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Hybrid Differential Inclusion Involving Two Multi-Valuedoperators with Nonlocal Multi-Valued Integral Condition

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Abstract: The present paper is devoted to the existence of solution for the Hybrid differential inclusions of the second type. Here, we present the inclusion problem with two multi-valued maps. In addition, it is considered with nonlocal integral boundary condition $\eta(0) \in \int_0^{\sigma} \Delta(s, \eta(s))ds$, where Δ is a multi-valued map. Relative compactness of the set $\int_0^{\sigma} \Delta(s, \eta(s))ds$ in $L^2((0, \varepsilon), \mathbb{R})$ is used to justify the condensing condition for some created operators. Fixed point theorems connected with the weak compactness manner is utilized to explore the results throughout this paper.

Keywords: hybrid fractional inclusion; existence and uniqueness; compactness and noncompactness; multi-operators; multi-condition

MSC: 26A33; 34A08; 34A12

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1. Introduction

Among of a large amount contributions dedicated to study the existence and uniqueness of solution for Hybrid differential equations and inclusions with one multi-valued map, it is worth mentioning the works of Dhage [1,2], Dhage and Lakishmikantham [3] that focused on fixed point theorems to Hybrid operators and their applications. For such example, Ahmad et al. [4] explore the solvability for first and second type of Hybrid equations and inclusions with one multi-valued map.

In fractional analysis field, we focus on studying and improving the solvability of some fractional differential problems with various conditions. For instance, in [5], we studied the existence results to the $(k, n-k)$ conjugate fractional differential inclusion type without continuity and compactness conditions which is not studied before. The guideline in this work is the monotonicity of multi-valued operators. In [6], the main results are devoted to three sides of generalization: the first is of antiperiodic, periodic, and almost periodic solutions. The second is the generalization of infinite countable system of fractional differential equation into inclusion type. The last one is the generalization of fractional differential operators with the Mittag-Leffler kernel. The authors in [7] were coming with a specific general formula of fractional differential equations and inclusions and they called this formula by equi-inclusion problem. Furthermore, it should be noted that Alzabut et al. [8] worked great to investigate the novel solvability techniques on the generalized φ-Caputo fractional inclusion boundary problem. It is also interesting to draw the attention to the work presented by Etemad et al. [9] on the fractional Caputo-Hadamard inclusion problem with sum boundary value conditions by using approximate endpoint property. Continuously, we consider the following Hybrid fractional differential inclusion associated with multi-valued maps Z and E.

$$ {}^{CF}D^{\alpha}\Big[\eta(t) - Z\Big(t,\eta(t),\psi\Big({}^{CF}D^{\rho}\eta(t)\Big)\Big)\Big] \subseteq E\Big(t,\eta(t),\psi\Big({}^{CF}D^{\rho}\eta(t)\Big)\Big) \tag{1} $$

where $\rho \in [0,\alpha]$, ${}^{CF}D^{\alpha}$ denotes the Caputo-Fabrizio time-fractional derivative, ψ is a given integrable bounded real valued function. That is: For all $R > 0$ there exists a nondecreasing function $\hat{\psi} \in L^1((0,\varepsilon),\mathbb{R})$ such that

$$ (\mathcal{K}^*):\ |\psi(r)| \le \hat{\psi}(\|r\|),\ \forall\, r \in \mathbb{R},\ \|r\| \le R. $$

Recently, Kamenskii et al. [10] studied one kind of the fractional inclusions satisfying the nonlocal boundary condition $x(0) \in \Delta(x)$ provided that Δ is a given multi-valued map. The idea here is to improve the boundary condition $x(0) \in \Delta(x)$ into the nonlocal integral boundary condition of the form $\eta(0) \in \int_0^{\sigma} \Delta(s,\eta(s))ds$, where Δ is a given multi-valued map. It means that the unknown function is probably starting from varied places.

The diversity of boundary conditions provides many ways to obtain different results. This fact draws the attention to investigate some results the hybrid fractional differential inclusion (1) under the nonlocal integral condition

$$ \eta(0) \in \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^{\sigma} \Theta(s,\eta(s))ds, \quad \alpha \in [0,1], \ \ t \in [0,\varepsilon], \tag{2} $$

where $M(\alpha)$ denotes the normalization constant depending on α using to define the corresponding integral, $0 \le \sigma < \varepsilon$ and $\Theta(t,\eta(t))$ is a multi-valued map.

Admittedly, the integral boundary conditions have various applications in chemical engineering, thermoplasticity, underground water flow, and population dynamics. Mias [11] studied the finite difference parabolic equation methods which used nonlocal boundary conditions (specially integral conditions) to model radiowave propagation over electrically large domains. These methods require the computation of time consuming spatial convolution integrals. Erofeenko [12] justified some results on boundary value problems with integral boundary conditions for the modeling of magnetic fields in cylindrical film shells. Arara and Benchohra [13] have focused on the behavior of fuzzy solution for boundary value problem with integral boundary conditions. In fact, boundary value problems involving integral boundary conditions have received considerable attention in recent years.

As far as we know, fractional calculus and its theorems and applications have strong impacts, getting many different results in a lot of modeling, and they are able to make changes to them. A huge number of contributors have been paying attention to some applications for the fractional differentiations and integrations. For example, Salem et al. studied Langevin equations in different situations according to the diversity of conditions (see [14–18]). In the inclusion field, there are some great literatures like the ones in [5–9,19].

For a long time, Krasonleskii Hybrid fixed point theorem and its extensions and generalizations play extremely roles to provide the sufficient conditions for the solvability to the mixed types of nonlinear differential and integral equations and inclusions (see [20–23] and the references therein).

It should be noted that Caputo-Fabrizio time-fractional derivative is mainly used to study the new fractional modeling with its amazing property (nonsingular kernels). It attracted the interest of the scientific society and numerous articles to skip analyzes of fractional derivatives with weakly singular memories of Riemann-Liouville and Caputo type [24]. In recent years, phenomena modelings have been dissipated by CF-derivative into the new ones that began after the seminal work of Caputo and Fabrizio [25]. On the other side, it is worth noting that the condensing and contraction conditions basically have a relationship with the measure of noncompactness sets. In our knowledge, non-compactness gives the concept of the lack of compactness for the given sets or multi-valued operators.

Here, we present some solvability results for one kind of hybrid inclusions subject to two multi-operators. These results are in case of satisfying nonlocal integral condition involving multi-valued map. The solution of the problem (1) and (2) is assumed in the Banach space $L^2[a,b]$ while Caputo-Fabrizio time-fractional derivative and integral are well-defined for the functions $\eta(t) \in H^1(a,b)$ (Hilbert space).

The reminder is organized to start first with a basic hypothesis presented in the next section. Then, the existence results are observed in the third section into two cases. Compactness case endowed with the variant of a fixed point theorem of O'Regan under slightly weaker condition is shown first. After that, we focus on noncompactness case via Leray-Schauder nonlinear alternatives type theorem. This theorem is associated with the multi-valued version of Krasnoselskii fixed point theorem and justified with the sum of two multi-valued operators in a Banach space. In section four, we explore some applications of adopted results. Finally, section five is formed as a conclusion of main illustrated facts in the present paper.

2. Basic Hypotheses

Ordinarily in this section, we present some basic acquaintances for the assumed Banach space, basics in fractional calculus, Multi-valued mapping and operators, condensing and contraction conditions, compactness and basics of the measure of noncompactness sets and some lemmas and fixed point theorems.

2.1. Setting of Banach Space

Here, we give some facts and properties of the Banach space that sculpt the area of solutions in case of existence. Let

$$X = L^p[a,b] = \left\{ \eta(t) | \int_a^b |\eta(s)|^p ds < \infty \right\}, \quad 1 \leq p < \infty$$

be a Banach space introduced with the norm

$$\|\eta\|_X = \left(\int_a^b |\eta(s)|^p ds \right)^{\frac{1}{p}}.$$

Then, we can get the following facts.

Definition 1 ([26]). *Given Ω to be an open subset of the real number set $\mathbb{R}$, $1 \leq p < \infty$ and k and ρ are non-negative integers. Then,*

(a) The separable Banach space $W^{k,p}(\Omega)$ is defined as follows

$$W^{k,p}(\Omega) = \{\eta(t) \in L^p(\Omega) | \, D^\rho \eta(t) \in L^p(\Omega), \quad \rho \leq k\}$$

where D^ρ is the ordinary derivative of order ρ.

(b) In particular, the Hilbert Banach space $H^1(\Omega)$ is confirmed by $W^{k,p}(\Omega)$ if we are taking $k = 1, \ p = 2$. Hence,

$$H^1(\Omega) = \left\{ \eta(t) \in L^2(\Omega) | \, \eta'(t) \in L^2(\Omega) \right\}.$$

Theorem 1 ([26]). *Adopt $1 \leq p,q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, the inequalities thereafter hold*

(i) Holder Inequality. If $\eta \in L^p$ and $\eta^* \in L^q$, then $\eta\eta^* \in L^1$ and

$$\|\eta\eta^*\|_{L^1} \leq \|\eta\|_{L^p} \|\eta^*\|_{L^q}.$$

(ii) *Minkowski Inequality. If η, $\eta^* \in L^p$, then $\eta + \eta^* \in L^p$ and*

$$\|\eta + \eta^*\|_{L^p} \leq \|\eta\|_{L^p} + \|\eta^*\|_{L^p}.$$

(iii) *Imbedding Theorem. If Ω has a finite positive measure and $q \leq p$, then $L_P(\Omega) \subseteq L_q(\Omega)$ and*

$$\|\eta\|_{L^q} \leq [\mu(\Omega)]^{\frac{1}{r}} \|\eta\|_{L^p},\ r > 0 \text{ for which } \frac{1}{q} - \frac{1}{p} = \frac{1}{r}.$$

(iv) $\lim_{p\to\infty} \|\eta\|_{L^p} = \|\eta\|_{L^\infty} = \|\eta\|_\infty = \sup_{t\in\Omega} |\eta(t)|$.

2.2. Fractional Calculus

Several scientific fields are actually affected by fractional calculus. Due to that, researchers have used this science as a generalization of ordinary calculus in order to get great results and applications (see for example [24,27] and the references therein). Of the utmost importance, we need to talk about some basic hypothesis from fractional calculus [4,28,29] that support our results.

Definition 2 (Riemann-Liouville Integral). *For the real order $\alpha > 0$, the Riemann-Liouville fractional integral of a piecewise continuous function $h(t) : [0, \infty) \to \mathbb{R}$ is defined by*

$$I^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds.$$

Definition 3 (Caputo Derivative). *The fractional derivative of order α for n-times differentiable map g is defined in Caputo sense by*

$${}^c D^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} g^{(n)}(s) ds, \quad n-1 < \alpha \leq n.$$

Caputo-Fabrizio derivative [30,31] is investigated by interchanging the singular kernel $(t-s)^\alpha$ by nonsingular kernel $\exp\left(-\frac{\alpha(t-s)}{1-\alpha}\right)$ in Caputo derivative. For the order $\alpha \in [0,1]$, the CF-derivative is endowed with the normalization constant map $M(\alpha)$ to get the following formula

Definition 4 (Caputo-Fabrizio Derivative). *CF-derivative for the order $\alpha \in [0,1]$ and $\eta(t) \in H^1(a,b)$ is given by*

$${}^{CF}D^\alpha \eta(t) = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)} \int_a^t e^{-\frac{\alpha(t-s)}{1-\alpha}} \eta'(s) ds.$$

Definition 5 (Caputo-Fabrizio Integral). *For the order $\alpha \in [0,1]$ and $\eta(t) \in H^1(a,b)$, CF-integral is presented by*

$${}^{CF}I^\alpha \eta(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \eta(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_a^t \eta(s) ds.$$

Lemma 1 ([30,31])**.** *Let ${}^{CF}I^\alpha$ and ${}^{CF}D^\alpha$ be CF-integral and CF-derivative of order $\alpha \in [0,1]$. Then, for $\eta(t) \in H^1(a,b)$, the facts come below hold*

1. ${}^{CF}I^\alpha\, {}^{CF}D^\alpha \eta(t) = \eta(t) - \eta(a) = \eta(t) - C$,
2. $\lim_{\alpha\to1} {}^{CF}D^\alpha \eta(t) = \eta'(t)$,
3. $\lim_{\alpha\to0} {}^{CF}D^\alpha \eta(t) = \eta(t) - \eta(a)$.

Lemma 2. *Let $\eta(t) \in H^1(a,b)$, $\nu \in [0,1]$ and $X = L^2[a,b]$ be the Banach space endowed with the norm*

$$\|\eta\|_X = \left(\int_a^b |\eta(s)|^2 ds\right)^{\frac{1}{2}}.$$

Then, we have the inequality

$$\left\| {}^{CF}D^{\nu}\eta \right\|_X \leq \frac{(2-\nu)M(\nu)}{1-\nu}\|\eta\|_X. \tag{3}$$

Proof. For $\eta(t) \in H^1(a,b)$ and $\nu \in [0,1]$, we have

$$\begin{aligned} {}^{CF}D^{\nu}\eta(t) &= \frac{(2-\nu)M(\nu)}{2(1-\nu)}\int_a^t e^{-\frac{\nu(t-s)}{1-\nu}}\eta'(s)ds \leq \frac{(2-\nu)M(\nu)}{2(1-\nu)}\int_a^t \eta'(s)ds \\ &= \frac{(2-\nu)M(\nu)}{2(1-\nu)}(\eta(t)-\eta(a)) \end{aligned}$$

which explains that

$$\left\| {}^{CF}D^{\nu}\eta \right\|_X \leq \frac{(2-\nu)M(\nu)}{2(1-\nu)}(2\|\eta\|_X) = \frac{(2-\nu)M(\nu)}{(1-\nu)}\|\eta\|_X.$$

This means that the proof is completed. □

2.3. Basics in Multi-Valued Maps

Some important precursors to the fractional differential inclusions formed by the multi-valued properties are brought in this subsection. These facts are confirmed in [32,33].

Let $(E, \|.\|)$ and $(H, \|.\|)$ be two Banach spaces. A multi-valued map $A : E \to P_{cl}(E)$ is said to be a convex (closed) if for every $e \in E$, then $A(e)$ is convex (closed). In addition, it is completely continuous if $A(B)$ is relatively compact for every $B \in P_b(E)$.

The map A is said to be upper semi-continuous if for each closed subset $W \subset E; A^{-1}(W)$ is closed subset of E. That means if the set $\{e \in E : A(e) \subseteq O\}$ is open for all open sets $O \subset E$. It is lower semi-continuous if for each open subset $Z \subset E; A^{-1}(Z)$ is an open subset of E. By other ward, A seems to be lower semi-continuous as long as the set $\{e \in E : A(e) \cap O \neq \emptyset\}$ is open for all open sets $O \subset E$.

A map $A : [0, \varepsilon] \to P_{cl}(E)$ is presented to be measurable multi-valued if for every $e \in E$, the function $t \to d(e, A(t)) = \inf\{d(e,a) : a \in A(t)\}$ is $\mathcal{L}-$measurable function.

Adopt A to be completely continuous function with nonempty compact values. Then, it is upper semi-continuous if and only if its graph is closed: The graph is said to be closed if $\nu_n \to \nu_*$ *and* $y_n \to y_*$, then $y_n \in A(\nu_n)$ implies that $y_* \in A(\nu_*)$.

Definition 6. *A multi-valued map $A : [a,b] \times \mathbb{R} \times \mathbb{R} \to P(\mathbb{R})$ is called a Caratheodory if for all $r, r^* \in \mathbb{R}$ and $t \in [a,b]$, we have*

(1) $t \to A(t,r,r^*)$ is measurable,
(2) $(r,r^*) \to A(t,r,r^*)$ is upper semi-continuous.

In addition of the assumptions (1) and (2), the map A is L^1- Caratheodory if for each $k > 0$ there exists $\phi_k \in L^1[a,b]$ satisfying $\sup_{t\geq 0}|\phi_k(t)| < +\infty$ and $\phi_k > 0$ and non-decreasing map Ł for which:

$$\|A(t,r,r^*)\| = \sup\{|a| : a(t) \in A(t,r,r^*)\} \leq \phi_k(t)Ł(\|r\| + \|r^*\|),$$

for all $\|r\|, \|r^*\| \leq k,\ t \in [a,b]$.

2.4. Basics of Contraction and Condensing

Let $\eta(t) \in C[0,\varepsilon] \cap L^2[0,\varepsilon]$ and $B \subset C\big([0,\varepsilon], P_{cp}(\mathbb{R})\big)$. Then, we define from [34,35] the measures of non-compactness sets χ and χ_2 in $C[0,\varepsilon]$ and $L^2[0,\varepsilon]$, respectively, as follows:

$$\chi(B) = \inf\{r > 0 | B \subseteq \cup_{i=1}^{n} B_i,\ \textit{such that}\ \delta(B_i) \leq r\} \tag{4}$$

where $\delta(B_i) = \sup d(x,y),\ x,y \in B_i$ and

$$\chi_2(B) = \left\{ \left(\int_0^{\varepsilon} [\chi(B(s))]^2 ds \right)^{\frac{1}{2}} \right\}, \tag{5}$$

where $B(t) = \{\eta(t) | \eta \in B\}$.

Let $A, B \subset P_{bd,cl}(Q)$ and $a \in A$. Define the metrics

$$\begin{aligned} \mathcal{D}(a,B) &= \inf\{\|a-b\|,\ b \in B\}, \\ \mathcal{O}(A,B) &= \sup \mathcal{D}(a,B), \\ \mathcal{H}(A,B) &= \max(\mathcal{O}(A,B), \mathcal{O}(B,A)). \end{aligned}$$

Then , it is interesting to have the following facts:

Definition 7 (Lipschitz Condition). *Take $(\Sigma, \|.\|)$ as a normed space, and d be the metric map confirmed from the norm. Then, a multi-valued map $\Psi : \Sigma \to P_{cl}(\Sigma)$ is adopted as:*

(1) γ-Lipschitz if there exists $\gamma > 0$ such that:

$$\mathcal{H}(\Psi(z), \Psi(w)) \leq \gamma d(z,w), \forall z, w \in \Sigma$$

(2) a contraction if the first statement is held with $\gamma < 1$.

Lemma 3. *Suppose that $A, B \subset P_{bd}(Q)$ where Q is a real Banach space and β is the measure of noncompactness sets in Q. Then, the upcoming properties are all satisfied:*

1. *$A \subset P_{cp}(Q) \iff \beta(A) = 0$.*
2. *$\beta(A) = \beta(\overline{A}) = \beta(coA)$ where $\overline{A}$ and coA are the closure and convex sets, respectively, of A.*
3. *$\beta(A) \leq \beta(B)$ when $A \subseteq B$.*
4. *$\beta(A+B) \leq \beta(A) + \beta(B)$ where $A + B = \{a + b | a \in A,\ b \in B\}$.*
5. *$\beta(rA) \leq |r|\beta(A)$ for all $r \in \mathbb{R}$.*
6. *The map $\Delta : D(\Delta) \subset Q \to Z \subset Q$ is $\beta-$contraction with constant k if $\beta(\Delta B) \leq k\beta(B)$ and $\beta-$condensing if $k = 1$.*
7. *If the map $\Delta : D(\Delta) \subset Q \to Z$ is Lipschitz contraction with constant k, then $\beta_Z(\Delta B) \leq k\beta(B)$.*
8. *If $W \subset C([0,\varepsilon], Q)$ is bounded, it follows*

$$\beta(W(t)) \leq \beta(W)$$

for all $t \in [0,\varepsilon]$ where $W(t) = \{\eta(t) | \eta \in W\}$. Furthermore, if W is equicontinuous on $[0,\varepsilon]$, then $\beta(W(t))$ is continuous on $[0,\varepsilon]$ and

$$\beta(W) = \sup\{\beta(W(t)), t \in [0,\varepsilon]\}.$$

9. *If $W \subset C([0,\varepsilon], Q)$ is bounded and equicontinuous, it follows*

$$\beta\left(\int_0^t W(s) ds\right) \leq \int_0^t \beta(W(s)) ds.$$

10. *All equicontinuous and contraction maps are condensing maps.*

11. *Let $p \in [1, \infty)$, Σ be a Banach space, $\Omega = (a, b)$ with $a, b \in \mathbb{R}^n (a_i < b_i)$ and $F \subset L^p(\Omega, \Sigma)$. Then, F is relatively compact in $L^p(\Omega, \Sigma)$ if and only if*
    (i) *for every rectangle $C \subset \Omega$ the set $\left\{\int_C f dx : f \in F\right\}$ is relatively compact in Σ*
    (ii) *for $z \in \mathbb{R}^n$ with $0 \leq z_i \leq b_i - a_i,\ i = 1, ..., n$, we have*

$$\sup_{f \in F} \|f(t+z) - f(t)\|_{L^p(\Omega_z, \Sigma)} \to 0, \text{ whenever } z \to 0.$$

2.5. Fixed Point Theorems and Some Basic Lemmas

Define the spaces

$$\begin{aligned}
S_{Z,\eta} &= \left\{z(t) | z(t) \in Z\left(t, \eta(t), \psi\left({}^{CF}D^{\rho}\eta(t)\right)\right) \cap L^1([0, \varepsilon], \mathbb{R})\right\}; \\
S_{E,\eta} &= \left\{e(t) | e(t) \in E\left(t, \eta(t), \psi\left({}^{CF}D^{\rho}\eta(t)\right)\right) \cap L^1([0, \varepsilon], \mathbb{R})\right\}; . \\
S_{\Theta,\eta} &= \left\{\theta(t) | \theta(t) \in \Theta(t, \eta(t)) \cap L^1([0, \varepsilon], \mathbb{R})\right\}
\end{aligned}$$

Then, we have the following Lemmas:

Lemma 4. *Let $z(t) \in \overline{S_{Z,\eta}}$, $e(t) \in \overline{S_{E,\eta}}$ and $\theta(t) \in \overline{S_{\Theta,\eta}}$. Then, $\eta(t) \in H^1(0, \varepsilon)$ satisfies (1)-(2) if the multi-map E is vanishing at $t = 0$ and we have*

$$\begin{aligned}
\eta(t) &= z(t) - z(0) + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} e(t) \\
&\quad + \frac{2\alpha}{(2-\alpha)M(\alpha)}\left[\int_0^t e(s)ds + \int_0^{\sigma} \theta(s)ds\right].
\end{aligned} \tag{6}$$

Proof. Since $\eta(t) \in H^1(0, \varepsilon)$ satisfies (1) and (2), then there exists $z(t) \in \overline{S_{Z,\eta}}$, $e(t) \in \overline{S_{E,\eta}}$ and $\theta(t) \in \overline{S_{\Theta,\eta}}$ such that

$${}^{CF}D^{\alpha}[\eta(t) - z(t)] = e(t), \tag{7}$$

$$\eta(0) = \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^{\sigma} \theta(s)ds, \quad \alpha \in [0, 1], \quad t \in [0, \varepsilon]. \tag{8}$$

Operating the CF-integral of order α to both sides and applying Lemma 1-(I) lead to

$$\eta(t) = [z(t) - z(0)] + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} e(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t e(s)ds + C.$$

Now, by using the integral condition in (5) we find that C is formulated by

$$C = \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^{\sigma} \theta(s)ds.$$

This explains the Equation (6).

Conversely, if $\eta(t) \in H^1(0, \varepsilon)$, then it is easy to see that $\eta(t)$ is satisfying (7) and (8) if the multi-map E is vanishing at $t = 0$ by noting that

$$\begin{aligned}
{}^{CF}D^{\alpha}\left({}^{CF}I^{\alpha}e(t)\right) &= \int_0^t \exp\left(-\alpha\frac{(t-s)}{1-\alpha}\right)e'(s)ds + \frac{\alpha}{1-\alpha}\int_0^t \exp\left(-\alpha\frac{(t-s)}{1-\alpha}\right)e(s)ds \\
&= \int_0^t \left[\exp\left(-\alpha\frac{(t-s)}{1-\alpha}\right)e(s)\right]' ds = \left[\exp\left(-\alpha\frac{(t-s)}{1-\alpha}\right)e(s)\right]_0^t
\end{aligned}$$

which drives to embrace (1) and (2). □

Lemma 5. *Taken Σ to be Banach space, $\Psi : [0,\varepsilon] \times \Sigma \to P_{cp,cv}(\Sigma)$ is a L^1-Caratheodory multi-valued map and $P : L^1([0,\varepsilon],\Sigma) \to C([0,\varepsilon],\Sigma)$ is a continuous and linear map. Then, the operator*

$$P \circ S_\Psi : C([0,\varepsilon],\Sigma) \to P_{cp,cv}(C([0,\varepsilon],\Sigma))$$

such that $(P \circ S_\Psi)(y) = P\big(S_{\Psi,y}\big)$ is an operator with closed graph in $C([0,\varepsilon],\Sigma) \times C([0,\varepsilon],\Sigma)$.

Theorem 2 ([36,37])**.** *Assuming that Σ is a Banach space and Ω and $\overline{\Omega}$ are, respectively, open and closed subsets of Σ with $0 \in \Omega$. If $\Psi : \overline{\Omega} \to P_{cp,cv}(\Sigma)$ is upper semi-continuous multi condensing mapping such that $\Psi\big(\overline{\Omega}\big)$ is bounded, then either*

(i) there exist $\omega \in \partial\Omega$, $\rho \in (0.1)$ such that $\omega \in \rho\Psi(\omega)$, or
(ii) there exists a fixed point $\omega \in \overline{\Omega}$.

Theorem 3 ([36,37])**.** *Assuming that Σ is a Banach space and Ω and $\overline{\Omega}$ are, respectively, open and closed subsets of Σ with $0 \in \Omega$. If $\Psi_1 : \overline{\Omega} \to P_{cl,bd,cv}(\Sigma)$, $\Psi_2 : \overline{\Omega} \to P_{cp,cv}(\Sigma)$ are two multi-valued operators such that $\Psi_1\big(\overline{\Omega}\big) + \Psi_2\big(\overline{\Omega}\big)$ is bounded and:*

(a_1) Ψ_1 is contraction with constant k,
(a_2) Ψ_2 is compact and upper semi-continuous,

then either

(i) there exist $\omega \in \partial\Omega$, $\rho \in (0.1)$ such that $\omega \in \rho(\Psi_1(\omega) + \Psi_2(\omega))$, or
(ii) there exists a fixed point $\omega \in \overline{\Omega}$. Hence, the inclusion

$$\omega \in \rho(\Psi_1(\omega) + \Psi_2(\omega))$$

has a solution with $\rho = 1$.

3. Presented Results

Two cases are particularized in the actual section. One of them is endowed with compact hypothesis and the non-compactness is the way to present the other case.

3.1. Compactness Case

The upshot proved here follows the techniques of Theorem 2. For the sake of that, we first need to define some needed concepts.

Let $\mathcal{M} : L^2([0,\varepsilon],\mathbb{R}) \to P(\mathbb{R})$ be the multi-operator defined for every $\eta(t) \in L^2([0,\varepsilon],\mathbb{R})$ by the relation:

$$(\mathcal{M}\eta)(t) = \{\mu(t)|\mu(t) = \Delta(z,e,\theta)(t),\ z(t) \in \overline{S_{Z,\eta}},\ e(t) \in \overline{S_{E,\eta}},\ \theta(t) \in \overline{S_{\Theta,\eta}}\} \tag{9}$$

where

$$\begin{aligned}\Delta(z,e,\theta)(t) =& [z(t) - z(0)] + \left[\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\right]e(t) \\ &+ \frac{2\alpha}{(2-\alpha)M(\alpha)}\left[\int_0^t e(s)ds + \int_0^\sigma \theta(s)ds\right].\end{aligned} \tag{10}$$

And define an open subset containing zero by:

$$\mathcal{O}_\varrho = \Big\{\eta(t)|\ \|\eta\|_{L^2(\Omega)} < \varrho,\ \Omega = [0,\varepsilon]\Big\}. \tag{11}$$

Then, the next theorem is embraced.

Theorem 4. *In the vision of assuming that Z, E and Θ are all L^1-Caratheodory multi- maps subject to some conditions as come in after:*

($\mathcal{K}_1$) $Z : [0,\varepsilon] \times \mathbb{R}^2 \to P_{cp,cv}(\mathbb{R})$ *satisfying there exist* $\Lambda_1 \in L^\infty([0,\varepsilon],\mathbb{R}_+)$ *and non-decreasing function* $\lambda_1(t) \in C([0,\varepsilon],\mathbb{R}_+)$ *with*

$$\|Z(t,r,r^*)\| \leq \lambda_1(t)\Lambda_1(\|r\| + \|r^*\|), \ \ \forall \|r\|, \|r^*\| \leq \varrho.$$

($\mathcal{K}_2$) $E : [0,\varepsilon] \times \mathbb{R}^2 \to P_{cp,cv}(\mathbb{R})$ *satisfying there exist* $\Lambda_2 \in L^\infty([0,\varepsilon],\mathbb{R}_+)$ *and non-decreasing function* $\lambda_2(t) \in C([0,\varepsilon],\mathbb{R}_+)$ *with*

$$\|E(t,r,r^*)\| \leq \lambda_2(t)\Lambda_2(\|r\| + \|r^*\|), \ \ \forall \|r\|, \|r^*\| \leq \varrho.$$

($\mathcal{K}_3$) $\Theta : [0,\varepsilon] \times \mathbb{R} \to P_{cl,bd,cv}(\mathbb{R})$ *satisfying there exist* $\Lambda_3 \in L^\infty([0,\varepsilon],\mathbb{R}_+)$ *and non-decreasing function* $\lambda_3(t) \in C([0,\varepsilon],\mathbb{R}_+)$ *with*

$$\|\Theta(t,r)\| \leq \lambda_3(t)\Lambda_3(\|r\|), \ \ \forall \|r\| \leq \varrho.$$

($\mathcal{K}_4$) *There exist positive constants* λ^* *and* K_ε *and map* $\Lambda \in L^\infty(\Omega)$ *such that*

$$\frac{\varrho}{\varepsilon^{\frac{1}{2}}\lambda^*\Lambda(\varrho)K_\varepsilon} > 1$$

where $\lambda^*\Lambda = \max\{\lambda_1^*\Lambda_1, \lambda_2^*\Lambda_2, \lambda_3^*\Lambda_3\}$, $\lambda_i^* = \sup|\lambda_i(t)|$ *and*

$$K_\varepsilon = 2 + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2\alpha(\varepsilon+\sigma)}{(2-\alpha)M(\alpha)}.$$

Then, the problem (1) and (2) is attainable for solving if E *is vanishing at* $t = 0$.

Lemma 6. *The set* $\int_0^\sigma S_{\Theta,\overline{\mathcal{O}_\varrho}}ds$ *is relatively compact in* $L^2((0,\varepsilon),\mathbb{R})$.

Proof. Using Lemma 3-(11), then we take

$$F = \int_0^\sigma S_{\Theta,\overline{\mathcal{O}_\varrho}}ds.$$

First, we prove that $\left\{\int_0^t f dt : f \in F\right\}$ is relatively compact in $\mathbb{R}$. For all $f \in F$ there exists $\theta \in S_{\Theta,\overline{\mathcal{O}_\varrho}}$ such that $f = \int_0^\sigma \theta(s,\eta(s))ds$, $\eta \in \overline{\mathcal{O}_\varrho}$. Thus,

$$\left|\int_0^t f dt\right| = \left|\int_0^t \int_0^\sigma \theta(s,\eta(s))dsdt\right| \leq t\sigma\lambda_3(t)\Lambda_3(\|\eta\|)$$

which tends to $\sup\left|\int_0^t f dt\right| \leq \varepsilon\sigma\lambda_3^*\Lambda_3(\varrho)$ that proves the boundedness.

To prove that $\left\{\int_0^t f dt : f \in F\right\}$ is equicontinuous, we take $t_1, t_2 \in (0,\varepsilon)$ such that $t_1 \to t_2 (t_2 > t_1)$. Whence

$$\begin{aligned}
\left|\int_0^{t_2} f dt - \int_0^{t_1} f dt\right| &= \left|\int_0^{t_2}\int_0^\sigma \theta(s,\eta(s))dsdt - \int_0^{t_1}\int_0^\sigma \theta(s,\eta(s))dsdt\right| \\
&= \left|\int_{t_1}^{t_2}\int_0^\sigma \theta(s,\eta(s))dsdt\right| \\
&\leq \int_{t_1}^{t_2}\left|\int_0^\sigma \theta(s,\eta(s))ds\right|dt \\
&\leq \sup\left|\int_0^\sigma \theta(s,\eta(s))ds\right|\int_{t_1}^{t_2} dt \\
&\leq (t_2 - t_1)\int_0^\sigma \|\theta(s,\eta(s))\|ds \to 0
\end{aligned}$$

as $t_1 \to t_2$, which shows the equicontinuity. Now, for $0 \leq z < \varepsilon$, we have

$$\sup_{f\in F}\|f(t+z)-f(t)\|_{L^2((0,\varepsilon),\mathbb{R})} = \sup_{f\in F}\left\|\int_0^\sigma \theta(s,\eta(s))ds - \int_0^\sigma \theta(s,\eta(s))ds\right\|_{L^2((0,\varepsilon),\mathbb{R})} = 0.$$

This inequality, boundedness and equicontinuity hypotheses altogether prove the relatively compact in $L^2((0,\varepsilon),\mathbb{R})$. □

Now, we ready to prove Theorem 4.

Proof of Theorem 4. Due to Theorem 2, we need to prove that the multi-operator $\mathcal{M}$ is required as convex, bounded, equicontinuous, upper semi-continuous and condensing in closed, bounded and convex subsets of $L^2[0,\varepsilon]$.

Consider the operator $\mathcal{M}$ as in (9) and (10) and $\mathcal{O}_\varrho$ formed by (11). Then,

C1: $\mathcal{M}$ should be convex. Let $\gamma \in (0,1)$ and $\mu_1,\mu_2 \in \mathcal{M}(\eta)$ which means that there exist"43" $z_1(t),z_2(t) \in \overline{S_{Z,\eta}}$, $e_1(t),e_2(t) \in \overline{S_{E,\eta}}$ and $\theta_1(t),\theta_2(t) \in \overline{S_{\Theta,\eta}}$ in which

$$\begin{aligned}\mu_i(t) &= [z_i(t)-z_i(0)] + \left[\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\right]e_i(t)\\ &+ \frac{2\alpha}{(2-\alpha)M(\alpha)}\left[\int_0^t e_i(s)ds + \int_0^\sigma \theta_i(s)ds\right], \qquad i=1,2.\end{aligned}$$

These imply

$$\begin{aligned}&\gamma\mu_1(t)+(1-\gamma)\mu_2(t) = [(\gamma z_1(t)+(1-\gamma)z_2(t)) - (\gamma z_1(0)+(1-\gamma)z_2(0))]\\ &+\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}[\gamma e_1(t)+(1-\gamma)e_2(t)]\\ &+\frac{2\alpha}{(2-\alpha)M(\alpha)}\left[\int_0^t(\gamma e_1(s)+(1-\gamma)e_2(s))ds + \int_0^\sigma(\gamma\theta_1(s)+(1-\gamma)\theta_2(s))ds\right]\end{aligned}.$$

Take

$$\begin{aligned}z &= \gamma z_1(t)+(1-\gamma)z_2(t) \in \overline{S_{Z,\eta}};\\ e &= \gamma e_1(t)+(1-\gamma)e_2(t) \in \overline{S_{E,\eta}}\\ \theta &= \gamma\theta_1(t)+(1-\gamma)\theta_2(t) \in \overline{S_{\Theta,\eta}}.\end{aligned}$$

implies

$$\gamma\mu_1(t)+(1-\gamma)\mu_2(t) = \Delta(z,e,\theta)(t) = \mu(t) \in \mathcal{M}(\eta)(t),$$

which the convexity of $\mathcal{M}$ since $\overline{S_{Z,\eta}}$, $\overline{S_{E,\eta}}$ and $\overline{S_{\Theta,\eta}}$ are all convex.

C2: $\mathcal{M}$ must be bounded. Let $\mu(t) \in \mathcal{M}(\overline{\mathcal{O}_\varrho})$. Then, there exist $z(t) \in \overline{S_{Z,\eta}}$, $e(t) \in \overline{S_{E,\eta}}$ and $\theta(t) \in \overline{S_{\Theta,\eta}}$ where $\mu(t) = \Delta(z,e,\theta)(t)$ defined in (10). By using Theorem 1, we get

$$|\mu(t)| \leq |z(t)-z(0)| + \frac{2}{(2-\alpha)M(\alpha)}\left((1-\alpha)|e(t)| + \alpha\int_0^t|e(s)|ds + \alpha\int_0^\sigma|\theta(s)|ds\right).$$

Using the statement $(\mathcal{K}^*)$, we get

$$\begin{aligned}|\mu(t)| &\leq 2|z(t)| + \frac{2}{(2-\alpha)M(\alpha)}\left((1-\alpha)|e(t)| + \alpha\int_0^t|e(s)|ds + \alpha\int_0^\sigma|\theta(s)|ds\right)\\ &\leq 2\lambda_1(t)\Lambda_1\left(\|\eta\|_{L^2(\Omega)} + \hat{\psi}\left(\frac{(2-\rho)M(\rho)}{1-\rho}\|\eta\|_{L^2(\Omega)}\right)\right)\\ &+\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\lambda_2(t)\Lambda_2\left(\|\eta\|_{L^2(\Omega)} + \hat{\psi}\left(\frac{(2-\rho)M(\rho)}{1-\rho}\|\eta\|_{L^2(\Omega)}\right)\right)\\ &+\frac{2\alpha}{(2-\alpha)M(\alpha)}\Lambda_2\left(\|\eta\|_{L^2(\Omega)} + \hat{\psi}\left(\frac{(2-\rho)M(\rho)}{1-\rho}\|\eta\|_{L^2(\Omega)}\right)\right)\int_0^t\lambda_2(s)ds\\ &+\frac{2\alpha}{(2-\alpha)M(\alpha)}\Lambda_3\left(\|\eta\|_{L^2(\Omega)}\right)\int_0^\sigma\lambda_3(s)ds.\end{aligned}$$

Hence,

$$
\begin{aligned}
|\mu(t)| \leq\ & 2\lambda_1^*\Lambda_1\left(\|\eta\|_{L^2(\Omega)} + \hat{\psi}\left(\frac{(2-\rho)M(\rho)}{1-\rho}\|\eta\|_{L^2(\Omega)}\right)\right) \\
& + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\lambda_2^*\Lambda_2\left(\|\eta\|_{L^2(\Omega)} + \hat{\psi}\left(\frac{(2-\rho)M(\rho)}{1-\rho}\|\eta\|_{L^2(\Omega)}\right)\right) \\
& + \frac{2\alpha\varepsilon\lambda_2^*}{(2-\alpha)M(\alpha)}\Lambda_2\left(\|\eta\|_{L^2(\Omega)} + \hat{\psi}\left(\frac{(2-\rho)M(\rho)}{1-\rho}\|\eta\|_{L^2(\Omega)}\right)\right) \\
& + \frac{2\alpha\sigma\lambda_3^*}{(2-\alpha)M(\alpha)}\Lambda_3\left(\|\eta\|_{L^2(\Omega)}\right),
\end{aligned}
$$

which drives by $(\mathcal{K}_4)$ to

$$\|\mu\|_{L^2(\Omega)} \leq \varepsilon^{\frac{1}{2}}\lambda^*\Lambda\left(\|\eta\|_{L^2(\Omega)}\right)K_\varepsilon \leq \varepsilon^{\frac{1}{2}}\lambda^*\Lambda(\varrho)K_\varepsilon \leq \varrho.$$

C3: $\mathcal{M}(\overline{\mathcal{O}_\varrho})$ should be equicontinuous. For $0 < t_1 < t_2 < \varepsilon$ and $\mu \in \mathcal{M}(\overline{\mathcal{O}_\varrho})$, we see that

$$
\begin{aligned}
|\mu(t_2)-\mu(t_1)| &\leq |z(t_2)-z(t_1)| + \frac{2}{(2-\alpha)M(\alpha)}\left[(1-\alpha)|e(t_2)-e(t_1)| + \alpha\int_{t_1}^{t_2}|e(s)|ds\right] \\
&\leq |z(t_2)-z(t_1)| + \frac{2}{(2-\alpha)M(\alpha)}[(1-\alpha)|e(t_2)-e(t_1)| + \alpha\lambda_2^*\Lambda_2(\varrho)(t_2-t_1)]
\end{aligned}
$$

tends to zero as $t_1 \to t_2$.

C4: $\mathcal{M}$ has an upper semi-continuous graph. Here we are going through the algorithm of Lemma 5. So, take the linear operator Δ given by (10). Suppose that $\mu_n(t) \in \mathcal{M}(\eta_n)$, $\mu_n(t) \to \underline{\mu_*(t)}$ and $\eta_n \to \eta_*$. Claim that $\mu_*(t) \in \mathcal{M}(\eta_*)$. In case of $\mu_n(t) \in \mathcal{M}(\eta_n)$, there exist $z_n(t) \in \overline{S_{Z,\eta_n}}$, $e_n(t) \in \overline{S_{E,\eta_n}}$ and $\theta_n(t) \in \overline{S_{\Theta,\eta_n}}$ such that $\mu_n(t) = \Delta(z_n, e_n, \theta_n)(t)$. Since μ_n is convergent and Δ has a closed graph, then there exist $z_*(t) \in \overline{S_{Z,\eta_*}}$, $e_*(t) \in \overline{S_{E,\eta_*}}$ and $\theta_*(t) \in \overline{S_{\Theta,\eta_*}}$ such that

$$\mu_n(t) = \Delta(z_n, e_n, \theta_n)(t) \to \Delta(z_*, e_*, \theta_*)(t).$$

Taking $\mu_*(t) = \Delta(z_*, e_*, \theta_*)(t)$ makes that $\mu_*(t) \in \mathcal{M}(\eta_*)$. Since $\mathcal{M}$ is equicontinuous and has a closed graph, then it is an upper semi-continuous operator.

C5: $\mathcal{M}$ satisfies the condensing condition. In view of Lemmas 3 and 6 with using the measure χ_2 defined by (4) and (5), we get

$$
\begin{aligned}
\chi_2(\mathcal{M}(\overline{\mathcal{O}_\varrho})) \leq\ & 2\chi_2\left(S_{Z,\overline{\mathcal{O}_\varrho}}\right) + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\chi_2\left(S_{E,\overline{\mathcal{O}_\varrho}}\right) \\
& + \frac{2\alpha}{(2-\alpha)M(\alpha)}\left[\int_0^t \chi_2\left(S_{E,\overline{\mathcal{O}_\varrho}}\right)ds + \chi_2\left(\int_0^\sigma S_{\Theta,\overline{\mathcal{O}_\varrho}}ds\right)\right] \leq 0.
\end{aligned}
$$

Now, from (C1)–(C5) and Theorem 2, the problem (1) and (2) is able to solve if there is no element $\eta \in \partial\mathcal{O}_\varrho$ such that $\gamma\eta \in \mathcal{M}(\eta)$ for all $\gamma \in (0,1)$. That absolutely holds if we take $\varrho = \varepsilon^{\frac{1}{2}}\lambda^*\Lambda(\varrho)K_\varepsilon + 1$. □

3.2. Noncompactness Case

Under this case, the result is surveyed by assuming that the map

$$Z : [0,\varepsilon] \times \mathbb{R}^2 \to P_{cl,bd,cv}(\mathbb{R})$$

is a contraction in measure. To apply Theorem 3, spilt the multi-operator $\mathcal{M}$ in (9) into two operators $\mathcal{M}_1,\ \mathcal{M}_2 : L^2([0,\varepsilon],\ \mathbb{R}) \to P(\mathbb{R})$ defined as

$$(\mathcal{M}_1\eta)(t) = \left\{\mu(t) = \Delta_1(z,e,\theta)(t) \,|\, z \in \overline{S_{Z,\eta}},\ e \in \overline{S_{E,\eta}},\ \theta \in \overline{S_{\Theta,\eta}}\right\}, \tag{12}$$

$$(\mathcal{M}_2\eta)(t) = \left\{\mu(t) \,|\, \mu(t) = \Delta_2(e)(t),\ e(t) \in \overline{S_{E,\eta}}\right\} \tag{13}$$

where

$$\Delta_1(z,e,\theta)(t) = [z(t) - z(0)] + \frac{2}{(2-\alpha)M(\alpha)}\left[(1-\alpha)e(t) + \alpha\int_0^{\sigma}\theta(s)ds\right]$$
$$\Delta_2(e)(t) = \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t e(s)ds.$$

Theorem 5. *Let Z, E and Θ be L^1-Caratheodory maps satisfying $(\mathcal{K}_1)$, $(\mathcal{K}_2)$ and $(\mathcal{K}_3)$, respectively. Moreover, consider that the following assumptions hold*

$(\mathcal{K}_5)$ *The map Z is contraction in measure with constant Ł (for B is bounded set), we have*

$$\chi(S_{Z,B}) \le Ł\chi_2(B), \;\; Ł,\; 2Ł\varepsilon^{\frac{1}{2}} < 1$$

$(\mathcal{K}_6)$ *For λ^* and Λ defined as in $(\mathcal{K}_4)$. There is a positive constant K such that*

$$\frac{\varrho}{\varepsilon^{\frac{1}{2}}\lambda^*\Lambda(\varrho)K} > 1,$$

where $K = K_0 + \frac{2\alpha\varepsilon}{(2-\alpha)M(\alpha)}$ and K_ε defined in $(\mathcal{K}_4)$.

Then, the given problem in (1)-(2) has at least one solution if E is vanishing at $t = 0$.

Proof. Let $\mathcal{O}_\varrho$ be an open bounded subset defined by (11) and $\mathcal{M}_1$ and $\mathcal{M}_2$ be defined by (12) and (13), respectively. Then, according to Theorem 3, we have

A1: $\mathcal{M}_1$ is convex. To explain that, let $\gamma \in (0,1)$ *and* $\mu_1,\ \mu_2 \in \mathcal{M}_1(\eta)$. It means that there exist $z_1(t),\ z_2(t) \in \overline{S_{Z,\eta}}$, $e_1(t),\ e_2(t) \in \overline{S_{E,\eta}}$ *and* $\theta_1(t),\ \theta_2(t) \in \overline{S_{\Theta,\eta}}$ for $\mu_1,\ \mu_2$, respectively, in which

$$\mu_i(t) = [z_i(t) - z_i(0)] + \left[\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\right]e_i(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^{\sigma}\theta_i(s)ds$$

for $i = 1,2$. These follow

$$\begin{aligned}\gamma\mu_1(t) + (1-\gamma)\mu_2(t) &= [(\gamma z_1(t) + (1-\gamma)z_2(t)) - (\gamma z_1(0) + (1-\gamma)z_2(0))] \\ &+ \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}[\gamma e_1(t) + (1-\gamma)e_2(t)] + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^{\sigma}(\gamma\theta_1(s) + (1-\gamma)\theta_2(s))ds,\end{aligned}$$

which implies the convexity of $\mathcal{M}_1$ since $\overline{S_{Z,\eta}}$, $\overline{S_{E,\eta}}$ *and* $\overline{S_{\Theta,\eta}}$ are all convex.

A2: $\mathcal{M}_1$ is bounded. Let $\mu \in \mathcal{M}_1(\overline{\mathcal{O}_\varrho})$. Then, by using the assumptions $(\mathcal{K}_1)$, $(\mathcal{K}_2)$ *and* $(\mathcal{K}_3)$, we have

$$\|\mu\| = \|\Delta_1(z,e,\theta)\| \le \varepsilon^{\frac{1}{2}}\lambda^*\Lambda(\varrho)K_0 \le \varrho.$$

A3: $\mathcal{M}_1$ is closed. Backing to Lemma 5. Suppose that $\mu_n(t) \in \mathcal{M}_1(\eta_n)$, $\mu_n(t) \to \mu_*(t)$ and $\eta_n \to \eta_*$. Our aim is to prove that $\mu_*(t) \in \mathcal{M}_1(\eta_*)$. In case that $\mu_n(t) \in \mathcal{M}_1(\eta_n)$, then there exist $z_n(t) \in \overline{S_{Z,\eta_n}}$, $e_n(t) \in \overline{S_{E,\eta_n}}$ and $\theta_n(t) \in \overline{S_{\Theta,\eta_n}}$ in which $\mu_n(t) = \Delta_1(z_n, e_n, \theta_n)(t)$. Since μ_n is convergent and Δ_1 has a closed graph. Then, there exist $z_*(t) \in \overline{S_{Z,\eta_*}}$, $e_*(t) \in \overline{S_{E,\eta_*}}$ and $\theta_*(t) \in \overline{S_{\Theta,\eta_*}}$ such that

$$\mu_n(t) = \Delta_1(z_n, e_n, \theta_n)(t) \to \Delta_1(z_*, e_*, \theta_*)(t).$$

Taking $\mu_*(t) = \Delta_1(z_*, e_*, \theta_*)(t)$, leads to $\mu_*(t) \in \mathcal{M}_1(\eta_*)$.

A4: $\mathcal{M}_1$ is a contraction in measure. For the sake of proving, we use the properties given in Lemmas 3 and 6. Thus,

$$
\begin{aligned}
\chi_2(\mathcal{M}_1(\overline{\mathcal{O}_\varrho})) &\le 2\chi_2\left(S_{Z,\overline{\mathcal{O}_\varrho}}\right) + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\chi_2\left(S_{E,\overline{\mathcal{O}_\varrho}}\right) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\chi_2\left(\int_0^\sigma S_{\Theta,\overline{\mathcal{O}_\varrho}}ds\right) \\
&\le 2\left(\int_0^\varepsilon\left[\chi\left(S_{Z,\overline{\mathcal{O}_\varrho}}\right)\right]^2\right)^{\frac{1}{2}} \le 2Ł\chi_2(\overline{\mathcal{O}_\varrho})\left(\int_0^\varepsilon ds\right)^{\frac{1}{2}} \\
&= 2Ł\varepsilon^{\frac{1}{2}}\chi_2(\overline{\mathcal{O}_\varrho}).
\end{aligned}
$$

By $(\mathcal{K}_6)$, we get $\mathcal{M}_1$ is a contraction in measure.

B1: $\mathcal{M}_2$ is convex. To explain that, let $\gamma \in (0,1)$, $\mu_1, \mu_2 \in \mathcal{M}_2(\eta)$, then there exist $e_1(t), e_2(t) \in \overline{S_{E,\eta}}$ for μ_1, μ_2 respectively in which

$$\mu_i(t) = \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t e_i(s)ds, \qquad i = 1,2.$$

These follow

$$\gamma\mu_1(t) + (1-\gamma)\mu_2(t) = \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t (\gamma e_1(s) + (1-\gamma)e_2(s))ds,$$

which leads to the convexity of $\mathcal{M}_2$ since $\overline{S_{E,\eta}}$ is convex.

B2: $\mathcal{M}_2$ is bounded. Let $\mu \in \mathcal{M}_2(\overline{\mathcal{O}_\varrho})$, then by using $(\mathcal{K}_2)$ we have

$$\|\mu\| = \|\Delta_2(z,e,\theta)\| \le \varepsilon^{\frac{1}{2}}\lambda_2^*\Lambda_2(\varrho)\frac{2\alpha\varepsilon}{(2-\alpha)M(\alpha)} \le \varrho.$$

B3: $\mathcal{M}_2$ is equicontinuous. So, for $o < t_1 < t_2 < \varepsilon \text{ such as } t_1 \to t_2$ and $\mu \in \mathcal{M}_2(\overline{\mathcal{O}_\varrho})$, we see that:

$$
\begin{aligned}
|\mu(t_2) - \mu(t_1)| &\le \frac{2\alpha}{(2-\alpha)M(\alpha)}\left|\int_0^{t_2} e(s)ds - \int_0^{t_1} e(s)ds\right| \\
&\le \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_{t_1}^{t_2}|e(s)|ds \\
&\le \frac{2\alpha}{(2-\alpha)M(\alpha)}\lambda_2^*\Lambda_2(\varrho)(t_2 - t_1)
\end{aligned}
$$

which tends to zero uniformly as $t_1 \to t_2$

B4: $\mathcal{M}_2$ is upper semi continuous. Here, we follow the algorithm of Lemma 5. Suppose that $\mu_n(t) \in \mathcal{M}_2(\eta_n)$, $\mu_n(t) \to \mu_*(t)$ and $\eta_n \to \eta_*$. Claim that $\mu_*(t) \in \mathcal{M}_2(\eta_*)$. In case that $\mu_n(t) \in \mathcal{M}_2(\eta_n)$, then there exists $e_n(t) \in \overline{S_{E,\eta_n}}$ where $\mu_n(t) = \Delta_2(e_n)(t)$. Since μ_n is convergent and Δ_2 has a closed graph, then there exists

$$e_*(t) \in \overline{S_{E,\eta_*}} \text{ with } \mu_n(t) = \Delta_2(e_n)(t) \to \Delta_2(e_*)(t).$$

Taking $\mu_*(t) = \Delta_2(e_*)(t)$ makes that $\mu_*(t) \in \mathcal{M}_2(\eta_*)$. Since $\mathcal{M}_2$ is equicontinuous and has a closed graph then, it is formed as upper semi-continuous operator.

It remains to show that the inclusion $\eta \in \gamma[\mathcal{M}_1\eta + \mathcal{M}_2\eta]$ has a solution with $\gamma = 1$. For that, conform the set $\mathcal{O}_\varrho$ as in (11) with:

$$\varrho = \varepsilon^{\frac{1}{2}}\lambda^*\Lambda(\varrho)K + 1$$

By $(\mathcal{K}_7)$, we have no element $\eta \in \partial\mathcal{O}_\varrho$ satisfying the inclusion $\eta \in \gamma[\mathcal{M}_1\eta + \mathcal{M}_2\eta]$ with $\gamma \in (0,1)$. Applying Theorem 3 together with the results in A and B on the set $\mathcal{O}_\varrho$, implies the existence of one or more solutions for the problems (1) and (2). □

3.3. Particular Case: Implicit Case

Corollary 1 (Compactness case)**.** *In case that $\rho = \alpha$ and $\psi = I$ (identity map), the problem (1) becomes of implicit type. Hence, we have*

$$^{CF}D^{\alpha}\Big[\eta(t) - Z\Big(t, \eta(t), \, ^{CF}D^{\alpha}\eta(t)\Big)\Big] \subseteq E\Big(t, \eta(t), \, ^{CF}D^{\alpha}\eta(t)\Big). \tag{14}$$

The problem (14) with (2) has at least one solution if Z, E and Θ are satisfying all hypothesis in Theorem 4.

Corollary 2 (Noncompactness case)**.** *The problem (14) with (2) has at least one solution if Z, E and Θ are satisfying all hypothesis in Theorem 5.*

With nonlocal integral condition, there is a huge number of contributions on implicit fractional differential boundary value problem. For such examples, Borisut et al. [38] explored the solvability of the implicit fractional problem:

$$\begin{aligned} &^{C}D^{q}_{0^+}u(t) = f(t, u(t), \, ^{C}D^{q}_{0^+}u(t)); \qquad t \in [0, T] \\ &u(0) = \eta; \; u(T) = \, _{RL}I^{p}_{0^+}u(\kappa), \; \kappa \in (0, T) \end{aligned}$$

where $1 < q \leq 2; 0 < p \leq 1; \eta \in \mathbb{R}$; $^{C}D^{q}_{0^+}u(t)$ is the Caputo fractional derivative of order q, $_{RL}I^{p}_{0^+}$ is the Riemann-Liouville fractional integral of order p and f is continuous function. Vivek et al. [39] studied the following implicit problem

$$\begin{aligned} &^{C}D^{\alpha,\beta}_{0^+}x(t) = f(t, x(t), \, ^{C}D^{\alpha,\beta}_{0^+}x(t)); \qquad t \in [0, T] \\ &I^{1-\gamma}_{0^+}x(0) = \sum_{i=0}^{m} c_i x(t_i), \; t_i \in [0, T] \end{aligned}$$

where $D^{\alpha,\beta}_{0^+}$ is the Hilfer fractional derivative, $0 < \alpha < 1$, $0 \leq \beta \leq 1$ and $\gamma = \alpha + \beta(1 - \alpha)$. Also, f is a given continuous function, $I^{1-\gamma}_{0^+}$ is the left-sided Riemann-Liouville fractional integral of order $1 - \gamma$, c_i are real numbers and t_i; $i = 1, 2, ..., m$ are prefixed points satisfying $0 < t_1 \leq t_2 \leq ...... \leq t_m < T$. Sousa et al. [40] considered the implicit problem:

$$\begin{aligned} &^{C}D^{\alpha,\beta;\psi}_{a^+}x(t) = f(t, x(t), \, ^{C}D^{\alpha,\beta;\psi}_{a^+}x(t)); \qquad t \in [a, T] \\ &I^{1-\gamma}_{a^+}x(a) = g_a \end{aligned}$$

where $D^{\alpha,\beta;\psi}_{a^+}$ is the $\psi -$ Hilfer fractional derivative, $0 < \alpha < 1$, $0 \leq \beta \leq 1$ and $\gamma = \alpha + \beta(1 - \alpha)$. Also, f is a given continuous function, $I^{1-\gamma;\psi}_{0^+}$ is the left-sided Riemann-Liouville fractional integral of order $1 - \gamma$.

In fact, our work presents new extents for some kinds of hybrid operators that have solutions starting from one place into the ones having solutions starting from different places. The secret here is using the non-local integral condition endowed with multi-valued map instead of singular-valued map. Note that, the implicit problems given by Corollaries 1 and 2 have solutions with multi-beginnings while all three implicit problems above have solutions with a unique beginning.

4. Applications

The interesting thing in the analogy is giving related examples of required facts.

Example 1. *Consider the problem*

$$\begin{cases} {}^{CF}D^{\frac{1}{2}}\Big[\eta(t) - Z\Big(t,\eta(t),\psi\Big({}^{CF}D^{\frac{1}{4}}\eta(t)\Big)\Big)\Big] \subseteq E\Big(t,\eta(t),\psi\Big({}^{CF}D^{\frac{1}{4}}\eta(t)\Big)\Big), \\ \eta(0) \in \dfrac{1}{\frac{3}{2}M(\frac{1}{2})}\displaystyle\int_0^{\sigma}\Theta(s,\eta(s))ds, \qquad t \in [0,\varepsilon] \end{cases}, \tag{15}$$

in case that

$$Z\Big(t,\eta(t),\psi\Big({}^{CF}D^{\frac{1}{4}}\eta(t)\Big)\Big) = \left[\frac{|\eta|^2 e^{-\left(|\eta|+\left|\psi\left({}^{CF}D^{\frac{1}{4}}\eta\right)\right|\right)}}{e(1+|\eta|^2)}, \frac{\left|\psi\left({}^{CF}D^{\frac{1}{4}}\eta\right)\right|}{e(1+|\eta|)}\right], \tag{16}$$

$$E\Big(t,\eta(t),\psi\Big({}^{CF}D^{\frac{1}{4}}\eta(t)\Big)\Big) = \left[\frac{t}{2e+t}\left(\frac{|\eta| + \left|\psi\left({}^{CF}D^{\frac{1}{4}}\eta\right)\right|}{e(1+|\eta|)}\right), \frac{t}{2e^{2t}}(1+e^{-2t})\right], \tag{17}$$

$$\Theta(t,\eta(t)) = \left[\frac{1}{n^2}\sin(-\eta)\right], \qquad n = 2,3,4 \tag{18}$$

with taking $\varepsilon = \pi$, $\psi(x) = x\exp(-|x|)$. *Then, by using the fact* $\exp(-x) \leq 1$ *for all* $x \in [0,\infty)$ *and Lemma 2, we have*

$$\left\|\psi\left({}^{CF}D^{\frac{1}{4}}\eta\right)\right\| \leq \frac{8}{3}\|\eta\| \quad \textit{and} \quad \|\eta\| + \left\|\psi\left({}^{CF}D^{\frac{1}{4}}\eta\right)\right\| \leq \frac{11}{3}\|\eta\|.$$

Therefore,

$$\|Z\| \leq \frac{1}{e}.\frac{8}{3} \Rightarrow \lambda_1(t) = \frac{1}{e},\ \Lambda_1 = \frac{8}{3},$$
$$\|E\| \leq \frac{1}{e}.\frac{11}{3} \Rightarrow \lambda_2(t) = \frac{1}{e},\ \Lambda_2 = \frac{11}{3},$$

and finally,

$$\|\Theta\| \leq \frac{1}{4}\|-sin(\eta)\| \leq \frac{1}{4} \Rightarrow \lambda_3(t) = \frac{1}{2},\ \Lambda_3 = \frac{1}{2},$$

which means that the assumptions $(\mathcal{K}_1)$, $(\mathcal{K}_2)$ *and* $(\mathcal{K}_3)$ *in Theorem 4 hold. We need to prove also that* $(\mathcal{K}_4)$ *hold. For that, we need to calculate some values*

$$\lambda^* = \max\{\lambda_1^*, \lambda_2^*, \lambda_3^*\} = \frac{1}{2},$$

$$\Lambda = \max\{\Lambda_1, \Lambda_2, \Lambda_3\} = \frac{11}{3},$$

According to C2 of in proof of Theorem 4, it follows that $\varrho = \sqrt{\pi}\left(\frac{11(5+\sigma+\pi)}{12}\right) + 1$. *By the previous results and Theorem 4, the problem (15) with respect to (16)–(18) can be solved.*

Example 2. *Consider the problem (15), take E,* Θ *defined, respectively, by (16) and (17) in Example 1 and*

$$Z = \left[-a_i\left(\frac{\eta}{2m+|\eta|}\right)\right]_{(i=0)}^{m-2}, \qquad m,i \in \mathbb{N} \tag{19}$$

where $2 \leq m = [b]$, $b \in \mathbb{Q}^c, a_i \in [0,b] \cap \mathbb{Q}$ *where* $[a_i] = i$, $i = 0,\ldots,m-2$. *Then, as in Example 1, E and* Θ *are satisfying* $(\mathcal{K}_2)$ *and* $(\mathcal{K}_3)$, *respectively, with*

$$\lambda_2(t) = \frac{1}{e}, \quad \Lambda_2 = \frac{11}{3}, \quad \lambda_3(t) = \frac{1}{2}, \qquad \Lambda_3 = \frac{1}{2}.$$

Now, for Z, we have

$$\|Z\| \leq \|-a_i\| \left\| \left(\frac{\eta}{2m + |\eta|} \right) \right\| \leq m - 1 \Rightarrow \lambda_1 = m - 1, \Lambda_1 = 1.$$

Due to that, $(\mathcal{K}_1)$ *hold with respect to Z, it is easy to see that*

$$\lambda^* = m - 1, \ \Lambda = \frac{11}{3}.$$

In addition and by Lemma 3-(7), since $H_d\left(Z_{\eta_2}, Z_{\eta_1}\right) \leq \frac{m-1}{2m} d(\eta_2, \eta_1)$, *we have*

$$\chi 2\left(S_{z,\overline{\mathcal{O}_\varrho}}\right) \leq \frac{m-1}{2m} \chi 2\left(\overline{\mathcal{O}_\varrho}\right)$$

Thus, $Ł = \frac{m-1}{2m} < 1$ *and* $2Ł\varepsilon^{\frac{1}{2}} = \frac{\varepsilon^{\frac{1}{2}}(m-1)}{m} < 1 \ \forall m \in \mathbb{N}$ *if and only if* $\varepsilon \in \left(0, \left(\frac{m}{m-1}\right)^2\right]$ *which means that* $(\mathcal{K}_5)$ *hold. For* $(\mathcal{K}_6)$, *take*

$$\varrho = \frac{11(m-1)\varepsilon^{\frac{1}{2}}}{6}(5 + \sigma + \varepsilon) + 1.$$

All these facts with Theorem 5 are showing the solvability of the problem 15 with respect to (16), (17) and (19).

5. Conclusions

The non-singularity of CF-fractional derivative's kernel, the basic concepts of Hybrid fractional inclusion and the measure of noncompactness sets have strong effects on the investigated facts. The amazing thing is linking the fractional inclusions to nonlocal-integral condition involving multi-valued map. Applying the relatively compact theorem in $L^p((0,\varepsilon),\mathbb{R})$, $1 \leq p < \infty$ gives direct answers for showing the condensing property of some needed operators. We always believe that every generalizations of theorems will make different and strong applications and extents into new fractional modeling.

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