K(t) > 0$ for all $t \ge 0$, Kirchhoff problems are said to be nondegenerate and this happens, for example, if $a > 0$ and $b \ge 0$ in the model case (1), see for instance [20,32,33]. Otherwise, if $K(0) = 0$ and $K(t) > 0$ for all $t > 0$, the Kirchhoff problems are called degenerate and this occurs in the model case (1) when $a = 0$ and $b > 0$, see also [34,35]. An interesting point regarding this problem is the involvement of comes from the fact that $logx$ is sign changing and behaving at the origin similar to the power function $-t^\alpha$ for $\alpha < 0$ with a slow growth. In addition, the logarithmic function is not invariant by scaling, which does not hold for the power function. Furthermore, the presence of the variable exponent makes the problem more significant.

To study our main result, we need to make further assumptions.

(i) $2 < q^- < q(x) < q^+ < p^- < p(x) < p^+ < (p^+)^3 < \theta p^- < \theta p(x) < \theta p^+ < p_s^*(x)$, $p(x,y)$ is symmetric for all $(x,y) \in \overline{\Omega} \times \overline{\Omega}$.

(ii) $K : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function that satisfies the condition: there exists $1 < a_3 < a_4$ with $a_4^2(p^+)^2 < \theta$ such that $a_3 t^\theta < c_2\hat{K}(t) \le K(t)t \le c_1 K'(t)t^2 \le a_4 t^\theta$, where $c_1 \in (p^+, \infty)$, $c_2 \in \left(\frac{1}{(p^+)^2 a_4}, \frac{1}{p^+ a_4}\right)$ and $\hat{K}(t) = \int_0^t K(\tau)d\tau$.

An example that satisfies our hypothesis could be $K(x,u) = |u|^{\theta-2}u$.

A function $u \in X_0 = W_0^{s,p(x,y)}(\overline{\Omega})$ is a weak solution to the problem (1), if

$$K(\gamma_{p(x,y)}(u)) \int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(\phi(x)-\phi(x))}{|x-y|^{N+sp(x,y)}} dxdy - \int_\Omega h(x)|u|^{\theta\overline{p}(x)-2} u \ln|u|\phi dx - \beta \int_\Omega |u|^{q(x)-2} u\phi dx = 0$$

for any $\phi \in W_0^{s,p(x,y)}(\overline{\Omega})$, where $\gamma_{p(x,y)}(u) = \int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}} dxdy$.

We are ready to state the main result of this paper.

Theorem 1. *Let $\|u\| > 1$. Assume that the assumptions (i) and (ii) hold. Then, there exists $\beta^{**} > 0$ such that for any $\beta \in (0, \beta^{**})$, problem (1) has at least two nontrivial weak solutions.*

2. Functional Analytic Setup

In this section, first of all, we review some basic properties about the variable exponent Lebesgue spaces as well as the fractional Sobolev spaces with variable exponents.
Set

$$C_+(\bar{\Omega}) = \left\{ p \in C(\bar{\Omega}) : 1 < p(x) \ \text{ for all } \ x \in \bar{\Omega} \right\}.$$

For any $p \in C_+(\bar{\Omega})$, we define the variable exponent Lebesgue space as

$$L^{p(x)}(\Omega) = \left\{ u : \ \Omega \to \mathbb{R} \ \text{ is measurable, } \ \int_\Omega |u(x)|^{p(x)} dx < \infty \right\},$$

and the Luxemburg norm defined on this space as,

$$|u|_{p(x)} = \inf \left\{ \mu > 0 : \int_\Omega \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\}.$$

Clearly, $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ is a separable reflexive Banach space, see [36] (Theorem 2.5 and Corollaries 2.7 and 2.12).

Lemma 1. *Hölder's inequality [14]: Let $L^{p'(x)}(\Omega)$ denote the conjugate space of $L^{p(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ and $p'(x) = (p(x)/(p(x)-1))$. If $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ then the following Hölder-type inequality holds:*

$$\left| \int_\Omega uv\, dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) |u|_{p(x)} |v|_{p'(x)}.$$

A modular of the space $L^{p(x)}(\Omega)$ is defined by

$$\varrho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$$

$$u \mapsto \varrho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)} dx.$$

Assume that $u \in L^{p(x)}(\Omega)$ and $\{u_k\} \subset L^{p(x)}(\Omega)$. Then the following assertions hold (see [2]):

$$
\begin{aligned}
&(a)\ \ |u|_{p(x)} < 1(\text{resp.}, = 1, > 1) \Leftrightarrow \varrho_{p(x)}(u) < 1(\text{resp.}, = 1, > 1),\\
&(b)\ \ |u|_{p(x)} < 1 \Rightarrow |u|^{p^+}_{p(x)} \leq \varrho_{p(x)}(u) \leq |u|^{p^-}_{p(x)},\\
&(c)\ \ |u|_{p(x)} > 1 \Rightarrow |u|^{p^-}_{p(x)} \leq \varrho_{p(x)}(u) \leq |u|^{p^+}_{p(x)},\\
&(d)\ \ \lim_{k\to\infty} |u_k|_{p(x)} = 0(\infty) \Leftrightarrow \lim_{k\to\infty} \varrho_{p(x)}(u_k) = 0(\infty),\\
&(e)\ \ \lim_{k\to\infty} |u_k - u|_{p(x)} = 0 \Leftrightarrow \lim_{k\to\infty} \varrho_{p(x)}(u_k - u) = 0.
\end{aligned}
$$

Let us set the fractional modular function $\varrho^s_{p(.)} : X_0 \to \mathbb{R}$ as

$$
\varrho^s_{p(.)}(u) = \int\int_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dxdy.
$$

Then the following assertions hold (see [2]):

Proposition 1. *Assume that $u \in X_0$ and $\{u_j\}_j \subset X_0$, then*

(1) $\|u\|_{X_0} < 1(\textit{resp.}, = 1, > 1) \Leftrightarrow \varrho^s_{p(.)}(u) < 1(\textit{resp.}, = 1, > 1)$,

(2) $\|u\|_{X_0} < 1 \Rightarrow \|u\|^{p^+}_{X_0} \leq \varrho^s_{p(.)}(u) \leq \|u\|^{p^-}_{X_0}$,

(3) $|u|_{p(x)} > 1 \Rightarrow \|u\|^{p^-}_{X_0} \leq \varrho^s_{p(.)}(u) \leq \|u\|^{p^+}_{X_0}$,

(4) $\lim_{k\to\infty} \|u_k\|_{X_0} = 0(\infty) \Leftrightarrow \lim_{k\to\infty} \varrho^s_{p(.)}(u_k) = 0(\infty)$,

(5) $\lim_{k\to\infty} \|u_k - u\|_{X_0} = 0 \Leftrightarrow \lim_{k\to\infty} \varrho^s_{p(.)}(u_k - u) = 0$.

For any $m \in C_+(\bar{\Omega})$, the fractional Sobolev space with variable exponent, is denoted by

$$
W^{s,m(x),p(x,y)}(\Omega) = \left\{ u \in L^{m(x)}(\Omega) : \int_\Omega\int_\Omega \frac{|u(x) - u(y)|^{p(x,y)}}{\mu^{p(x,y)}|x - y|^{N+sp(x,y)}} dxdy < +\infty, \text{ for some } \mu > 0 \right\}
$$

with the norm $\|u\|_{s,m(x),p(x,y)} = \|u\|_{L^{m(x)}(\Omega)} + [u]^{s,p(x,y)}(\Omega)$, where

$$
[u]^{s,p(x,y)}_\Omega = \inf\left\{ \mu > 0 : \int_\Omega\int_\Omega \frac{|u(x) - u(y)|^{p(x,y)}}{\mu^{p(x,y)}|x - y|^{N+sp(x,y)}} dxdy \leq 1 \right\}.
$$

Readers may refer to [13,16] for more information related to this space. Define $X = W^{s,m(x),p(x,y)}(\Omega)$ over $T = \mathbb{R}^{2N} \setminus (\Omega^c \times \Omega^c)$ as the space

$$
\left\{ u : \mathbb{R}^N \to \mathbb{R} : u|_\Omega \in L^{m(x)}(\Omega), \int_T \frac{|u(x) - u(y)|^{p(x,y)}}{\mu^{p(x,y)}|x - y|^{N+sp(x,y)}} dxdy < +\infty, \text{ for some } \mu > 0 \right\}
$$

and our *solution space* X_0 is defined as the space $\{u \in X : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}$, which is a convex, reflexive and separable Banach space (see [13]) with respect to the norm

$$
\|u\|_{X_0} = \inf\left\{ \lambda > 0 : \int_T \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x - y|^{N+sp(x,y)}} dxdy \leq 1 \right\}.
$$

We will denote $\|u\|_{X_0} = \|u\|$ in all the upcoming results.

Theorem 2. *[14] Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ denotes a smooth bounded domain and $s \in (0,1)$. Let $m(x), p(x,y)$ be continuous variable exponents with $sp(x,y) < N$ for $(x,y) \in \bar{\Omega} \times \bar{\Omega}$ and $m(x) > p(x,x)$ for $x \in \bar{\Omega}$. Assume that $r : \bar{\Omega} \to (1,\infty)$ is a continuous function such that $p_s^*(x) > r(x) \geq r^- > 1$, for $x \in \bar{\Omega}$. Then there exists a constant $C = C(N,s,p,q,r,\Omega)$ such that*

$$\|f\|_{L^{r(\cdot)}(\Omega)} \leq C\|f\|_{W^{s,m(\cdot),p(\cdot,\cdot)}(\Omega)}, \ \textit{for any} \ f \in W^{s,m(\cdot),p(\cdot,\cdot)}(\Omega).$$

Thus, the space $W^{s,m(\cdot),p(\cdot,\cdot)}(\Omega)$ is continuously embedded in $L^{r(\cdot)}(\Omega)$ for any $r \in (1, p_s^*)$. Furthermore, this embedding is compact and the result also holds for the space $X_0 = W_0^{s,m(\cdot),p(\cdot,\cdot)}(\Omega)$.

3. The Proof of Result

The functional corresponding to the problem (1) is defined as

$$I(u) = \hat{K}\Big(\gamma_{p(x,y)}(u)\Big) - \beta \int_\Omega \frac{|u|^{q(x)}}{q(x)} dx - \int_\Omega \frac{h(x)|u|^{\theta\overline{p}(x)} \ln |u|}{\theta\overline{p}(x)} dx + \int_\Omega \frac{h(x)|u|^{\theta\overline{p}(x)}}{(\theta p(x))^2} dx,$$

which is well defined and of class C^1 on X_0. Next, we show the necessity of considering the Nehari manifold.

Lemma 2. *The functional I is not bounded below over X_0.*

Proof. Let $u(\neq 0) \in X_0$.

$$\begin{aligned} I(ru) &= \hat{K}(\gamma_{p(x,y)}(ru)) - \beta \int_\Omega \frac{|ru|^{q(x)}}{q(x)} dx - \int_\Omega \frac{h(x)|ru|^{\theta\overline{p}(x)} ln|ru|}{\theta\overline{p}(x)} dx + \int_\Omega \frac{h(x)|ru|^{\theta\overline{p}(x)}}{(\theta\overline{p}(x))^2} dx \\ &\leq \frac{a_4}{c_2}(\gamma_{p(x,y)}(ru))^\theta - \frac{\beta}{q^+}\int_\Omega |ru|^{q(x)} dx - \frac{1}{\theta p^+}\int_\Omega h(x)|ru|^{\theta\overline{p}(x)} ln|ru| dx + \frac{1}{(\theta p^-)^2}\int_\Omega h(x)|ru|^{\theta\overline{p}(x)} \\ &\leq \frac{a_4 r^{\theta\overline{p}(x)}\|u\|^{\theta p^+}}{c_2(p^-)^\theta} - \frac{\beta r^{q(x)}}{q^+}\int_\Omega |u|^{q(x)} dx - \frac{r^{\theta\overline{p}(x)} ln\ r}{\theta p^+}\int_\Omega h(x)|u|^{\theta\overline{p}(x)} dx - \frac{r^{\theta\overline{p}(x)}}{\theta p^+}\int_\Omega h(x)|u|^{\theta\overline{p}(x)} ln|u| dx \\ &+ \frac{r^{\theta\overline{p}(x)}}{(\theta p^-)^2}\int_\Omega h(x)|u|^{\theta\overline{p}(x)} dx. \end{aligned}$$

Assumption (i) implies that, $q(x) < \theta p(x)$. Therefore, on passing the limit $r \to \infty$ we conclude that the functional I is not bounded below over X_0. □

Hence, we will seek weak solutions over the Nehari manifold. Define the Nehari manifold as $\mathcal{N} = \{u \in X_0 \setminus 0 : \langle I'(u), u\rangle = 0\}$. In particular, $u \in \mathcal{N}$ if and only if $B_\beta(u) = \langle I'(u), u\rangle = 0$.

Lemma 3. *The functional I is coercive and bounded below over $\mathcal{N}$.*

Proof. Since, $u \in \mathcal{N}$ so $\langle I'(u), u\rangle = 0$. This implies that

$$K(\gamma_{p(x,y)}(u)) \int\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}} dxdy - \beta \int_\Omega |u|^{q(x)} dx - \int_\Omega h(x)|u|^{\theta\overline{p}(x)} ln|u| = 0. \tag{2}$$

Now using above equation, we obtain

$$\begin{aligned}
I(u) &\geq \hat{K}(\gamma_{p(x,y)}(u)) - \frac{\beta}{q^-}\int_\Omega |u|^{q(x)}dx + \frac{1}{(\theta p^+)^2}\int_\Omega h(x)|u|^{\theta\overline{p}(x)}dx \\
&+ \frac{1}{\theta p^-}\left[-K(\gamma_{p(x,y)}(u))\int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy + \beta\int_\Omega |u|^{q(x)}dx\right] \\
&= \hat{K}(\gamma_{p(x,y)}(u)) - \beta q^-\int_\Omega |u|^{q(x)}dx - \frac{1}{\theta p^-}K(\gamma_{p(x,y)}(u))\int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} \\
&+ \frac{\beta}{\theta p^-}\int_\Omega |u|^{q(x)}dx + \frac{1}{(\theta p^+)^2}\int_\Omega h(x)|u|^{\theta\overline{p}(x)}dx \\
&> \hat{K}(\gamma_{p(x,y)}(u)) - K(\gamma_{p(x,y)}(u))\int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy + \beta\left(\frac{1}{\theta p^-} - \frac{1}{q^-}\right)\int_\Omega |u|^{q(x)}dx.
\end{aligned}$$

Now using assumption (ii) and the Theorem 2, we obtain

$$\begin{aligned}
I(u) &> \frac{a_3}{c_2}(\gamma_{p(x,y)}(u))^\theta - p^+a_4(\gamma_{p(x,y)}(u))^\theta - \beta\left(\frac{1}{q^-} - \frac{1}{\theta p^-}\right)\int_\Omega |u|^{q(x)}dx \\
&= \left(\frac{a_3}{c_2} - p^+a_4\right)(\gamma_{p(x,y)}(u))^\theta - \beta\left(\frac{1}{q^-} - \frac{1}{\theta p^-}\right)\int_\Omega |u|^{q(x)}dx \\
&\geq \left(\frac{a_3}{c_2} - p^+a_4\right)\|u\|^{\theta p^-} - \beta\left(\frac{1}{q^-} - \frac{1}{\theta p^-}\right)\rho_{q(.)}(u) \\
&\geq \left(\frac{a_3}{c_2} - p^+a_4\right)\|u\|^{\theta p^-} - \beta\left(\frac{1}{q^-} - \frac{1}{\theta p^-}\right)C\|u\|^{q^+}.
\end{aligned}$$

Since, $a_3 > 1$, $\frac{1}{c_2} > p^+a_4$ and $\theta p^- > q^+$, hence we can conclude that the functional I is coercive and bounded below. □

Now we will divide the Nehari manifold into three sets

$$\begin{aligned}
\mathcal{N}^+ &= \{u \in N : \langle B'_\beta(u), u\rangle > 0\}, \\
\mathcal{N}^0 &= \{u \in N : \langle B'_\beta(u), u\rangle = 0\}, \\
\mathcal{N}^- &= \{u \in N : \langle B'_\beta(u), u\rangle < 0\},
\end{aligned}$$

where,

$$\begin{aligned}
\langle B'_\beta(u), u\rangle &= K'(\gamma_{p(x,y)}(u))\left(\int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dx\right)^2 dy - \beta\int_\Omega q(x)|u|^{q(x)}dx \\
&+ K(\gamma_{p(x,y)}(u))\int\int_{\mathbb{R}^{2N}} \frac{p(x,y)|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy - \theta\int_\Omega h(x)\overline{p}(x)|u|^{\theta\overline{p}(x)}ln|u|dx \\
&- \int_\Omega h(x)|u|^{\theta\overline{p}(x)}dx.
\end{aligned} \tag{3}$$

Lemma 4. *There exists $\overline{\beta}$ such that for $0 < \beta < \overline{\beta}$, the set $\mathcal{N}^0$ is empty .*

Proof. Let $u(\neq 0) \in \mathcal{N}^0$. We will prove the result by contradiction.

$$\begin{aligned}
0 = \langle B'_\beta(u), u\rangle &\geq (p^-)^2K'(\gamma_{p(x,y)}(u))(\gamma_{p(x,y)}(u))^2 + p^-K(\gamma_{p(x,y)}(u))(\gamma_{p(x,y)}(u)) - \beta q^+\int_\Omega |u|^{q(x)}dx \\
&- \theta p^+\int_\Omega h(x)|u|^{\theta\overline{p}(x)}ln|u|dx - \int_\Omega h(x)|u|^{\theta\overline{p}(x)}dx.
\end{aligned}$$

Since $u \in \mathcal{N}^0 \subset \mathcal{N}$, so $\langle I'(u), u\rangle = 0$. Using this fact, Theorem 2 and Lemma 1, we obtain

$$
\begin{aligned}
0 &\geq \frac{a_3(p^-)^2}{c_1}(\gamma_{p(x,y)}(u))^\theta + p^- a_3(\gamma_{p(x,y)}(u))^\theta - \beta q^+ \int_\Omega |u|^{q(x)}dx - \int_\Omega h(x)|u|^{\theta\overline{p}(x)}dx \\
&+ \theta p^+\left[-K(\gamma_{p(x,y)}(u)) \int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy + \beta\int_\Omega |u|^{q(x)}dx\right] \\
&\geq \frac{a_3(p^-)^2}{c_1}(\gamma_{p(x,y)}(u))^\theta + p^- a_3(\gamma_{p(x,y)}(u))^\theta - \theta(p^+)^2 K(\gamma_{p(x,y)}(u))(\gamma_{p(x,y)}(u)) - \int_\Omega h(x)|u|^{\theta\overline{p}(x)}dx \\
&+ \beta(\theta p^+ - q^+)\int_\Omega |u|^{q(x)}dx \\
&\geq \frac{a_3(p^-)^2}{c_1}(\gamma_{p(x,y)}(u))^\theta + p^- a_3(\gamma_{p(x,y)}(u))^\theta - \theta(p^+)^2 a_4(\gamma_{p(x,y)}(u))^\theta - \int_\Omega h(x)|u|^{\theta\overline{p}(x)}dx \\
&+ \beta(\theta p^+ - q^+)\int_\Omega |u|^{q(x)}dx \\
&\geq \frac{a_3(p^-)^2}{c_1}(\gamma_{p(x,y)}(u))^\theta + p^- a_3(\gamma_{p(x,y)}(u))^\theta - \theta(p^+)^2 a_4(\gamma_{p(x,y)}(u))^\theta - C\rho_{\theta\overline{p}(.)}(u) \\
&+ \beta(\theta p^+ - q^+)\int_\Omega |u|^{q(x)}dx \\
&\geq \frac{a_3(p^-)^2}{c_1}(\gamma_{p(x,y)}(u))^\theta + p^- a_3(\gamma_{p(x,y)}(u))^\theta - \theta(p^+)^2 a_4(\gamma_{p(x,y)}(u))^\theta - C\|u\|^{\theta p^+} \\
&\geq \left[\frac{a_3(p^-)^2}{c_1} + p^- a_3\right]\frac{\|u\|^{\theta p^-}}{(p^+)^\theta} - \left[C + \frac{a_4\theta(p^+)^2}{(p^-)^\theta}\right]\|u\|^{\theta p^+} \\
&\geq \frac{\left(\frac{1}{c_1}+1\right)}{(p^+)^\theta}\|u\|^{\theta p^-} - \left[C + \frac{a_4\theta(p^+)^2}{(p^-)^\theta}\right]\|u\|^{\theta p^+}.
\end{aligned}
$$

This implies that

$$
\left[C + \frac{a_4\theta(p^+)^2}{(p^-)^\theta}\right]\|u\|^{\theta p^+} > \frac{\left(\frac{1}{c_1}+1\right)}{(p^+)^\theta}\|u\|^{\theta p^-}. \tag{4}
$$

This further implies that

$$
\|u\| > \left(\frac{A}{B}\right)^{\frac{1}{\theta p^+ - \theta p^-}} > 0 \tag{5}
$$

where, $A = \frac{\left(\frac{1}{c_1}+1\right)}{(p^+)^\theta}$ and $B = \left[C + \frac{a_4\theta(p^+)^2}{(p^-)^\theta}\right]$.

Again,

$$
\begin{aligned}
0 = \langle B'_\beta(u), u\rangle &= K'(\gamma_{p(x,y)}(u))\left(\int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy\right)^2 - \theta\int_\Omega h(x)\overline{p}(x)|u|^{\theta\overline{p}(x)} ln|u|dx \\
&- \int_\Omega h(x)|u|^{\theta\overline{p}(x)}dx - \beta\int_\Omega q(x)|u|^{q(x)}dx + K(\gamma_{p(x,y)}(u))\int\int_{\mathbb{R}^{2N}} \frac{p(x,y)|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy.
\end{aligned}
$$

Using $\langle I'(u), u\rangle = 0$ and assumption (ii), we obtain

$$0 \leq \frac{a_4(p^+)^2}{c_1}(\gamma_{p(x,y)}(u))^\theta + a_4(p^+)^2(\gamma_{p(x,y)}(u))^\theta + \beta(\theta p^- - q^-)\int_\Omega |u|^{q(x)}dx$$
$$- \theta p^- K(\gamma_{p(x,y)}(u)) \int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy.$$

Since, $c_1 > p^+$ so

$$\begin{aligned}
0 &< \frac{a_4(p^+)^2}{c_1}(\gamma_{p(x,y)}(u))^\theta + a_4(p^+)^2(\gamma_{p(x,y)}(u))^\theta + \beta(\theta p^- - q^-)\int_\Omega |u|^{q(x)}dx - a_3\theta(p^-)^2(\gamma_{p(x,y)}(u))^\theta \\
&= \left[\frac{a_4(p^+)^2}{c_1} + a_4(p^+)^2 - a_3\theta(p^-)^2\right](\gamma_{p(x,y)}(u))^\theta + \beta(\theta p^- - q^-)\int_\Omega |u|^{q(x)}dx \\
&< \left[2(p^+)^2 a_4 - a_3\theta(p^-)^2\right](\gamma_{p(x,y)}(u))^\theta + \beta(\theta p^- - q^-)\int_\Omega |u|^{q(x)}dx.
\end{aligned}$$

Since, coefficient of $(\gamma_{p(x,y)}(u))^\theta$ is negative as $\theta > a_4(p^+)^2$ so using Theorem 2 and Proposition 1

$$\left[\theta a_3(p^-)^2 - 2a_4(p^+)^2\right]\frac{\|u\|^{\theta p^-}}{(p^+)^\theta} < \beta(\theta p^- - q^-)\int_\Omega |u|^{q(x)}dx$$
$$\left[\theta a_3(p^-)^2 - 2a_4(p^+)^2\right]\frac{\|u\|^{\theta p^-}}{(p^+)^\theta} < \beta(\theta p^- - q^-)\rho_{q(.)}(u)$$
$$\left[\theta a_3(p^-)^2 - 2a_4(p^+)^2\right]\frac{\|u\|^{\theta p^-}}{(p^+)^\theta} < \beta(\theta p^- - q^-)C_1\|u\|^{q+}$$
$$\|u\| < \left[\frac{\beta(\theta p^- - q^-)C_1(p^+)^\theta}{\theta a_3(p^-)^2 - 2a_4(p^+)^2}\right]^{\frac{1}{\theta p^- - q^+}}.$$

Choosing β small enough, say $\overline{\beta}$, so that $\left[\frac{\beta(\theta p^- - q^-)C_1(p^+)^\theta}{\theta a_3(p^-)^2 - 2a_4(p^+)^2}\right]^{\frac{1}{\theta p^- - q^+}} < \left(\frac{A}{B}\right)^{\frac{1}{\theta p^+ - \theta p^-}}$ we obtain a contradiction to (5) for $\beta \in (0, \overline{\beta})$. Hence, the set $\mathcal{N}^0$ is empty. $\square$

Since, $\mathcal{N}^0 = \phi$, so $\mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^-$ by Lemma 4. Define $i^+ = \inf_{u\in\mathcal{N}^+} I(u)$ and $i^- = \inf_{u\in\mathcal{N}^-} I(u)$.

Lemma 5. *If $0 < \beta < \beta^*$, then we have*
(i) $i^+ < 0$
(ii) $i^- > 0$.

Proof. (i) Let $u \in \mathcal{N}^+$.

$$\begin{aligned}
I(u) =& \hat{K}\left(\gamma_{p(x,y)}(u)\right) - \beta\int_\Omega \frac{|u|^{q(x)}}{q(x)}dx - \int_\Omega \frac{h(x)|u|^{\theta\overline{p}(x)} ln|u|}{\theta\overline{p}(x)}dx + \int_\Omega \frac{h(x)|u|^{\theta\overline{p}(x)}}{(\theta p(x))^2}dx \\
\leq& \hat{K}\left(\gamma_{p(x,y)}(u)\right) - \frac{1}{\theta p^+}\int_\Omega h(x)|u|^{\theta\overline{p}(x)} ln|u|dx + \frac{1}{(\theta p^-)^2}\int_\Omega h(x)|u|^{\theta\overline{p}(x)}dx - \frac{\beta}{q^+}\int_\Omega |u|^{q(x)}dx \qquad (6)\\
\leq& \hat{K}\left(\gamma_{p(x,y)}(u)\right) - \frac{1}{\theta p^+}\int_\Omega h(x)|u|^{\theta\overline{p}(x)} ln|u|dx + \int_\Omega h(x)|u|^{\theta\overline{p}(x)}dx - \frac{\beta}{q^+}\int_\Omega |u|^{q(x)}dx.
\end{aligned}$$

Since, $u \in \mathcal{N}^+$ so $\langle B'_\beta(u), u\rangle > 0$ which implies that

$$K'(\gamma_{p(x,y)}(u))\left(\int\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dx\right)^2+K(\gamma_{p(x,y)}(u))\int\int_{\mathbb{R}^{2N}}\frac{p(x,y)|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy$$
$$-\beta\int_\Omega q(x)|u|^{q(x)}dx-\theta\int_\Omega h(x)\overline{p}(x)|u|^{\theta\overline{p}(x)}ln|u|dx-\int_\Omega h(x)|u|^{\theta\overline{p}(x)}dx>0.$$

This further implies

$$\begin{aligned}&K'(\gamma_{p(x,y)}(u))\left(\int\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dx\right)^2+K(\gamma_{p(x,y)}(u))\int\int_{\mathbb{R}^{2N}}\frac{p(x,y)|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy\\&-\beta q^-\int_\Omega|u|^{q(x)}dx-\theta p^-\int_\Omega h(x)|u|^{\theta\overline{p}(x)}ln|u|dx-\int_\Omega h(x)|u|^{\theta\overline{p}(x)}dx>0.\end{aligned}\tag{7}$$

Furthermore,

$$\begin{aligned}&\langle I'(u),u\rangle=0\\&\Rightarrow K(\gamma_{p(x,y)}(u))\int\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy-\beta\int_\Omega|u|^{q(x)}dx-\int_\Omega h(x)|u|^{\theta\overline{p}(x)}ln|u|dx=0.\end{aligned}\tag{8}$$

Multiplying (8) by $(-\theta p^-)$ and adding to (7), we obtain

$$K'(\gamma_{p(x,y)}(u))\left(\int\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dx\right)^2+K(\gamma_{p(x,y)}(u))\int\int_{\mathbb{R}^{2N}}\frac{p(x,y)|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy$$
$$+\beta(\theta p^--q^-)\int_\Omega|u|^{q(x)}dx-\int_\Omega h(x)|u|^{\theta\overline{p}(x)}dx-\theta p^-K(\gamma_{p(x,y)}(u))\int\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy>0.$$

This implies that

$$\begin{aligned}\int_\Omega h(x)|u|^{\theta\overline{p}(x)}dx<{}&K'(\gamma_{p(x,y)}(u))\left(\int\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dx\right)^2+\beta(\theta p^--q^-)\int_\Omega|u|^{q(x)}dx\\&+K(\gamma_{p(x,y)}(u))\int\int_{\mathbb{R}^{2N}}\frac{p(x,y)|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy\\&-\theta p^-K(\gamma_{p(x,y)}(u))\int\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy.\end{aligned}\tag{9}$$

Using (9) in (6), we obtain

$$\begin{aligned}I(u)\le{}&\hat{K}(\gamma_{p(x,y)}(u))-\frac{\beta}{q^+}\int_\Omega|u|^{q(x)}dx-\frac{1}{\theta p^+}\int_\Omega h(x)|u|^{\theta\overline{p}(x)}ln|u|dx+\beta(\theta p^--q^-)\int_\Omega|u|^{q(x)}dx\\&+K'(\gamma_{p(x,y)}(u))\left(\int\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dx\right)^2+K(\gamma_{p(x,y)}(u))\int\int_{\mathbb{R}^{2N}}\frac{p(x,y)|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy\\&-\theta p^-K(\gamma_{p(x,y)}(u))\int\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy\\&<\hat{K}(\gamma_{p(x,y)}(u))+(p^+)^2K'(\gamma_{p(x,y)}(u))(\gamma_{p(x,y)}(u))^2+(p^+)^2K(\gamma_{p(x,y)}(u))(\gamma_{p(x,y)}(u))\\&-\theta(p^-)^2K(\gamma_{p(x,y)}(u))(\gamma_{p(x,y)}(u))+\beta\left(\theta p^--q^--\frac{1}{q^+}\right)\int_\Omega|u|^{q(x)}dx\\&\le\left[\frac{a_4}{c_2}+\frac{a_4(p^+)^2}{c_1}+a_4(p^+)^2-a_3\theta(p^-)^2\right](\gamma_{p(x,y)}(u))^\theta+\beta\left(\theta p^--q^--\frac{1}{q^+}\right)\int_\Omega|u|^{q(x)}dx.\end{aligned}$$

Now using assumptions on c_1 and c_2 from (ii), we obtain

$$I(u) < \left[3a_4^2(p^+)^2 - a_3\theta(p^-)^2\right](\gamma_{p(x,y)}(u))^\theta + \beta\left(\theta p^- - q^- - \frac{1}{q^+}\right)\int_\Omega |u|^{q(x)}dx$$
$$< \left[3a_4^2(p^+)^2 - a_3\theta(p^-)^2\right](\gamma_{p(x,y)}(u))^\theta + \beta\left(\theta p^- - q^- - \frac{1}{q^+}\right)\rho_{q(.)}u.$$

Since, $\theta > a_4^2(p^+)^2$ from assumption (ii) so coefficient of $(\gamma_{p(x,y)}(u))^\theta$ is negative which along with using Theorem 2 further implies that

$$I(u) < \left[3a_4^2(p^+)^2 - a_3\theta(p^-)^2\right]\|u\|^{\theta p^-} + \beta\left(\theta p^- - q^- - \frac{1}{q^+}\right)C\|u\|^{q+}.$$

For β say in the range of $(0,\beta^*)$, we obtain $I(u) < 0$ and hence $i^+ = \underset{u\in N^+}{inf}\ I(u) < 0$.

(ii) Let $u \in \mathcal{N}^-$. Then, $\langle B'_\beta(u), u\rangle < 0$. This implies that

$$K'(\gamma_{p(x,y)}(u))\left(\int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dx\right)^2 + K(\gamma_{p(x,y)}(u))\int\int_{\mathbb{R}^{2N}} \frac{p(x,y)|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy$$
$$-\beta\int_\Omega q(x)|u|^{q(x)}dx - \theta\int_\Omega h(x)\overline{p}(x)|u|^{\theta\overline{p}(x)}ln|u|dx - \int_\Omega h(x)|u|^{\theta\overline{p}(x)}dx < 0.$$

This further implies that

$$K'(\gamma_{p(x,y)}(u))\left(\int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dx\right)^2 + K(\gamma_{p(x,y)}(u))\int\int_{\mathbb{R}^{2N}} \frac{p(x,y)|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy$$
$$-\beta q^+\int_\Omega |u|^{q(x)}dx - \theta p^+\int_\Omega h(x)|u|^{\theta\overline{p}(x)}ln|u|dx - \int_\Omega h(x)|u|^{\theta\overline{p}(x)}dx < 0. \tag{10}$$

Multiplying $\langle I'(u), u\rangle$ by $-(\theta p^+)^2$ and adding from (10) we obtain,

$$K'(\gamma_{p(x,y)}(u))\left(\int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dx\right)^2 + K(\gamma_{p(x,y)}(u))\int\int_{\mathbb{R}^{2N}} \frac{p(x,y)|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy$$
$$-\beta q^+\int_\Omega |u|^{q(x)}dx - \int_\Omega h(x)|u|^{\theta\overline{p}(x)}dx - \theta p^+\int_\Omega h(x)|u|^{\theta\overline{p}(x)}ln|u|dx$$
$$-(\theta p^+)^2\left[K(\gamma_{p(x,y)}(u))\int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy - \beta\int_\Omega |u|^{q(x)}dx - \int_\Omega h(x)|u|^{\theta\overline{p}(x)}ln|u|dx\right] < 0.$$

This implies

$$\int_\Omega h(x)|u|^{\theta\overline{p}(x)}dx > K'(\gamma_{p(x,y)}(u))\left(\int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dx\right)^2 - \theta p^+\int_\Omega h(x)|u|^{\theta\overline{p}(x)}ln|u|dx$$
$$-\beta q^+\int_\Omega |u|^{q(x)}dx + K(\gamma_{p(x,y)}(u))\int\int_{\mathbb{R}^{2N}} \frac{p(x,y)|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy \tag{11}$$
$$-(\theta p^+)^2\left[K(\gamma_{p(x,y)}(u))\int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy - \beta\int_\Omega |u|^{q(x)}dx - \int_\Omega h(x)|u|^{\theta\overline{p}(x)}ln|u|dx\right].$$

Using (11) and Proposition 1, we obtain

$$
\begin{aligned}
I(u) &\geq \hat{K}(\gamma_{p(x,y)}(u)) - \frac{1}{\theta p^-}\int_\Omega h(x)|u|^{\theta\overline{p}(x)} ln|u|dx - \frac{\beta}{q^-}\int_\Omega |u|^{q(x)}dx + \frac{1}{(\theta p^+)^2}\int_\Omega h(x)|u|^{\theta\overline{p}(x)}dx \\
&> \hat{K}(\gamma_{p(x,y)}(u)) - \frac{1}{\theta p^-}\int_\Omega h(x)|u|^{\theta\overline{p}(x)} ln|u|dx - \frac{\beta}{q^-}\int_\Omega |u|^{q(x)}dx \\
&+ \frac{1}{(\theta p^+)^2}\Bigg[K'(\gamma_{p(x,y)}(u))\left(\int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dx\right)^2 + \beta\Big((\theta p^+)^2 - q^+\Big)\int_\Omega |u|^{q(x)}dx \\
&+ p^- K(\gamma_{p(x,y)}(u))\int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy - (\theta p^+)^2 K(\gamma_{p(x,y)}(u))\int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy \\
&+ \Big((\theta p^+)^2 - (\theta p^+)\Big)\int_\Omega h(x)|u|^{\theta\overline{p}(x)} ln|u|dx\Bigg] \\
&> \hat{K}(\gamma_{p(x,y)}(u)) + \frac{p^-}{(\theta p^+)^2}K(\gamma_{p(x,y)}(u))\int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy \\
&+ \frac{1}{(\theta p^+)^2}K'(\gamma_{p(x,y)}(u))\left(\int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dx\right)^2 - K(\gamma_{p(x,y)}(u))\int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy \\
&+ \beta\left((\theta p^+)^2 - q^+ - \frac{1}{q^-}\right)\int_\Omega |u|^{q(x)}dx \\
&\geq \hat{K}(\gamma_{p(x,y)}(u)) + \frac{(p^-)^2}{(\theta p^+)^2}K'(\gamma_{p(x,y)}(u))(\gamma_{p(x,y)}(u))^2 - p^+\left(1 - \frac{1}{(\theta p^+)^2}\right)K(\gamma_{p(x,y)}(u))(\gamma_{p(x,y)}(u)) \\
&> \frac{a_3}{c_2}(\gamma_{p(x,y)}(u))^\theta + \frac{a_3(p^-)^2}{c_1(\theta p^+)^2}(\gamma_{p(x,y)}(u))^\theta - a_4 p^+\left(1 - \frac{1}{(\theta p^+)^2}\right)(\gamma_{p(x,y)}(u))^\theta \\
&> \left[a_3 a_4 p^+ + \frac{a_3(p^-)^2}{c_1(\theta p^+)^2} - a_4 p^+\right](\gamma_{p(x,y)}(u))^\theta \\
&\geq \frac{\left[a_3 a_4 p^+ + \frac{a_3(p^-)^2}{c_1(\theta p^+)^2} - a_4 p^+\right]}{(p^+)^\theta}\|u\|^{\theta p^-} > 0.
\end{aligned}
$$

Thus, $I(u) > 0$, and hence, $i^- = \inf\limits_{u\in\mathcal{N}^-} I(u) > 0$. □

Lemma 6. *If $0 < \beta < \beta_1$, where $\beta_1 = min\{\beta^*, \overline{\beta}\}$ then the functional I has a minimizer u_0^+ in $\mathcal{N}^+$ and $I(u_0^+) = i^+$.*

Proof. Since I is bounded below on $\mathcal{N}$ and so on $\mathcal{N}^+$, there exists a minimizing sequence $(u_n^+) \subset \mathcal{N}^+$ such that $\lim\limits_{n\to\infty} I(u_n^+) = \inf\limits_{u\in\mathcal{N}^+} I(u) = i^+ < 0$ from Lemma 5. Furthermore, I is coercive so u_n^+ is bounded in $\mathcal{N}$ from Lemma 3 and hence $u_n^+ \rightharpoonup u_0^+$ in $\mathcal{N} \subset X_0$ up to a subsequence. By compact embedding, $u_n^+ \to u_0^+$ in $L^{q(x)}(\Omega)$ for $q \in (1, p_s^*)$ (Theorem 2). Since $\theta p^\pm < p_s^*$ from assumption (i), so $|u_n^+|^{\theta p^\pm} \to |u_0^+|^{\theta p^\pm}$ by compact embedding. Thus, as $h(x) \in C(\bar{\Omega})$, we obtain $\lim\limits_{n\to\infty}\int_\Omega h(x)|u_n^+|^{\theta p^\pm} ln|u_n^+|dx = \int_\Omega h(x)|u_0^+|^{\theta p^\pm} ln|u_0^+|dx$ and $\lim\limits_{n\to\infty}\int_\Omega h(x)|u_n^+|^{\theta p^\pm}dx = \int_\Omega h(x)|u_0^+|^{\theta p^\pm}dx$ (refer [26]).

Now, we need to show that $u_n^+ \to u_0^+$ in X_0. We will prove it by contradiction. Let $u_n^+ \not\to u_0^+$ in X_0 then

$$
\int\int_{\mathbb{R}^{2N}} \frac{|u_0^+(x)-u_0^+(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy < \liminf_{n\to\infty}\int\int_{\mathbb{R}^{2N}} \frac{|u_n^+(x)-u_n^+(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy. \tag{12}
$$

Furthermore, $\langle I'(u_n^+), (u_n^+)\rangle = 0$. Hence,

$$
\begin{aligned}
I(u_n^+) &= I(u_n^+) - \frac{1}{\theta p^-}\langle I'(u_n^+),(u_n^+)\rangle \\
&\geq \hat{K}(\gamma_{p(x,y)}(u_n^+)) - \frac{\beta}{q^-}\int_\Omega |u_n^+(x)|^{q(x)}dx - \frac{1}{\theta p^-}\int_\Omega h(x)|u_n^+|^{\theta\overline{p}(x)} ln|u_n^+|dx \\
&+ \frac{1}{(\theta p^+)^2}\int_\Omega h(x)|u_n^+|^{\theta\overline{p}(x)}dx - \frac{1}{\theta p^-}\left[K(\gamma_{p(x,y)}(u_n^+))\int\int_{\mathbb{R}^{2N}}\frac{|u_n^+(x)-u_n^+(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy\right. \\
&\left.-\beta\int_\Omega |u_n^+|^{q(x)}dx - \int_\Omega h(x)|u_n^+|^{\theta\overline{p}(x)} ln|u_n^+|dx\right] \\
&> \hat{K}(\gamma_{p(x,y)}(u_n^+)) - \frac{1}{\theta p^-}K(\gamma_{p(x,y)}(u_n^+))\int\int_{\mathbb{R}^{2N}}\frac{|u_n^+(x)-u_n^+(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy \\
&- \beta\left(\frac{1}{q^-}-\frac{1}{\theta p^-}\right)\int_\Omega |u_n^+|^{q(x)}dx \\
&\geq \hat{K}(\gamma_{p(x,y)}(u_n^+)) - \frac{p^+}{\theta p^-}K(\gamma_{p(x,y)}(u_n^+))(\gamma_{p(x,y)}(u_n^+)) - \beta\left(\frac{1}{q^-}-\frac{1}{\theta p^-}\right)\int_\Omega |u_n^+|^{q(x)}dx \\
&> \frac{a_3}{c_2}(\gamma_{p(x,y)}(u_n^+))^\theta - \frac{a_4 p^+}{\theta p^-}(\gamma_{p(x,y)}(u_n^+))^\theta - \beta\left(\frac{1}{q^-}-\frac{1}{\theta p^-}\right)\int_\Omega |u_n^+|^{q(x)}dx \\
&= \left[\frac{a_3}{c_2} - \frac{a_4 p^+}{\theta p^-}\right](\gamma_{p(x,y)}(u_n^+))^\theta - \beta\left(\frac{1}{q^-}-\frac{1}{\theta p^-}\right)\int_\Omega |u_n^+|^{q(x)}dx.
\end{aligned}
$$

Since $c_2 < \frac{1}{a_4 p^+}$ and $a_3 > 1$ from assumption (ii), we obtain

$$
\begin{aligned}
I(u) &> a_4 p^+\left[a_3 - \frac{1}{\theta p^-}\right](\gamma_{p(x,y)}(u_n^+))^\theta - \beta\left(\frac{1}{q^-}-\frac{1}{\theta p^-}\right)\int_\Omega |u_n^+|^{q(x)}dx \\
&\geq \frac{C_2}{p^+}\left(\int\int_{\mathbb{R}^{2N}}\frac{|u_n^+(x)-u_n^+(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy\right)^\theta - \beta\left(\frac{1}{q^-}-\frac{1}{\theta p^-}\right)\int_\Omega |u_n^+|^{q(x)}dx,
\end{aligned}
$$

where, $C_2 = a_4 p^+\left[a_3 - \frac{1}{\theta p^-}\right]$. Now taking limit infimum both sides and using (11) and Theorem 2, we obtain

$$
\begin{aligned}
i^+ &> \frac{C_2}{p^+}\left(\int\int_{\mathbb{R}^{2N}}\frac{|u_0^+(x)-u_0^+(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy\right)^\theta - \beta\left(\frac{1}{q^-}-\frac{1}{\theta p^-}\right)\int_\Omega |u_n^+|^{q(x)}dx \\
&\geq \frac{C_2\|u_0^+\|^{\theta p^-}}{p^+} - \beta\left(\frac{1}{q^-}-\frac{1}{\theta p^-}\right)\int_\Omega |u_n^+|^{q(x)}dx \\
&\geq \frac{C_2\|u_0^+\|^{\theta p^-}}{p^+} - \beta\left(\frac{1}{q^-}-\frac{1}{\theta p^-}\right)\rho_{q(.)}(u_n^+) \\
&\geq \frac{C_2\|u_0^+\|^{\theta p^-}}{p^+} - \beta\left(\frac{1}{q^-}-\frac{1}{\theta p^-}\right)C\|u_n^+\|^{q+} \\
&> 0.
\end{aligned}
$$

This is a contradiction to $i^+ < 0$ for β small enough. Hence, $u_n^+ \to u_0^+$ in X_0 and $I(u_0^+) = \lim_{n\to\infty} I(u_n^+) = \inf_{u\in\mathcal{N}^+} I(u)$. Thus, u_0^+ is a minimizer for I on $\mathcal{N}^+$. □

Lemma 7. *If $0 < \beta < \beta_2$ then the functional I has a minimizer u_0^- in $\mathcal{N}^-$ and $I(u_0^-) = i^-$.*

Proof. Since I is bounded below on $\mathcal{N}$ and so on $\mathcal{N}^-$, there exists a minimizing sequence $(u_n^-) \subset \mathcal{N}^-$ such that $\lim_{n\to\infty} I(u_n^-) = \inf_{u\in\mathcal{N}^-} I(u) = i^- > 0$ from Lemma 5. Furthermore, I

is coercive so u_n^- is bounded in $\mathcal{N}$ from Lemma 3, and hence, $u_n^- \rightharpoonup u_0^-$ in $\mathcal{N}^- \subset X_0$ up to a subsequence. By compact embedding, $u_n^- \to u_0^-$ in $L^{q(x)}(\Omega)$. Furthermore, $\lim\limits_{n\to\infty} \int_\Omega h(x)|u_n^-|^{\theta p^\pm} ln|u_n^-|dx = \int_\Omega h(x)|u_0^-|^{\theta p^\pm} ln|u_0^-|dx$ and $\lim\limits_{n\to\infty} \int_\Omega h(x)|u_n^-|^{\theta p^\pm} dx = \int_\Omega h(x)|u_0^-|^{\theta p^\pm} dx$.

Moreover, there exists a constant $t > 0$ such that $tu_0^- \in \mathcal{N}^-$. This can be verified as follows:

$$\begin{aligned}\langle B'_\beta(tu_0^-),(tu_0^-)\rangle &= K'(\gamma_{p(x,y)}(tu_0^-))\left(\int\int_{\mathbb{R}^{2N}} \frac{|tu_0^-(x)-tu_0^-(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy\right)^2 - \beta\int_\Omega q(x)|tu_0^-|^{q(x)}dx \\ &+ K(\gamma_{p(x,y)}(tu_0^-))\int\int_{\mathbb{R}^{2N}} \frac{p(x,y)|tu_0^-(x)-tu_0^-(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy \\ &- \theta\int_\Omega h(x)\overline{p}(x)|tu_0^-|^{\theta\overline{p}(x)}ln|tu_0^-|dx - \int_\Omega h(x)|tu_0^-|^{\theta\overline{p}(x)}dx.\end{aligned}$$

For, $tu_0^- \in \mathcal{N}$ we have $\langle I'(tu_0^-),(tu_0^-) = 0\rangle$ i.e.

$$K(\gamma_{p(x,y)}(tu_0^-))\int\int_{\mathbb{R}^{2N}} \frac{|tu_0^-(x)-tu_0^-(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy - \beta\int_\Omega |tu_0^-|^{q(x)}dx - \int_\Omega h(x)|tu_0^-|^{\theta\overline{p}(x)}ln|tu_0^-|dx = 0.$$

Now,

$$\begin{aligned}\langle B'_\beta(tu_0^-),(tu_0^-)\rangle &= \langle B'_\beta(tu_0^-),(tu_0^-)\rangle - \theta p^+ I'(tu_0^-),(tu_0^-) \\ &\le (p^+)^2K'(\gamma_{p(x,y)}(tu_0^-))(\gamma_{p(x,y)}(tu_0^-))^2 + (p^+)^2K(\gamma_{p(x,y)}(tu_0^-))(\gamma_{p(x,y)}(tu_0^-)) \\ &+ \beta(\theta p^+ - q^-)\int_\Omega |tu_0^-|^{q(x)}dx - \theta p^+p^-K(\gamma_{p(x,y)}(tu_0^-))(\gamma_{p(x,y)}(tu_0^-)) \\ &< \frac{a_4(p^+)^2}{c_1}(\gamma_{p(x,y)}(tu_0^-))^\theta + a_4(p^+)^2(\gamma_{p(x,y)}(tu_0^-))^\theta - a_3\theta p^+p^-(\gamma_{p(x,y)}(tu_0^-))^\theta \\ &+ \beta(\theta p^+ - q^-)\int_\Omega |tu_0^-|^{q(x)}dx.\end{aligned}$$

Using $c_1 > p^+$ from assumption (i) and $\theta > a_4^2(p^+)^2$ from assumption (ii), we obtain

$$\langle B'_\beta(tu_0^-),(tu_0^-)\rangle < \left[a_4p^+ + a_4(p^+)^2 - a_3\theta p^+p^-\right](\gamma_{p(x,y)}(tu_0^-))^\theta + \beta(\theta p^+ - q^-)\int_\Omega |tu_0^-|^{q(x)}dx.$$

There arises two cases $t < 1$ and $t > 1$. When $0 < t < 1$, we obtain

$$\begin{aligned}\langle B'_\beta(tu_0^-),(tu_0^-)\rangle &< \left[a_4p^+ + a_4(p^+)^2 - a_3\theta p^+p^-\right]t^{\theta p^+}(\gamma_{p(x,y)}(u_0^-))^\theta + \beta(\theta p^- - q^-)\int_\Omega t^{q^-}|u_0^-|^{q(x)}dx \\ &< \left[a_4p^+ + a_4(p^+)^2 - a_3\theta p^+p^-\right]t^{\theta p^+}(\gamma_{p(x,y)}(u_0^-))^\theta + \beta(\theta p^- - q^-)t^{q^-}\rho_{q(.)}(u_0^-) \\ &\le \left[a_4p^+ + a_4(p^+)^2 - a_3\theta p^+p^-\right]t^{\theta p^+}\|u_0^-\|^{\theta p^-} + \beta(\theta p^- - q^-)t^{q-}C\|u_0^-\|^{q+}.\end{aligned}$$

Choosing β small enough, say β_3, we obtain $\langle B'_\beta(tu_0^-),(tu_0^-)\rangle < 0$.
Let $t > 1$, then

$$\begin{aligned}\langle B'_\beta(tu_0^-),(tu_0^-)\rangle &< \left[a_4p^+ + a_4(p^+)^2 - a_3\theta p^+p^-\right]t^{\theta p^-}(\gamma_{p(x,y)}(u_0^-))^\theta + \beta(\theta p^- - q^-)\int_\Omega t^{q+}|u_0^-|^{q(x)}dx \\ &< \left[a_4p^+ + a_4(p^+)^2 - a_3\theta p^+p^-\right]t^{\theta p^-}(\gamma_{p(x,y)}(u_0^-))^\theta + \beta(\theta p^- - q^-)t^{q+}\rho_{q(.)}(u_0^-) \\ &\le \left[a_4p^+ + a_4(p^+)^2 - a_3\theta p^+p^-\right]t^{\theta p^-}\|u_0^-\|^{\theta p^-} + C\beta(\theta p^- - q^-)t^{q+}\|u_0^-\|^{q+}.\end{aligned}$$

Choosing β small enough, say β_4, we obtain $\langle B'_\beta(tu_0^-),(tu_0^-)\rangle < 0$.
Hence, $tu_0^- \in \mathcal{N}^-$ for $\beta_2 = \min\{\beta_3,\beta_4\}$.

Now, we will prove that $u_n^- \to u_0^-$ in X_0. Since $u_n^- \rightharpoonup u_0^-$ in X_0 so $tu_n^- \rightharpoonup tu_0^-$ in X_0 and $tu_n^- \nrightarrow tu_0^-$ in X_0. Hence, $\hat{K}(\gamma_{p(x,y)}(tu_0^-)) < \liminf_{n\to\infty} \hat{K}(\gamma_{p(x,y)}(tu_n^-))$. Therefore,

$$
\begin{aligned}
I(tu_0^-) &= \hat{K}\Big(\gamma_{p(x,y)}(tu_0^-)\Big) - \beta\int_\Omega \frac{|tu_0^-|^{q(x)}}{q(x)}dx - \int_\Omega \frac{h(x)|tu_0^-|^{\theta\overline{p}(x)} ln|tu_0^-|}{\theta\overline{p}(x)}dx \\
&+ \int_\Omega \frac{h(x)|tu_0^-|^{\theta\overline{p}(x)}}{(\theta p(x))^2}dx \\
&= \hat{K}\left(\int\int_{\mathbb{R}^{2N}} \frac{|tu_0^-(x) - tu_0^-(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}dxdy\right) - \beta\int_\Omega \frac{|tu_0^-|^{q(x)}}{q(x)}dx - \int_\Omega \frac{h(x)|tu_0^-|^{\theta\overline{p}(x)} ln|tu_0^-|}{\theta\overline{p}(x)}dx \\
&+ \int_\Omega \frac{h(x)|tu_0^-|^{\theta\overline{p}(x)}}{(\theta p(x))^2}dx \\
&< \liminf_{n\to\infty}\left[\hat{K}\left(\int\int_{\mathbb{R}^{2N}} \frac{|tu_n^-(x) - tu_n^-(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}dxdy\right) - \beta\int_\Omega \frac{|tu_n^-|^{q(x)}}{q(x)}dx\right. \\
&\left. - \int_\Omega \frac{h(x)|tu_n^-|^{\theta\overline{p}(x)} ln|tu_n^-|}{\theta\overline{p}(x)}dx + \int_\Omega \frac{h(x)|tu_n^-|^{\theta\overline{p}(x)}}{(\theta p(x))^2}dx\right] \\
&\le \lim_{n\to\infty}\left[\hat{K}\left(\int\int_{\mathbb{R}^{2N}} \frac{|tu_n^-(x) - tu_n^-(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}dxdy\right) - \beta\int_\Omega \frac{|tu_n^-|^{q(x)}}{q(x)}dx\right. \\
&\left. - \int_\Omega \frac{h(x)|tu_n^-|^{\theta\overline{p}(x)} ln|tu_n^-|}{\theta\overline{p}(x)}dx + \int_\Omega \frac{h(x)|tu_n^-|^{\theta\overline{p}(x)}}{(\theta p(x))^2}dx\right] \\
&= \lim_{n\to\infty} I(tu_n^-).
\end{aligned}
$$

Furthermore, since $u_n^- \rightharpoonup u_0^-$ and $(u_n^-) \subset \mathcal{N}^-$, by using continuity of the function K, we obtain

$$
\begin{aligned}
\langle I_\beta'(u_0^-), (u_0^-)\rangle &= K(\gamma_{p(x,y)}(u_0^-))\int\int_{\mathbb{R}^{2N}} \frac{|u_0^-(x) - u_0^-(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy - \beta\int_\Omega |u_0^-|^{q(x)}dx \\
&- \int_\Omega h(x)|u_0^-|^{\theta\overline{p}(x)} ln|u_0^-|dx \\
&< \liminf_{n\to\infty} K(\gamma_{p(x,y)}(u_n^-))\int\int_{\mathbb{R}^{2N}} \frac{|u_n^-(x) - u_n^-(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy - \beta\int_\Omega |u_n^-|^{q(x)}dx \\
&- \int_\Omega h(x)|u_n^-|^{\theta\overline{p}(x)} ln|u_n^-|dx \\
&= \langle I_\beta'(u_n^-), (u_n^-)\rangle = 0,
\end{aligned}
$$

which is a contradiction to $u_0^- \in \mathcal{N}^-$ and hence $t \neq 1$. Furthermore, observe that the function $I(tu_n^-)$ attains its maximum at $t = 1$. Thus, we have

$$I(tu_0^-) < \lim_{n\to\infty} I(tu_n^-) \le \lim_{n\to\infty} I(u_n^-) = \inf_{u\in\mathcal{N}^-} I(u) = i^-,$$

which is absurd. Hence, $u_n^- \to u_0^-$ in X_0 and therefore $I(u_0^-) = \lim_{n\to\infty} I(u_n^-) = \inf_{u\in\mathcal{N}^-} I(u)$. Thus, u_0^- is a minimizer for I on $\mathcal{N}^-$. □

Proof of Theorem 1. By Lemmas 6 and 7, we conclude that there exist $u_0^+ \in \mathcal{N}^+$ and $u_0^- \in \mathcal{N}^-$ such that $I(u_0^+) = \inf_{u\in\mathcal{N}^+} I(u) < 0$ and $I(u_0^-) = \inf_{u\in\mathcal{N}^-} I(u) > 0$. Hence, we obtain at least two distinct nontrivial weak solutions of the considered problem for $\beta \in (0, \beta^{**})$, where $\beta^{**} = \min\{\beta_1, \beta_2\}$.

4. Conclusions

In this article, we address the multiplicity of the solutions of an elliptic problem with variable exponents involving logarithmic nonlinearity and a nonlocal term using the analysis of the fibering map and Nehari manifold. The Nehari manifold technique via the fibering map applied for the variable exponents problem is interesting because of the non-homogeneity that arises from the variable exponents. It is likewise well worth citing that due to the presence of the variable exponents, most of the estimates are not maintained straight away, unlike inside the regular exponent set-up. Hence, to overcome this problem, some rigorous analysis has been performed.

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