fractal and fractional

MDPI

Article

Inclusion Relations for Dini Functions Involving Certain Conic Domains

Bilal Khan [1], Shahid Khan [2], Jong-Suk Ro [3,4,*], Serkan Araci [5], Nazar Khan [6] and Nasir Khan [7]

[1] School of Mathematical Sciences and Shanghai Key Laboratory of PMMP, East China Normal University, 500 Dongchuan Road, Shanghai 200241, China; bilalmaths789@gmail.com
[2] Department of Mathematics and Statistics, Riphah International University, Islamabad 44000, Pakistan; shahidmath761@gmail.com
[3] School of Electrical and Electronics Engineering, Chung-Ang University, Dongjak-gu, Seoul 06974, Korea
[4] Department of Intelligent Energy and Industry, Chung-Ang University, Dongjak-gu, Seoul 06974, Korea
[5] Department of Economics, Faculty of Economics Administrative and Social Sciences, Hasan Kalyoncu University, Gaziantep TR-27410, Turkey; serkan.araci@hku.edu.tr
[6] Department of Mathematics, Abbottabad University of Science and Technology, Abbottabad 22010, Pakistan; nazarmaths@gmail.com
[7] Department of Mathematics, FATA University, Akhorwal (Darra Adam Khel), Kohat FR-26000, Pakistan; dr.nasirkhan@fu.edu.pk
* Correspondence: jongsukro@gmail.com

Abstract: In recent years, special functions such as Bessel functions have been widely used in many areas of mathematics and physics. We are essentially motivated by the recent development; in our present investigation, we make use of certain conic domains and define a new class of analytic functions associated with the Dini functions. We derive inclusion relationships and certain integral preserving properties. By applying the Bernardi-Libera-Livingston integral operator, we obtain some remarkable applications of our main results. Finally, in the concluding section, we recall the attention of curious readers to studying the q-generalizations of the results presented in this paper. Furthermore, based on the suggested extension, the $(\mathfrak{p}, q)$-extension will be a relatively minor and unimportant change, as the new parameter $\mathfrak{p}$ is redundant.

Keywords: analytic function; univalent function; starlike and convex function; subordination; quasi-convex functions; Bessel functions; Dini functions

MSC: Primary 30C45; 30C50; 30C80; Secondary 11B65; 47B38

Citation: Khan, B.; Khan, S.; Ro, J.-S.; Araci, S.; Khan, N.; Khan, N. Inclusion Relations for Dini Functions Involving Certain Conic Domains. *Fractal Fract.* **2022**, *6*, 118. https://doi.org/10.3390/fractalfract6020118

Academic Editors: Acu Mugur Alexandru and Shahram Najafzadeh

Received: 18 January 2022
Accepted: 15 February 2022
Published: 17 February 2022

1. Introduction, Definitions and Motivation

The theory of special functions is an important component in most branches of mathematics. Special functions are applied in complicated mathematical calculations by engineers and scientists. The applications are covering a wide range of fields in physics, engineering and computer sciences. In the field of computer science, special functiona known as activation functions have a very important place. The widespread usage of these functions has attracted a large number of researchers to work in many areas. The study of the geometric properties of special functions such as Bessel functions, hypergeometric functions, Mittag-Leffler functions, Struve functions, Wright functions and other related functions is a continuing aspect of geometric function theory research. Some geometric properties of these functions can be found in [1–3].

Bessel functions are a set of solutions to a second-order differential equation that can appear in a variety of contexts. Bessel functions were initially employed by Bessel to explain three-body motion, with the Bessel functions appearing in the series expansion on planetary perturbation [4]. Euler, Lagrange and the Bernoullis also examined the Bessel functions. Now, consider the equation

$$\zeta^2 w''(\zeta) + \zeta w'(\zeta) + \left(\zeta^2 - u^2\right) w(\zeta) = 0, \qquad u, \zeta \in \mathbb{C}, \tag{1}$$

which is a second-order linear homogeneous differential equation. The Equation (1) is well-known Bessel's differential equation. Its solution is known as the Bessel function and is represented by $J_u(\zeta)$, where $J_u(\zeta)$ is given as follows:

$$J_u(\zeta) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+u+1)} \left(\frac{\zeta}{2}\right)^{u+2n}, \qquad \zeta \in \mathbb{C}, \tag{2}$$

where Γ represents the well-known Euler Gamma function; also, the well-known Pochhammar symbol $(\gamma)_n$ is given by the relation

$$(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)},$$

or

$$(\gamma)_n = \begin{cases} 1 & (n=0) \\ \gamma(\gamma+1)(\gamma+2)...(\gamma+n-1) & (n \in \mathbb{N} \text{ and } \gamma \in \mathbb{C}). \end{cases}$$

See [5] for a thorough examination of the first-order Bessel function.

The Dini function is a kind of special function and is the combination of the Bessel function of the first kind and is defined by

$$r_u(\zeta) = (1-u)J_u(\zeta) + \zeta J_u'(\zeta)$$

Moreover, the normalized Dini functions $d_u(\zeta) : E \to \mathbb{C}$ can be defined as (see [6])

$$\begin{aligned} d_u(\zeta) &= 2^u \Gamma(u+1) \zeta^{1-\frac{u}{2}} \left((1-u)\varphi_u\left(\sqrt{\zeta}\right) + \sqrt{\zeta}\varphi_u'\left(\sqrt{\zeta}\right) \right) \\ &= \zeta + \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)\Gamma(u+1)}{4^n n! \Gamma(u+n+1)} \zeta^{n+1} \\ &= \zeta + \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)}{4^n n! (k_1)_n} \zeta^{n+1}, \end{aligned}$$

where

$$\varphi_u(\zeta) = \zeta + \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n n! (k_1)_n} \zeta^{n+1}$$

$$u \in \mathbb{R}, \quad n \in \mathbb{N} \quad \text{and} \quad k_1 = u \notin \mathbb{Z}_0^{-1} = \{0, -1, -2, ...\}.$$

Let the symbol $\mathcal{A}$ mean the class of all analytic functions t in the open unit disk:

$$E = \{\zeta : \zeta \in \mathbb{C} \text{ and } |\zeta| < 1\}$$

and having a Taylor-Maclaurin form as

$$t(\zeta) = \zeta + \sum_{n=2}^{\infty} a_n \zeta^n. \tag{3}$$

Let $\mathcal{S} \subset \mathcal{A}$, consisting of univalent functions in E and with the condition

$$t(0) = 0 = t'(0) - 1.$$

Moreover, all normalized univalent functions in E are contained in the set $\mathcal{A} \subset \mathcal{S}$. For two given functions $g_1, g_2 \in \mathcal{A}$, we say that g_1 is subordinate to g_2, written symbolically as $g_1 \prec g_2$, if there exists a Schwarz function w, which is holomorphic in the open unit disk E with

$$w(0) = 0 \quad \text{and} \quad |w(\zeta)| < 1,$$

so that

$$g_1(\zeta) = g_2(w(\zeta)) \qquad (\zeta \in E).$$

Moreover, if the function g_2 is univalent in E, then the following equivalence hold true:

$$g_1 \prec g_2 \iff g_1(0) = g_2(0) \text{ and } g_1(E) \subset g_2(E). \tag{4}$$

For two given functions t and g from the class $\mathcal{A}$, the convolution (or Hadamard product) of these functions is given by

$$(t * g)(\zeta) = \zeta + \sum_{n=2}^{\infty} a_n b_n \zeta^n$$

Let $\mathcal{P}$ be the class of Carathéodory functions, consisting of all analytic functions p satisfying the conditions

$$\Re(p(\zeta)) > 0 \quad (\zeta \in E)$$

and

$$p(\zeta) = 1 + \sum_{n=1}^{\infty} c_n \zeta^n.$$

Robertson [7] introduced and studied the classes of starlike $\mathcal{S}^*(\gamma)$ and convex $\mathcal{C}(\gamma)$ functions of order γ as follows

$$t \in \mathcal{A} \text{ and } \Re\left(\frac{\zeta t'(\zeta)}{t(\zeta)}\right) > \gamma \quad (0 \leq \gamma < 1)$$

and

$$t \in \mathcal{A} \text{ and } \Re\left(1 + \frac{\zeta t''(\zeta)}{t'(\zeta)}\right) > \gamma \quad (0 \leq \gamma < 1),$$

respectively. Note that $t \in \mathcal{C}(\gamma)$ if and only if $\zeta t' \in \mathcal{S}^*(\gamma)$ (see also Srivastava and Owa [8]).

In 1964, Libera [9] introduced the class $\mathcal{K}(\gamma, \alpha)$ of close to convex functions of order γ and type α $(0 \leq \alpha < 1)$, which is defined by

$$\mathcal{K}(\gamma, \alpha) = \left\{ t \in \mathcal{A} \text{ and } g \in \mathcal{S}^*(\alpha) : \Re\left(\frac{\zeta t'(\zeta)}{g(\zeta)}\right) > \gamma \right\} \quad (0 \leq \gamma < 1).$$

Many researchers have recently examined different classes of analytic and univalent functions in various areas; see for more information [10–18]. By taking inspiration from the above-cited work, Shams et al. [19] introduced the domain $\Omega_{k,\gamma}$ for $0 \leq k < \infty$ as follows:

$$\Omega_{k,\gamma} = \left\{ u + iv : (u - \gamma)^2 > k^2\left((u-1)^2 + v^2\right) \right\}.$$

Note that, for $0 < k < 1$,

$$\Omega_{k,\gamma} = \left\{ u + iv : \left(\frac{u + \frac{k^2 - \gamma}{1 - k^2}}{k\left(\frac{1-\gamma}{1-k^2}\right)}\right)^2 - \left(\frac{v}{\frac{1-\gamma}{\sqrt{k^2-1}}}\right) > 1 \right\},$$

for $k > 1$,

$$\Omega_{k,\gamma} = \left\{ u + iv : \left(\frac{u + \frac{k^2 - \gamma}{k^2 - 1}}{k\left(\frac{1-\gamma}{k^2-1}\right)}\right)^2 - \left(\frac{v}{\frac{1-\gamma}{\sqrt{k^2-1}}}\right) < 1 \right\}.$$

Here, $Q_{k,\gamma}(\zeta)$ are the extremal functions for the conic domain, $\Omega_{k,\gamma}$, and are given below:

$$Q_{k,\gamma}(\zeta) = \begin{cases} Q_1(\zeta) & (k=0) \\ Q_2(\zeta) & (k=1) \\ Q_3(\zeta) & (0<k<1) \\ Q_4(\zeta) & (k>1) \end{cases} \tag{5}$$

where

$$Q_1(\zeta) = \frac{1+(1-2\gamma)\zeta}{1-\zeta},$$

$$Q_2(\zeta) = 1 + \frac{2(1-\gamma)}{\pi^2}\left(\log\frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}}\right)^2,$$

$$Q_3(\zeta) = 1 + \frac{2(1-\gamma)}{1-k^2} \times \sinh^2\left\{\begin{array}{c} \left(\frac{2}{\pi}\arccos k\right) \\ \operatorname{arctanh}\sqrt{\zeta} \end{array}\right\},$$

$$Q_4(\zeta) = \frac{(1-\gamma)}{k^2-1}\sin\left(\frac{\pi}{2K(i)}\int_0^{\frac{u(\zeta)}{\sqrt{t}}}\frac{1}{\sqrt{1-x^2}\sqrt{1-(ix)^2}}dx\right) + \frac{k^2-\gamma}{k^2-1},$$

and $i \in (0,1)$, $k = \cosh\left(\frac{\pi K'(i)}{K(i)}\right)$, $K(i)$ is the first kind of Legendre's complete elliptic integral.

We now give the following well-known subclasses of the analytic function.

Definition 1 (see [19])**.** *A function $t \in \mathcal{A}$ is said to be in the class $t \in k\text{-}\mathcal{ST}(\gamma)$ if and only if*

$$\frac{\zeta t'(\zeta)}{t(\zeta)} \prec Q_{k,\gamma}(\zeta). \tag{6}$$

Definition 2 (see [19])**.** *A function $t \in \mathcal{A}$ is said to be in the class $t \in k\text{-}\mathcal{UCV}(\gamma)$ if and only if*

$$1 + \frac{\zeta t''(\zeta)}{t'(\zeta)} \prec Q_{k,\gamma}(\zeta).$$

We note that

$$\Re(p(\zeta)) > \Re(Q_{k,\gamma}(\zeta)) > \frac{k+\gamma}{k+1}.$$

Definition 3 (see [20])**.** *A function $t \in \mathcal{A}$ is said to be in the class $k\text{-}\mathcal{UCC}(\gamma)$ if and only if*

$$\frac{\zeta t'(\zeta)}{g(\zeta)} \prec Q_{k,\gamma}(\zeta), \quad \text{for some } \ g(\zeta) \in k\text{-}\mathcal{ST}(\gamma) \quad (0 \le \gamma < 1).$$

Definition 4 (see [20])**.** *A function $t \in \mathcal{A}$ is said to be in the class $k\text{-}\mathcal{UQC}(\gamma)$ if and only if*

$$\frac{(\zeta t'(\zeta))'}{g'(\zeta)} \prec Q_{k,\gamma}(\zeta), \quad \text{for some } \ g(\zeta) \in k\text{-}\mathcal{UCV}(\gamma) \quad (0 \le \gamma < 1).$$

The functions defined in (5) play the role of extremal functions for each of the above-defined function classes.

Remark 1. *First of all, it is easy to see that*

$$0\text{-}\mathcal{UCV}(\gamma) = \mathcal{C}(\gamma) \quad \text{and} \quad 0\text{-}\mathcal{ST}(\gamma) = \mathcal{S}^*(\gamma),$$

where $\mathcal{C}(\gamma)$ and $\mathcal{S}^*(\gamma)$ are the functions classes introduced and studied by Robertson (see [7]). Secondly, we have

$$k\text{-}\mathcal{UCV}(0) = k\text{-}\mathcal{UCV} \quad \text{and} \quad 0\text{-}\mathcal{ST}(\gamma) = k\text{-}\mathcal{ST},$$

where k-$\mathcal{UCV}$ and k-$\mathcal{ST}$ are the functions classes introduced and studied by Kanas and Wiśniowska (see [21]). Thirdly, we have

$$1\text{-}\mathcal{UCV}(0) = \mathcal{UCV} \quad \text{and} \quad 1\text{-}\mathcal{ST}(0) = \mathcal{ST},$$

where $\mathcal{UCV}$ and $\mathcal{ST}$ are the function classes introduced by Goodman [22] and also studied in [23,24].

Here, we remark that quantum calculus or simply q-calculus has many applications in different branches of mathematics and physics. Geometric function theory is one among them. In geometric function theory, the role of q-calculus is very important. Many new subclasses of analytic functions have been generalized and studied by giving it q-extension. Furthermore, by using the q-approach, deferent subclasses, which involve the conic domains, have been generalized. Some significant works on q-calculus in geometric function theory can be found in [25–33]. In particular, we call the attention of curious readers to the prospect of studying the q-generalizations of the results presented in this paper, influenced by a newly published survey-cum-expository review article by Srivastava [17]; see also [16].

The theory of the operator plays a vital role in the development of geometric function theory. Many new convolution operators have been defined and studied by well-known mathematicians. We are motivated by the above-mentioned works and the recent research; here, we now use the idea of convolution and introduce the operator d_{k_1} as follows:

$$d_{k_1}t(\zeta) = d_{k_1} = \varphi_u(\zeta) * t(\zeta) = \zeta + \sum_{n=1}^{\infty} \frac{(-1)^n(2n+1)}{4^n n!(k_1)_n} a_{n+1}\zeta^{n+1}. \tag{7}$$

The following recursive relation can be easily verified from the definition (7):

$$\zeta\big(d_{k_1+1}t(\zeta)\big)' = k_1 d_{k_1}t(\zeta) - (k_1-1)d_{k_1+1}t(\zeta), \tag{8}$$

where

$$k_1 = u \notin Z_0^-.$$

Here, in [34], the generalized Bernardi-Libera-Livingston integral operator is defined as follows:

$$\mathcal{L}_\lambda(t) = \frac{\lambda+1}{\zeta^\lambda}\int_0^\zeta \frac{t(t)}{t^{1-\lambda}}dt\,, \qquad (\lambda > -1).$$

In our present investigation, we make use of certain conic domains and define a new class of analytic functions associated with the Dini functions. We derive inclusion relationships and certain integral preserving properties. By applying the Bernardi-Libera-Livingston integral operator, we obtain some remarkable applications of our main results.

2. Preliminaries Results

In order to prove our main results, we need the following Lemmas.

Lemma 1 (In [35])**.** *In For a convex univalent function h in E with the following conditions,*

$$h(0) = 1 \quad \text{and} \quad \Re(vh(\zeta)+\mu) > 0 \quad (v,\mu \in \mathbb{C}).$$

If the function p with $p(0) = 1$ is analytic in E, then we have

$$p(\zeta) + \frac{\zeta p'(\zeta)}{vp(\zeta)+\mu} \prec h(\zeta) \Rightarrow p(\zeta) \prec h(\zeta) \qquad (\zeta \in E).$$

Lemma 2 (In [36]). *Let h be convex in the open unit disk E, let $U \geq 0$ and $B(\zeta)$ is analytic in E with*

$$\Re(B(\zeta)) > 0.$$

If $g(\zeta)$ is analytic in E with $h(0) = g(0)$, then

$$U\zeta^2 g''(\zeta) + B(\zeta)g(\zeta) \prec h(\zeta) \Longrightarrow g(\zeta) \prec h(\zeta).$$

3. Main Results

Theorem 1. *Let $c \geq 1$ and h be convex univalent in E with*

$$h(0) = 1 \quad and \quad \Re(h(\zeta)) > 0.$$

If the following condition is satisfied by a function $t \in \mathcal{A}$

$$\frac{1}{1-\gamma}\left(\frac{\zeta\left(d_{k_1}t(\zeta)\right)'}{d_{k_1}t(\zeta)} - \gamma\right) \prec h(\zeta) \qquad (0 \leq \gamma < 1 \quad and \quad \zeta \in E),$$

then, we have

$$\frac{1}{1-\gamma}\left(\frac{\zeta\left(d_{k_1+1}t(\zeta)\right)'}{d_{k_1+1}t(\zeta)} - \gamma\right) \prec h(\zeta) \qquad (0 \leq \gamma < 1 \quad and \quad \zeta \in E),$$

where d_{k_1} is defined in (7).

Proof. Suppose that

$$p(\zeta) = \frac{1}{1-\gamma}\left(\frac{\zeta\left(d_{k_1+1}t(\zeta)\right)'}{d_{k_1+1}t(\zeta)} - \gamma\right),$$

where the function p belongs to the family of analytic functions in E, with the condition

$$p(0) = 1.$$

Now, making use of (8), we have

$$\gamma + (1-\gamma)p(\zeta) = k_1\left(\frac{\zeta d_{k_1}t(\zeta)}{d_{k_1+1}t(\zeta)}\right) - (k_1 - 1).$$

Upon taking the logarithmic differentiation of the above equation with regard to ζ, we find that

$$p(\zeta) + \frac{\zeta p'(\zeta)}{(1-\gamma)p(\zeta) + \gamma + k_1 - 1} = \frac{1}{1-\gamma}\left(\frac{\zeta d_{k_1}t(\zeta)}{d_{k_1+1}t(\zeta)} - \gamma\right).$$

Now, by applying Lemma 1 in conjunction with the above equation, one can easily get the required result. □

Theorem 2. *Let $t \in \mathcal{A}$. If $d_{k_1}t(\zeta) \in k\text{-}\mathcal{ST}(\gamma)$, then $d_{k_1+1}t(\zeta) \in k\text{-}\mathcal{ST}(\gamma)$.*

Proof. Suppose that

$$q(\zeta) = \frac{\zeta\left(d_{k_1+1}t(\zeta)\right)'}{d_{k_1+1}t(\zeta)},$$

where the function q is from the family of analytic functions in E with the condition $q(0) = 1$. From (8), we can write

$$k_1\left(\frac{d_{k_1}t(\zeta)}{d_{k_1+1}t(\zeta)}\right) = q(\zeta) + k_1 - 1.$$

Taking the logarithmic differentiation of the above equation with regard to ζ, we find that
$$\frac{\zeta\left(d_{k_1}t(\zeta)\right)'}{d_{k_1}t(\zeta)} = q(\zeta) + \frac{\zeta q'(\zeta)}{q(\zeta)+k_1-1} \prec Q_{k,\gamma}(\zeta).$$

Since $Q_{k,\gamma}(\zeta)$ is convex univalent in E and
$$\Re\left(Q_{k,\gamma}(\zeta)\right) > \frac{k+\gamma}{k+1}.$$

By making use of by Theorem 1, in conjunction with (6), we can get the desired result. □

Theorem 3. *Suppose that $t \in \mathcal{A}$. If $d_{k_1}t(\zeta) \in k\text{-}\mathcal{UCV}(\gamma)$, then $d_{k_1+1}t(\zeta) \in k\text{-}\mathcal{UCV}(\gamma)$.*

Proof. Making use of Definitions 1 and 2 in conjunction with Theorem 2, we have
$$\begin{aligned} d_{k_1}t(\zeta) \in k\text{-}\mathcal{UCV}(\gamma) &\Longleftrightarrow \zeta\left(d_{k_1}t(\zeta)\right)' \in k\text{-}\mathcal{ST}(\gamma) \\ &\Longleftrightarrow d_{k_1}\zeta t'(\zeta) \in k\text{-}\mathcal{ST}(\gamma) \\ &\Longleftrightarrow d_{k_1+1}\zeta t'(\zeta) \in k\text{-}\mathcal{ST}(\gamma) \\ &\Longleftrightarrow d_{k_{1+1}}t(\zeta) \in k\text{-}\mathcal{UCV}(\gamma). \end{aligned}$$

Hence, Theorem 3 is completed. □

Theorem 4. *Let $t \in \mathcal{A}$. If $d_{k_1}t(\zeta) \in k\text{-}\mathcal{UCC}(\gamma)$, then $d_{k_1+1}t(\zeta) \in k\text{-}\mathcal{UCC}(\gamma)$.*

Proof. Since we see that
$$d_{k_1}t(\zeta) \in k\text{-}\mathcal{UCC}(\gamma)$$
then, we have
$$\frac{\zeta\left(d_{k_1}t(\zeta)\right)'}{d_{k_1}g(\zeta)} \prec Q_{k,\gamma}(\zeta) \quad \text{for some} \quad d_{k_1}g(\zeta) \in k\text{-}\mathcal{ST}(\gamma). \tag{9}$$

Letting
$$h(\zeta) = \frac{\zeta\left(d_{k_1+1}t(\zeta)\right)'}{d_{k_1+1}g(\zeta)} \quad \text{and} \quad \mathcal{H}(\zeta) = \frac{\zeta\left(d_{k_1+1}g(\zeta)\right)'}{d_{k_1+1}g(\zeta)}.$$

It could be seen that both functions $h(\zeta)$ and $\mathcal{H}(\zeta)$ are analytic in E with the following condition:
$$h(0) = \mathcal{H}(0) = 1.$$

The following relation now holds true according to Theorem 2:
$$d_{k_1+1}g(\zeta) \in k\text{-}\mathcal{ST}(\gamma) \quad \text{and} \quad \Re(\mathcal{H}(\zeta)) > \frac{k+\gamma}{k+1}.$$

We observed that
$$\zeta\left(d_{k_1+1}t(\zeta)\right)' = h(\zeta)\left(d_{k_1+1}g(\zeta)\right). \tag{10}$$

Taking the differentiation of (10), we obtain
$$\begin{aligned} \frac{\zeta\left(\zeta\left(d_{k_1+1}t(\zeta)\right)'\right)'}{d_{k_1+1}g(\zeta)} &= \zeta\frac{\left(d_{k_1+1}g(\zeta)\right)'}{d_{k_1+1}g(\zeta)}h(\zeta) + \zeta h'(\zeta) \\ &= \mathcal{H}(\zeta)h(\zeta) + \zeta h'(\zeta) \end{aligned} \tag{11}$$

By using (8), we obtain

$$\begin{aligned}\frac{\zeta\left(d_{k_1}t(\zeta)\right)'}{d_{k_1}g(\zeta)} &= \frac{d_{k_1}\zeta t'(\zeta)}{d_{k_1}g(\zeta)} \\ &= \frac{\zeta\left(d_{k_1+1}\zeta t'(\zeta)\right)' + (k_1-1)d_{k_1+1}(\zeta t'(\zeta))}{\zeta\left(d_{k_1+1}g(\zeta)\right)' + (k_1-1)d_{k_1+1}g(\zeta)} \\ &= \frac{\frac{\zeta\left(d_{k_1+c}\zeta t'(\zeta)\right)'}{d_{k_1+1}g(\zeta)} + (k_1-1)\frac{d_{k_1+c}(\zeta t'(\zeta))}{d_{k_1+1}g(\zeta)}}{\frac{\zeta\left(d_{k_1+1}g(\zeta)\right)'}{d_{k_1+1}g(\zeta)} + (k_1-1)} \\ &= h(\zeta) + \frac{\zeta h'(\zeta)}{\mathcal{H}(\zeta)+k_1-1}. \end{aligned} \tag{12}$$

From (9), (11) and (12), we have

$$h(\zeta) + \frac{\zeta h'(\zeta)}{\mathcal{H}(\zeta)+k_1-1} \prec Q_{k,\gamma}(\zeta)$$

By taking $B(\zeta) = \frac{1}{\mathcal{H}(\zeta)+k_1-1}$ and $U = 0$, we obtain

$$\Re(B(\zeta)) = \frac{\Re(\mathcal{H}(\zeta)+k_1-1)}{|\mathcal{H}(\zeta)+k_1-1|^2} > 0$$

By making use of Lemma 2, we have

$$h(\zeta) \prec Q_{k,\gamma}(\zeta).$$

Then, by Definition 3, we get

$$d_{k_1+1}t(\zeta) \in k\text{-}\mathcal{UCC}(\gamma).$$

Thus, we have now completed the proof of our Theorem 4. □

If we put $k = 0$ in the above Theorem, we get the following Corollary.

Corollary 1. *Let $t \in \mathcal{A}$. If $d_{k_1}t(\zeta) \in 0\text{-}\mathcal{UCC}(\gamma)$, then $d_{k_1+1}t(\zeta) \in 0\text{-}\mathcal{UCC}(\gamma)$.*

The next result (Theorem 5) can be proved similarly as we proved Theorem 4; therefore, we have chosen to omit the details involved in the proof of Theorem 5.

Theorem 5. *Let $t \in \mathcal{A}$. If $d_{k_1}t(\zeta) \in k\text{-}\mathcal{UQC}(\gamma)$, then $d_{k_1+1}t(\zeta) \in k\text{-}\mathcal{UQC}(\gamma)$.*

In the next result, for the generalized Bernardi-Libera-Livingston integral operator L_λ, certain closure properties are investigated.

Theorem 6. *Let $\lambda > -\frac{k+\gamma}{k+1}$. If $d_{k_1} \in k\text{-}\mathcal{ST}(\gamma)$, then $\mathcal{L}_\lambda\left(d_{k_1}\right) \in k\text{-}\mathcal{ST}(\gamma)$.*

Proof. It could be seen that the following relation is due to applying the definition of $\mathcal{L}_\lambda(t)$ and the linearity property of the operator d_{k_1}

$$\zeta\left(d_{k_1}\mathcal{L}_\lambda t(\zeta)\right)' = (\lambda+1)d_{k_1}t(\zeta) - \lambda d_{k_1}\mathcal{L}_\lambda t(\zeta). \tag{13}$$

Substituting

$$\frac{\zeta\left(d_{k_1}\mathcal{L}_\lambda t(\zeta)\right)'}{d_{k_1}\mathcal{L}_\lambda t(\zeta)} = p(\zeta)$$

in (13), we have

$$p(\zeta) = (\lambda+1)\frac{d_{k_1}t(\zeta)}{d_{k_1}\mathcal{L}_\lambda t(\zeta)} - \lambda. \tag{14}$$

p is analytic in E with $p(0) = 1$.
On differentiating (14), we get

$$\frac{\zeta\left(d_{k_1}(t(\zeta))\right)'}{d_{k_1}t(\zeta)} = \frac{\zeta\left(d_{k_1}\mathcal{L}_\lambda t(\zeta)\right)'}{d_{k_1}\mathcal{L}_\lambda t(\zeta)} + \frac{\zeta p'(\zeta)}{p(\zeta)+\lambda} = p(\zeta) + \frac{\zeta p'(\zeta)}{p(\zeta)+\lambda}.$$

By using Lemma 1, we have

$$p(\zeta) \prec Q_{k,\gamma}(\zeta),$$

since

$$(Q_{k,\gamma}(\zeta) + \lambda) > 0.$$

Hence, the proof of Theorem 6 is completed. □

If we put $k = 0$, in the above Theorem, we get the following Corollary.

Corollary 2. *Let $\lambda > -\gamma$. If $d_{k_1} \in 0\text{-}\mathcal{ST}(\gamma)$, then $\mathcal{L}_\lambda\left(d_{k_1}\right) \in 0\text{-}\mathcal{ST}(\gamma)$.*

The following Theorem (Theorem 7) can be proved by using argumenty similar to those that are used in the proof of Theorem 6. Therefore, we choose to omit the details involved.

Theorem 7. *Let $\lambda > -\frac{k+\gamma}{k+1}$. If $d_{k_1} \in k\text{-}\mathcal{UCV}(\gamma)$, then $\mathcal{L}_\lambda\left(d_{k_1}\right) \in k\text{-}\mathcal{UCV}(\gamma)$.*

Theorem 8. *Let $\lambda > -\frac{k+\gamma}{k+1}$. If $d_{k_1} \in k\text{-}\mathcal{UCC}(\gamma)$, then $\mathcal{L}_\lambda\left(d_{k_1}\right) \in k\text{-}\mathcal{UCC}(\gamma)$.*

Proof. By definition, we have

$$\frac{\zeta\left(d_{k_1}(t(\zeta))\right)'}{d_{k_1}g(\zeta)} \prec Q_{k,\gamma}(\zeta) \quad \text{for some} \quad d_{k_1}g(\zeta) \in k\text{-}\mathcal{ST}(\gamma). \tag{15}$$

Now, from (13), we have

$$\begin{aligned} \frac{\zeta\left(d_{k_1}t(\zeta)\right)'}{d_{k_1}g(\zeta)} &= \frac{\zeta\left(d_{k_1}\mathcal{L}_\lambda(\zeta t'(\zeta))\right)' + \lambda d_{k_1}\mathcal{L}_\lambda(\zeta t'(\zeta))}{\zeta\left(d_{k_1}\mathcal{L}_\lambda(g(\zeta))\right)' + \lambda d_{k_1}\mathcal{L}_\lambda(g(\zeta))} \\ &= \frac{\frac{\zeta\left(d_{k_1}(\zeta t'(\zeta))\right)'}{d_{k_1}\mathcal{L}_\lambda(g(\zeta))} + \frac{\lambda d_{k_1}(\zeta t'(\zeta))}{d_{k_1}\mathcal{L}_\lambda(g(\zeta))}}{\frac{\zeta\left(d_{k_1}\mathcal{L}_\lambda(g(\zeta))\right)'}{d_{k_1}\mathcal{L}_\lambda(g(\zeta))} + \lambda}. \end{aligned} \tag{16}$$

Since $d_{k_1}g(\zeta) \in k\text{-}\mathcal{ST}(\gamma)$, by Theorem 6, we have $\mathcal{L}_\lambda d_{k_1}g(\zeta) \in k\text{-}\mathcal{ST}(\gamma)$.
Let

$$\frac{\zeta\left(d_{k_1}\mathcal{L}_\lambda(g(\zeta))\right)'}{d_{k_1}\mathcal{L}_\lambda(g(\zeta))} = \mathcal{H}(\zeta)$$

and

$$\Re(\mathcal{H}(\zeta)) > \frac{k+\gamma}{k+1}.$$

Now,

$$h(\zeta) = \frac{\zeta\left(d_{k_1}\mathcal{L}_\lambda(t(\zeta))\right)'}{d_{k_1}\mathcal{L}_\lambda(g(\zeta))}.$$

We now have that

$$\zeta\left(d_{k_1}\mathcal{L}_\lambda(t(\zeta))\right)' = h(\zeta)d_{k_1}\mathcal{L}_\lambda(g(\zeta)). \tag{17}$$

Taking the differentiation of both sides of Equation (17), we have

$$\begin{aligned}\frac{\zeta\Big(d_{k_1}(\zeta\mathcal{L}_\lambda(t(\zeta)))'\Big)'}{d_{k_1}\mathcal{L}_\lambda(g(\zeta))} &= \zeta h'(\zeta) + h(\zeta)\frac{\zeta\left(d_{k_1}\mathcal{L}_\lambda(g(\zeta))\right)'}{d_{k_1}\mathcal{L}_\lambda(g(\zeta))} \\ &= \zeta h'(\zeta) + h(\zeta)\mathcal{H}(\zeta).\end{aligned} \tag{18}$$

Thus, from (16) and (18), we arrive at the following:

$$\frac{\zeta\left(d_{k_1}t(\zeta)\right)'}{d_{k_1}g(\zeta)} = \frac{\zeta h'(\zeta) + h(\zeta)\mathcal{H}(\zeta) + \lambda h(\zeta)}{\mathcal{H}(\zeta) + \lambda}.$$

By using the above equation along with (15), we have

$$h(\zeta) + \frac{\zeta h'(\zeta)}{\mathcal{H}(\zeta) + \lambda} \prec Q_{k,\gamma}(\zeta). \tag{19}$$

By supposing $U = 0$ along with the following

$$B(\zeta) = \frac{1}{\mathcal{H}(\zeta) + \lambda} \quad \text{and} \quad \Re(B(\zeta)) > 0 \quad \text{as} \quad \lambda > -\frac{k+\gamma}{k+1}.$$

Finally, by making use of Lemma 2, we have our desired result. □

If we put $k = 0$, in the above Theorem, we have the following Corollary.

Corollary 3. *Let $\lambda > -\gamma$. If $d_{k_1} \in 0\text{-}\mathcal{UCC}(\gamma)$, then $\mathcal{L}_\lambda\left(d_{k_1}\right) \in 0\text{-}\mathcal{UCC}(\gamma)$.*

The following Theorem (Theorem 9) can be proved by using arguments similar to those that are used in the proof of Theorem 6. Therefore, we choose to omit the details involved.

Theorem 9. *Let $\lambda > -\frac{k+\gamma}{k+1}$. If $d_{k_1} \in k\text{-}\mathcal{UQC}(\gamma)$, then $\mathcal{L}_\lambda\left(d_{k_1}\right) \in k\text{-}\mathcal{UQC}(\gamma)$.*

4. Concluding Remarks and Observations

In our present investigation, we have studied some remarkable subclasses of analytic functions involving the Dini functions in conic domains. We have derived certain inclusion type results and the integral preserving properties for our defined function classes. We have also applied the well-known integral operator *Bernardi-Libera-Livingston* and have discussed some interesting applications of our main results.

Moreover, we recall the attention of curious readers to the prospect of studying the q-generalizations of the results reported in this paper, influenced by a newly published survey-cum-expository review article by Srivastava [17]; see also [3,16,37–39]. Furthermore, based on the suggested extension, the $(\mathfrak{p}, q)$-extension will be a relatively minor and unimportant change, as the new parameter $\mathfrak{p}$ is redundant (see, for details, Srivastava [17], p. 340). Furthermore, in light of Srivastava's recent result [5], the interested reader's attention is brought to further investigation into the (k, s)-extension of the *Riemann-Liouville fractional integral*.

Author Contributions: Conceptualization, S.K.; methodology, S.K., J.-S.R., S.A., N.K. (Nazar Khan) and N.K. (Nasir Khan); validation, B.K., J.-S.R., S.A., N.K. (Nazar Khan) and N.K. (Nasir Khan); formal analysis, B.K., S.K., J.-S.R., S.A. and N.K. (Nasir Khan); investigation, S.K. and N.K. (Nazar Khan); writing—original draft preparation, B.K., S.K., J.-S.R., S.A. and N.K. (Nazar Khan); writing—review and editing, B.K., J.-S.R., S.A., N.K. (Nazar Khan) and N.K. (Nasir Khan); funding acquisition, J.-S.R. All authors have read and agreed to the published version of the manuscript.

Funding: This research was supported by (1) Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education (2016R1D1A1B01008058). (2) Competency Development Program for Industry Specialists of the Korean Ministry of Trade, Industry and Energy (MOTIE), operated by the Korea Institute for Advancement of Technology (KIAT) (No. P0002397, HRD program for Industrial Convergence of Wearable Smart Devices).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to thank the reviewers for their valuable suggestions and comments, which have improved the presentation of the paper substantially.

Conflicts of Interest: The authors declare that they have no competing interest.

References

1. Din, M.U.; Raza, M.; Yagmur, N.; Malik, S.N. On partial sums of Wright functions. *UPB Sci. Bull. Ser. A* **2018**, *80*, 79–90.
2. Baricz, Á. Bessel transforms and Hardy space of generalized Bessel functions. *Mathematica* **2006**, *48*, 127–136.
3. Raza, M.; Din, M.U.; Malik, S.N. Certain Geometric Properties of Normalized Wright Functions. *J. Funct. Spaces* **2016**, *2016*, 1–8. [CrossRef]
4. Connor, J.J.O. *Friedrich Wilhelm Bessel*; School of Mathematics and Statistics University of St Andrews: St. Andrews, UK, 1997.
5. Srivastava, H.M. Some parametric and argument variations of the operators of fractional calculus and related special functions and integral transformatioons. *J. Nonlinear Convex Anal.* **2021**, *22*, 1501–1520.
6. Baricz, Á.; Deniz, E.; Yağmur, N. Close-to-convexity of normalized Dini functions. *Math. Nachr.* **2016**, *289*, 1721–1726. [CrossRef]
7. Robertson, M.I.S. On the Theory of Univalent Functions. *Ann. Math.* **1936**, *37*, 374. [CrossRef]
8. Srivastava, H.M.; Owa, S. (Eds.) *Current Topics in Analytic Function Theory*; World Scientific: River Edge, NJ, USA, 1992.
9. Libera, R.J. Some radius of convexity problems. *Duke Math. J.* **1964**, *31*, 143–158. [CrossRef]
10. Kanas, S. Alternative characterization of the class k-$\mathcal{UCV}$ and related classes of univalent functions. *Serdica Math. J.* **1999**, *25*, 341–350.
11. Kanas, S. Techniques of the differential subordination for domains bounded by conic sections. *Int. J. Math. Math. Sci.* **2003**, *2003*, 2389–2400. [CrossRef]
12. Kanas, S. Differential subordination related to conic sections. *J. Math. Anal. Appl.* **2006**, *317*, 650–658. [CrossRef]
13. Kanas, S. Subordination for domains bounded by conic sections. *Bull. Belg. Math. Soc. Simon Stevin* **2008**, *15*, 589–598. [CrossRef]
14. Kanas, S.; Srivastava, H.M. Linear operators associated with k-uniform convex functions. *Integral Transform. Spec. Funct.* **2000**, *9*, 121–132. [CrossRef]
15. Kanas, S.; Wisniowska, A. Conic regions and k-starlike function. *Rev. Roum. Math. Pures Appl.* **2000**, *45*, 647–657.
16. Khan, B.; Srivastava, H.M.; Tahir, M.; Darus, M.; Ahmad, Q.Z.; Khan, N. Applications of a certain q-integral operator to the subclasses of analytic and bi-univalent functions. *AIMS Math.* **2021**, *6*, 1024–1039. [CrossRef]
17. Srivastava, H.M. Operators of Basic (or q-) Calculus and Fractional q-Calculus and Their Applications in Geometric Function Theory of Complex Analysis. *Iran. J. Sci. Technol. Trans. A Sci.* **2020**, *44*, 327–344. [CrossRef]
18. Srivastava, H.M.; Arif, M.; Raza, M. Convolution properties of meromorphically harmonic functions defined by a generalized convolution q-derivative operator. *AIMS Math.* **2021**, *6*, 5869–5885. [CrossRef]
19. Shams, S.; Kulkarni, S.R.; Jahangiri, J.M. Classes of uniformly starlike and convex functions. *Int. J. Math. Math. Sci.* **2004**, *2004*, 2959–2961. [CrossRef]
20. Al-Kharsani, H.A.; Al-Hajiry, S.S. Subordination results for the family of uniformly convex p-valent functions. *J. Inequal. Pure Appl. Math.* **2006**, *7*, 1–9.
21. Kanas, S.; Wisniowska, A. Conic regions and k-uniform convexity. *J. Comput. Appl. Math.* **1999**, *105*, 327–336. doi:10.1016/s0377-0427(99)00018-7. [CrossRef]
22. Goodman, W. On uniformly starlike functions. *J. Math. Anal. Appl.* **1991**, *155*, 364–370. [CrossRef]
23. Ma, W.C.; Minda, D. Uniformly convex functions. *Ann. Pol. Math.* **1992**, *57*, 165–175. [CrossRef]
24. Rønning, F. A survey on uniformly convex and uniformly starlike functions. *Ann. Univ. Mariae Curie-Skłodowska Sect. A* **1993**, *47*, 123–134.

25. Khan, Q.; Arif, M.; Raza, M.; Srivastava, G.; Tang, H.; Rehman, S.U. Some Applications of a New Integral Operator in q-Analog for Multivalent Functions. *Mathematics* **2019**, *7*, 1178. [CrossRef]
26. Khan, B.; Srivastava, H.M.; Arjika, S.; Khan, S.; Khan, N.; Ahmad, Q.Z. A certain q-Ruscheweyh type derivative operator and its applications involving multivalent functions. *Adv. Differ. Equ.* **2021**, *2021*, 1–14. [CrossRef]
27. Khan, B.; Srivastava, H.; Khan, N.; Darus, M.; Tahir, M.; Ahmad, Q.Z. Coefficient Estimates for a Subclass of Analytic Functions Associated with a Certain Leaf-Like Domain. *Mathematics* **2020**, *8*, 1334. [CrossRef]
28. Khan, B.; Liu, Z.-G.; Srivastava, H.; Khan, N.; Darus, M.; Tahir, M. A Study of Some Families of Multivalent q-Starlike Functions Involving Higher-Order q-Derivatives. *Mathematics* **2020**, *8*, 1470. [CrossRef]
29. Khan, B.; Srivastava, H.; Khan, N.; Darus, M.; Ahmad, Q.Z.; Tahir, M. Applications of Certain Conic Domains to a Subclass of q-Starlike Functions Associated with the Janowski Functions. *Symmetry* **2021**, *13*, 574. [CrossRef]
30. Shi, L.; Khan, Q.; Srivastava, G.; Liu, J.-L.; Arif, M. A Study of Multivalent q-starlike Functions Connected with Circular Domain. *Mathematics* **2019**, *7*, 670. [CrossRef]
31. Shi, L.; Ahmad, B.; Khan, N.; Khan, M.G.; Araci, S.; Mashwani, W.K.; Khan, B. Coefficient Estimates for a subclass of meromorphic multivalent q-Close-to-convex functions. *Symmetry* **2021**, *13*, 1840. [CrossRef]
32. Shi, L.; Khan, M.G.; Ahmad, B.; Mashwani, W.K.; Agarwal, P.; Momani, S. Certain coefficient estimates problems for three-leaf-type starlike functions. *Fractal Fract.* **2021**, *5*, 137. [CrossRef]
33. Hu, Q.; Srivastava, H.M.; Ahmad, B.; Khan, N.; Khan, M.G.; Mashwani, W.K.; Khan, B. A subclass of multivalent Janowski type q-starlike functions and its consequences. *Symmetry* **2021**, *13*, 1275. [CrossRef]
34. Bernardi, S.D. Convex and starlike univalent functions. *Trans. Am. Math. Soc.* **1969**, *135*, 429–446. [CrossRef]
35. Eeinigenburg, P.; Miller, S.S.; Mocanu, P.T.; Reade, M.D. General inequalities. *Birkhäuser* **1983**, *64*, 339–348.
36. Miller, S.S.; Mocanu, P.T. Differential subordinations and inequalities in the complex plane. *J. Differ. Equ.* **1987**, *67*, 199–211. [CrossRef]
37. Ahmad, Q.Z.; Khan, N.; Raza, M.; Tahir, M.; Khan, B. Certain q-difference operators and their applications to the subclass of meromorphic q-starlike functions. *Filomat* **2019**, *33*, 3385–3397. [CrossRef]
38. Malik, S.N.; Mahmood, S.; Raza, M.; Farman, S.; Zainab, S. Coefficient Inequalities of Functions Associated with Petal Type Domains. *Mathematics* **2018**, *6*, 298. [CrossRef]
39. Ul-Haq, M.; Raza, M.; Arif, M.; Khan, Q.; Tang, H.; Ul-Haq, M. q-Analogue of Differential Subordinations. *Mathematics* **2019**, *7*, 724. [CrossRef]