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On a Nonlocal Problem for Mixed-Type Equation with Partial Riemann-Liouville Fractional Derivative

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Abstract: The present paper presents a study on a problem with a fractional integro-differentiation operator in the boundary condition for an equation with a partial Riemann-Liouville fractional derivative. The unique solvability of the problem is proved. In the hyperbolic part of the considered domain, the functional equation is solved by the iteration method. The problem is reduced to solving the Volterra integro-differential equation.

Keywords: fractional order derivative; Riemann-Liouville operator; boundary value problem; singular coefficient; mixed-type equation

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1. Introduction and Formulation of a Problem

Boundary value problems for the mixed-type equations of fractional order were investigated in [1–4]. In [5], the unique solvability was investigated for the problem of an equation with the partial fractional derivative of Riemann-Liouville and a boundary condition that contains the generalized operator of fractional integro-differentiation. A problem, in which the boundary condition contains a linear combination of generalized fractional operators with a Gauss hypergeometric function for a mixed-type equation with a Riemann-Liouville partial fractional derivative, was studied in [6]. The nonlocal boundary value problem for mixed-type equations with singular coefficients was considered in [7]. The Gellerstedt-type problem, with nonlocal boundary and integral gluing conditions for the parabolic-hyperbolic-type equation, with nonlinear terms and Gerasimov-Caputo operator of differentiation, was studied in [8]. The work [9] is devoted to study the boundary value problems for a mixed type fractional differential equation with Caputo operator. A nonlocal boundary value problem for weak nonlinear partial differential equations of mixed type, with a fractional Hilfer operator, was solved in [10]. The work [11] is concerned with the existence and uniqueness of solutions for a Hilfer-Hadamard fractional differential equation.

Let D be a finite domain bounded by segments AA_0, BB_0, and A_0B_0 of lines $x = 0$, $x = 1$, and $y = 1$, respectively, lying in the half-plane $y > 0$, and characteristics $AC = \{(x,y) : x - \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = 0\}$, $BC = \{(x,y) : x + \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = 1\}$ of the following equation:

$$\begin{cases} u_{xx} - D^{\gamma}_{0,y}u = 0,\ \gamma \in (0,1), y > 0, \\ -(-y)^m u_{xx} + u_{yy} + \dfrac{\alpha_0}{(-y)^{1-\frac{m}{2}}}u_x + \dfrac{\beta_0}{y}u_y = 0,\ \ y < 0, \end{cases} \tag{1}$$

in the half-plane, $y < 0$, and the interval, AB, of the straight line, $y = 0$. In (1) $m > 0$, $|\alpha_0| < (m+2)/2$, $1 < \beta_0 < (m+4)/2$. Here, $D^{\gamma}_{0,y}$ is the partial fractional Riemann-Liouville derivative [3]

$$(D^{\gamma}_{0,y}u)(x,y) = \frac{\partial}{\partial y}\frac{1}{\Gamma(1-\gamma)}\int\limits_0^y \frac{u(x,t)dt}{(y-t)^{\gamma}}, (0 < \gamma < 1, y > 0).$$

Let $D^{+} = D \cap (y > 0)$, $D^{-} = D \cap (y < 0)$.

Problem 1. *Find a solution $u = u(x,y)$ of Equation (1) in the domain D, satisfying the following boundary conditions:*

$$u(0,y) = \varphi_1(y), u(1,y) = \varphi_2(y), 0 \le y \le 1, \tag{2}$$

$$\begin{gathered} x^{\bar{\alpha}} D_{0,x}^{1-\bar{\beta}} x^{1-\bar{\alpha}-\bar{\beta}} u[\theta_0(x)] + (1-x)^{\bar{\beta}} \rho(x) D_{x,1}^{1-\bar{\alpha}} (1-x)^{1-\bar{\alpha}-\bar{\beta}} u[\theta_k(x)] \\ = \mu_1 D_{0,x}^{1-\bar{\alpha}-\bar{\beta}} \tau(x) - \mu_2 D_{x,1}^{1-\bar{\alpha}-\bar{\beta}} \tau(x) + f(x), x \in [0,1] \end{gathered} \tag{3}$$

and the following conjugation conditions:

$$\lim_{y \to +0} y^{1-\gamma} u(x,y) = \lim_{y \to -0} (-y)^{\beta_0 - 1} u(x,y), \; x \in [0,1],$$

$$\lim_{y \to +0} y^{1-\gamma} (y^{1-\gamma} u(x,y))_y = \lim_{y \to -0} (-y)^{2-\beta_0} ((-y)^{\beta_0-1} u(x,y))_y, x \in I = (0,1). \tag{4}$$

Here, $\bar{\alpha} = \frac{m+2(2-\beta_0+\alpha_0)}{2(m+2)}$, $\bar{\beta} = \frac{m+2(2-\beta_0-\alpha_0)}{2(m+2)}$, $\tau(x) = \lim\limits_{y \to -0} (-y)^{\beta_0-1} u(x,y)$, $\varphi_1(y)$, $\varphi_2(y)$, $\rho(x)$, $f(x)$ are given functions; moreover, $y^{1-\gamma}\varphi_1(y)$, $y^{1-\gamma}\varphi_2(y) \in C([0,1])$, $\varphi_1(0) = \varphi_2(0) = 0$, $f(0) = 0$, μ_1 and μ_2 are constants; $\theta_0(x_0) = \left(\frac{x_0}{2}, -(\frac{m+2}{4} x_0)^{2/(m+2)}\right)$ is a point of intersection of characteristics of Equation (1), outgoing from the point $(x_0, 0)$ $(x_0 \in I)$, with the characteristic AC; $\theta_k(x_0) = \left(\frac{x_0+k}{1+k}, -(\frac{(m+2)(1-x_0)}{2(1+k)})^{\frac{2}{m+2}}\right)$ is the intersection point of the curve $x - \frac{2k}{m+2}(-y)^{\frac{m+2}{2}} = x_0$, $k = const > 1$, with the characteristic BC, $D_{x,1}^{1-\bar{\alpha}} f(x) = -\frac{d}{dx} D_{x,1}^{-\bar{\alpha}} f(x) = -\frac{d}{dx} \frac{1}{\Gamma(\bar{\alpha})} \int\limits_x^1 \frac{f(t)dt}{(t-x)^{1-\bar{\alpha}}}$. We are looking for a solution, $u(x,y)$, of the problem in the class of twice differentiable functions in the domain D, such that $y^{1-\gamma} u \in C(\bar{D^{+}})$, $u(x,y) \in C(\bar{D^{-}} \setminus \overline{OB})$, $y^{1-\gamma}(y^{1-\gamma}u)_y \in C(D^{+} \cup \{(x,y) : 0 < x < 1, y = 0\})$, $u_{xx} \in C(D^{+} \cup D^{-})$, $u_{yy} \in C(D^{-})$.

Note that from Equation (1), at $m = 2$, $\beta_0 = 0$, we obtain the moisture transfer Equation [12], and at $\alpha_0 = 0$, $\beta_0 = 0$, Equation (1) passes to the Gellerstedt equation, which finds application in the problem of determining the shape of the dam slot.

In this article, we study a problem with a shift for Equation (1), in which some part of the characteristic BC is freed from nonlocal boundary conditions.

2. Main Results

We denote that $\lim\limits_{y \to +0} y^{1-\gamma} u(x,y) = \tau(x)$, $\lim\limits_{y \to +0} y^{1-\gamma}(y^{1-\gamma} u(x,y))_y = \nu(x)$. The solution of Equation (1) in the domain D^{+}, satisfying condition (2) and condition $\lim\limits_{y \to +0} y^{1-\gamma} u(x,y) = \tau(x), x \in \bar{I}$, has the following form [13]:

$$u(x,y) = \int\limits_0^y \frac{\partial G}{\partial \xi}|_{\xi=0} \varphi_1(\eta) d\eta - \int\limits_0^y \frac{\partial G}{\partial \xi}|_{\xi=1} \varphi_2(\eta) d\eta - \int\limits_0^1 G(x,y;\xi,0)\tau(\xi) d\xi,$$

where

$$G(x,y;\xi,\eta) = \frac{\Gamma(\gamma)}{2}(y-\eta)^{\delta-1} \sum_{n=-\infty}^{\infty} \left[e_{1,\delta}^{1,\delta}\left(-\frac{|x-\xi+2n|}{(y-\eta)^{\delta}}\right) - e_{1,\delta}^{1,\delta}\left(-\frac{|x+\xi+2n|}{(y-\eta)^{\delta}}\right)\right],$$

$$e_{b,c}^{p,q}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(p+kb)\Gamma(q-kc)}, b > c, b > 0, z \in C, \delta = \frac{\gamma}{2}, \gamma > 0,$$

$$e^{1,\delta}_{1,\delta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\delta - k\delta)k!}, \delta < 1.$$

The functional relation between $\tau = \tau(x)$ and $\nu = \nu(x)$, transferred from the parabolic part, D^+, to the line, $y = 0$, has the following form [1]:

$$\nu(x) = 1/(\Gamma(1+\gamma))\tau''(x). \tag{5}$$

Applying the Darboux formula, given in the domain D^-, the solution of the modified Cauchy problem with the initial data $\lim\limits_{y\to 0-}(-y)^{\beta_0-1}u(x,y) = \tau(x)$,

$\lim\limits_{y\to -0}(-y)^{2-\beta_0}((-y)^{\beta_0-1}u(x,y))_y = \nu(x), x \in I$, has the following form:

$$\begin{aligned} u(x,y) = \bar{\gamma}_1(-y)^{1-\beta_0} \int_0^1 \tau\Big(x + \tfrac{2}{m+2}(2t-1)(-y)^{\frac{m+2}{2}}\Big) t^{\bar{\beta}-1}(1-t)^{\bar{\alpha}-1}dt \\ +\bar{\gamma}_2 \int_0^1 \nu\Big(x + \tfrac{2}{m+2}(2t-1)(-y)^{\frac{m+2}{2}}\Big) t^{-\bar{\alpha}}(1-t)^{-\bar{\beta}}dt, \end{aligned} \tag{6}$$

where $\bar{\gamma}_1 = \frac{\Gamma(\bar{\alpha}+\bar{\beta})}{\Gamma(\bar{\alpha})\Gamma(\bar{\beta})}$, $\bar{\gamma}_2 = -\frac{\Gamma(2-\bar{\alpha}-\bar{\beta})}{(\beta_0-1)\Gamma(1-\bar{\alpha})\Gamma(1-\bar{\beta})}$.

From (6) we have the following:

$$u[\theta_0(x)] = \bar{\gamma}_1 \left(\frac{m+2}{4}\right)^{\bar{\alpha}+\bar{\beta}-1} \Gamma(\bar{\alpha}) D^{-\bar{\alpha}}_{0,x} x^{\bar{\beta}-1}\tau(x) + \bar{\gamma}_2 x^{\bar{\alpha}+\bar{\beta}-1}\Gamma(1-\bar{\beta}) D^{\bar{\beta}-1}_{0,x} x^{-\bar{\alpha}}\nu(x). \tag{7}$$

Multiplying both sides of (7) by $x^{1-\bar{\alpha}-\bar{\beta}}$, we have the following:

$$\begin{aligned} x^{1-\bar{\alpha}-\bar{\beta}}u[\theta_0(x)] = \bar{\gamma}_1\Gamma(\bar{\alpha})\left(\tfrac{m+2}{4}\right)^{\bar{\alpha}+\bar{\beta}-1} x^{1-\bar{\alpha}-\bar{\beta}} D^{-\bar{\alpha}}_{0,x} x^{\bar{\beta}-1}\tau(x) \\ +\Gamma(1-\bar{\beta})\bar{\gamma}_2 D^{\bar{\beta}-1}_{0,x} x^{-\bar{\alpha}}\nu(x). \end{aligned} \tag{8}$$

Applying the operator $D^{1-\bar{\beta}}_{0,x}$ to both sides of relation (8) we obtain the following:

$$\begin{aligned} D^{1-\bar{\beta}}_{0,x} x^{1-\bar{\alpha}-\bar{\beta}}u[\theta_0(x)] = \bar{\gamma}_1((m+2)/4)^{\bar{\alpha}+\bar{\beta}-1}\Gamma(\bar{\alpha}) D^{1-\bar{\beta}}_{0,x} x^{1-\bar{\alpha}-\bar{\beta}} D^{-\bar{\alpha}}_{0,x} x^{\bar{\beta}-1}\tau(x) \\ +\Gamma(1-\bar{\beta})\bar{\gamma}_2 D^{1-\bar{\beta}}_{0,x} D^{\bar{\beta}-1}_{0,x} x^{-\bar{\alpha}}\nu(x). \end{aligned} \tag{9}$$

Equalities are true, as follows:

$$D^{1-\bar{\beta}}_{0,x} x^{1-\bar{\alpha}-\bar{\beta}} D^{-\bar{\alpha}}_{0,x} x^{\bar{\beta}-1}\tau(x) = x^{-\bar{\alpha}} D^{1-\bar{\alpha}-\bar{\beta}}_{0,x}\tau(x) \tag{10}$$

$$D^{1-\bar{\beta}}_{0,x} D^{\bar{\beta}-1}_{0,x} x^{-\bar{\alpha}}\nu(x) = x^{-\bar{\alpha}}\nu(x). \tag{11}$$

Let us show Relation (10).

Denoting the left-hand side of Equality(10) by $g_1(x)$, we obtain the following:

$$\begin{aligned} g_1(x) = D^{1-\bar{\beta}}_{0,x} x^{1-\bar{\alpha}-\bar{\beta}} D^{-\bar{\alpha}}_{0,x} x^{\bar{\beta}-1}\tau(x) = \frac{d}{dx} D^{-\bar{\beta}}_{0,x} x^{1-\bar{\alpha}-\bar{\beta}} D^{-\bar{\alpha}}_{0,x} x^{\bar{\beta}-1}\tau(x) \\ = \frac{1}{\Gamma(\bar{\beta})\Gamma(\bar{\alpha})} \frac{d}{dx} \int_0^x \frac{\xi^{1-\bar{\alpha}-\bar{\beta}} d\xi}{(x-\xi)^{1-\bar{\beta}}} \int_0^\xi \frac{t^{\bar{\beta}-1}\tau(t)dt}{(\xi-t)^{1-\bar{\alpha}}}. \end{aligned}$$

Changing the order of integration, we obtain the following:

$$g_1(x) = \frac{1}{\Gamma(\bar{\beta})\Gamma(\bar{\alpha})} \frac{d}{dx} \int_0^x \tau(t) t^{\bar{\beta}-1} dt \int_t^x \xi^{1-\bar{\alpha}-\bar{\beta}}(x-\xi)^{\bar{\beta}-1}(\xi-t)^{\bar{\alpha}-1} d\xi.$$

Setting in the inner integral, $\xi = t + (x-t)\sigma$, we have the following:

$$g_1(x) = \frac{1}{\Gamma(\bar{\beta})\Gamma(\bar{\alpha})}\frac{d}{dx}\int_0^x \tau(t)t^{-\bar{\alpha}}(x-t)^{\bar{\alpha}+\bar{\beta}-1}dt\int_0^1 \sigma^{\bar{\alpha}-1}(1-\sigma)^{\bar{\beta}-1}$$
$$\times\left(1-\frac{t-x}{t}\sigma\right)^{1-\bar{\alpha}-\bar{\beta}}d\sigma.$$

Using the Euler hypergeometric integral [14], as follows:

$$\int_0^1 t^{\mu-1}(1-t)^{k-\mu-1}(1-zt)^{-\lambda}dt = \frac{\Gamma(\mu)\Gamma(k-\mu)}{\Gamma(k)}F(\mu,\lambda,k;z), 0<\mu<k,$$

we obtain the following:

$$g_1(x) = \frac{1}{\Gamma(\bar{\beta})\Gamma(\bar{\alpha})}\frac{d}{dx}\int_0^x \tau(t)t^{-\bar{\alpha}}(x-t)^{\bar{\alpha}+\bar{\beta}-1}\frac{\Gamma(\bar{\alpha})\Gamma(\bar{\beta})}{\Gamma(\bar{\alpha}+\bar{\beta})}F\left(\bar{\alpha},\bar{\alpha}+\bar{\beta}-1,\bar{\alpha}+\bar{\beta};\frac{t-x}{t}\right)dt.$$

From here, taking into account [14], as follows:

$$F(\mu,\lambda,k;z) = (1-z)^{-\lambda}F\left(k-\mu,\lambda,k;\frac{z}{z-1}\right)$$

we have the following:

$$g_1(x) = 1/(\Gamma(\bar{\alpha}+\bar{\beta}))\frac{d}{dx}\int_0^x \tau(t)t^{\bar{\beta}-1}\left(\frac{x-t}{x}\right)^{\bar{\alpha}+\bar{\beta}-1}F\left(\bar{\beta},\bar{\alpha}+\bar{\beta}-1,\bar{\alpha}+\bar{\beta};\frac{x-t}{x}\right)dt.$$

Consider the following function:

$$g_\varepsilon(x) = 1/(\Gamma(\bar{\alpha}+\bar{\beta}))\frac{d}{dx}\int_0^{x-\varepsilon} \tau(t)t^{\bar{\beta}-1}\left(\frac{x-t}{x}\right)^{\bar{\alpha}+\bar{\beta}-1}F\left(\bar{\beta},\bar{\alpha}+\bar{\beta}-1,\bar{\alpha}+\bar{\beta};\frac{x-t}{x}\right)dt.$$

Differentiating the right-hand side of this equality and using the formulas from [14], as follows:

$$\frac{d}{dz}[z^\mu F(\mu,\lambda,k;z)] = \mu z^{\mu-1}F(\mu+1,\lambda,k;z), F(\mu,\lambda,\lambda;z) = (1-z)^{-\mu},$$

we obtain the following:

$$g_\varepsilon(x) = \frac{1}{\Gamma(\bar{\alpha}+\bar{\beta})}(x-\varepsilon)^{\bar{\beta}-1}\left(\frac{\varepsilon}{x}\right)^{\bar{\alpha}+\bar{\beta}-1}F\left(\bar{\beta},\bar{\alpha}+\bar{\beta}-1,\bar{\alpha}+\bar{\beta};\frac{\varepsilon}{x}\right)\tau(x-\varepsilon)$$
$$+\frac{\bar{\alpha}+\bar{\beta}-1}{\Gamma(\bar{\alpha}+\bar{\beta})}x^{-\bar{\alpha}}\int_0^{x-\varepsilon}(x-t)^{\bar{\alpha}+\bar{\beta}-2}\tau(t)dt.$$

Now, taking into account that

$$(\bar{\alpha}+\bar{\beta}-1)\int_0^{x-\varepsilon}(x-t)^{\bar{\alpha}+\bar{\beta}-2}\tau(t)dt = \frac{d}{dx}\int_0^{x-\varepsilon}(x-t)^{\bar{\alpha}+\bar{\beta}-1}\tau(t)dt - \varepsilon^{\bar{\alpha}+\bar{\beta}-1}\tau(x-\varepsilon),$$

we find the following:

$$g_\varepsilon(x) = \frac{\varepsilon^{\bar{\alpha}+\bar{\beta}-1}}{\Gamma(\bar{\alpha}+\bar{\beta})} x^{-\bar{\alpha}} \left[\left(\frac{x}{x-\varepsilon} \right)^{1-\bar{\beta}} F\left(\bar{\beta}, \bar{\alpha}+\bar{\beta}-1, \bar{\alpha}+\bar{\beta}; \frac{\varepsilon}{x}\right) - 1 \right] \tau(x-\varepsilon)$$
$$+ \frac{x^{-\bar{\alpha}}}{\Gamma(\bar{\alpha}+\bar{\beta})} \frac{d}{dx} \int_0^{x-\varepsilon} (x-t)^{\bar{\alpha}+\bar{\beta}-1} \tau(t) dt.$$

Passing in this equality to limit as $\varepsilon \to 0$, by virtue of the formula $F(\mu, \lambda, k; 0) = 1$ we obtain the following:

$$g_1(x) = g_\varepsilon(x) = \frac{x^{-\bar{\alpha}}}{\Gamma(\bar{\alpha}+\bar{\beta})} \frac{d}{dx} \int_0^x \frac{\tau(t)dt}{(x-t)^{1-\bar{\alpha}+\bar{\beta}}} = x^{-\bar{\alpha}} \frac{d}{dx} D^{-\bar{\alpha}-\bar{\beta}} \tau(x) = x^{-\bar{\alpha}} D^{1-\bar{\alpha}-\bar{\beta}} \tau(x).$$

Thus, let us be convinced of the validity of Equality (10).

By virtue of (10) and (11), Equality (9) can be written in the following form:

$$\begin{aligned} D_{0,x}^{1-\bar{\beta}} x^{1-\bar{\alpha}-\bar{\beta}} u[\theta_0(x)] = \bar{\gamma}_1((m+2)/4)^{\bar{\alpha}+\bar{\beta}-1} \Gamma(\bar{\alpha}) x^{-\bar{\alpha}} D_{0,x}^{1-\bar{\alpha}-\bar{\beta}} \tau(x) \\ + \Gamma(1-\bar{\beta}) \bar{\gamma}_2 x^{-\bar{\alpha}} \nu(x). \end{aligned} \tag{12}$$

Now, from (6) we obtain the following:

$$\begin{aligned} u[\theta_k(x)] = a^{1-\bar{\alpha}-\bar{\beta}} \bar{\gamma}_1 \left(\tfrac{m+2}{2(1+k)} \right)^{\bar{\alpha}+\bar{\beta}-1} \Gamma(\bar{\beta}) D_{ax+b,1}^{-\bar{\beta}} (1-x)^{\bar{\alpha}-1} \tau(x) \\ + \Gamma(1-\bar{\alpha}) \bar{\gamma}_2 a^{\bar{\alpha}+\bar{\beta}-1} (1-x)^{\bar{\alpha}+\bar{\beta}-1} D_{ax+b,1}^{\bar{\alpha}-1} (1-x)^{-\bar{\beta}} \nu(x). \end{aligned} \tag{13}$$

Multiplying both sides of (13) by $(1-x)^{1-\bar{\alpha}-\bar{\beta}}$ and applying the operator $D_{x,1}^{1-\bar{\alpha}}$ we obtain the following:

$$\begin{aligned} D_{x,1}^{1-\bar{\alpha}} (1-x)^{1-\bar{\alpha}-\bar{\beta}} u[\theta_k(x)] = \Gamma(\bar{\beta}) \bar{\gamma}_1 \left(\tfrac{m+2}{2(1+k)} \right)^{\bar{\alpha}+\bar{\beta}-1} a^{1-\bar{\alpha}-\bar{\beta}} D_{x,1}^{1-\bar{\alpha}} (1-x)^{1-\bar{\alpha}-\bar{\beta}} \\ \times D_{ax+b,1}^{-\bar{\beta}} (1-x)^{\bar{\alpha}-1} \tau(x) + \bar{\gamma}_2 a^{\bar{\alpha}+\bar{\beta}-1} \Gamma(1-\bar{\alpha}) D_{x,1}^{1-\bar{\alpha}} D_{ax+b,1}^{\bar{\alpha}-1} (1-x)^{-\bar{\beta}} \nu(x). \end{aligned} \tag{14}$$

It is easy to show the following:

$$D_{x,1}^{1-\bar{\alpha}} (1-x)^{1-\bar{\alpha}-\bar{\beta}} D_{ax+b,1}^{-\bar{\beta}} (1-x)^{\bar{\alpha}-1} \tau(x) = (1-x)^{-\beta} D_{ax+b,1}^{1-\bar{\alpha}-\bar{\beta}} \tau(x),$$

$$D_{x,1}^{1-\bar{\alpha}} D_{ax+b,1}^{\bar{\alpha}-1} (1-x)^{-\bar{\beta}} \nu(x) = a^{1-\bar{\alpha}-\bar{\beta}} \nu(ax+b)(1-x)^{-\bar{\beta}}.$$

Then, from (14) we obtain the following:

$$\begin{aligned} D_{x,1}^{1-\bar{\alpha}} (1-x)^{1-\bar{\alpha}-\bar{\beta}} u[\theta_k(x)] = a^{1-\bar{\alpha}-\bar{\beta}} \bar{\gamma}_1 \left(\tfrac{m+2}{2(1+k)} \right)^{\bar{\alpha}+\bar{\beta}-1} \Gamma(\bar{\beta}) \\ \times (1-x)^{-\bar{\beta}} D_{ax+b,1}^{1-\bar{\alpha}-\bar{\beta}} \tau(x) + \bar{\gamma}_2 \Gamma(1-\bar{\alpha}) \nu(ax+b)(1-x)^{-\bar{\beta}}. \end{aligned} \tag{15}$$

Now, substituting (12) and (15) into (3), we obtain the following:

$$\begin{aligned} \bar{\gamma}_1((m+2)/4)^{\bar{\alpha}+\bar{\beta}-1} \Gamma(\bar{\alpha}) D_{0,x}^{1-\bar{\alpha}-\bar{\beta}} \tau(x) + \Gamma(1-\bar{\beta}) \bar{\gamma}_2 \nu(x) \\ + a^{1-\bar{\alpha}-\bar{\beta}} \left(\tfrac{m+2}{2(1+k)} \right)^{\bar{\alpha}+\bar{\beta}-1} \Gamma(\bar{\beta}) \bar{\gamma}_1 \rho(x) D_{ax+b,1}^{1-\bar{\alpha}-\bar{\beta}} \tau(x) \\ + \bar{\gamma}_2 \Gamma(1-\bar{\alpha}) \rho(x) \nu(ax+b) = \mu_1 D_{0,x}^{1-\bar{\alpha}-\bar{\beta}} \tau(x) - \mu_2 D_{x,1}^{1-\bar{\alpha}-\bar{\beta}} \tau(x) + f(x), \\ x \in [0,1]. \end{aligned} \tag{16}$$

Let $\mu_1 = \frac{\Gamma(\bar{\alpha}+\bar{\beta})}{\Gamma(\bar{\beta})}\left(\frac{m+2}{4}\right)^{\bar{\alpha}+\bar{\beta}-1}$. Then, from (16) we obtain the following:

$$\begin{gathered}\bar{\gamma_2}\Gamma(1-\bar{\beta})\nu(x) + \left(\frac{m+2}{2(1+k)}\right)^{\bar{\alpha}+\bar{\beta}-1} a^{1-\bar{\alpha}-\bar{\beta}}\bar{\gamma_1}\Gamma(\bar{\beta})\rho(x)D^{1-\bar{\alpha}-\bar{\beta}}_{ax+b,1}\tau(x) \\ +\rho(x)\bar{\gamma_2}\Gamma(1-\bar{\alpha})\nu(ax+b) + \mu_2 D^{1-\bar{\alpha}-\bar{\beta}}_{x,1}\tau(x) = f(x), x \in [0,1].\end{gathered} \tag{17}$$

Dividing both sides of (17) by $\bar{\gamma_2}\Gamma(1-\bar{\beta})$, by virtue of $a = \frac{2}{1+k}$, we obtain the following:

$$\begin{gathered}\nu(x) + \frac{\Gamma(\bar{\alpha}+\bar{\beta})}{\bar{\gamma_2}\Gamma(1-\bar{\beta})\Gamma(\bar{\alpha})}\left(\frac{m+2}{4}\right)^{\bar{\alpha}+\bar{\beta}-1}\rho(x)D^{1-\bar{\alpha}-\bar{\beta}}_{ax+b,1}\tau(x) \\ +\frac{\Gamma(1-\bar{\alpha})}{\Gamma(1-\bar{\beta})}\rho(x)\nu(ax+b) + \frac{\mu_2 D^{1-\bar{\alpha}-\bar{\beta}}_{x,1}\tau(x)}{\bar{\gamma_2}\Gamma(1-\bar{\beta})} = \frac{f(x)}{\bar{\gamma_2}\Gamma(1-\bar{\beta})}.\end{gathered} \tag{18}$$

We write Equality (18) in the following form:

$$\begin{gathered}\nu(x) + \frac{\mu_2}{\bar{\gamma_2}\Gamma(1-\bar{\beta})}D^{1-\bar{\alpha}-\bar{\beta}}_{x,1}\tau(x) = -\frac{\Gamma(1-\bar{\alpha})}{\Gamma(1-\bar{\beta})}\rho(x) \\ \times\left[\nu(ax+b) + \frac{\left(\frac{4}{m+2}\right)^{1-\bar{\alpha}-\bar{\beta}}}{\bar{\gamma_2}\Gamma(1-\bar{\beta})}\frac{\Gamma(\bar{\alpha}+\bar{\beta})\Gamma(1-\bar{\beta})}{\Gamma(\bar{\alpha})\Gamma(1-\bar{\alpha})}D^{1-\bar{\alpha}-\bar{\beta}}_{ax+b,1}\tau(x)\right] + f(x)\bar{\gamma_0},\end{gathered} \tag{19}$$

where $\bar{\gamma_0} = \frac{1}{\bar{\gamma_2}\Gamma(1-\bar{\beta})}$.

Let $\mu_2 = \left(\frac{4}{m+2}\right)^{1-\bar{\alpha}-\bar{\beta}}(\Gamma(\bar{\alpha}+\bar{\beta})\Gamma(1-\bar{\beta}))/(\Gamma(\bar{\alpha})\Gamma(1-\bar{\alpha}))$.

We introduce the following notation:

$$\Phi(x) := \nu(x) + \frac{\Gamma(\bar{\alpha}+\bar{\beta})\Gamma(1-\bar{\beta})}{\Gamma(\bar{\alpha})\Gamma(1-\bar{\alpha})}\left(\frac{4}{m+2}\right)^{1-\bar{\alpha}-\bar{\beta}}\bar{\gamma_0}D^{1-\bar{\alpha}-\bar{\beta}}_{x,1}\tau(x). \tag{20}$$

Then, from (19), we obtain the following functional equation:

$$\Phi(x) = \Phi(ax+b)\omega(x) + f_1(x), \tag{21}$$

where $\omega(x) = -\frac{\Gamma(1-\bar{\alpha})}{\Gamma(1-\bar{\beta})}\rho(x)$, $f_1(x) = \bar{\gamma_0}f(x)$.

Applying the iteration method to (21), after simple calculations, we obtain the following:

$$\begin{gathered}\Phi(x) = \omega(x)A_{n-1}(x)\Phi(a^n x + 1 - a^n) \\ +\omega(x)\sum_{j=1}^{n-1} f_1(a^j x + 1 - a^j)A_{j-1}(x) + f_1(x),\end{gathered} \tag{22}$$

where

$$A_{j-1}(x) = \omega(ax+1-a)\omega(a^2x+1-a^2)\cdots\omega(a^{j-1}x+1-a^{j-1}), A_0 = 1.$$

By $|\omega(x)| \leq 1$, for $x \in [1-\varepsilon, 1]$, here, ε is a sufficiently small positive number; therefore, we obtain the following:

$$|A_{j-1}(x)| \leq M_0^{n_0}, \tag{23}$$

where $M_0 = \max\limits_{[1-\varepsilon,1]} \omega(x)$, and $n_0 = [log_a(a\varepsilon)]$. We seek a solution of Equation (21), in the class of functions bounded at the point $x = 1$.

After passing to the limit, $n \to \infty$, in (22), and considering that, by virtue of $\rho(x), f(x) \in C(\bar{I}) \cap C^2(I)$ and (23), the series on the right-hand side of (22) is uniformly converging on the interval $[0,1]$, we obtain the following:

$$\Phi(x) = P(x), \tag{24}$$

$$P(x) = \omega(x)\sum_{j=1}^{\infty} f_1(a^j x + 1 - a^j)A_{j-1}(x) + f_1(x).$$

Thus, according to (20) from (24), we have

$$\nu(x) + \bar{\gamma}_0(\Gamma(\bar{\alpha}+\bar{\beta})\Gamma(1-\bar{\beta}))/(\Gamma(\bar{\alpha})\Gamma(1-\bar{\alpha}))\left(\frac{4}{m+2}\right)^{1-\bar{\alpha}-\bar{\beta}} D_{x,1}^{1-\bar{\alpha}-\bar{\beta}}\tau(x) = P(x). \quad (25)$$

Thus, we obtained the functional relation between the functions $\tau = \tau(x)$ and $\nu = \nu(x)$, on I, from the domain D^-.

By virtue of (4), substituting (5) into (25) we obtain the following:

$$1/(\Gamma(1+\gamma))\tau''(x) + \bar{\gamma}_3 D_{x,1}^{1-\bar{\alpha}-\bar{\beta}}\tau(x) = P(x), \quad (26)$$

where $\bar{\gamma}_3 = \bar{\gamma}_0(4/(m+2))^{1-\bar{\alpha}-\bar{\beta}}(\Gamma(\bar{\alpha}+\bar{\beta})\Gamma(1-\bar{\beta}))/(\Gamma(\bar{\alpha})\Gamma(1-\bar{\alpha}))$.

Equation (26) can be written as follows:

$$1/(\Gamma(1+\gamma))\tau''(x) - \bar{\gamma}_3/(\Gamma(\bar{\alpha}+\bar{\beta}))\frac{d}{dx}\int_x^1 \frac{\tau(t)dt}{(t-x)^{1-\bar{\alpha}-\bar{\beta}}} = P(x).$$

We have obtained the Volterra integro-differential equation, which is uniquely solvable, see [15].

3. Discussion

The properties of solutions of Equation (1) at $y < 0$ essentially depend on the coefficients α_0 and β_0, at the lowest terms of Equation (1). If $\beta_0 < 1$, then the solution of Equation (1) on the parabolic degeneration line is bounded. In this case, a problem for an elliptic-hyperbolic-type equation with singular coefficients was studied in [7]. If $\beta_0 = 1$, then the solution to Equation (1) on the parabolic degeneracy line has a logarithmic singularity. $\beta_0 = \frac{m+4}{2}$ is the limiting case. In these cases, boundary value problems for Equation (1) are studied with different conditions.

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