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Existence Results of Global Solutions for a Coupled Implicit Riemann-Liouville Fractional Integral Equation via the Vector Kuratowski Measure of Noncompactness

Noura Laksaci [1], Ahmed Boudaoui [1], Wasfi Shatanawi [2,3,*] and Taqi A. M. Shatnawi [3]

1 Laboratory of Mathematics Modeling and Applications, Department of Mathematics and Computer Science, University of Adrar, National Road No. 06, Adrar 01000, Algeria; nor.laksaci@univ-adrar.edu.dz (N.L.); ahmedboudaoui@univ-adrar.edu.dz (A.B.)
2 Department of General Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia
3 Department of Mathematics, Faculty of Science, The Hashemite University, P.O. Box 330127, Zarqa 13133, Jordan; taqi_shatnawi@hu.edu.jo
* Correspondence: wshatanawi@psu.edu.sa

Abstract: The main goal of this study is to demonstrate an existence result of a coupled implicit Riemann-Liouville fractional integral equation. First, we prove a new fixed point theorem in spaces with an extended norm structure. That theorem generalized Darbo's theorem associated with the vector Kuratowski measure of noncompactness. Second, we employ our obtained fixed point theorem to investigate the existence of solutions to the coupled implicit fractional integral equation on the generalized Banach space $\mathcal{C}([0,1],\mathbb{R}) \times \mathcal{C}([0,1],\mathbb{R})$.

Keywords: coupled implicit Riemann-Liouville fractional integral equation; generalized Banach space; fixed point theorems; M-set contractive; generalized measure of noncompactness

MSC: 47H08, 26A33

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1. Introduction

Fractional calculus is considered as a generalization of the differentiation and integration since the order is not necessarily an integer. At the end of the XIXth century, pioneering mathematicians Liouville [1–4] and Riemann [5] developed the theoretical analysis of fractional calculus. Over the past three decades, the field of fractional calculus has been studied by many authors due to its potential to be applied to many problems from several areas of scientific disciplines; see [6–9] and the references therein.

The measures of noncompactness play an important role in showing that there are solutions to differential equations, especially for implicit differential equations, integral equations, and integro-differential equations (see, for example, [10–13]).

This notion was introduced by Kuratowski in [14] with the function α, which determines the degree of noncompactness of a bounded set B in complete metric spaces by the infimum of the numbers $\varepsilon > 0$ such that B admits a finite covering by sets of diameter smaller than ε, i.e.,

$$\alpha(B) = \inf\left\{ \varepsilon > 0 : B \subset \bigcup_{i=1}^{n} S_i : S_i \subset X,\ \mathrm{diam}(S_i) < \varepsilon,\ i = 1,2,\ldots,n,\ n \in \mathbb{N} \right\}. \quad (1)$$

In 1964, Perov [15] initiated the study of fixed point theorems in complete vector-valued metric spaces by extending and proving the Banach contraction principle in complete vector-valued metric spaces. Viorel in [16] studied the topological fixed point theorem of type Schauder in generalized Banach spaces. Graef et al. in [17] presented many topological methods for differential equations and inclusions in complete vector-valued metrics.

Precup in [18] showed that using the vector-valued norm is more appropriate when treating systems of equations. Several authors have recently studied the existence of solutions to systems of the differential and integral equations in generalized Banach using appropriate fixed point theorems; for example, see [16,17,19–25].

This paper is organized as follows: Section 2 is devoted to providing some preliminary remarks, lemmas, and definitions regarding generalized Banach spaces, the generalized measure of noncompactness, and the Riemann-Liouville fractional integrals. In Section 3, we present a generalization of the generalized Darbo's fixed point theorem [17]. Then, we prove that the following system of implicit fractional integral equations

$$\begin{cases} \varrho_1(\iota) = \int_0^\iota \mathbb{K}_1(\iota,\varsigma)\mathbb{P}_1(\iota,\varrho_1(\varsigma),\varrho_2(\varsigma),{}^{RL}\mathcal{I}_{0^+}^{\gamma_1}\varrho_1(\varsigma))d\varsigma, \\ \varrho_2(\iota) = \int_0^\iota \mathbb{K}_2(\iota,\varsigma)\mathbb{P}_2(\iota,\varrho_1(\varsigma),\varrho_2(\varsigma),{}^{RL}\mathcal{I}_{0^+}^{\gamma_2}\varrho_2(\varsigma))d\varsigma, \end{cases} \tag{2}$$

has a solution in the generalized Banach space $\mathcal{C}([0,1],\mathbb{R}) \times \mathcal{C}([0,1],\mathbb{R})$.

2. Preliminaries

In this section, we present some basic concepts that will be used to gain our own results. In the rest of this paper, $\mathcal{M}_{m\times n}(\mathbb{R}_+)$ is denoted to family of matrices of positive real numbers with dimension $m \times n$. We start our work by defining on $\mathcal{M}_{m\times n}(\mathbb{R}_+)$ a partial order relation as follows: Let $M,\ N \in \mathcal{M}_{m\times n}(\mathbb{R}_+), m \geq 1$ and $n \geq 1$. Put $M = (M_{i,j})_{\substack{1\leq j\leq m\\ 1\leq i\leq n}}$ and $N = (N_{i,j})_{\substack{1\leq j\leq m\\ 1\leq i\leq n}}$. Then, we define $\preccurlyeq$ on $\mathcal{M}_{m\times n}(\mathbb{R}_+)$ by

$$\begin{aligned} M \preccurlyeq N \text{ if } N_{i,j} \geq M_{i,j} &\quad \text{for all } j = 1,\cdots,m,\ i = 1,\cdots,n. \\ M \prec N \text{ if } N_{i,j} > M_{i,j} &\quad \text{for all } j = 1,\cdots,m,\ i = 1,\cdots,n. \end{aligned}$$

Let $\Omega = \prod_{i=1}^n \Omega_i$ be a bounded set of $\mathbb{R}^n$. The following vectors

$$\widehat{\sup}\{\lambda : \lambda \in \Omega\} := \begin{pmatrix} \sup\{\lambda_1 : \lambda_1 \in \Omega_1\} \\ \vdots \\ \sup\{\lambda_n : \lambda_n \in \Omega_n\} \end{pmatrix},\ \widehat{\inf}\{\lambda : \lambda \in \Omega\} := \begin{pmatrix} \inf\{\lambda_1 : \lambda_1 \in \Omega_1\} \\ \vdots \\ \inf\{\lambda_n : \lambda_n \in \Omega_n\} \end{pmatrix}$$

represent the supremum and infimum bounds of Ω.

Definition 1. *Let $\mathcal{E}$ be a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A vector-valued norm (generalized norm) on $\mathcal{E}$ is a map*

$$\begin{aligned} \|\cdot\|_G : \mathcal{E} &\longrightarrow [0,+\infty)^n \\ \eta &\mapsto \|\eta\|_G = \begin{pmatrix} \|\eta\|_1 \\ \vdots \\ \|\eta\|_n \end{pmatrix} \end{aligned}$$

with the following properties

(i) *For all $\eta \in \mathcal{E}$; if $\|\eta\|_G = 0_{\mathbb{R}_+^n}$, then $\eta = 0$,*
(ii) *$\|\lambda\eta\|_G = |\lambda|\|\eta\|_G$ for all $\eta \in \mathcal{E}$ and $\lambda \in \mathbb{K}$, and*
(iii) *$\|\eta + v\|_G \preccurlyeq \|\eta\|_G + \|v\|_G$ for all $\eta, v \in \mathcal{E}$.*

The pair $(\mathcal{E}, \|\cdot\|_G)$ is called a vector (generalized) normed space. Furthermore, $(\mathcal{E}, \|\cdot\|_G)$ is called a generalized Banach space (in short, GBS). Every vector metric space that is generated by its vector metric is complete.

Proposition 1. [17] The definitions of convergence sequence, continuity, open subsets, and closed subsets in the sense of Perov for a GBS are similar to those for usual Banach spaces.

Let $(\mathcal{E}, \|\cdot\|_G)$ be a generalized Banach space, $r = (r_1, \cdots, r_n) \in \mathbb{R}_+^n$, and $\eta_0 \in \mathcal{E}$. Throughout this paper, the open balls centered at η_0 with radius r (resp., r_i) are defined by

$$B(\eta_0, r) = \{\eta \in \mathcal{E} : \|\eta_0 - \eta\|_G \prec r\} \ \ (\text{resp. } B_i(\eta_0, r_i) = \{\eta \in \mathcal{E} : \|\eta_0 - \eta\|_i < r_i\}), i = 1, \cdots, n.$$

Further, the closed balls centered at η_0 with radius r (resp. r_i) are defined by

$$\bar{B}(\eta_0, r) = \{\eta \in \mathcal{E} : \|\eta_0 - \eta\|_G \preccurlyeq r\} \ \ (\text{resp. } \bar{B}_i(\eta_0, r_i) = \{\eta \in \mathcal{E} : \|\eta_0 - \eta\|_i \leq r_i\}).$$

If $\eta_0 = 0$, then B_r and $\bar{B}_r$ are denoted to $B_r = B(0, r)$ and $\bar{B}_r = \bar{B}(0, r)$. Finally, the closure and the convex hull of an arbitrary subset $\mathcal{K}$ of $\mathcal{E}$ are denoted by $\overline{\mathcal{K}}$ and $co(\mathcal{K})$, respectively.

Definition 2. *A matrix $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ is said to be convergent to zero if*

$$M^m \longrightarrow 0, \quad \text{as} \quad m \longrightarrow \infty.$$

Lemma 1. [26] Let $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$. The following assertions are equivalent:

(i) $M^m \longrightarrow 0, \quad \text{as} \quad m \longrightarrow \infty.$

(ii) The matrix $I - M$ is invertible, and $(I - M)^{-1} \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$.

(iii) The eigenvalues of M lie in the open unit disc of $\mathbb{C}$.

Definition 3. *Let $(\mathcal{E}, \|\cdot\|_G)$ be a GBS and let $\mathcal{K}$ be a subset of $\mathcal{E}$. Then, $\mathcal{K}$ is said to be G-bounded if there exists a vector $A \in \mathbb{R}_+^n$ such that*

$$\text{for all } \eta \in \mathcal{K}, \qquad \|\eta\|_G \preccurlyeq A,$$

and we write

$$\|\mathcal{K}\|_G := \widehat{\sup}\{\|\eta\|_G : \eta \in \mathcal{K}\} = \begin{pmatrix} \sup\limits_{\eta\in\mathcal{K}} \|\eta\|_1 \\ \vdots \\ \sup\limits_{\eta\in\mathcal{K}} \|\eta\|_n \end{pmatrix} \preccurlyeq A.$$

Definition 4. Let $(\mathcal{E}, \|\cdot\|_G)$ be a GBS. A subset $\mathcal{K}$ of $\mathcal{E}$ is called G-compact if every open cover of $\mathcal{K}$ has a finite subcover. The subset $\mathcal{K}$ is said to be relatively G-compact if its closure is G-compact.

Notation: We denote by $\mathcal{N}_G(\mathcal{E})$ the family of all relatively G-compact subsets of $\mathcal{E}$.

Now, we present a definition of an axiomatic measure of noncompactness for generalized Banach spaces similar to that given by Banaś and Goebel [10] in 1980.

Definition 5. *Let $(\mathcal{E}, \|\cdot\|_G)$ be a GBS and let $\mathcal{B}_G(\mathcal{E})$ be the family of G-bounded subsets of $\mathcal{E}$. A map*

$$\mu_G : \mathcal{B}_G(\mathcal{E}) \longrightarrow [0, +\infty)^n$$

$$\Omega \mapsto \mu_G(\Omega) = \begin{pmatrix} \mu_1(\Omega) \\ \vdots \\ \mu_n(\Omega) \end{pmatrix}$$

is called a generalized measure of noncompactness (for short, G-MNC) defined on $\mathcal{E}$ if it satisfies the following conditions:

(i) *The family* $\ker \mu_G(\mathcal{E}) = \{\Omega \in \mathcal{B}_G(\mathcal{E}) : \mu_G(\Omega) = 0\} \neq \emptyset$ *and* $\ker \mu_G(\mathcal{E}) \subset \mathcal{N}_G(\mathcal{E})$.

(ii) *Monotonicity:* $\Omega_1 \subseteq \Omega_2 \Rightarrow \mu_G(\Omega_1) \preccurlyeq \mu_G(\Omega_2)$, *for all* $\Omega_1, \Omega_2 \in \mathcal{B}_G(\mathcal{E})$.

(iii) *Invariance under closure and convex hull:* $\mu_G(\Omega) = \mu_G(\overline{\Omega}) = \mu_G(co(\Omega))$, *for all* $\Omega \in \mathcal{B}_G(\mathcal{E})$.

(iv) *Convexity:* $\mu_G(\lambda\Omega_1 + (1-\lambda)\Omega_2) \preccurlyeq \lambda\mu_G(\Omega_1) + (1-\lambda)\mu_G(\Omega_2)$, *for all* $\Omega_1,\ \Omega_2 \in \mathcal{B}_G(\mathcal{E})$, *and* $\lambda \in [0,1]$.

(v) *Generalized Cantor intersection property: If* $(\Omega_m)_{m\geq 1}$ *is a sequence of nonempty, closed subsets of* $\mathcal{E}$ *such that* Ω_1 *is G-bounded,* $\Omega_1 \supseteq \Omega_2 \supseteq \ldots \supseteq \Omega_m \ldots$, *and* $\lim\limits_{m\to+\infty} \mu_G(\Omega_m) = 0_{\mathbb{R}^n_+}$, *then the set* $\Omega_\infty := \bigcap\limits_{m=1}^{\infty} \Omega_m$ *is nonempty and is G-compact.*

Example 1. *Let* $\mathcal{E}$ *be a GBS that is defined as the Cartesian product of Banach spaces* $\mathcal{E}_i$, $i = 1,\ldots,n$. *Then, the generalized Kuratowski measure* α_G *is an example of a G-MNC, which is defined as follows:*
For a G-bounded subset $\Omega := \prod_{i=1}^{n} \Omega_i$ *of* $\mathcal{E}$, *we let*

$$\alpha_G(\Omega) := \widehat{\inf}\left\{ \varepsilon \in \mathbb{R}^n_+ : \Omega \subset \bigcup_{k=1}^{m} S_k : S_k \subset \mathcal{E},\ \delta_G(S_k) \prec \varepsilon,\ k = 1,\ldots,m,\ m \in \mathbb{N} \right\},$$

$$= \begin{pmatrix} \inf\{\varepsilon_1 > 0 : \Omega_1 \subseteq \bigcup_{k=1}^{m} S_k^1,\ \delta_1(S_k^1) < \varepsilon_1,\ k = 1,\ldots,m,\ m \in \mathbb{N}\} \\ \vdots \\ \inf\{\varepsilon_n > 0 : \Omega_n \subseteq \bigcup_{k=1}^{m} S_k^n,\ \delta_n(S_k^n) < \varepsilon_n,\ k = 1,\ldots,m,\ m \in \mathbb{N}\} \end{pmatrix}$$
$$= \begin{pmatrix} \alpha_1(\Omega_1) \\ \vdots \\ \alpha_n(\Omega_n) \end{pmatrix}.$$

Here,

$$\delta_G(\Omega) = \begin{cases} 0_{\mathbb{R}^n}, \text{ if } \Omega \text{ is empty} \\ \widehat{\sup}\{\|\eta - v\|_G : \eta, v \in \Omega\}, \text{ otherwise} \end{cases}$$

and for each $i = 1, \cdots, n$

$$\delta_i(\Omega_i) = \begin{cases} 0, \text{ if } \Omega_i \text{ is empty} \\ \sup\{\|\eta_i - v_i\|_i : \eta_i, v_i \in \Omega_i\}, \text{ otherwise.} \end{cases}$$

Remark 1. *It is clear that if* $n = 1$ *in the previous example, then we obtain the usual Kuratowski measure of noncompactness* α (1).

Definition 6. *Let* $(\mathcal{E}, \|.\|_G)$ *be a GBS, and let* μ_G *be a G-MNC. Let M be a matrix that belongs to* $\mathcal{M}_{n\times n}(\mathbb{R}^+)$. *A self-mapping* $S : \mathcal{E} \to \mathcal{E}$ *is said to be an M-set contractive mapping with respect to* μ_G *if S maps G-bounded sets into G-bounded sets in such a way that*

$$\mu_G(S(\Omega)) \preccurlyeq M\mu_G(\Omega),$$

for every nonempty G-bounded subset Ω *of* $\mathcal{E}$. *If the matrix M converges to zero, then we say that S satisfies the generalized Darbo condition.*

Now, we give some basics of the fractional calculus.

Definition 7. *[27,28] Let $J = [a, b]$, $(-\infty < a < b < +\infty)$ be a finite interval on the real axis $\mathbb{R}$. The Riemann-Liouville fractional integral of the function $f \in L^1([a, b])$ of order $\gamma \in \mathbb{C}$ is defined by*

$$^{RL}\mathcal{I}^{\gamma}_{a^+} f(t) = \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} f(s) ds \ \ (a < t, \mathcal{R}(\gamma) > 0),$$

where the function

$$\gamma \longmapsto \Gamma(\gamma) = \int_0^\infty t^{\gamma-1} e^{-t} dt, \qquad \mathcal{R}(\gamma) > 0$$

is Euler's gamma function.

Lemma 2. [27] Let $\gamma \in \mathbb{R}$ with $0 < \gamma < 1$. Then, the fractional integration operator $\mathcal{I}^{\gamma}_{a^+}$ is bounded in $\mathcal{C}([a, b])$. Moreover,

$$\left\| ^{RL}\mathcal{I}^{\gamma}_{a^+} f \right\|_\infty \leq \frac{(b-a)^\gamma}{\Gamma(1+\gamma)} \|f\|_\infty.$$

Lemma 3. [12] Let $\Omega \subset \mathcal{C}(a, b)$ be bounded and equicontinuous. Then, $\overline{\text{co}}(\Omega) \subset \mathcal{C}(a, b)$ is also bounded and equicontinuous.

Lemma 4. [29] Let $\Omega \subset \mathcal{C}(a, b)$ be bounded and equicontinuous, and let α the Kuratowski's measure of noncompactness. Then, $u(\iota) = \alpha(\Omega(\iota))$ is continuous on J and

$$\alpha\left(\int_a^b \Omega(\varsigma) d\varsigma\right) \leqslant \int_a^b \alpha(\Omega(\varsigma)) d\varsigma.$$

Now, we recall the Schauder-type theorem for generalized Banach spaces.

Theorem 1. [16] Let $(\mathcal{E}, \|\cdot\|_G)$ be a GBS; $\mathcal{K}$ a closed, convex subset of $\mathcal{E}$; and let $N : \mathcal{K} \longrightarrow \mathcal{K}$ be a continuous operator such that $N(\mathcal{K})$ is relatively G-compact. Then, N has at least one fixed point in $\mathcal{K}$.

3. Main Results

We start our work by expanding Lemma 2.4 from [12] into a more general form.

Theorem 2. Let $\mathcal{K}$ be a closed, G-bounded, and convex subset of a GBS $\mathcal{E}$, and let $N : \mathcal{K} \to \mathcal{K}$ be a continuous operator. For any subset Ω of $\mathcal{K}$, set

$$\tilde{N}^1\Omega = N\Omega, \quad \tilde{N}^n\Omega = N\left(\overline{\text{co}}\left(\tilde{N}^{n-1}\Omega\right)\right), \quad n = 2, 3, \ldots \tag{3}$$

Suppose there exists a matrix M that approaches zero and a positive integer n_0 such that for any subset Ω of $\mathcal{K}$, we have

$$\mu_G(\tilde{N}^{n_0}\Omega) \preccurlyeq M\mu_G(\Omega), \tag{4}$$

where μ_G is an arbitrary generalized measure of noncompactness. Then, N has at least one fixed point in $\mathcal{K}$.

Proof. Consider the sequence of sets

$$\mathcal{K}_n = \begin{cases} \mathcal{K}_0 = \mathcal{K} & \text{if } n = 0, \\ \overline{\text{co}}(\tilde{N}^{n_0}(\mathcal{K}_{n-1})) & \text{otherwise.} \end{cases}$$

First, we will prove that this sequence is decreasing. We know that

$$\mathcal{K}_1 = \overline{\text{co}}(\tilde{N}^{n_0}(\mathcal{K}_0)) \subset \mathcal{K},$$

hence,

$$\tilde{N}^{n_0}(\mathcal{K}_1) \subset \tilde{N}^{n_0}(\mathcal{K}_0),$$

that means

$$\mathcal{K}_2 = \overline{\text{co}}\big(\tilde{N}^{n_0}(\mathcal{K}_1)\big) \subset \overline{\text{co}}\big(\tilde{N}^{n_0}(\mathcal{K}_0)\big) = \mathcal{K}_1.$$

By induction, we find $\mathcal{K}_n \subset \mathcal{K}_{n-1}$, with $n \in \mathbb{N}$. Thus, $(\mathcal{K}_n)_{n\in\mathbb{N}}$ is a decreasing sequence. So, (4) implies that

$$\begin{aligned}
\mu_G(\mathcal{K}_n) &= \mu_G\big(\overline{\text{co}}\big(\tilde{N}^{n_0}(\mathcal{K}_{n-1})\big)\big) \\
&\preccurlyeq M\mu_G(\mathcal{K}_{n-1}) = M\mu_G\big(\overline{\text{co}}\big(\tilde{N}^{n_0}(\mathcal{K}_{n-2})\big)\big) \\
&\preccurlyeq M^2\mu_G(\mathcal{K}_{n-2}) \\
&\ \ \vdots \\
&\preccurlyeq M^n\mu_G(\mathcal{K}) \to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

By keeping in mind that the mapping μ_G generalizes the Cantor's intersection theorem, it follows that the intersection $\mathcal{K}_\infty = \bigcap_{n=0}^{\infty} \mathcal{K}_n$ is a convex closed compact set, and we can prove that $N(\mathcal{K}_\infty) \subset \mathcal{K}_\infty$. So, Theorem 1 ensures that N has at least one fixed point in $\mathcal{K}_\infty \subset \mathcal{K}$. $\square$

Remark 2. *Theorem 2 is a generalization of the generalized Darbo's fixed point theorem [17] with* $n_0 = 1$.

Now, we are concerned with the existence of solutions to the implicit system of fractional integral equations in (2). First, we present the next useful lemmas.

Lemma 5. Assume for all $r \succ 0_{\mathbb{R}^2}$, the function $\mathbb{P}_i$ is bounded and uniformly continuous on $[0,1] \times B_{r_1} \times B_{r_2} \times B_{r_i}$ and $\Omega \subset \mathcal{C}(0,1) \times \mathcal{C}(0,1)$ is bounded and equicontinuous. Then, the family $\{\mathbb{P}_i(\iota, \varrho_1(\varsigma), \varrho_2(\varsigma), {}^{RL}\mathcal{I}_{0^+}^{\gamma_2}\varrho_i(\varsigma)) : \varrho \in \Omega\}$ is bounded and equicontinuous in $\mathcal{C}(0,1)$.

Remark 3. *The previous lemma is a corollary of Lemma 1 in [30].*

Lemma 6. The space $\mathcal{E} = \mathcal{C}(0,1) \times \mathcal{C}(0,1)$ equipped with the generalized norm

$$\|f\|_G := \begin{pmatrix} \|f_1\|_\infty \\ \\ \|f_2\|_\infty \end{pmatrix},$$

for each $f := (f_1, f_2) \in \mathcal{C}(0,1) \times \mathcal{C}(0,1)$ is a generalized Banach space.

We consider the following assumptions:

$(\mathcal{H}_1)$ $\mathbb{K}_1,\ \mathbb{K}_2 \in \mathcal{C}([0,1]^2, \mathbb{R})$.

$(\mathcal{H}_2)$ The functions $\mathbb{P}_i : [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, $i = 1,2$ are supposed to satisfy the following:

(a) There exists a matrix $M_1 = \begin{pmatrix} \vartheta_{1,1} & \vartheta_{1,2} & \vartheta_{1,3} \\ \vartheta_{2,1} & \vartheta_{2,2} & \vartheta_{2,3} \end{pmatrix} \in \mathcal{M}_{2\times 3}(\mathbb{R}_+)$ such that for each $(\iota, \varrho_1, \varrho_2, \varrho_3), (\iota, \bar{\varrho}_1, \bar{\varrho}_2, \bar{\varrho}_3) \in (a,b) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, and for $i \in \{1,2\}$, we have

$$|\mathbb{P}_i(\iota, \varrho_1, \varrho_2, \varrho_3) - \mathbb{P}_i(\iota, \bar{\varrho}_1, \bar{\varrho}_2, \bar{\varrho}_3)| \le \vartheta_{i,1}|\varrho_1 - \bar{\varrho}_1| + \vartheta_{i,2}|\varrho_2 - \bar{\varrho}_2| + \vartheta_{i,3}|\varrho_3 - \bar{\varrho}_3|.$$

(b) The functions $\mathbb{P}_1$ and $\mathbb{P}_2$ are continuous with respect to the first variable, and there exists functions $\theta_i \in \mathcal{C}([0,1], \mathbb{R}_+)$ and a vector $r \in \mathbb{R}^2_+$ such that

$$|\mathbb{P}_i(\iota, \varrho_1(\varsigma), \varrho_2(\varsigma), {}^{RL}\mathcal{I}^{\gamma_i}_{0^+}\varrho_i(\varsigma))| \leq \theta_i(\varsigma),\ i = 1, 2$$

for $\varrho \in \mathcal{E}$ with $\|\varrho\|_{G_\mathcal{E}} \preccurlyeq r$.

$(\mathcal{H}_3)$ There exists a matrix $M_2 = \begin{pmatrix} \kappa_{1,1} & \kappa_{1,2} & \kappa_{1,3} \\ \kappa_{2,1} & \kappa_{2,2} & \kappa_{2,3} \end{pmatrix} \in \mathcal{M}_{2\times 3}(\mathcal{C}^1([0,1], \mathbb{R}_+))$ such that for any bounded sets $\mathbb{D}_j \subset \mathbb{R}$, $j = 1, 2, 3$ and $\iota \in [0,1]$ we have

$$\alpha_i(\mathbb{P}_i(\iota, \mathbb{D}_1, \mathbb{D}_2, \mathbb{D}_3)) \leq \sum_{j=1}^{3} \kappa_{i,j}(\iota)\alpha_i(\mathbb{D}_j). \tag{5}$$

Theorem 3. Suppose that the assumptions $(\mathcal{H}_1) - (\mathcal{H}_3)$ are satisfied. Then, the coupled implicit fractional integral Equation (2) has a solution in $\mathcal{E}$ provided that there exists a vector $\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$ of positive numbers fulfilling the following inequality

$$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \succcurlyeq K\left[\begin{pmatrix} \vartheta_{1,1} + \dfrac{\vartheta_{1,3}}{\Gamma(1+\gamma_1)} & \vartheta_{1,2} \\ \vartheta_{2,1} & \vartheta_{2,2} + \dfrac{\vartheta_{2,3}}{\Gamma(1+\gamma_2)} \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} + \begin{pmatrix} \|\mathbb{P}_1(.,0,0,0)\|_\infty \\ \|\mathbb{P}_2(.,0,0,0)\|_\infty \end{pmatrix}\right]. \tag{6}$$

Here,

$$K := \max_{i\in\{1,2\}}\left\{\sup_{(\iota,\varsigma)\in[0,1]^2} |\mathbb{K}_i(\iota,\varsigma)|\right\}.$$

Proof. We define the operator $\Xi : \mathcal{C}([0,1], \mathbb{R}) \times \mathcal{C}([0,1], \mathbb{R}) \to \mathcal{C}([0,1], \mathbb{R}) \times \mathcal{C}([0,1], \mathbb{R})$ by

$$\begin{aligned} \Xi(\varrho)(\iota) &= \begin{pmatrix} \Xi_1(\varrho)(\iota) \\ \Xi_2(\varrho)(\iota) \end{pmatrix} \\ &= \begin{pmatrix} \int_0^\iota \mathbb{K}_1(\iota,\varsigma)\mathbb{P}_1(\iota, \varrho_1(\varsigma), \varrho_2(\varsigma), {}^{RL}\mathcal{I}^{\gamma_1}_{0^+}\varrho_1(\varsigma))d\varsigma \\ \int_0^\iota \mathbb{K}_2(\iota,\varsigma)\mathbb{P}_2(\iota, \varrho_1(\varsigma), \varrho_2(\varsigma), {}^{RL}\mathcal{I}^{\gamma_2}_{0^+}\varrho_2(\varsigma))d\varsigma \end{pmatrix}. \end{aligned}$$

Let $\bar{B}_r$ be the closed ball on $\mathcal{E}$ centered at the origin of radius $r \succcurlyeq 0_{\mathbb{R}^2}$, where the positive vector $r \in \mathbb{R}^2_+$ has the properties as described above in Hypothesis $(\mathcal{H}_2)(b)$ and satisfies the inequality (6).

We focus on applying Theorem 2 to find a fixed point for the operator Ξ in $\mathcal{E}$. The proof will be divided into the following steps:
Step 1: First, we shall show that the mapping

$$\Xi : \mathcal{C}([0,1], \mathbb{R}) \times \mathcal{C}([0,1], \mathbb{R}) \to \mathcal{C}([0,1], \mathbb{R}) \times \mathcal{C}([0,1], \mathbb{R})$$

is well-defined. To see this, let $\varrho \in \mathcal{E}$ and $\{\iota_n\}_{n\in\mathbb{N}}$ be a sequence in $[0,1]$ converging to a point ι in $[0,1]$. Hence, for each $i = 1, 2$, we have

$$
\begin{aligned}
|(\Xi_i\varrho)(\iota_n)-(\Xi_i\varrho)(\iota)| \leq{}& \int_0^{\iota_n}\Big|\mathbb{K}_i(\iota_n,\varsigma)\mathbb{P}_i(\iota_n,\varrho_1(\varsigma),\varrho_2(\varsigma),{}^{RL}\mathcal{I}_{0^+}^{\gamma_i}\varrho_i(\varsigma))\\
&-\mathbb{K}_i(\iota,\varsigma)\mathbb{P}_i(\iota,\varrho_1(\varsigma),\varrho_2(\varsigma),{}^{RL}\mathcal{I}_{0^+}^{\gamma_i}\varrho_i(\varsigma))\Big|d\varsigma\\
&+\int_{\iota}^{\iota_n}\Big|\mathbb{K}_i(\iota,\varsigma)\mathbb{P}_i(\iota,\varrho_1(\varsigma),\varrho_2(\varsigma),{}^{RL}\mathcal{I}_{0^+}^{\gamma_i}\varrho_i(\varsigma))\Big|d\varsigma\\
\leq{}& \int_0^1|\mathbb{K}_i(\iota_n,\varsigma)-\mathbb{K}_i(\iota,\varsigma)|\Big|\mathbb{P}_i(\iota,\varrho_1(\varsigma),\varrho_2(\varsigma),{}^{RL}\mathcal{I}_{0^+}^{\gamma_i}\varrho_i(\varsigma))\Big|d\varsigma\\
&+\int_0^1|\mathbb{K}_i(\iota,\varsigma)|\Big|\mathbb{P}_i(\iota_n,\varrho_1(\varsigma),\varrho_2(\varsigma),{}^{RL}\mathcal{I}_{0^+}^{\gamma_i}\varrho_i(\varsigma))\\
&-\mathbb{P}_i(\iota,\varrho_1(\varsigma),\varrho_2(\varsigma),{}^{RL}\mathcal{I}_{0^+}^{\gamma_i}\varrho_i(\varsigma))\Big|d\varsigma\\
&+K\sup_{\iota\in[0,1]}|\mathbb{P}_i(\iota,\varrho_1(\varsigma),\varrho_2(\varsigma),{}^{RL}\mathcal{I}_{0^+}^{\gamma_i}\varrho_i(\varsigma))||\iota_n-\iota|\\
\leq{}& \theta_i(\varsigma)\int_0^1|\mathbb{K}_i(\iota_n,\varsigma)-\mathbb{K}_i(\iota,\varsigma)|d\varsigma+K\theta_i(\varsigma)|\iota_n-\iota|\\
&+K\int_0^1\Big|\mathbb{P}_i(\iota_n,\varrho_1(\varsigma),\varrho_2(\varsigma),{}^{RL}\mathcal{I}_{0^+}^{\gamma_i}\varrho_i(\varsigma))\\
&-\mathbb{P}_i(\iota,\varrho_1(\varsigma),\varrho_2(\varsigma),{}^{RL}\mathcal{I}_{0^+}^{\gamma_i}\varrho_i(\varsigma))\Big|d\varsigma.
\end{aligned}
\tag{7}
$$

The Hypotheses $(\mathcal{H}_1)$, $(\mathcal{H}_2)(b)$, and the dominated convergence theorem imply that $\Xi_i\varrho\in\mathcal{C}([0,1],\mathbb{R})$ for each $i\in\{1,2\}$. Hence, Ξ is well-defined. Furthermore, if $\varrho\in\bar{B}_r$, then for each $i=1,2$, we have

$$
\begin{aligned}
|(\Xi_i\varrho)(\iota)| \leq{}& \int_0^1|\mathbb{K}_i(\iota,\varsigma)|\Big(|\mathbb{P}_i(\iota,\varrho_1(\varsigma),\varrho_2(\varsigma),{}^{RL}\mathcal{I}_{0^+}^{\gamma_i}\varrho_i(\varsigma))-\mathbb{P}_i(\iota,0,0,0)|+|\mathbb{P}_i(\iota,0,0,0)|\Big)d\varsigma\\
\leq{}& K\int_0^1\vartheta_{i,1}|\varrho_1(\varsigma)|+\vartheta_{i,2}|\varrho_2(\varsigma)|+\frac{\vartheta_{i,3}}{\Gamma(1+\gamma_i)}|\varrho_i(\varsigma)|+|\mathbb{P}_i(\iota,0,0,0)|d\varsigma\\
\leq{}& K(\vartheta_{i,1}\|\varrho_1\|_\infty+\vartheta_{i,2}\|\varrho_2\|_\infty+\frac{\vartheta_{i,3}}{\Gamma(1+\gamma_i)}\|\varrho_i\|_\infty+\|\mathbb{P}_i(.,0,0,0)\|_\infty).
\end{aligned}
$$

Since the vector r fulfills (6), we derive that for each $\varrho\in\bar{B}_r$ and $i=1,2$

$$
\begin{aligned}
\|(\Xi_i\varrho)\|_G \preccurlyeq{}& K\left[\begin{pmatrix}\vartheta_{1,1}+\dfrac{\vartheta_{1,3}}{\Gamma(1+\gamma_1)} & \vartheta_{1,2}\\ \vartheta_{2,1} & \vartheta_{2,2}+\dfrac{\vartheta_{2,3}}{\Gamma(1+\gamma_2)}\end{pmatrix}\begin{pmatrix}r_1\\ r_2\end{pmatrix}\right.\\
&\left.+\begin{pmatrix}\|\mathbb{P}_1(.,0,0,0)\|_\infty\\ \|\mathbb{P}_2(.,0,0,0)\|_\infty\end{pmatrix}\right]\\
\preccurlyeq{}& \begin{pmatrix}r_1\\ r_2\end{pmatrix}.
\end{aligned}
\tag{8}
$$

Due to (8), we conclude that Ξ is a mapping from $\bar{B}_r$ into $\bar{B}_r$.

Step 2: Now, our aim is to show that the map Ξ is continuous. Let $(\varrho_n)_{n\in\mathbb{N}} := (\varrho_{1,n}, \varrho_{2,n})_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}([0,1],\mathbb{R}) \times \mathcal{C}([0,1],\mathbb{R})$ that converges to $\varrho := (\varrho_1, \varrho_2)$. Then, ϱ belongs to the same space, and for each $i \in \{1,2\}$, we have

$$\begin{aligned}|(\Xi_i\varrho)(\iota) - (\Xi_i\varrho_n)(\iota)| \leq & \int_0^\iota \Big(|\mathbb{K}_i(\iota,\varsigma)|\mathbb{P}_i(\iota, \varrho_1(\varsigma), \varrho_2(\varsigma), {}^{RL}\mathcal{I}_{0^+}^{\gamma_i}\varrho_i(\varsigma)) \\ & - \mathbb{P}_i(\iota, \varrho_{1,n}(\varsigma), \varrho_{2,n}(\varsigma), {}^{RL}\mathcal{I}_{0^+}^{\gamma_i}\varrho_{i,n}(\varsigma))| \Big) d\varsigma, \\ \leq & K \int_0^\iota \Big(\vartheta_{i,1}|\varrho_1(\varsigma) - \varrho_{1,n}(\varsigma)| + \vartheta_{i,2}|\varrho_2(\varsigma) - \varrho_{2,n}(\varsigma)| \\ & + \frac{\vartheta_{i,3}}{\Gamma(1+\gamma_i)}|\varrho_i(\varsigma) - \varrho_{i,n}(\varsigma)| \Big) d\varsigma.\end{aligned}$$

By taking the supremum over ι, we find

$$\begin{aligned}\|(\Xi_i\varrho) - (\Xi_i\varrho_n)\|_G \leq & K \int_0^1 \Big(\vartheta_{i,1}|\varrho_1(\varsigma) - \varrho_{1,n}(\varsigma)| + \vartheta_{i,2}|\varrho_2(\varsigma) - \varrho_{2,n}(\varsigma)| \\ & + \frac{\vartheta_{i,3}}{\Gamma(1+\gamma_i)}|\varrho_i(\varsigma) - \varrho_{i,n}(\varsigma)| \Big) d\varsigma \\ \leq & K\Big(\vartheta_{i,1}\|\varrho_{1,n} - \varrho_1\|_G + \vartheta_{i,2}\|\varrho_{2,n} - \varrho_2\|_G + \frac{\vartheta_{i,3}}{\Gamma(1+\gamma_i)}\|\varrho_{i,n} - \varrho_i\|_G \Big).\end{aligned}$$

Thus, Ξ is continuous on $\mathcal{E}$.

Step 3: Here, we shall show that Ξ satisfies the inequality (4), which is related to Kuratowski's generalized measure of noncompactness. We begin by proving the equicontinuity of $\Xi_i(\bar{B}_r) \subset \mathcal{C}([0,1],\mathbb{R})$ for each $i = 1,2$. Let $\varrho \in \bar{B}_r, \tau, \iota \in [0,1]$, with $\iota < \tau$. Then, by using our hypotheses, we obtain

$$\begin{aligned}|(\Xi_i\varrho)(\tau) - (\Xi_i\varrho)(\iota)| \leq & \int_0^\tau \Big|\mathbb{K}_i(\tau,\varsigma)\mathbb{P}_i(\tau, \varrho_1(\varsigma), \varrho_2(\varsigma), {}^{RL}\mathcal{I}_{0^+}^{\gamma_i}\varrho_i(\varsigma)) \\ & - \mathbb{K}_i(\iota,\varsigma)\mathbb{P}_i(\iota, \varrho_1(\varsigma), \varrho_2(\varsigma), {}^{RL}\mathcal{I}_{0^+}^{\gamma_i}\varrho_i(\varsigma))\Big| d\varsigma \\ & + \int_\iota^\tau \Big|\mathbb{K}_i(\iota,\varsigma)\mathbb{P}_i(\iota, \varrho_1(\varsigma), \varrho_2(\varsigma), {}^{RL}\mathcal{I}_{0^+}^{\gamma_i}\varrho_i(\varsigma))\Big| d\varsigma \\ \leq & \theta_i(\varsigma) \int_0^1 |\mathbb{K}_i(\tau,\varsigma) - \mathbb{K}_i(\iota,\varsigma)| d\varsigma + K\theta_i(\varsigma)|\tau - \iota| \\ & + K \int_0^1 \Big|\mathbb{P}_i(\tau, \varrho_1(\varsigma), \varrho_2(\varsigma), {}^{RL}\mathcal{I}_{0^+}^{\gamma_i}\varrho_i(\varsigma)) - \mathbb{P}_i(\iota, \varrho_1(\varsigma), \varrho_2(\varsigma), {}^{RL}\mathcal{I}_{0^+}^{\gamma_i}\varrho_i(\varsigma))\Big| d\varsigma.\end{aligned}$$

So, the hypotheses $(\mathcal{H}_1)$ and $(\mathcal{H}_2)(b)$ imply that $\Xi_i(\bar{B}_r)$, $i = 1,2$ are equicontinuous.

Let $\mathcal{F}_i = \overline{\text{co}}\,(\Xi_i(\bar{B}_r))$ $i = 1,2$. Lemma 3 implies that $\mathcal{F}_i \subset \bar{B}_{r_i}, i = 1,2$ are bounded and equicontinuous, and $\Xi : \mathcal{F}_1 \times \mathcal{F}_2 \to \mathcal{F}_1 \times \mathcal{F}_2$ is a continuous and G-bounded operator. For any $\Omega \subset \mathcal{F}_1 \times \mathcal{F}_2$, we obtain that $\Xi_i\Omega, i = 1,2$ are bounded and equicontinuous on $[0,1]$. Now, using the Lemmas 3, 5 and Equation (3), we conclude that $\tilde{\Xi}_i^m\Omega, i = 1,2$ are bounded and equicontinuous. For each $m = 1,2,\ldots$, we have

$$\alpha_i\big(\tilde{\Xi}_i^m\Omega\big) = \max_{\iota\in[0,1]} \alpha_i\big((\tilde{\Xi}_i^m\Omega)(\iota)\big), \quad m = 1,2,\ldots, i = 1,2.$$

Now, we are going to prove that there are a positive integer n_0 and a matrix M that approaches zero such that for any $\Omega \subset \mathcal{F} := \mathcal{F}_1 \times \mathcal{F}_2$, the following inequality

$$\alpha_G\big(\tilde{\Xi}^{n_0}\Omega\big) \preccurlyeq M\alpha_G(\Omega)$$

holds. By using condition $(\mathcal{H}_3)$ and Lemma 4, we obtain

$$\begin{aligned}
\alpha_i\Big(\Big(\tilde{\Xi}_i{}^1\Omega\Big)(\iota)\Big) &= \alpha_i((\Xi_i\Omega)(\iota)) \\
&= \alpha_i\bigg(\int_0^\iota \mathbb{K}_i(\iota,\varsigma)\mathbb{P}_i(\iota,\Omega_1(\varsigma),\Omega_2(\varsigma),({}^{RL}\mathcal{I}_{0^+}^{\gamma_i}\Omega_i)(\varsigma))d\varsigma\bigg) \\
&\le K\int_0^\iota \alpha_i\Big(\mathbb{P}_i(\iota,\Omega_1(\varsigma),\Omega_2(\varsigma),({}^{RL}\mathcal{I}_{0^+}^{\gamma_i}\Omega_i)(\varsigma))\Big)d\varsigma \\
&\le K\int_0^\iota \bigg(\kappa_{i,1}(\iota)\alpha_1(\Omega_1(\varsigma)) + \kappa_{i,2}(\iota)\alpha_2(\Omega_2(\varsigma)) + \frac{\kappa_{i,3}(\iota)}{\Gamma(1+\gamma_i)}\alpha_i(\Omega_i(\varsigma))\bigg)d\varsigma \\
&\le K\iota\bigg(\|\kappa_{i,1}\|_\infty\alpha_1(\Omega_1) + \|\kappa_{i,2}\|_\infty\alpha_2(\Omega_2) + \frac{\|\kappa_{i,3}\|_\infty}{\Gamma(1+\gamma_i)}\alpha_i(\Omega_i)\bigg).
\end{aligned}$$

Suppose that

$$\alpha_i\Big(\Big(\tilde{\Xi}_i{}^m\Omega\Big)(\iota)\Big) \le \frac{(K\iota)^m L^{m-1}}{m!}\bigg(\|\kappa_{i,1}\|_\infty\alpha_1(\Omega_1) + \|\kappa_{i,2}\|_\infty\alpha_2(\Omega_2) + \frac{\|\kappa_{i,3}\|_\infty}{\Gamma(1+\gamma_i)}\alpha_i(\Omega_i)\bigg),$$

where

$$L = \|\kappa_{i,1}\|_\infty + \|\kappa_{i,2}\|_\infty + \frac{\|\kappa_{i,3}\|_\infty}{\Gamma(1+\gamma_i)}.$$

Hence, for any $\iota \in [0,1]$, we obtain

$$\begin{aligned}
\alpha_i\Big(\Big(\tilde{\Xi}_i{}^{m+1}\Omega\Big)(\iota)\Big) &= \alpha_i(\overline{\mathrm{co}}(\tilde{\Xi}_i{}^m\Omega)(\iota)) \\
&= \alpha_i\bigg(\int_0^\iota \mathbb{K}_i(\iota,\varsigma)\mathbb{P}_i(\iota,(\overline{\mathrm{co}}(\tilde{\Xi}_i{}^m\Omega))(\varsigma),(\overline{\mathrm{co}}(\tilde{\Xi}_i{}^m\Omega))(\varsigma),({}^{RL}\mathcal{I}_{0^+}^{\gamma_i}(\overline{\mathrm{co}}(\tilde{\Xi}_i{}^m\Omega)))(\varsigma))d\varsigma\bigg) \\
&\le K\int_0^\iota \alpha_i\Big(\mathbb{P}_i(\iota,(\overline{\mathrm{co}}(\tilde{\Xi}_i{}^m\Omega))(\varsigma),(\overline{\mathrm{co}}(\tilde{\Xi}_i{}^m\Omega))(\varsigma),({}^{RL}\mathcal{I}_{0^+}^{\gamma_i}(\overline{\mathrm{co}}(\tilde{\Xi}_i{}^m\Omega)))(\varsigma))\Big)d\varsigma \\
&\le K\int_0^\iota \kappa_{i,1}(\iota)\alpha_i(\overline{\mathrm{co}}(\tilde{\Xi}_i{}^m\Omega))(\varsigma) + \kappa_{i,2}(\iota)\alpha_i(\overline{\mathrm{co}}(\tilde{\Xi}_i{}^m\Omega))(\varsigma) \\
&+ \frac{\kappa_{i,3}(\iota)}{\Gamma(1+\gamma_i)}\alpha_i(\overline{\mathrm{co}}(\tilde{\Xi}_i{}^m\Omega))(\varsigma)d\varsigma \\
&= K\int_0^\iota \kappa_{i,1}(\iota)\alpha_i(\tilde{\Xi}_i{}^m\Omega(\varsigma)) + \kappa_{i,2}(\iota)\alpha_i(\tilde{\Xi}_i{}^m\Omega(\varsigma)) + \frac{\kappa_{i,3}(\iota)}{\Gamma(1+\gamma_i)}\alpha_i(\tilde{\Xi}_i{}^m\Omega(\varsigma)d\varsigma \\
&\le K\int_0^\iota \kappa_{i,1}(\iota)\bigg(\frac{(K\varsigma)^m L^{m-1}}{m!}\bigg(\|\kappa_{i,1}\|_\infty\alpha_1(\Omega_1) + \|\kappa_{i,2}\|_\infty\alpha_2(\Omega_2) \\
&+ \frac{\|\kappa_{i,3}\|_\infty}{\Gamma(1+\gamma_i)}\alpha_i(\Omega_i)\bigg)\bigg) \\
&+ \kappa_{i,2}(\iota)\bigg(\frac{(K\varsigma)^m L^{m-1}}{m!}\bigg(\|\kappa_{i,1}\|_\infty\alpha_1(\Omega_1) + \|\kappa_{i,2}\|_\infty\alpha_2(\Omega_2) \\
&+ \frac{\|\kappa_{i,3}\|_\infty}{\Gamma(1+\gamma_i)}\alpha_i(\Omega_i)\bigg)\bigg) \\
&+ \frac{\kappa_{i,3}(\iota)}{\Gamma(1+\gamma_i)}\bigg(\frac{(K\varsigma)^m L^{m-1}}{m!}\bigg(\|\kappa_{i,1}\|_\infty\alpha_1(\Omega_1) + \|\kappa_{i,2}\|_\infty\alpha_2(\Omega_2) \\
&+ \frac{\|\kappa_{i,3}\|_\infty}{\Gamma(1+\gamma_i)}\alpha_i(\Omega_i)\bigg)\bigg)d\varsigma \\
&\le \frac{(K\iota)^{m+1}L^m}{(m+1)!}\bigg(\|\kappa_{i,1}\|_\infty\alpha_1(\Omega_1) + \|\kappa_{i,2}\|_\infty\alpha_2(\Omega_2) + \frac{\|\kappa_{i,3}\|_\infty}{\Gamma(1+\gamma_i)}\alpha_i(\Omega_i)\bigg) \\
&\le \frac{K^{m+1}L^m}{(m+1)!}\bigg(\|\kappa_{i,1}\|_\infty\alpha_1(\Omega_1) + \|\kappa_{i,2}\|_\infty\alpha_2(\Omega_2) + \frac{\|\kappa_{i,3}\|_\infty}{\Gamma(1+\gamma_i)}\alpha_i(\Omega_i)\bigg).
\end{aligned}$$

Then, it follows that

$$\alpha_G\left(\tilde{\Xi}^m\Omega\right) \preccurlyeq \frac{K^m L^{m-1}}{(m)!}\begin{pmatrix} \|\kappa_{1,1}\|_\infty + \dfrac{\|\kappa_{1,3}\|_\infty}{\Gamma(1+\gamma_1)} & \|\kappa_{1,2}\|_\infty \\ \|\kappa_{2,1}\|_\infty & \|\kappa_{2,2}\|_\infty + \dfrac{\|\kappa_{2,3}\|_\infty}{\Gamma(1+\gamma_2)} \end{pmatrix}\begin{pmatrix} \alpha_1(\Omega_1) \\ \alpha_2(\Omega_2) \end{pmatrix}$$
$$=: M_m \begin{pmatrix} \alpha_1(\Omega_1) \\ \alpha_2(\Omega_2) \end{pmatrix}.$$

Notice that

$$\lim_{m\to\infty} \frac{K^m L^{m-1}}{(m)!} = 0.$$

Now, put

$$n_0 = \min\left\{m \in \mathbb{N} : \|M_m\|_{\mathcal{M}_{2\times 2}(\mathbb{R}_+)} < 1\right\}.$$

Notice that n_0 is a finite positive integer. Moreover, since $\rho(M_{n_0}) \leq \|M_{n_0}\|_{\mathcal{M}_{2\times 2}(\mathbb{R}_+)} < 1$, we conclude that M_{n_0} converges to zero. In other words, there are $n_0 \in \mathbb{N}$ and a matrix M_{n_0} converges to zero such that for any subset $\Omega \subset \bar{B}_r$,

$$\alpha_G\left(\tilde{\Xi}^{n_0}\Omega\right) \preccurlyeq M_{n_0}\alpha_G(\Omega).$$

By applying Theorem 1, we obtain that the problem (2) has a solution in $\mathcal{E}$. □

4. Conclusions

In this work, we gave a new fixed point result in spaces with a vector-norm structure related to a generalized measure of noncompactness. The conditions of that theorem are weaker compared with the conditions of the generalized Darbo theorem [17]. On the other hand, we applied our theorem to prove that there exist solutions for a coupled implicit Riemann-Liouville fractional integral equation with the help of the vector Kuratowski measure of noncompactness and its properties.

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