



Article

Hermite-Hadamard Fractional Inequalities for Differentiable Functions

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Abstract: In this article, we look at a variety of mean-type integral inequalities for a well-known Hilfer fractional derivative. We consider twice differentiable convex and s -convex functions for $s \in (0, 1]$ that have applications in optimization theory. In order to infer more interesting mean inequalities, some identities are also established. The consequences for Caputo fractional derivative are presented as special cases to our general conclusions.

Keywords: Hermite-Hadamard-type inequalities; Hilfer fractional derivative; Hölder's inequality

MSC: 26D15; 26D07; 26D10; 26A33; 26A51



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1. Introduction

The subject of fractional calculus has achieved a significant prominence during the most recent couple of years due to its demonstrated applications in the field of science and engineering. This offers useful strategies to solve differential and integral equations, see the books [1,2] and articles [3–5]. Fractional calculus has been applied in different areas of science, engineering, financial mathematics, applied sciences, bio engineering, etc.

Mathematical inequalities are significantly important in the study of mathematics and related fields. Nowadays, fractional integral inequalities are fruitful in generating the uniqueness of solutions for fractional partial differential equations. They also provide boundedness of the solutions of fractional boundary value problems. These recommendations have inspired various researchers in the field of integral inequalities to inquire the extensions by involving fractional calculus operators. Recently, Peter Korus presented a class of Hermite-Hadamard inequalities by considering the class of convex or generalized convex derivative in [6], Farid et al. explored Fejér-Hadamard type inequalities [7] for $(\alpha, h - m) - p$ -convex functions by involving the fractional operators. We further refer the reader to, e.g., [8–10].

The convex functions are utilized to create numerous inequalities like Alomari et al. [11] present Ostrowski's inequalities via s convexity in second sense, Dragomir et al. discuss some properties of convex functions in [12] and explored some important quadrature rules in [13]. More applications can be observed from literature [14–16] on convex functions and inequalities. Hermite-Hadamard's inequality [17] is one of the most important classical inequalities, as it has a rich geometrical meaning and applications [18–20]. Hermite-Hadamard's double inequality is one of the most widely studied concerning convex functions. The inequality is defined as follows:

Let $\psi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex mapping and $\theta, \zeta \in I$ with $\theta < \zeta$. Then,

$$\psi\left(\frac{\theta + \zeta}{2}\right) \leq \frac{1}{\zeta - \theta} \int_{\theta}^{\zeta} \psi(v) dv \leq \frac{\psi(\theta) + \psi(\zeta)}{2}. \quad (1)$$

If ψ is concave, then the inequalities (1) hold in reverse direction. For particular choices of function ψ , some classical inequalities for means can be derived from (1) (see [21]). The principle point of this paper is to infer Hermite-Hadamard-type integral inequalities for Hilfer fractional derivative. Such inequalities were proved by many scientists for different convexities and for many fractional operators, but the main results of this paper are more general than the existing literature.

2. Preliminaries

In this section, we recall some basic preliminary results.

Definition 1 ([22]). Let $\psi : [\theta, \zeta] \rightarrow \mathbb{R}$ is said to be convex if the inequality

$$\psi(v\gamma + (1-v)\beta) \leq v\psi(\gamma) + (1-v)\psi(\beta),$$

holds for $\gamma, \beta \in [\theta, \zeta]$ and $v \in [0, 1]$.

The definition of classical Riemann–Liouville fractional derivative (see [23] (Chapter 4)) is given as follows.

Definition 2. Let $\Phi \in L^1[\theta, \zeta]$, then the right-sided and left-sided Riemann–Liouville fractional derivative of order $\alpha > 0$ are defined by

$$D_{\theta+}^{\gamma} \psi(v) = \frac{1}{\Gamma(n-\gamma)} \left(\frac{d}{dv}\right)^n \int_{\theta}^v (v-\tau)^{n-\gamma-1} \psi(\tau) d\tau,$$

and

$$D_{\zeta-}^{\gamma} \psi(v) = \frac{1}{\Gamma(n-\gamma)} \left(\frac{d}{dv}\right)^n \int_v^{\zeta} (\tau-v)^{n-\gamma-1} \psi(\tau) d\tau,$$

where $n = [\gamma] + 1$, $v \in [\theta, \zeta]$.

Let $x > \theta > 0$ and $L^1(\theta, x)$, denote the space of all Lebesgue integrable functions on the interval (θ, x) . Then, for any $\psi \in L^1(\theta, x)$ the Riemann–Liouville fractional integral of order γ is defined by

$$(I_{\theta+}^{\gamma} \psi)(v) = \frac{1}{\Gamma(\gamma)} \int_{\theta}^v (v-\tau)^{\gamma-1} \psi(\tau) d\tau = (\psi * K_{\gamma})(v), \quad v \in [\theta, x], \quad (\gamma > 0), \quad (2)$$

where $K_{\gamma}(v) = \frac{v^{\gamma-1}}{\Gamma(\gamma)}$. The integral on the right side of (2) exists for almost $v \in [\theta, x]$ and $I_{\theta+}^{\gamma} \psi \in L^1(\theta, x)$.

Throughout this paper, the space of all continuous differentiable functions up to order m , on $[\theta, x]$ is presented by $C^m[\theta, x]$. By $AC[\theta, x]$, we mean the space of all absolutely continuous functions on $[\theta, x]$ and the space $AC^m[\theta, x]$, denote the space of all such functions $\psi \in C^m[\theta, x]$ with $\psi^{(m-1)} \in AC[\theta, x]$. By $L_{\infty}(\theta, x)$, we denote the space of all measurable functions essentially bounded on $[\theta, x]$. Let $\mu > 0$, $m = [\mu] + 1$ and $f \in AC^m[a, b]$. The Caputo derivative of order $\gamma > 0$ is defined as

$$({}^C D_{\theta+}^{\gamma} \psi)(v) = \left(I_{\theta+}^{m-\gamma} \frac{d^m}{dv^m} \psi\right)(v) = \frac{1}{\Gamma(m-\gamma)} \int_{\theta}^v (v-\tau)^{m-\gamma-1} \frac{d^m}{dv^m} \psi(\tau) d\tau.$$

Definition 3 ([24]). Let $\psi \in L^1[\theta, \zeta]$, $\psi * K_{(1-\beta)(1-\gamma)} \in AC^1[\theta, \zeta]$. The fractional derivative operator $D_{\theta+}^{\gamma, \beta}$ of order $0 < \gamma < 1$ and type $0 < \beta \leq 1$ with respect to $v \in [\theta, \zeta]$ is defined by

$$\left(D_{\theta+}^{\gamma, \beta} \psi\right)(v) := I_{\theta+}^{\beta(1-\gamma)} \frac{d}{dv} \left(I_{\theta+}^{(1-\beta)(1-\gamma)} \psi(v)\right). \quad (3)$$

The derivative (3) is usually called Hilfer fractional derivative.

The more general integral representation of Equation (3) given in [24] is defined as follows:

Let $\psi \in L^1[\theta, \zeta]$, $\psi * K_{(1-\beta)(n-\gamma)} \in AC^n[\theta, \zeta]$, $n-1 < \gamma < n$, $0 < \beta \leq 1$, $n \in \mathbb{N}$. Then,

$$\left(D_{\theta+}^{\gamma, \beta} \psi\right)(v) = \left(I_{\theta+}^{\beta(n-\gamma)} \frac{d^n}{dv^n} \left(I_{\theta+}^{(1-\beta)(n-\gamma)} \psi(v)\right)\right), \quad (4)$$

which coincide with (3) for $n = 1$.

Specially for $\beta = 0$, $D_{\theta+}^{\gamma, 0} \psi = D_{\theta+}^{\gamma} \psi$ is Riemann–Liouville fractional derivative of order γ and for $\beta = 1$ it is Caputo fractional derivative $D_{\theta+}^{\gamma, 1} \psi = {}^C D_{\theta+}^{\gamma} \psi$ of order γ . Applying the properties of Riemann–Liouville integral the relation (4) can be rewritten in the form

$$\begin{aligned} \left(D_{\theta+}^{\gamma, \beta} \psi\right)(v) &= \left(I_{\theta+}^{\beta(n-\gamma)} \left(D_{\theta+}^{n-(1-\beta)(n-\gamma)} \psi\right)(v)\right) \\ &= \frac{1}{\Gamma(\beta(n-\gamma))} \int_{\theta}^v (v-\tau)^{\beta(n-\gamma)-1} \left(D_{\theta+}^{\gamma+\beta(n-\gamma)} \psi\right)(\tau) d\tau. \end{aligned} \quad (5)$$

The geometric arithmetically s -convex function given in [25] presented in the following definition.

Definition 4. Let $\psi : I \subset \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and $s \in (0, 1]$. A function ψ is geometric-arithmetically s -convex function on I if for every $\gamma, \beta \in I$ and $v \in [0, 1]$, we have

$$\psi(\gamma^v \beta^{1-v}) \leq v^s (\psi(\gamma)) + (1-v)^s \psi(\beta).$$

The following lemma was given by Liao et al. [25].

Lemma 1. For $\theta \in [0, 1]$, $\gamma, \beta > 0$, we have

$$\theta\gamma + (1-\theta)\beta \geq \beta^{1-\theta} \gamma^{\theta}.$$

Deng et al. [26] prove the following lemma.

Lemma 2. For $\theta \in [0, 1]$, we have

$$\begin{aligned} (1-\theta)^{\gamma} &\leq 2^{1-\gamma} - \theta^{\gamma}, \quad \gamma \in [0, 1], \\ (1-\theta)^{\gamma} &\geq 2^{1-\gamma} - \theta^{\gamma}, \quad \gamma \in [1, \infty). \end{aligned}$$

3. Main Results

This section includes several mean-type fractional integral inequalities involving Hilfer fractional derivative. The first main result for the fractional derivative is presented in the following theorem.

Theorem 1. Let $\psi \in L^1[\theta, \zeta]$, $\psi * K_{(1-\beta)(n-\gamma)} \in AC^n[\theta, \zeta]$, $n \in \mathbb{N}$ and $D_{(\theta, \zeta)}^{\gamma+\beta(n-\gamma)} \psi : [\theta, \zeta] \rightarrow \mathbb{R}$ be a positive function with $0 \leq \theta < \zeta$, $n-1 < \gamma < n$, $0 < \beta \leq 1$ and $D_{(\theta, \zeta)}^{\gamma+\beta(n-\gamma)} \psi \in$

$L^1[\theta, \zeta]$. If $D_{(\theta, \zeta)}^{\gamma+\beta(n-\gamma)}\psi$ is convex function on $[\theta, \zeta]$, then the following inequality for fractional derivative holds.

$$\begin{aligned} & D_{(\theta, \zeta)}^{\gamma+\beta(n-\gamma)}\Phi\left(\frac{\theta+\zeta}{2}\right) \\ & \leq \frac{\Gamma(\beta(n-\gamma)+1)}{(\zeta-\theta)^{\beta(n-\gamma)}} \left[D_{\theta+}^{\gamma, \beta}\Phi(\zeta) + D_{\zeta-}^{\gamma, \beta}\Phi(\theta) \right] \\ & \leq D_{\theta+}^{\gamma+\beta(n-\gamma)}\Phi(\zeta) + D_{\zeta-}^{\gamma+\beta(n-\gamma)}\Phi(\theta). \end{aligned} \quad (6)$$

Proof. We define functions $\tilde{\psi}(\nu) = \psi(\theta + \zeta - \nu)$, $\nu \in [\theta, \zeta]$ and $\Phi(\nu) = \psi(\nu) + \tilde{\psi}(\nu)$, $\nu \in [\theta, \zeta]$. Since $D_{\theta+}^{\gamma+\beta(n-\gamma)}\psi$ is convex on $[\theta, \zeta]$, therefore with $\mu = \frac{1}{2}$, we have

$$D_{\theta+}^{\gamma+\beta(n-\gamma)}\psi\left(\frac{x+y}{2}\right) \leq \frac{D_{\theta+}^{\gamma+\beta(n-\gamma)}\psi(x) + D_{\theta+}^{\gamma+\beta(n-\gamma)}\psi(y)}{2}.$$

Choosing $x = \nu\theta + (1-\nu)\zeta$ and $y = (1-\nu)\theta + \nu\zeta$, we get

$$\begin{aligned} & 2D_{\theta+}^{\gamma+\beta(n-\gamma)}\psi\left(\frac{\theta+\zeta}{2}\right) \\ & \leq D_{\theta+}^{\gamma+\beta(n-\gamma)}\psi(\nu\theta + (1-\nu)\zeta) + D_{\theta+}^{\gamma+\beta(n-\gamma)}\psi((1-\nu)\theta + \nu\zeta) \\ & = D_{\theta+}^{\gamma+\beta(n-\gamma)}\Phi(\nu\theta + (1-\nu)\zeta). \end{aligned}$$

Now, we multiply both sides of above inequality by $\nu^{\beta(n-\gamma)-1}$ and then integrating the resulting inequality with respect to ν over $[0, 1]$, we have

$$\frac{1}{\beta(n-\gamma)} D_{\theta+}^{\gamma+\beta(n-\gamma)}\Phi\left(\frac{\theta+\zeta}{2}\right) \leq \int_0^1 \nu^{\beta(n-\gamma)-1} D_{\theta+}^{\gamma+\beta(n-\gamma)}\Phi(\nu\theta + (1-\nu)\zeta) d\nu. \quad (7)$$

By substituting $u = \nu\theta + (1-\nu)\zeta$, the inequality (7) becomes

$$D_{\theta+}^{\gamma+\beta(n-\gamma)}\Phi\left(\frac{\theta+\zeta}{2}\right) \leq \frac{\Gamma(\beta(n-\gamma)+1)}{(\zeta-\theta)^{\beta(n-\gamma)}} D_{\theta+}^{\gamma, \beta}\Phi(\zeta). \quad (8)$$

Similarly, for the choice

$$D_{\zeta-}^{\gamma+\beta(n-\gamma)}\psi\left(\frac{x+y}{2}\right) \leq \frac{D_{\zeta-}^{\gamma+\beta(n-\gamma)}\psi(x) + D_{\zeta-}^{\gamma+\beta(n-\gamma)}\psi(y)}{2},$$

we get

$$D_{\zeta-}^{\gamma+\beta(n-\gamma)}\Phi\left(\frac{\theta+\zeta}{2}\right) \leq \frac{\Gamma(\beta(n-\gamma)+1)}{(\zeta-\theta)^{\beta(n-\gamma)}} D_{\zeta-}^{\gamma, \beta}\Phi(\theta). \quad (9)$$

By adding (8) and (9), we obtain

$$\begin{aligned} & D_{\theta+}^{\gamma+\beta(n-\gamma)}\Phi\left(\frac{\theta+\zeta}{2}\right) + D_{\zeta-}^{\gamma+\beta(n-\gamma)}\Phi\left(\frac{\theta+\zeta}{2}\right) \\ & \leq \frac{\Gamma(\beta(n-\gamma)+1)}{(\zeta-\theta)^{\beta(n-\gamma)}} \left[D_{\theta+}^{\gamma, \beta}\Phi(\zeta) + D_{\zeta-}^{\gamma, \beta}\Phi(\theta) \right], \end{aligned} \quad (10)$$

which proves the left half part of inequality (6).

For the proof of the second half, we first note that if $D_{\theta^+}^{\gamma+\beta(n-\gamma)}\psi$ is convex, then for $\nu \in [0, 1]$, yields

$$\begin{aligned} D_{\theta^+}^{\gamma+\beta(n-\gamma)}\psi(\nu\theta + (1-\nu)\zeta) &\leq \nu D_{\theta^+}^{\gamma+\beta(n-\gamma)}\tilde{\psi}(\zeta) + (1-\nu)D_{\theta^+}^{\gamma+\beta(n-\gamma)}\psi(\zeta) \\ D_{\theta^+}^{\gamma+\beta(n-\gamma)}\psi((1-\nu)\theta + \nu\zeta) &\leq (1-\nu)D_{\theta^+}^{\gamma+\beta(n-\gamma)}\tilde{\psi}(\zeta) + \nu D_{\theta^+}^{\gamma+\beta(n-\gamma)}\psi(\zeta). \end{aligned}$$

By adding above two inequalities, we have

$$D_{\theta^+}^{\gamma+\beta(n-\gamma)}\Phi(\nu\theta + (1-\nu)\zeta) \leq D_{\theta^+}^{\gamma+\beta(n-\gamma)}\Phi(\zeta). \quad (11)$$

Similarly,

$$D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\Phi((1-\nu)\theta + \nu\zeta) \leq D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\Phi(\theta). \quad (12)$$

From (11) and (12), we get

$$\begin{aligned} D_{\theta^+}^{\gamma+\beta(n-\gamma)}\Phi(\nu\theta + (1-\nu)\zeta) + D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\Phi((1-\nu)\theta + \nu\zeta) \\ \leq D_{\theta^+}^{\gamma+\beta(n-\gamma)}\Phi(\zeta) + D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\Phi(\theta). \end{aligned} \quad (13)$$

Now, first, we multiply both sides of (13) by $\nu^{\beta(n-\gamma)-1}$, and then we integrate the resulting inequality with respect to ν over $[0, 1]$, we have

$$\begin{aligned} &\int_0^1 \nu^{\beta(n-\gamma)-1} D_{\theta^+}^{\gamma+\beta(n-\gamma)}\Phi(\nu\theta + (1-\nu)\zeta) d\nu \\ &+ \int_0^1 \nu^{\beta(n-\gamma)-1} D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\Phi((1-\nu)\theta + \nu\zeta) d\nu \\ &\leq [D_{\theta^+}^{\gamma+\beta(n-\gamma)}\Phi(\zeta) + D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\Phi(\theta)] \int_0^1 \nu^{\beta(n-\gamma)-1} d\nu. \end{aligned}$$

By substituting $u = \nu\theta + (1-\nu)\zeta$ and $v = (1-\nu)\theta + \nu\zeta$, the above inequality becomes

$$\frac{\Gamma(\beta(n-\gamma)+1)}{(\zeta-\theta)^{\beta(n-\gamma)}} [D_{\theta^+}^{\gamma,\beta}\Phi(\zeta) + D_{\zeta^-}^{\gamma,\beta}\Phi(\theta)] \leq D_{\theta^+}^{\gamma+\beta(n-\gamma)}\Phi(\zeta) + D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\Phi(\theta). \quad (14)$$

From (10) and (14), we get inequality (6). $\square$

The special case of Theorem 1 presented in [27] (Theorem 2.3) is given as follows.

Corollary 1. If we choose $\beta = 1$ and ψ is symmetric about $\frac{\theta+\zeta}{2}$ in Theorem 1, we get

$$\psi^n\left(\frac{\theta+\zeta}{2}\right) \leq \frac{\Gamma(n-\gamma+1)}{2(\zeta-\theta)^{n-\gamma}} [{}^C D_{\theta^+}^{\gamma}\psi(\zeta) + (-1)^n {}^C D_{\zeta^-}^{\gamma}\psi(\theta)] \leq \frac{\psi^n(\theta) + \psi^n(\zeta)}{2}.$$

Lemma 3. Let $\psi \in L^1[\theta, \zeta]$, $\psi * K_{(1-\beta)(n-\gamma)} \in AC^n[\theta, \zeta]$, $n \in \mathbb{N}$. For the differentiable function $D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)}\psi : [\theta, \zeta] \rightarrow \mathbb{R}$ with $n-1 < \gamma < n$, $0 < \beta \leq 1$ and $D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+1}\psi \in L^1[\theta, \zeta]$ the following equality

$$\begin{aligned} &\frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)}\Phi(\zeta) + D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\Phi(\theta)}{2} - \frac{\Gamma(\beta(n-\gamma)+1)}{2(\zeta-\theta)^{\beta(n-\gamma)}} [D_{\theta^+}^{\gamma,\beta}\Phi(\zeta) + D_{\zeta^-}^{\gamma,\beta}\Phi(\theta)] \\ &= \frac{\zeta-\theta}{2} \int_0^1 [(1-\nu)^{\beta(n-\gamma)} - \nu^{\beta(n-\gamma)}] D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+1}\psi(\nu\theta + (1-\nu)\zeta) d\nu, \end{aligned}$$

holds.

Proof. Consider

$$\begin{aligned}
 I &= \int_0^1 \left[(1-\nu)^{\beta(n-\gamma)} - \nu^{\beta(n-\gamma)} \right] D_{(\theta, \zeta)}^{\gamma+\beta(n-\gamma)+1} \psi(\nu\theta + (1-\nu)\zeta) d\nu \\
 &= \int_0^1 \left[(1-\nu)^{\beta(n-\gamma)} - \nu^{\beta(n-\gamma)} \right] D_{\theta^+}^{\gamma+\beta(n-\gamma)+1} \psi(\nu\theta + (1-\nu)\zeta) d\nu \\
 &\quad + \int_0^1 \left[(1-\nu)^{\beta(n-\gamma)} - \nu^{\beta(n-\gamma)} \right] D_{\zeta^-}^{\gamma+\beta(n-\gamma)+1} \psi(\nu\theta + (1-\nu)\zeta) d\nu \\
 &= I_1 + I_2.
 \end{aligned} \tag{15}$$

Integrating I_1 by parts, we get

$$\begin{aligned}
 I_1 &= (1-\nu)^{\beta(n-\gamma)} \frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)} \psi(\nu\theta + (1-\nu)\zeta)}{\theta - \zeta} \Big|_0^1 \\
 &\quad + \int_0^1 \beta(n-\gamma)(1-\nu)^{\beta(n-\gamma)-1} \frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)} \psi(\nu\theta + (1-\nu)\zeta)}{\theta - \zeta} d\nu \\
 &\quad - \nu^{\beta(n-\gamma)} \frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)} \psi(\nu\theta + (1-\nu)\zeta)}{\theta - \zeta} \Big|_0^1 \\
 &\quad + \int_0^1 \beta(n-\gamma)\nu^{\beta(n-\gamma)-1} \frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)} \psi(\nu\theta + (1-\nu)\zeta)}{\theta - \zeta} d\nu.
 \end{aligned}$$

By substituting $x = \nu\theta + (1-\nu)\zeta$, we obtain

$$\begin{aligned}
 I_1 &= \frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)} \psi(\zeta) + D_{\theta^+}^{\gamma+\beta(n-\gamma)} \tilde{\psi}(\zeta)}{\zeta - \theta} - \frac{\beta(n-\gamma)}{\zeta - \theta} \left[\int_{\zeta}^{\theta} \left(\frac{\theta - x}{\theta - \zeta} \right)^{\beta(n-\gamma)-1} \right. \\
 &\quad \times \left. \frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)} \psi(x)}{\theta - \zeta} dx + \int_{\zeta}^{\theta} \left(\frac{\zeta - x}{\zeta - \theta} \right)^{\beta(n-\gamma)-1} \frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)} \psi(x)}{\theta - \zeta} dx \right] \\
 &= \frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)} \Phi(\zeta)}{\zeta - \theta} - \frac{\beta(n-\gamma)}{(\zeta - \theta)^{\beta(n-\gamma)+1}} \int_{\theta}^{\zeta} (\zeta - x)^{\beta(n-\gamma)-1} D_{\theta^+}^{\gamma+\beta(n-\gamma)} \Phi(x) dx \\
 &= \frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)} \Phi(\zeta)}{\zeta - \theta} - \frac{\Gamma(\beta(n-\gamma) + 1)}{(\zeta - \theta)^{\beta(n-\gamma)+1}} D_{\theta^+}^{\gamma, \beta} \Phi(\zeta).
 \end{aligned} \tag{16}$$

Similarly, integrating I_2 by parts, we get

$$\begin{aligned}
 I_2 &= (1-\nu)^{\beta(n-\gamma)} \frac{D_{\zeta^-}^{\gamma+\beta(n-\gamma)} \psi(\nu\theta + (1-\nu)\zeta)}{\theta - \zeta} \Big|_0^1 \\
 &\quad + \int_0^1 \beta(n-\gamma)(1-\nu)^{\beta(n-\gamma)-1} \frac{D_{\zeta^-}^{\gamma+\beta(n-\gamma)} \psi(\nu\theta + (1-\nu)\zeta)}{\theta - \zeta} d\nu \\
 &\quad - \nu^{\beta(n-\gamma)} \frac{D_{\zeta^-}^{\gamma+\beta(n-\gamma)} \psi(\nu\theta + (1-\nu)\zeta)}{\theta - \zeta} \Big|_0^1 \\
 &\quad + \int_0^1 \beta(n-\gamma)\nu^{\beta(n-\gamma)-1} \frac{D_{\zeta^-}^{\gamma+\beta(n-\gamma)} \psi(\nu\theta + (1-\nu)\zeta)}{\theta - \zeta} d\nu.
 \end{aligned}$$

By substituting again $x = \nu\theta + (1-\nu)\zeta$, we get

$$I_2 = \frac{D_{\zeta^-}^{\gamma+\beta(n-\gamma)} \Phi(\theta)}{\zeta - \theta} - \frac{\Gamma(\beta(n-\gamma) + 1)}{(\zeta - \theta)^{\beta(n-\gamma)+1}} D_{\zeta^-}^{\gamma, \beta} \Phi(\theta). \tag{17}$$

Using (16) and (17) in (15), we have

$$I = \frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)}\Phi(\zeta) + D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\Phi(\theta)}{\zeta - \theta} - \frac{\Gamma(\beta(n-\gamma)+1)}{(\zeta - \theta)^{\beta(n-\gamma)+1}} \left[D_{\theta^+}^{\gamma,\beta}\Phi(\zeta) + D_{\zeta^-}^{\gamma,\beta}\Phi(\theta) \right].$$

Thus, by multiplying both sides with $\frac{\zeta-\theta}{2}$, we get the desired result. $\square$

The following special case of Lemma 3 was proved by Farid et al. in [27] (Lemma 2.2).

Corollary 2. If we take $\beta = 1$ and ψ is symmetric about $\frac{\theta+\zeta}{2}$ in Lemma 3, we obtain

$$\begin{aligned} & \frac{\psi^n(\theta) + \psi^n(\zeta)}{2} - \frac{\Gamma(n-\gamma+1)}{2(\zeta-\theta)^{n-\gamma}} \left[{}^C D_{\theta^+}^{\gamma} \psi(\zeta) + (-1)^n {}^C D_{\zeta^-}^{\gamma} \psi(\theta) \right] \\ &= \frac{\zeta-\theta}{2} \int_0^1 \left[(1-\nu)^{n-\gamma} - \nu^{n-\gamma} \right] \psi^{n+1}(\nu\theta + (1-\nu)\zeta) d\nu, \end{aligned}$$

for Caputo fractional derivatives.

Theorem 2. Let $\psi \in L^1[\theta, \zeta]$, $\psi * K_{(1-\beta)(n-\gamma)} \in AC^n[\theta, \zeta]$, $n \in \mathbb{N}$ and $D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)}\psi : [\theta, \zeta] \rightarrow \mathbb{R}$ be a differentiable function with $n-1 < \gamma < n$ and $0 < \beta \leq 1$. If $D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+1}\psi$ is convex on $[\theta, \zeta]$, then the following inequality is true

$$\begin{aligned} & \left| \frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)}\Phi(\zeta) + D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\Phi(\theta)}{2} - \frac{\Gamma(\beta(n-\gamma)+1)}{2(\zeta-\theta)^{\beta(n-\gamma)}} \left[D_{\theta^+}^{\gamma,\beta}\Phi(\zeta) + D_{\zeta^-}^{\gamma,\beta}\Phi(\theta) \right] \right| \\ & \leq \frac{\zeta-\theta}{2(\beta(n-\gamma)+1)} \left(1 - \frac{1}{2^{\beta(n-\gamma)}} \right) \left(|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+1}\psi(\zeta)| + |D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+1}\psi(\theta)| \right). \end{aligned}$$

Proof. By using Lemma 3 and Definition 1, we get

$$\begin{aligned} & \left| \frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)}\Phi(\zeta) + D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\Phi(\theta)}{2} - \frac{\Gamma(\beta(n-\gamma)+1)}{2(\zeta-\theta)^{\beta(n-\gamma)}} \left[D_{\theta^+}^{\gamma,\beta}\Phi(\zeta) + D_{\zeta^-}^{\gamma,\beta}\Phi(\theta) \right] \right| \\ & \leq \frac{\zeta-\theta}{2} \int_0^1 |(1-\nu)^{\beta(n-\gamma)} - \nu^{\beta(n-\gamma)}| \\ & \quad \times \left(\nu |D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+1}\psi(\theta)| + (1-\nu) |D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+1}\psi(\zeta)| \right) d\nu \\ & = \frac{\zeta-\theta}{2} \int_0^{\frac{1}{2}} \left[(1-\nu)^{\beta(n-\gamma)} - \nu^{\beta(n-\gamma)} \right] \\ & \quad \times \left(\nu |D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+1}\psi(\theta)| + (1-\nu) |D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+1}\psi(\zeta)| \right) d\nu \\ & \quad + \int_{\frac{1}{2}}^1 \left[\nu^{\beta(n-\gamma)} - (1-\nu)^{\beta(n-\gamma)} \right] \left(\nu |D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+1}\psi(\theta)| + (1-\nu) |D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+1}\psi(\zeta)| \right) d\nu \\ & = \frac{\zeta-\theta}{2} \left[|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+1}\psi(\zeta)| \int_0^{\frac{1}{2}} \left[(1-\nu)^{\beta(n-\gamma)+1} - (1-\nu)\nu^{\beta(n-\gamma)} \right] d\nu \right. \\ & \quad \left. + |D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+1}\psi(\theta)| \int_0^{\frac{1}{2}} \left[\nu(1-\nu)^{\beta(n-\gamma)} - \nu^{\beta(n-\gamma)+1} \right] d\nu \right. \\ & \quad \left. + |D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+1}\psi(\zeta)| \int_{\frac{1}{2}}^1 \left[(1-\nu)\nu^{\beta(n-\gamma)} - (1-\nu)^{\beta(n-\gamma)+1} \right] d\nu \right. \\ & \quad \left. + |D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+1}\psi(\theta)| \int_{\frac{1}{2}}^1 \left[\nu^{\beta(n-\gamma)+1} - \nu(1-\nu)^{\beta(n-\gamma)} \right] d\nu \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{\zeta - \theta}{2} \left[|D_{(\theta, \zeta)}^{\gamma+\beta(n-\gamma)+1} \psi(\zeta)| \left(\frac{1}{\beta(n-\gamma)+1} - \frac{1}{(\beta(n-\gamma)+1)2^{\beta(n-\gamma)}} \right) \right. \\
&\quad \left. + |D_{(\theta, \zeta)}^{\gamma+\beta(n-\gamma)+1} \psi(\theta)| \left(\frac{1}{\beta(n-\gamma)+1} - \frac{1}{(\beta(n-\gamma)+1)2^{\beta(n-\gamma)}} \right) \right] \\
&= \frac{\zeta - \theta}{2(\beta(n-\gamma)+1)} \left(1 - \frac{1}{2^{\beta(n-\gamma)}} \right) \left(|D_{(\theta, \zeta)}^{\gamma+\beta(n-\gamma)+1} \psi(\zeta)| + |D_{(\theta, \zeta)}^{\gamma+\beta(n-\gamma)+1} \psi(\theta)| \right).
\end{aligned}$$

Hence, the proof is complete. $\square$

The corollary given below presented in [27] (Theorem 2.4) is a special case of Theorem 2.

Corollary 3. If we choose $\beta = 1$ and ψ is symmetric about $\frac{\theta+\zeta}{2}$ in Theorem 2, we get

$$\begin{aligned}
&\frac{\psi^n(\zeta) + \psi^n(\theta)}{2} - \frac{\Gamma(n-\gamma+1)}{2(\zeta-\theta)^{n-\gamma}} \left[{}^C D_{\theta^+}^{\gamma} \psi(\zeta) + (-1)^n {}^C D_{\zeta^-}^{\gamma} \psi(\theta) \right] \\
&\leq \frac{\zeta - \theta}{2(n-\gamma+1)} \left(1 - \frac{1}{2^{n-\gamma}} \right) \left(|\psi^{n+1}(\zeta)| + |\psi^{n+1}(\theta)| \right).
\end{aligned}$$

Lemma 4. Let $\psi \in L^1[\theta, \zeta]$, $\psi * K_{(1-\beta)(n-\gamma)} \in AC^n[\theta, \zeta]$, $n \in \mathbb{N}$ and $D_{(\theta, \zeta)}^{\gamma+\beta(n-\gamma)} \psi : [\theta, \zeta] \rightarrow \mathbb{R}$ be twice differential mapping on (θ, ζ) with $n-1 < \gamma < n$ and $0 < \beta \leq 1$. If $D_{(\theta, \zeta)}^{\gamma+\beta(n-\gamma)+2} \psi \in L^1[\theta, \zeta]$, then we have the following equality.

$$\begin{aligned}
&\frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)} \Phi(\zeta) + D_{\zeta^-}^{\gamma+\beta(n-\gamma)} \Phi(\theta)}{2} - \frac{\Gamma(\beta(n-\gamma)+1)}{2(\zeta-\theta)^{\beta(n-\gamma)}} \left[D_{\theta^+}^{\gamma, \beta} \Phi(\zeta) + D_{\zeta^-}^{\gamma, \beta} \Phi(\theta) \right] \\
&= \frac{(\zeta - \theta)^2}{2} \int_0^1 \frac{1 - (1-\nu)^{\beta(n-\gamma)+1} - \nu^{\beta(n-\gamma)+1}}{\beta(n-\gamma)+1} D_{(\theta, \zeta)}^{\gamma+\beta(n-\gamma)+2} \psi(\nu\theta + (1-\nu)\zeta) d\nu.
\end{aligned}$$

Proof. By using Lemma 3, we get

$$\begin{aligned}
&\frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)} \Phi(\zeta) + D_{\zeta^-}^{\gamma+\beta(n-\gamma)} \Phi(\theta)}{2} - \frac{\Gamma(\beta(n-\gamma)+1)}{2(\zeta-\theta)^{\beta(n-\gamma)}} \left[D_{\theta^+}^{\gamma, \beta} \Phi(\zeta) + D_{\zeta^-}^{\gamma, \beta} \Phi(\theta) \right] \\
&= \frac{\zeta - \theta}{2} \left[\int_0^1 \left[(1-\nu)^{\beta(n-\gamma)} - \nu^{\beta(n-\gamma)} \right] D_{\theta^+}^{\gamma+\beta(n-\gamma)+1} \psi(\nu\theta + (1-\nu)\zeta) d\nu \right. \\
&\quad \left. + \int_0^1 \left[(1-\nu)^{\beta(n-\gamma)} - \nu^{\beta(n-\gamma)} \right] D_{\zeta^-}^{\gamma+\beta(n-\gamma)+1} \psi(\nu\theta + (1-\nu)\zeta) d\nu \right].
\end{aligned}$$

Integrating by parts, we get

$$\begin{aligned}
&= \frac{\zeta - \theta}{2} \left[\frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)+1} \psi(\zeta) - D_{\theta^+}^{\gamma+\beta(n-\gamma)+1} \psi(\theta) + D_{\zeta^-}^{\gamma+\beta(n-\gamma)+1} \psi(\zeta) - D_{\zeta^-}^{\gamma+\beta(n-\gamma)+1} \psi(\theta)}{\beta(n-\gamma)+1} \right. \\
&\quad \left. - \frac{\zeta - \theta}{\beta(n-\gamma)+1} \int_0^1 \left[(1-\nu)^{\beta(n-\gamma)+1} + \nu^{\beta(n-\gamma)+1} \right] D_{(\theta, \zeta)}^{\gamma+\beta(n-\gamma)+2} \psi(\nu\theta + (1-\nu)\zeta) d\nu \right]. \quad (18)
\end{aligned}$$

Since

$$D_{\theta^+}^{\gamma+\beta(n-\gamma)+1} \psi(\zeta) - D_{\theta^+}^{\gamma+\beta(n-\gamma)+1} \psi(\theta) = \int_{\theta}^{\zeta} D_{\theta^+}^{\gamma+\beta(n-\gamma)+2} \psi(u) du.$$

By substituting $u = \nu\theta + (1 - \nu)\zeta$, we get

$$\begin{aligned} & D_{\theta^+}^{\gamma+\beta(n-\gamma)+1}\psi(\zeta) - D_{\theta^+}^{\gamma+\beta(n-\gamma)+1}\psi(\theta) \\ &= (\zeta - \theta) \int_0^1 D_{\theta^+}^{\gamma+\beta(n-\gamma)+2}\psi(\nu\theta + (1 - \nu)\zeta) d\nu, \end{aligned} \quad (19)$$

and

$$D_{\zeta^-}^{\gamma+\beta(n-\gamma)+1}\psi(\zeta) - D_{\zeta^-}^{\gamma+\beta(n-\gamma)+1}\psi(\theta) = \int_{\theta}^{\zeta} D_{\zeta^-}^{\gamma+\beta(n-\gamma)+2}\psi(u) du.$$

By substituting again $u = \nu\theta + (1 - \nu)\zeta$, we get

$$D_{\zeta^-}^{\gamma+\beta(n-\gamma)+1}\psi(\zeta) - D_{\zeta^-}^{\gamma+\beta(n-\gamma)+1}\psi(\theta) = (\zeta - \theta) \int_0^1 D_{\zeta^-}^{\gamma+\beta(n-\gamma)+2}\psi(\nu\theta + (1 - \nu)\zeta) d\nu. \quad (20)$$

By adding (19) and (20), we obtain

$$\begin{aligned} & D_{\theta^+}^{\gamma+\beta(n-\gamma)+1}\psi(\zeta) - D_{\theta^+}^{\gamma+\beta(n-\gamma)+1}\psi(\theta) + D_{\zeta^-}^{\gamma+\beta(n-\gamma)+1}\psi(\zeta) - D_{\zeta^-}^{\gamma+\beta(n-\gamma)+1}\psi(\theta) \\ &= (\zeta - \theta) \int_0^1 D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\nu\theta + (1 - \nu)\zeta) d\nu. \end{aligned} \quad (21)$$

Using Equation (21) into (18), we get the required result. $\square$

Corollary 4. If we take $\beta = 1$ and ψ is symmetric about $\frac{\theta+\zeta}{2}$ in Lemma 4, we get the following equality for Caputo fractional derivatives:

$$\begin{aligned} & \frac{\psi^n(\theta) + \psi^n(\zeta)}{2} - \frac{\Gamma(n - \gamma + 1)}{2(\zeta - \theta)^{n-\gamma}} \left[{}^C D_{\theta^+}^{\gamma} \psi(\zeta) + (-1)^n {}^C D_{\zeta^-}^{\gamma} \psi(\theta) \right] \\ &= \frac{(\zeta - \theta)^2}{2} \int_0^1 \frac{1 - (1 - \nu)^{n-\gamma+1} - \nu^{n-\gamma+1}}{n - \gamma + 1} \psi^{n+2}(\nu\theta + (1 - \nu)\zeta) d\nu. \end{aligned}$$

Lemma 5. Let $\psi \in L^1[\theta, \zeta]$, $\psi * K_{(1-\beta)(n-\gamma)} \in AC^n[\theta, \zeta]$, $n \in \mathbb{N}$ and $D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)}\psi : [\theta, \zeta] \rightarrow \mathbb{R}$ is twice differentiable and measurable on $[\theta, \zeta]$, $n - 1 < \gamma < n$ and $0 < \beta \leq 1$, then the equation

$$\begin{aligned} & \frac{\Gamma(\beta(n - \gamma) + 1)}{2(\zeta - \theta)^{\beta(n-\gamma)}} \left[D_{\theta^+}^{\gamma,\beta} \Phi(\zeta) + D_{\zeta^-}^{\gamma,\beta} \Phi(\theta) \right] - D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)}\psi\left(\frac{\theta + \zeta}{2}\right) \\ &= \frac{(\zeta - \theta)^2}{2} \int_0^1 m(\nu) D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\nu\theta + (1 - \nu)\zeta) d\nu, \end{aligned}$$

$$\text{holds for } m(\nu) = \begin{cases} \nu - \frac{1 - (1-\nu)^{\beta(n-\gamma)+1} - \nu^{\beta(n-\gamma)+1}}{\beta(n-\gamma)+1}, & \nu \in [0, \frac{1}{2}); \\ 1 - \nu - \frac{1 - (1-\nu)^{\beta(n-\gamma)+1} - \nu^{\beta(n-\gamma)+1}}{\beta(n-\gamma)+1}, & \nu \in [\frac{1}{2}, 1]. \end{cases}$$

Proof. Consider

$$\begin{aligned} & \frac{(\zeta - \theta)^2}{2} \int_0^1 m(\nu) D_{(\theta, \zeta)}^{\gamma + \beta(n-\gamma) + 2} \psi(\nu\theta + (1-\nu)\zeta) d\nu \\ &= \frac{(\zeta - \theta)^2}{2} \left[\int_0^{\frac{1}{2}} \nu \left(D_{\theta^+}^{\gamma + \beta(n-\gamma) + 2} \psi(\nu\theta + (1-\nu)\zeta) \right. \right. \\ & \quad \left. \left. + D_{\zeta^-}^{\gamma + \beta(n-\gamma) + 2} \psi(\nu\theta + (1-\nu)\zeta) \right) d\nu \right. \\ & \quad \left. + \int_{\frac{1}{2}}^1 (1-\nu) \left(D_{\theta^+}^{\gamma + \beta(n-\gamma) + 2} \psi(\nu\theta + (1-\nu)\zeta) + D_{\zeta^-}^{\gamma + \beta(n-\gamma) + 2} \psi(\nu\theta + (1-\nu)\zeta) \right) d\nu \right. \\ & \quad \left. - \int_0^1 \left(\frac{1 - (1-\nu)^{\beta(n-\gamma)+1} - \nu^{\beta(n-\gamma)+1}}{\beta(n-\gamma) + 1} \right) D_{(\theta, \zeta)}^{\gamma + \beta(n-\gamma) + 2} \psi(\nu\theta + (1-\nu)\zeta) d\nu \right]. \end{aligned}$$

Let

$$\begin{aligned} I &= \int_0^{\frac{1}{2}} \nu \left(D_{\theta^+}^{\gamma + \beta(n-\gamma) + 2} \psi(\nu\theta + (1-\nu)\zeta) + D_{\zeta^-}^{\gamma + \beta(n-\gamma) + 2} \psi(\nu\theta + (1-\nu)\zeta) \right) d\nu \\ &+ \int_{\frac{1}{2}}^1 (1-\nu) \left(D_{\theta^+}^{\gamma + \beta(n-\gamma) + 2} \psi(\nu\theta + (1-\nu)\zeta) + D_{\zeta^-}^{\gamma + \beta(n-\gamma) + 2} \psi(\nu\theta + (1-\nu)\zeta) \right) d\nu \\ &= I_1 + I_2. \end{aligned} \quad (22)$$

Integrating I_1 by parts, we get

$$\begin{aligned} I_1 &= \frac{D_{\theta^+}^{\gamma + \beta(n-\gamma) + 1} \psi\left(\frac{\theta + \zeta}{2}\right) + D_{\zeta^-}^{\gamma + \beta(n-\gamma) + 1} \psi\left(\frac{\theta + \zeta}{2}\right)}{2(\theta - \zeta)} \\ &- \frac{\left[D_{\theta^+}^{\gamma + \beta(n-\gamma)} \psi\left(\frac{\theta + \zeta}{2}\right) + D_{\zeta^-}^{\gamma + \beta(n-\gamma)} \psi\left(\frac{\theta + \zeta}{2}\right) - D_{\theta^+}^{\gamma + \beta(n-\gamma)} \psi(\zeta) - D_{\zeta^-}^{\gamma + \beta(n-\gamma)} \tilde{\psi}(\theta) \right]}{(\theta - \zeta)^2}. \end{aligned} \quad (23)$$

Now integrating I_2 by parts, we get

$$\begin{aligned} I_2 &= - \frac{D_{\theta^+}^{\gamma + \beta(n-\gamma) + 1} \psi\left(\frac{\theta + \zeta}{2}\right) + D_{\zeta^-}^{\gamma + \beta(n-\gamma) + 1} \psi\left(\frac{\theta + \zeta}{2}\right)}{2(\theta - \zeta)} \\ &- \frac{\left[D_{\theta^+}^{\gamma + \beta(n-\gamma)} \psi\left(\frac{\theta + \zeta}{2}\right) + D_{\zeta^-}^{\gamma + \beta(n-\gamma)} \psi\left(\frac{\theta + \zeta}{2}\right) - D_{\theta^+}^{\gamma + \beta(n-\gamma)} \tilde{\psi}(\zeta) - D_{\zeta^-}^{\gamma + \beta(n-\gamma)} \psi(\theta) \right]}{(\theta - \zeta)^2}. \end{aligned} \quad (24)$$

By substituting (23) and (24) to (22), we get

$$I = \frac{D_{\theta^+}^{\gamma + \beta(n-\gamma)} \Phi(\zeta) + D_{\zeta^-}^{\gamma + \beta(n-\gamma)} \Phi(\theta)}{(\zeta - \theta)^2} - \frac{2D_{\theta^+}^{\gamma + \beta(n-\gamma)} \psi\left(\frac{\theta + \zeta}{2}\right) + 2D_{\zeta^-}^{\gamma + \beta(n-\gamma)} \psi\left(\frac{\theta + \zeta}{2}\right)}{(\zeta - \theta)^2}.$$

Thus

$$\begin{aligned} & \frac{(\zeta - \theta)^2}{2} \int_0^1 m(\nu) D_{(\theta, \zeta)}^{\gamma + \beta(n-\gamma) + 2} \psi(\nu\theta + (1-\nu)\zeta) d\nu \\ &= \frac{D_{\theta^+}^{\gamma + \beta(n-\gamma)} \Phi(\zeta) + D_{\zeta^-}^{\gamma + \beta(n-\gamma)} \Phi(\theta)}{2} - D_{(\theta, \zeta)}^{\gamma + \beta(n-\gamma)} \psi\left(\frac{\theta + \zeta}{2}\right) \end{aligned}$$

$$-\frac{(\zeta - \theta)^2}{2} \int_0^1 \left(\frac{1 - (1 - \nu)^{\beta(n-\gamma)+1} - \nu^{\beta(n-\gamma)+1}}{\beta(n-\gamma) + 1} \right) \\ \times D_{(\theta, \zeta)}^{\gamma+\beta(n-\gamma)+2} \psi(\nu\theta + (1-\nu)\zeta) d\nu.$$

By using Lemma (4), we arrive at the desired result. $\square$

Corollary 5. If we take $\beta = 1$ and ψ is symmetric about $\frac{\theta+\zeta}{2}$ in Lemma 5, then the following equality for Caputo fractional derivatives

$$\frac{\Gamma(n-\gamma+1)}{2(\zeta-\theta)^{n-\gamma}} \left[{}^C D_{\theta^+}^{\gamma} \psi(\zeta) + (-1)^n {}^C D_{\zeta^-}^{\gamma} \psi(\theta) \right] - \psi^n \left(\frac{\theta+\zeta}{2} \right) \\ = \frac{(\zeta-\theta)^2}{2} \int_0^1 m(\nu) \psi^{n+2}(\nu\theta + (1-\nu)\zeta) d\nu,$$

$$\text{holds, where } m(\nu) = \begin{cases} \nu - \frac{1-(1-\nu)^{n-\gamma+1} - \nu^{n-\gamma+1}}{n-\gamma+1}, & \nu \in [0, \frac{1}{2}); \\ 1 - \nu - \frac{1-(1-\nu)^{n-\gamma+1} - \nu^{n-\gamma+1}}{n-\gamma+1}, & \nu \in [\frac{1}{2}, 1). \end{cases}$$

Theorem 3. Let $\psi \in L^1[\theta, \zeta]$, $\psi * K_{(1-\beta)(n-\gamma)} \in AC^n[\theta, \zeta]$, $n \in \mathbb{N}$ and $D_{(\theta, \zeta)}^{\gamma+\beta(n-\gamma)} \psi : [\theta, \zeta] \rightarrow \mathbb{R}$ be a twice differentiable function with $n-1 < \gamma < n$ and $0 < \beta \leq 1$. If $|D_{(\theta, \zeta)}^{\gamma+\beta(n-\gamma)+2} \psi|$ is measurable, decreasing and geometric-arithmetically s -convex on $[\theta, \zeta]$ for some fixed $\gamma \in (0, \infty)$, $s \in (0, 1]$, $0 \leq \theta < \zeta$, then the inequality

$$\left| \frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)} \Phi(\zeta) + D_{\zeta^-}^{\gamma+\beta(n-\gamma)} \Phi(\theta)}{2} - \frac{\Gamma(\beta(n-\gamma)+1)}{2(\zeta-\theta)^{\beta(n-\gamma)}} \left[D_{\theta^+}^{\gamma, \beta} \Phi(\zeta) + D_{\zeta^-}^{\gamma, \beta} \Phi(\theta) \right] \right| \\ \leq \frac{(\zeta-\theta)^2 \left(|D_{(\theta, \zeta)}^{\gamma+\beta(n-\gamma)+2} \psi(\theta)| + |D_{(\theta, \zeta)}^{\gamma+\beta(n-\gamma)+2} \psi(\zeta)| \right)}{2(\beta(n-\gamma)+1)} \\ \times \left(\frac{1}{s+1} - \frac{1}{\beta(n-\gamma)+s+2} \right),$$

holds.

Proof. By using Lemmas 1, 4 and Definition 4, we have

$$\left| \frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)} \Phi(\zeta) + D_{\zeta^-}^{\gamma+\beta(n-\gamma)} \Phi(\theta)}{2} - \frac{\Gamma(\beta(n-\gamma)+1)}{2(\zeta-\theta)^{\beta(n-\gamma)}} \left[D_{\theta^+}^{\gamma, \beta} \Phi(\zeta) + D_{\zeta^-}^{\gamma, \beta} \Phi(\theta) \right] \right| \\ \leq \frac{(\zeta-\theta)^2}{2(\beta(n-\gamma)+1)} \int_0^1 |1 - (1-\nu)^{\beta(n-\gamma)+1} - \nu^{\beta(n-\gamma)+1}| \\ \times |D_{(\theta, \zeta)}^{\gamma+\beta(n-\gamma)+2} \psi(\nu\theta + (1-\nu)\zeta)| d\nu \\ \leq \frac{(\zeta-\theta)^2}{2(\beta(n-\gamma)+1)} \int_0^1 |1 - (1-\nu)^{\beta(n-\gamma)+1} - \nu^{\beta(n-\gamma)+1}| |D_{(\theta, \zeta)}^{\gamma+\beta(n-\gamma)+2} \psi(\theta^\nu \zeta^{1-\nu})| d\nu$$

$$\begin{aligned}
&\leq \frac{(\zeta - \theta)^2}{2(\beta(n - \gamma) + 1)} \int_0^1 \left(1 - (1 - \nu)^{\beta(n - \gamma) + 1} - \nu^{\beta(n - \gamma) + 1}\right) \left[\nu^s |D_{(\theta, \zeta)}^{\gamma + \beta(n - \gamma) + 2} \psi(\theta)| \right. \\
&\quad \left. + (1 - \nu)^s |D_{(\theta, \zeta)}^{\gamma + \beta(n - \gamma) + 2} \psi(\zeta)|\right] d\nu \\
&= \frac{(\zeta - \theta)^2}{2(\beta(n - \gamma) + 1)} \left[\int_0^1 \left[\nu^s |D_{(\theta, \zeta)}^{\gamma + \beta(n - \gamma) + 2} \psi(\theta)| + (1 - \nu)^s |D_{(\theta, \zeta)}^{\gamma + \beta(n - \gamma) + 2} \psi(\zeta)|\right] d\nu \right. \\
&\quad \left. - \int_0^1 \left[\nu^s (1 - \nu)^{\beta(n - \gamma) + 1} |D_{(\theta, \zeta)}^{\gamma + \beta(n - \gamma) + 2} \psi(\theta)| + (1 - \nu)^{\beta(n - \gamma) + s + 1} |D_{(\theta, \zeta)}^{\gamma + \beta(n - \gamma) + 2} \psi(\zeta)|\right] d\nu \right. \\
&\quad \left. - \int_0^1 \left[\nu^{\beta(n - \gamma) + s + 1} |D_{(\theta, \zeta)}^{\gamma + \beta(n - \gamma) + 2} \psi(\theta)| + \nu^{\beta(n - \gamma) + 1} (1 - \nu)^s |D_{(\theta, \zeta)}^{\gamma + \beta(n - \gamma) + 2} \psi(\zeta)|\right] d\nu \right].
\end{aligned}$$

By using the definition of the beta function, we get

$$\begin{aligned}
&\left| \frac{D_{\theta^+}^{\gamma + \beta(n - \gamma)} \Phi(\zeta) + D_{\zeta^-}^{\gamma + \beta(n - \gamma)} \Phi(\theta)}{2} - \frac{\Gamma(\beta(n - \gamma) + 1)}{2(\zeta - \theta)^{\beta(n - \gamma)}} \left[D_{\theta^+}^{\gamma, \beta} \Phi(\zeta) + D_{\zeta^-}^{\gamma, \beta} \Phi(\theta) \right] \right| \\
&\leq \frac{(\zeta - \theta)^2}{2(\beta(n - \gamma) + 1)} \left[\frac{|D_{(\theta, \zeta)}^{\gamma + \beta(n - \gamma) + 2} \psi(\theta)|}{s + 1} + \frac{|D_{(\theta, \zeta)}^{\gamma + \beta(n - \gamma) + 2} \psi(\zeta)|}{s + 1} \right. \\
&\quad \left. - \frac{|D_{(\theta, \zeta)}^{\gamma + \beta(n - \gamma) + 2} \psi(\theta)|}{\beta(n - \gamma) + s + 2} - |D_{(\theta, \zeta)}^{\gamma + \beta(n - \gamma) + 2} \psi(\zeta)| B(s + 1, \beta(n - \gamma) + 2) \right. \\
&\quad \left. - |D_{(\theta, \zeta)}^{\gamma + \beta(n - \gamma) + 2} \psi(\theta)| B(s + 1, \beta(n - \gamma) + 2) - \frac{|D_{(\theta, \zeta)}^{\gamma + \beta(n - \gamma) + 2} \psi(\zeta)|}{\beta(n - \gamma) + s + 2} \right] \\
&\leq \frac{(\zeta - \theta)^2 \left(|D_{(\theta, \zeta)}^{\gamma + \beta(n - \gamma) + 2} \psi(\zeta)| + |D_{(\theta, \zeta)}^{\gamma + \beta(n - \gamma) + 2} \psi(\theta)| \right)}{2(\beta(n - \gamma) + 1)} \\
&\quad \times \left(\frac{1}{s + 1} - \frac{1}{\beta(n - \gamma) + s + 2} \right).
\end{aligned}$$

Which completes the proof of the result. $\square$

Corollary 6. If we take $\beta = 1$ and ψ is symmetric about $\frac{\theta + \zeta}{2}$ in Theorem 3, then the following result for Caputo fractional derivatives holds.

$$\begin{aligned}
&\left| \frac{\psi^n(\theta) + \psi^n(\zeta)}{2} - \frac{\Gamma(n - \gamma + 1)}{2(\zeta - \theta)^{n - \gamma}} \left[{}^C D_{\theta^+}^{\gamma} \psi(\zeta) + {}^C D_{\zeta^-}^{\gamma} \psi(\theta) \right] \right| \\
&\leq \frac{(\zeta - \theta)^2 \left(|\psi^{n+2}(\theta)| + |\psi^{n+2}(\zeta)| \right)}{2(n - \gamma + 1)} \left(\frac{1}{s + 1} - \frac{1}{n - \gamma + s + 2} \right).
\end{aligned}$$

Theorem 4. Let $\psi \in L^1[\theta, \zeta]$, $\psi * K_{(1 - \beta)(n - \gamma)} \in AC^n[\theta, \zeta]$, $n \in \mathbb{N}$. Consider $D_{(\theta, \zeta)}^{\gamma + \beta(n - \gamma)} \psi : [\theta, \zeta] \rightarrow \mathbb{R}$ to be a twice differentiable function with $n - 1 < \gamma < n$ and $0 < \beta \leq 1$. If $|D_{(\theta, \zeta)}^{\gamma + \beta(n - \gamma) + 2} \psi|^q$ is measurable, decreasing and geometric arithmetically s -convex on $[\theta, \zeta]$ for some fixed $\gamma \in (0, \infty)$, $s \in (0, 1]$, $0 \leq \theta < \zeta$, then the inequality

$$\begin{aligned}
& \left| \frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)}\Phi(\zeta) + D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\Phi(\theta)}{2} - \frac{\Gamma(\beta(n-\gamma)+1)}{2(\zeta-\theta)^{\beta(n-\gamma)}} [D_{\theta^+}^{\gamma,\beta}\Phi(\zeta) + D_{\zeta^-}^{\gamma,\beta}\Phi(\theta)] \right| \\
& \leq \frac{(\zeta-\theta)^2 \max(1-2^{1-\beta(n-\gamma)}, 2^{1-\beta(n-\gamma)}-1)}{2(\beta(n-\gamma)+1)} \\
& \times \left(\frac{|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\zeta)|^q + |D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\theta)|^q}{s+1} \right)^{\frac{1}{q}},
\end{aligned}$$

is true.

Proof. We shall prove this theorem in two cases:

Case 1: Let $\gamma \in (0, 1)$ and $\beta(n-\gamma) \in [0, 1]$, then by using Lemma 4, Holder's inequality, Lemma 1, Definition 4 and Lemma 2, we obtain

$$\begin{aligned}
& \left| \frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)}\Phi(\zeta) + D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\Phi(\theta)}{2} - \frac{\Gamma(\beta(n-\gamma)+1)}{2(\zeta-\theta)^{\beta(n-\gamma)}} [D_{\theta^+}^{\gamma,\beta}\Phi(\zeta) + D_{\zeta^-}^{\gamma,\beta}\Phi(\theta)] \right| \\
& \leq \frac{(\zeta-\theta)^2}{2(\beta(n-\gamma)+1)} \left(\int_0^1 |1-(1-\nu)^{\beta(n-\gamma)+1} - \nu^{\beta(n-\gamma)+1}|^p d\nu \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 |D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\nu\theta + (1-\nu)\zeta)|^q d\nu \right)^{\frac{1}{q}} \\
& \leq \frac{(\zeta-\theta)^2}{2(\beta(n-\gamma)+1)} \left(\int_0^1 |1-(1-\nu)^{\beta(n-\gamma)+1} - \nu^{\beta(n-\gamma)+1}|^p d\nu \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 |D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\theta^\nu \zeta^{1-\nu})|^q d\nu \right)^{\frac{1}{q}} \\
& \leq \frac{(\zeta-\theta)^2}{2(\beta(n-\gamma)+1)} \left(\int_0^1 |1-(1-\nu)^{\beta(n-\gamma)+1} - \nu^{\beta(n-\gamma)+1}|^p d\nu \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 [\nu^s |D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\theta)|^q + (1-\nu)^s |D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\zeta)|^q] d\nu \right)^{\frac{1}{q}} \\
& \leq \frac{(\zeta-\theta)^2}{2(\beta(n-\gamma)+1)} \left(\frac{|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\theta)|^q + |D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\zeta)|^q}{s+1} \right)^{\frac{1}{q}} \\
& \times \left(\int_0^1 [(1-\nu)^{\beta(n-\gamma)} + \nu^{\beta(n-\gamma)} - 1]^p d\nu \right)^{\frac{1}{p}} \\
& \leq \frac{(\zeta-\theta)^2}{2(\beta(n-\gamma)+1)} \left(\frac{|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\theta)|^q + |D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\zeta)|^q}{s+1} \right)^{\frac{1}{q}} \\
& \times \left(\int_0^1 (2^{1-\beta(n-\gamma)} - 1)^p d\nu \right)^{\frac{1}{p}} \\
& = \frac{(\zeta-\theta)^2}{2(\beta(n-\gamma)+1)} \left(\frac{|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\theta)|^q + |D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\zeta)|^q}{s+1} \right)^{\frac{1}{q}} \\
& \times (2^{1-\beta(n-\gamma)} - 1).
\end{aligned} \tag{25}$$

Case 2: Let $\gamma \in [1, \infty)$ and $\beta(n - \gamma) \in [1, \infty)$. By using Lemma 4, Holder's inequality, Lemma 1, Definition 4 and Lemma 2, we obtain

$$\begin{aligned}
 & \left| \frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)}\Phi(\zeta) + D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\Phi(\theta)}{2} - \frac{\Gamma(\beta(n-\gamma)+1)}{2(\zeta-\theta)^{\beta(n-\gamma)}} [D_{\theta^+}^{\gamma,\beta}\Phi(\zeta) + D_{\zeta^-}^{\gamma,\beta}\Phi(\theta)] \right| \\
 & \leq \frac{(\zeta-\theta)^2}{2(\beta(n-\gamma)+1)} \left(\frac{|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\theta)|^q + |D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\zeta)|^q}{s+1} \right)^{\frac{1}{q}} \\
 & \times \left(\int_0^1 (1-2^{1-\beta(n-\gamma)})^p dv \right)^{\frac{1}{p}} \\
 & = \frac{(\zeta-\theta)^2}{2(\beta(n-\gamma)+1)} \left(\frac{|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\theta)|^q + |D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\zeta)|^q}{s+1} \right)^{\frac{1}{q}} \\
 & \times (1-2^{1-\beta(n-\gamma)}). \tag{26}
 \end{aligned}$$

Now, from (25) and (26), we obtain the required result. $\square$

Corollary 7. If we take $\beta = 1$ and ψ is symmetric about $\frac{\theta+\zeta}{2}$ in Theorem 4, we get the following inequality for Caputo fractional derivatives:

$$\begin{aligned}
 & \left| \frac{\psi^n(\theta) + \psi^n(\zeta)}{2} - \frac{\Gamma(n-\gamma+1)}{2(\zeta-\theta)^{n-\gamma}} [{}^C D_{\theta^+}^{\gamma}\psi(\zeta) + (-1)^n {}^C D_{\zeta^-}^{\gamma}\psi(\theta)] \right| \\
 & \leq \frac{(\zeta-\theta)^2 \max(1-2^{1-n+\gamma}, 2^{1-n+\gamma}-1)}{2(n-\gamma+1)} \left(\frac{|\psi^{n+2}(\theta)|^q + |\psi^{n+2}(\zeta)|^q}{s+1} \right)^{\frac{1}{q}}.
 \end{aligned}$$

Theorem 5. Let $\psi \in L^1[\theta, \zeta]$, $\psi * K_{(1-\beta)(n-\gamma)} \in AC^n[\theta, \zeta]$, $n \in \mathbb{N}$ and $D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)}\psi : [\theta, \zeta] \rightarrow \mathbb{R}$ be differentiable function with $n-1 < \gamma < n$ and $0 < \beta \leq 1$. If $|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi|^q$ is measurable for $1 < q < \infty$, decreasing and geometric arithmetically s -convex on $[0, \zeta]$ for some fixed $\gamma \in (0, \infty)$, $s \in (0, 1]$, $0 \leq \theta < \zeta$, then the following fractional inequality holds.

$$\begin{aligned}
 & \frac{\Gamma(\beta(n-\gamma)+1)}{2(\zeta-\theta)^{\beta(n-\gamma)}} [D_{\theta^+}^{\gamma,\beta}\Phi(\zeta) + D_{\zeta^-}^{\gamma,\beta}\Phi(\theta)] - D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)}\psi\left(\frac{\theta+\zeta}{2}\right) \\
 & \leq \frac{(\zeta-\theta)^2}{2(\beta(n-\gamma)+1)} \left(\frac{|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\theta)|^q + |D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\zeta)|^q}{s+1} \right)^{\frac{1}{q}} \\
 & \times \left(\frac{(\beta(n-\gamma)+1)2^{-p-1} + (\beta(n-\gamma)+0.5)^{p+1} - (\beta(n-\gamma))^{p+1}}{p+1} \right)^{\frac{1}{p}},
 \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By using Lemmas 1 and 5, Holder's inequality and Definition 4, we get

$$\begin{aligned}
& \left| \frac{\Gamma(\beta(n-\gamma)+1)}{2(\zeta-\theta)^{\beta(n-\gamma)}} \left[D_{\theta^+}^{\gamma,\beta} \Phi(\zeta) + D_{\zeta^-}^{\gamma,\beta} \Phi(\theta) \right] - D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)} \psi\left(\frac{\theta+\zeta}{2}\right) \right| \\
& \leq \frac{(\zeta-\theta)^2}{2} \int_0^1 |m(\nu)| |D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2} \psi(\nu\theta + (1-\nu)\zeta)| d\nu \\
& \leq \frac{(\zeta-\theta)^2}{2} \int_0^1 |m(\nu)| |D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2} \psi(\theta^\nu \zeta^{1-\nu})| d\nu \\
& \leq \frac{(\zeta-\theta)^2}{2} \left(\int_0^1 |m(\nu)|^p d\nu \right)^{\frac{1}{p}} \left(\int_0^1 |D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2} \psi(\theta^\nu \zeta^{1-\nu})|^q d\nu \right)^{\frac{1}{q}} \\
& \leq \frac{(\zeta-\theta)^2}{2} \left(\int_0^1 |m(\nu)|^p d\nu \right)^{\frac{1}{p}} \left(\int_0^1 \left[\nu^s |D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2} \psi(\theta)|^q \right. \right. \\
& \quad \left. \left. + (1-\nu)^s |D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2} \psi(\zeta)|^q \right] d\nu \right)^{\frac{1}{q}} \\
& = \frac{(\zeta-\theta)^2}{2} \left(\frac{|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2} \psi(\theta)|^q + |D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2} \psi(\zeta)|^q}{s+1} \right)^{\frac{1}{q}} \\
& \quad \times \left[\int_0^{\frac{1}{2}} \left| \nu - \frac{1 - (1-\nu)^{\beta(n-\gamma)+1} - \nu^{\beta(n-\gamma)+1}}{\beta(n-\gamma)+1} \right|^p d\nu \right. \\
& \quad \left. + \int_{\frac{1}{2}}^1 \left| (1-\nu) - \frac{1 - (1-\nu)^{\beta(n-\gamma)+1} - \nu^{\beta(n-\gamma)+1}}{\beta(n-\gamma)+1} \right|^p d\nu \right]^{\frac{1}{p}} \\
& = \frac{(\zeta-\theta)^2}{2(\beta(n-\gamma)+1)} \left(\frac{|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2} \psi(\theta)|^q + |D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2} \psi(\zeta)|^q}{s+1} \right)^{\frac{1}{q}} \\
& \quad \times \left[\int_0^{\frac{1}{2}} \left| \beta(n-\gamma)\nu - 1 + (1-\nu)^{\beta(n-\gamma)+1} + \nu^{\beta(n-\gamma)+1} \right|^p d\nu \right. \\
& \quad \left. + \int_{\frac{1}{2}}^1 \left| \beta(n-\gamma) + 1 - \beta(n-\gamma)\nu - \nu - 1 + (1-\nu)^{\beta(n-\gamma)+1} + \nu^{\beta(n-\gamma)+1} \right|^p d\nu \right]^{\frac{1}{p}} \\
& \leq \frac{(\zeta-\theta)^2}{2(\beta(n-\gamma)+1)} \left(\frac{|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2} \psi(\theta)|^q + |D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2} \psi(\zeta)|^q}{s+1} \right)^{\frac{1}{q}} \\
& \quad \times \left(\int_0^{\frac{1}{2}} ((\beta(n-\gamma)+1)\nu)^p d\nu + \int_{\frac{1}{2}}^1 (\beta(n-\gamma) - \nu + 1)^p d\nu \right)^{\frac{1}{p}} \\
& \leq \frac{(\zeta-\theta)^2}{2(\beta(n-\gamma)+1)} \left(\frac{|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2} \psi(\theta)|^q + |D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2} \psi(\zeta)|^q}{s+1} \right)^{\frac{1}{q}} \\
& \quad \times \left((\beta(n-\gamma)+1) \int_0^{\frac{1}{2}} \nu^p d\nu + \int_{\frac{1}{2}}^1 (\beta(n-\gamma) - \nu + 1)^p d\nu \right)^{\frac{1}{p}} \\
& = \frac{(\zeta-\theta)^2}{2(\beta(n-\gamma)+1)} \left(\frac{|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2} \psi(\theta)|^q + |D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2} \psi(\zeta)|^q}{s+1} \right)^{\frac{1}{q}} \\
& \quad \times \left(\frac{(\beta(n-\gamma)+1)2^{-p-1} + (\beta(n-\gamma)+0.5)^{p+1} - (\beta(n-\gamma))^{p+1}}{p+1} \right)^{\frac{1}{p}}.
\end{aligned}$$

Which completes the proof of the result. $\square$

Corollary 8. If we take $\beta = 1$ and ψ is symmetric about $\frac{\theta+\zeta}{2}$ in Theorem 5, then the inequality

$$\begin{aligned} & \frac{\Gamma(n-\gamma+1)}{2(\zeta-\theta)^{n-\gamma}} \left[{}^C D_{\theta+}^{\gamma} \psi(\zeta) + (-1)^n {}^C D_{\zeta-}^{\gamma} \psi(\theta) \right] - \psi^n \left(\frac{\theta+\zeta}{2} \right) \\ & \leq \frac{(\zeta-\theta)^2}{2(n-\gamma+1)} \left(\frac{|\psi^{n+2}(\theta)|^q + |\psi^{n+2}(\zeta)|^q}{s+1} \right)^{\frac{1}{q}} \\ & \times \left(\frac{(n-\gamma+1)2^{-p-1} + (n-\gamma+0.5)^{p+1} - (n-\gamma)^{p+1}}{p+1} \right)^{\frac{1}{p}}, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, holds for Caputo fractional derivatives.

4. Concluding Remarks

The Hilfer fractional derivative has been used to set up a class of Hermite-Hadamard-type inequalities by involving convexity theory. Our results present many of the earlier inequalities that exist in the literature. The methodology used to generate the new inequalities is based on Hilfer's fractional derivative and skillful use of Hölder's inequality that has a wide range of applications in optimization theory. The findings of this work may stimulate the interest of researchers working in this field can pursue further investigation.

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References

- Guo, B.; Pu, X.; Huang, F. *Fractional Partial Differential Equations and Their Numerical Solution*; World Scientific: Hong Kong, China, 2015.
- Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; North-Holland Mathematics Studies; Elsevier Science: Amsterdam, The Netherlands, 2006.
- Dokuyucua, M.A. Caputo and Atangana-Baleanu-Caputo Fractional Derivative Applied to Garden Equation. *Turk. J. Sci.* **2020**, *5*, 1–7.
- Samraiz, M.; Perveen, Z.; Abdeljawad, T.; Iqbal, S.; Naheed, S. On certain fractional calculus operators and applications in mathematical physics. *Phys. Scr.* **2020**, *95*, 115210.
- Samraiz, M.; Perveen, Z.; Rahman, G.; Nisar, K.S.; Kumar, D. On the (k, s) -Hilfer-Prabhakar Fractional Derivative with Applications to Mathematical Physics. *Front. Phys.* **2020**, *8*, 309. [\[CrossRef\]](#)
- Korus, P. Some Hermite-Hadamard type inequalities for functions of generalized convex derivative. *Acta Math. Hung.* **2021**, *165*, 463–473. [\[CrossRef\]](#)
- Farid, G.; Yousaf, M.; Nonlaopon, K. Fejér-Hadamard type inequalities for $(a, h - m) - p$ -convex functions via extended generalized fractional integrals. *Fractal Fract.* **2021**, *5*, 253. [\[CrossRef\]](#)
- Ekinci, A.; Özdemir, M.E. Some new integral inequalities via Riemann-Liouville integral operators. *Appl. Comput. Math.* **2019**, *18*, 288–295. [\[CrossRef\]](#)
- Butt, S.I.; Nadeem, M.; Farid, G. On Caputo fractional derivatives via exponential s -convex functions. *Turk. J. Sci.* **2020**, *5*, 140–146.
- Zhou, S.-S.; Rashid, S.; Parveen, S.; Akdemir, A.O.; Hammouch, Z. New computations for extended weighted functionals within the Hilfer generalized proportional fractional integral operators. *AIMS Math.* **2021**, *6*, 507–4525.
- Alomari, M.; Darus, M.; Dragomir, S.S.; Seron, P. Ostrowski's inequalities for functions whose derivatives are s -convex in the second sense. *Appl. Math. Lett.* **2010**, *23*, 1071–1076. [\[CrossRef\]](#)

12. Dragomir, S.S.; Ionescu, N.M. Some remarks on convex functions. *Rev. d'analyse Numer. Theor. Approx. Tome* **1992**, *21*, 31–36. [[CrossRef](#)]
13. Dragomir, S.S.; Wang, S. A new inequality of Ostrowski's type in L_p norm and applications to some special means and to some numerical quadrature rules. *Indian J. Math.* **2008**, *40*, 299–304.
14. McShane, E.J. Jensen's inequality. *Bull. Am. Math. Soc.* **1937**, *43*, 521–527.
15. Samraiz, M.; Nawaz, F.; Iqbal, S.; Abdeljawad, T.; Rahman, G.; Nisar, K.S. Certain mean-type fractional integral inequalities via different convexities with applications. *J. Inequal. Appl.* **2020**, *2020*, 208. [[CrossRef](#)]
16. Sarikaya, M.Z.; Set, E.; Yaldiz, H.; Basak, B. Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities. *Math. Comp. Model.* **2013**, *57*, 2403–2407. [[CrossRef](#)]
17. Hadamard, J. Etude sur les proprietes des fonctions entieres et en particulier d'une fonction considerree par Riemann. *J. Math. Pures Appl.* **1893**, *58*, 171–216. [[CrossRef](#)]
18. Lyu, S.L. On the Hermite-Hadamard inequality for convex functions of two variables. *Numer. Algebra Control Optim.* **2014**, *4*, 1–8.
19. Niculescu, C.P.; Persson, L.E. Old and new on the Hermite Hadamard inequality. *Real Anal. Exch.* **2003**, *29*, 663–683. [[CrossRef](#)]
20. Wang, J.; Li, X.; Feckan, M.; Zhou, Y. Hermite-Hadamard-type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity. *Appl. Anal.* **2013**, *92*, 2241–2253. [[CrossRef](#)]
21. Mihai, M.V.; Awan, M.U.; Noor, M.A.; Kim, J.K. Hermite-Hadamard inequalities and their applications. *J. Inequal. Appl.* **2018**, *2018*, 309. [[CrossRef](#)]
22. Okur, N.; Yalcin, F.B.; Karahan, V. Some Hermite-Hadamard Type Integral Inequalities for Multidimensional Preinvex Functions. *Turk. J. Ineq.* **2019**, *3*, 54–63. [[CrossRef](#)] [[PubMed](#)]
23. Anastassiou, G.A. *Fractional Differentiation Inequalities*; Springer: New York, NY, USA, 2009.
24. Hilfer, R.; Luchko, Y.; Tomovski, Z. Operational method for solution of fractional differential equations with generalized Riemann-Liouville fractional derivative. *Fract. Calc. Appl. Anal.* **2009**, *12*, 299–318.
25. Liao, Y.; Deng, J.; Wang, J. Riemann-Liouville fractional Hermite-Hadamard inequalities, part I, for twice differentiable geometric-arithmetically s -convex functions. *J. Inequal. Appl.* **2013**, *2013*, 517.
26. Deng, J.; Wang, J. Fractional Hermite-Hadamard's inequalities for (α, m) -logarithmically convex functions. *J. Inequal. Appl.* **2013**, *2013*, 364. [[CrossRef](#)]
27. Farid, G.; Naqvi, S.; Javed, A. Hadamard and Fejer-Hadamard inequalities and related results via Caputo fractional derivative. *Bull. Math. Anal. Appl.* **2017**, *9*, 16–30. [[CrossRef](#)]