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Initial Value Problems of Fuzzy Fractional Coupled Partial Differential Equations with Caputo gH-Type Derivatives

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Abstract: The purpose of this paper is to investigate a class of initial value problems of fuzzy fractional coupled partial differential equations with Caputo gH-type derivatives. Firstly, using Banach fixed point theorem and the mathematical inductive method, we prove the existence and uniqueness of two kinds of gH-weak solutions of the coupled system for fuzzy fractional partial differential equations under Lipschitz conditions. Then we give an example to illustrate the correctness of the existence and uniqueness results. Furthermore, because of the coupling in the initial value problems, we develop Gronwall inequality of the vector form, and creatively discuss continuous dependence of the solutions of the coupled system for fuzzy fractional partial differential equations on the initial values and ε-approximate solution of the coupled system. Finally, we propose some work for future research.

Keywords: existence and uniqueness; continuous dependence and ε-approximation; coupled system of fuzzy fractional partial differential equations; Banach fixed point theorem; Gronwall inequality of the vector form

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1. Introduction

It is well known that fuzzy set theory naturally simulates uncertain systems [1,2] and has been probed into in linguistics, psychology, data sciences, decision-making and other related engineering and applied science fields; this is as a result of its tremendous adaptability and functionality (see [2]). Since there is still the possibility of ambiguity in real life, one needs to consider fuzzy uncertainty in order to better apply theory to life [3]. One of the basic characteristics of fuzzy numbers is to prevent the loss of information by using membership functions around crisp data [4]. Thus, in order to take into account the deliberately ignored uncertainties in the models, we introduce a fuzzy concept to make it possible for relatively complex systems to quantitatively describe and study things and concepts that are not deterministic. In fact, as Shah et al. [5] clearly indicated, "the modeling of some real world problems keeping uncertainty in data has given rise to fuzzy partial differential equations (PDEs)". In other words, fuzzy PDEs are usually used to deal with multi-dimensional dynamic systems of realistic problems in fuzzy environments [6] and have been developed more rapidly with the great expansion of research fields such as physical science, population dynamics, station elasticity, and so on. See, for example, [2,7–12] and the references therein.

In recent decades, firstly, the fractional differential operators as a kind of absolute operator provide a greater degree of freedom [5]. As we all know, the concept of Caputo fractional derivative was first proposed by Caputo in 1967. A lesser-known fact is that the Russian (Soviet) mathematician Gerasimov introduced the concept of fractional derivative 20 years before Caputo. So, it is also called the Gerasimov–Caputo derivative [13]. Secondly, fractional-order differential equations merge and describe problems more accurately [4] and accumulate the whole information of functions in a weighted form [14]. Thus, fractional-order differential equations have been widely used in simulating viscoelastic, turbulent, nonlinear biological systems and other real-world phenomena, especially in describing memory and genetic characteristics and so on, and promote the development of important disciplines such as physics and biology [15]. That is to say, the real-world problems can be fully described theoretically through fractional PDEs, and they can help us obtain more accurate results [16]. Relevant work can be found in [14–18] and their references.

In 2021, Niazi and Iqbal studied a class of Caputo fuzzy fractional evolution equations and obtained some important conclusions such as precise controllability of the evolution equations and existence and uniqueness of mild solutions (see [19–21]). About 10 years ago, Agarwal et al. [22] and Arshad and Lupulescu [23] applied different methods to prove the existence and uniqueness of solutions for fuzzy fractional ordinary differential equations. However, it is far from enough to solve practical problems using ordinary differential equations. Thus, PDEs were proposed. While thermal diffusion equations and Laplace equations can be well described by the classical PDEs, only mathematical models for describing real world problems with uncertainty can be successfully solved base on the introduction of fuzzy fractional PDEs. Hence, fuzzy fractional PDEs play a significant role in science and engineering. As a matter of fact, for solving nonlinear problems arising in environmental, medical, economical, social, physical and decision-making sciences, many scholars have developed some new concepts, methods and tools, which include integral transform of Fourier, Laplace, Sumudu, etc. (see [5]). Recently, Rashid et al. [24] studied a new method called EADM, which has a powerful function in the configuration of numerical solutions for nonlinear fuzzy fractional PDEs generated in physics and complex structures. However, as Bede and Stefanini [25] pointed out, it is well known that the usual Hukuhara difference (H-difference) between two fuzzy numbers exists only under very restrictive conditions and the generalized Hukuhara type (gH-type) difference of two fuzzy numbers exists under much less restrictive conditions, and the gH-type difference of intervals always exists. Thereupon, based on the concepts of gH-type differentiability and some properties due to Bede and Stefanini [25], Long et al. [26] defined fuzzy fractional integral and Caputo gH-type derivative for fuzzy-valued multivariable functions under H-difference and gH-type difference existing sorts. Next, they developed the concept of fuzzy Caputo derivatives from one-variable functions to fuzzy-valued multivariable functions, and stated that "it is important to think about the value of embedding our results within fractional calculus for fuzzy-valued multivariable functions in the sense of the gH-type derivative". Further, Long et al. [26] introduced and studied the following fuzzy hyperbolic Darboux problem under Caputo fractional gH-type derivative:

$$ {}^{C}_{gH}\mathcal{D}^{\hbar}_{k}u(x,y) = f(x,y,u(x,y)), \quad \forall (x,y) \in [0,a] \times [0,b], \quad k = 1,2 \tag{1} $$

with initial conditions $u(x,0) = \eta_1(x)$ for any $x \in [0,a]$ and $u(0,y) = \eta_2(y)$ for each $y \in [0,b]$, where $\hbar = (\hbar_1, \hbar_2) \in (0,1] \times (0,1]$ is the fractional order of Caputo gH-type derivative operator ${}^{C}_{gH}\mathcal{D}^{\hbar}_{k}$. Moreover, the existence and uniqueness results of two classes of fuzzy solutions for (1) are given by applying Banach and Schauder fixed point theorems, respectively. We note that the operator ${}^{C}_{gH}\mathcal{D}^{\hbar}_{k}$ in (1) and the main results of [26] presuppose the existence of gH-type difference and H-difference, respectively. Long et al. [26] indicated that "when we fuzzify these models to adopt real-world problems containing uncertainties, we find that there has been no paper developed on this subject for fuzzy fractional PDEs up to now". Recently, based on the gH-type differentiability, Senol et al. [4] exploited a perturbation-iterative algorithm for numerical solutions of fuzzy fractional PDEs under Caputo's gH-type derivative; here, Caputo time-fractional derivative was formalized for fuzzy numbers in the Hukuhara sense. Further, Shahsavari et al. [12] obtained fuzzy traveling wave solutions in special cases such as fuzzy convection–diffusion–reaction equations, fuzzy Klein–Gordon equations and others.

On the other hand, biodiversity is the most essential characteristic of an ecosystem. However, previous researchers mainly considered the survival and development of a single species and did not pay attention to the competition caused by the existence of multiple species. This class of relationship is called "coupling" if two or more things interact and influence each other [27]. These universal realistic problems have aroused the interest of many researchers, who claim that a complex system and process cannot be depicted by a single differential equation, so coupled systems have received extensive attention. For more details, one can refer to [3,28] and the references therein. In particular, in the sense of Caputo fractional derivatives, Dong et al. [29] proved the existence and uniqueness of solutions for a coupled system of nonlinear implicit fractional differential equations as follows:

$$\begin{cases} {}^{C}\mathcal{D}^{\alpha}x(t) = f(t, y(t), {}^{C}\mathcal{D}^{\alpha}x(t)), & 0 \le t \le 1, \\ {}^{C}\mathcal{D}^{\beta}y(t) = g(t, x(t), {}^{C}\mathcal{D}^{\beta}y(t)), & 0 \le t \le 1 \end{cases}$$

with initial conditions $x(0) = x_0$ and $y(0) = y_0$. Actually, one can see that it is a worth studying hotspot to employ fuzzy fractional PDE systems concerning the coupling systems, and it is very valuable and of great significance to extend the corresponding methods to study the coupled systems for fuzzy fractional PDEs.

Inspired by the work of predecessors such as Long et al. [26], Dong et al. [29] and other pioneers, in this paper, we consider the following coupled system of fuzzy fractional PDEs: For all $(x, y) \in J = [0, a] \times [0, b]$ and $k = 1, 2$,

$$\begin{cases} {}^{C}_{gH}\mathcal{D}^{\alpha}_{k}u(x,y) = f(x, y, v(x,y)), \\ {}^{C}_{gH}\mathcal{D}^{\beta}_{k}v(x,y) = g(x, y, u(x,y)) \end{cases} \tag{2}$$

with initial conditions

$$\begin{cases} u(x,0) = \xi_1(x),\ v(x,0) = \eta_1(x), & \forall x \in [0,a], \\ u(0,y) = \xi_2(y),\ v(0,y) = \eta_2(y), & \forall y \in [0,b], \end{cases} \tag{3}$$

where $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2) \in (0,1] \times (0,1]$ are fractional orders, and Caputo gH-type derivative operators ${}^{C}_{gH}\mathcal{D}^{\alpha}_{k}$ and ${}^{C}_{gH}\mathcal{D}^{\beta}_{k}$ are the same as in (1). This problem is a new fuzzy hyperbolic coupled system.

Remark 1. (i) *If* $\beta = (1,1)$, *then* (2) *with* (3) *becomes an initial problem as follows:*

$$\begin{cases} {}^{C}_{gH}\mathcal{D}^{\alpha}_{k}u(x,y) = f(x, y, v(x,y)), \\ \dfrac{\partial v(x,y)}{\partial x \partial y} = g(x, y, u(x,y)), \\ u(x,0) = \xi_1(x),\ u(0,y) = \xi_2(y), \\ v(x,0) = \eta_1(x),\ v(0,y) = \eta_2(y) \end{cases} \tag{4}$$

for any $(x, y) \in [0, a] \times [0, b]$ *and each* $k = 1, 2$, *where* α *is the same as in* (2). *Further, if* $\alpha = (1, 1)$, *then* (4) *reduces to the form*

$$\begin{cases} \dfrac{\partial u(x,y)}{\partial x \partial y} = f(x, y, v(x,y)), \\ \dfrac{\partial v(x,y)}{\partial x \partial y} = g(x, y, u(x,y)), \\ u(x,0) = \xi_1(x),\ u(0,y) = \xi_2(y), \\ v(x,0) = \eta_1(x),\ v(0,y) = \eta_2(y) \end{cases} \tag{5}$$

for every $(x, y) \in [0, a] \times [0, b]$.

(ii) *As far as we know, a problem similar to* (5) *was investigated more than 100 years ago by Riquier* [30]. *In the 20th century, French, Japanese, and Russian mathematicians published numerous publications on similar subjects (see, for example,* [31,32]). *Recently, in Kazakov* [33,34], *concerning the PDE problems consisting of two equations, where the right side depends on the unknown function, which is not differentiated in this equation, both independent variables and boundary conditions are specified on two coordinate axes as the "Generalized Cauchy problem". In* [34], *Kazakov and Lempert introduced applications of the generalized Cauchy problem.*

(iii) *While* (4) *and* (5) *are similar in form to the problems studied by Riquier* [30] *and Kazakov* [33], *they rely on gH-type derivatives. So we register that* (4) *and* (5) *are brand new and have not been reported in the literature.*

The remainder of this paper is organized as follows. In Section 2, we set out some necessary concepts and other preliminaries. We prove the existence and uniqueness of two kinds of gH-weak solutions for (2) with (3) using Banach fixed point theorem and give a numerical example in Section 3. In Section 4, on the basis of modifying the initial conditions, (2) with (3) shall be equivalent to a class of new nonlinear fractional order coupled Volterra integro-differential systems and the results that the solutions of (2) with (3) depend continuously on the initial values and ε-approximate solutions of (2) with (3) are given. Finally, some conclusions and future work are discussed in Section 5.

2. Preliminaries

In order to dispose of (2) with (3), we firstly follow the versions of some concepts introduced by Long et al. [26] for fractional integral and fractional Caputo gH-derivative of fuzzy valued multivariable functions.

Throughout this paper, let E_1 and E_2 be the spaces of fuzzy numbers from $\mathbb{R}$ into $[0,1]$, the mappings in which they are normal, fuzzy convex, upper semi-continuous and compactly supported. Define τ-level sets of fuzzy number $\varpi : \mathbb{R} \to [0,1]$ as follows

$$[\varpi]^\tau = \begin{cases} \{x \in \mathbb{R} : \varpi(x) \geq \tau\}, & \text{if} \quad 0 < \tau \leq 1, \\ cl(supp\,\varpi), & \text{if} \quad \tau = 0, \end{cases}$$

where cl is the closure of the set, and $supp\,\varpi$ denotes the support of the fuzzy number ϖ, which is defined by $supp\,\varpi = cl\{x \in \mathbb{R}^n | \varpi(x) > 0\}$. For any $\varpi \in E_i$ $(i = 1,2)$ and $\tau \in [0,1]$, the closed and bounded interval $[\varpi_\tau^-, \varpi_\tau^+]$ is the τ-level set of the fuzzy number ϖ, where ϖ_τ^- and ϖ_τ^+ are separately called the left-hand endpoint and the right-hand endpoint of ϖ, and $len[\varpi]^\tau = \varpi_\tau^+ - \varpi_\tau^-$ represents the diameter of the τ-level set of ϖ. The supremum metric on E_i for $i = 1,2$ is defined by

$$d_\infty(\varpi, \omega) = \sup_{0 \leq \tau \leq 1} \max\{|\varpi_\tau^- - \omega_\tau^-|, |\varpi_\tau^+ - \omega_\tau^+|\}, \quad \forall \varpi, \omega \in E_i.$$

For all ϖ, $\omega \in E_1$, $\tau \in [0,1]$, we have

$$[\varpi + \omega]^\tau = [\varpi]^\tau + [\omega]^\tau \tag{6}$$

and if $\varpi \ominus \omega$ exists, where $\ominus$ is the H-difference defined in [35], then

$$[\varpi \ominus \omega]^\tau = [\varpi_\tau^- - \omega_\tau^-, \varpi_\tau^+ - \omega_\tau^+]. \tag{7}$$

Lemma 1. ([10]) *For all $\nu, \theta, \omega, e \in E_i$, $i = 1,2$, we have the following presentations:*

(i) $d_\infty(\nu + \theta, \omega + e) \leq d_\infty(\nu, \omega) + d_\infty(\theta, e)$.

(ii) *If $\nu \ominus \theta$ and $\omega \ominus e$ exist, then $d_\infty(\nu \ominus \theta, \omega \ominus e) \leq d_\infty(\nu, \omega) + d_\infty(\theta, e)$.*

Remark 2. *The conclusions of Lemma* 1 (ii) *are conditional on existence of H-difference, which will be used to prove our main results.*

Definition 1. ([36]) *A mapping $w \in C(J, E_1)$ is said to be gH-type differentiable with respect to x at $(x_0, y_0) \in J$, if there exists an element $\frac{\partial w(x_0,y_0)}{\partial x} \in E_1$ such that $(x_0 + h, y_0) \in J$ holds for all sufficiently small h, $w(x_0 + h, y_0) \ominus_{gH} w(x_0 + h, y_0)$ and*

$$\lim_{h\to 0} \frac{w(x_0 + h, y_0) \ominus_{gH} w(x_0, y_0)}{h} = \frac{\partial w(x_0, y_0)}{\partial x},$$

where $w \ominus_{gH} \varpi$ denotes the gH-type difference ([35]) of $w \in E_1$ and $\varpi \in E_1$, which is the fuzzy number ν if it exists such that

$$w \ominus_{gH} \varpi = \nu \Longleftrightarrow \begin{cases} (\dagger)\, w = \varpi + \nu & \text{or} \\ (\ddagger)\, \varpi = w + (-1)\nu. \end{cases} \tag{8}$$

In this case, $\frac{\partial w(x_0,y_0)}{\partial x} \in E_1$ is called the gH-type derivative of w at (x_0, y_0) with respect to x, as long as the left-hand limit exists.

The gH-type derivative of w at (x_0, y_0) with respect to y and the higher fuzzy partial derivative of w are defined similarly.

Remark 3. *From Definition 1, one can see that the gH-type derivative of fuzzy number w with respect to x or y, which will support the concept of Caputo gH-type derivative in (2) and corresponding conclusions presented in this paper, has existence of the gH-type difference as a prerequisite.*

Based on the work of [26], for the space $C(J, E_i)$ of all fuzzy-valued continuous functions and the space $L^1(J, E_i)$ of Lebesque integrable fuzzy-valued functions on $J = [0, a] \times [0, b]$; here $i = 1, 2$; now we give the following other necessary definitions and lemmas.

Definition 2. *Let $J = [0, a] \times [0, b]$, $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in (0, 1] \times (0, 1]$, $u \in C(J, E_1) \bigcap L^1(J, E_1)$, $v \in C(J, E_2) \bigcap L^1(J, E_2)$, $[u(x, y)]^\tau = [u^-_\tau(x, y),\ u^+_\tau(x, y)]$ and $[v(x, y)]^\lambda = [v^-_\lambda(x, y), v^+_\lambda(x, y)]$.*

Then, based on level set-wise as follows

$$[{}^{RL}_{F}\mathcal{I}^{\alpha}_{0^+} u(x, y)]^\tau = [{}^{RL}_{F}\mathcal{I}^{\alpha}_{0^+} u^-_\tau(x, y), {}^{RL}_{F}\mathcal{I}^{\alpha}_{0^+} u^+_\tau(x, y)]$$

and

$$[{}^{RL}_{F}\mathcal{I}^{\beta}_{0^+} v(x, y)]^\lambda = [{}^{RL}_{F}\mathcal{I}^{\beta}_{0^+} v^-_\lambda(x, y), {}^{RL}_{F}\mathcal{I}^{\beta}_{0^+} v^+_\lambda(x, y)],$$

the mixed Riemann–Liouville fractional integral of orders α and β for fuzzy-valued multivariable functions $u(x, y)$ and $v(x, y)$ are, respectively, defined by

$${}^{RL}_{F}\mathcal{I}^{\alpha}_{0^+} u(x, y) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x - s)^{\alpha_1 - 1} (y - t)^{\alpha_2 - 1} u(s, t)dt\, ds, \tag{9}$$

$${}^{RL}_{F}\mathcal{I}^{\beta}_{0^+} v(x, y) = \frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_0^x \int_0^y (x - s)^{\beta_1 - 1} (y - t)^{\beta_2 - 1} v(s, t)dt\, ds. \tag{10}$$

Definition 3. *If for all $\epsilon > 0$, there exist δ_1, $\delta_2 > 0$ such that for any $(x, y, u) \in J \times C(J, E_1)$ and $(x, y, v) \in J \times C(J, E_2)$ with $|x - x_0| + |y - y_0| + d_\infty(u, \psi) < \delta_1$ and $|x - x_0| + |y - y_0| + d_\infty(v, \varphi) < \delta_2$, $d_\infty(f(x, y, v), f(x_0, y_0, \varphi)) < \epsilon$ and $d_\infty(g(x, y, u), g(x_0, y_0, \psi)) < \epsilon$, then the mappings $f : J \times C(J, E_2) \to E_1$ and $g : J \times C(J, E_1) \to E_2$ are called jointly continuous at point $(x_0, y_0, \varphi) \in J \times C(J, E_2)$ and $(x_0, y_0, \psi) \in J \times C(J, E_1)$, respectively.*

For all $(x,y) \in J = [0,a] \times [0,b]$, let

$$\psi(x,y) = \xi_2(y) + [\xi_1(x) \ominus \xi_1(0)], \tag{11}$$

$$\varphi(x,y) = \eta_2(y) + [\eta_1(x) \ominus \eta_1(0)], \tag{12}$$

where $\xi_1 \in C([0,a],E_1)$, $\eta_1 \in C([0,a],E_2)$, $\xi_2 \in C([0,b],E_1)$ and $\eta_2 \in C([0,b],E_2)$ are the given functions such that $\xi_2(y) \ominus \xi_1(0)$ and $\eta_2(y) \ominus \eta_1(0)$ exist, respectively. Then, we say

$$\begin{aligned}\widehat{C}^f_\psi(J,E_2) \quad &= \Big\{ v \in C(J,E_2) : \psi(x,y) \ominus (-1) {}^{RL}_{F}\mathcal{I}^{\alpha}_{0^+} f(x,y,v(x,y)) \\ &\qquad\qquad \text{exists}, \quad \forall (x,y) \in J \Big\},\end{aligned} \tag{13}$$

$$\begin{aligned}\widehat{C}^g_\varphi(J,E_1) \quad &= \Big\{ u \in C(J,E_1) : \varphi(x,y) \ominus (-1) {}^{RL}_{F}\mathcal{I}^{\beta}_{0^+} g(x,y,u(x,y)) \\ &\qquad\qquad \text{exists}, \quad \forall (x,y) \in J \Big\},\end{aligned} \tag{14}$$

where $\psi(\cdot,\cdot)$ and $\varphi(\cdot,\cdot)$ are defined by (11) and (12), respectively. Furthermore, denote $C_{\mathcal{J}}(J,E_m,E_n) = \{h : J \times C(J,E_m) \to E_n \,|\, h \text{ as jointly continuous}\}$ for each $m,n = 1,2$ $(m \neq n)$ and for $k,j = 0,1,2$ and $i = 1,2$, $C^{k,j}_{gH}(\jmath,E_i)$ by a set of all functions $\phi : \jmath \subset \mathbb{R}^2 \to E_i$, which have partial gH-type derivatives up to order k with respect to x and up to order j with respect to y in $\jmath$. In $C(J,E_i)$, we consider supremum metrics ρ defined by

$$\rho(u,v) = \sup_{(x,y)\in J} d_\infty(u(x,y),v(x,y)), \tag{15}$$

and stipulate the weighted metric d_r for $r = (r_1,r_2) \in [0,1] \times [0,1]$ as follows

$$d_r(\phi,v) = \sup_{(x,y)\in J} \{x^{r_1} y^{r_2} d_\infty(\phi(x,y),v(x,y))\}. \tag{16}$$

Definition 4. *Let $\alpha = (\alpha_1,\alpha_2), \beta = (\beta_1,\beta_2) \in (0,1] \times (0,1]$, $u \in C^{2,2}_{gH}(J,E_1)$ and $v \in C^{2,2}_{gH}(J,E_2)$. We define the Caputo gH-type derivatives of order α with respect to x and y of the function u as*

$$\begin{aligned}{}^{C}_{gH}\mathcal{D}^{\alpha} u(x,y) &= {}^{RL}_{F}\mathcal{I}^{1-\alpha}_{0^+}\left(\frac{\partial^2 u(x,y)}{\partial x \partial y}\right) \\ &= \frac{1}{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)} \int_0^x \int_0^y (x-s)^{-\alpha_1}(y-t)^{-\alpha_2} \frac{\partial^2 u(s,t)}{\partial s \partial t} dt\, ds\end{aligned}$$

and formulate the Caputo gH-type derivatives of order β in relation to x and y for the function v by

$$\begin{aligned}{}^{C}_{gH}\mathcal{D}^{\beta} v(x,y) &= {}^{RL}_{F}\mathcal{I}^{1-\beta}_{0^+}\left(\frac{\partial^2 v(x,y)}{\partial x \partial y}\right) \\ &= \frac{1}{\Gamma(1-\beta_1)\Gamma(1-\beta_2)} \int_0^x \int_0^y (x-s)^{-\beta_1}(y-t)^{-\beta_2} \frac{\partial^2 v(s,t)}{\partial s \partial t} dt\, ds,\end{aligned}$$

if the expressions on the right hand side are defined, where $1-\alpha = (1-\alpha_1, 1-\alpha_2), 1-\beta = (1-\beta_1, 1-\beta_2) \in [0,1) \times [0,1)$.

In particular, we distinguish two cases homologizing to $(\dagger)$ and $(\ddagger)$ in (8), and $u \in E_1$ is called

(i) *$(\dagger)$-Caputo gH-differentiable of order α with respect to x and y, which denotes ${}^{C}_{gH}\mathcal{D}^{\alpha}_1 u(x,y)$, if $\frac{\partial^2 u}{\partial x \partial y}(\cdot,\cdot)$ as a gH-type derivative in type 1 (i.e., $k = 1$ in (2)) at (x,y).*

(ii) ($\ddagger$)-*Caputo gH-differentiable of order α with respect to x and y when $\frac{\partial^2 u}{\partial x \partial y}(\cdot,\cdot)$ is a gH-type derivative in type* 2 (*i.e.*, $k=2$ *in* (2)) *at* (x,y). *This is indicated by* ${}^{C}_{gH}\mathcal{D}^{\alpha}_{2}u(x,y)$.

Remark 4. *If* $\alpha=\beta=(1,1)$ *in Definition 4, then we have*

$$ {}^{C}_{gH}\mathcal{D}^{\alpha}u(x,y)=\frac{\partial^2 u}{\partial x\partial y}(x,y), \quad {}^{C}_{gH}\mathcal{D}^{\beta}v(x,y)=\frac{\partial^2 v}{\partial x\partial y}(x,y) $$

for almost all $(x,y)\in J$.

Lemma 2. *Suppose that* $\psi(\cdot,\cdot)$ *and* $\varphi(\cdot,\cdot)$ *are the same as in* (11) *and* (12) *in several, and* $z_i(x,y)\in C(J,E_i)$ *is continuous for* $i=1,2$. *Then the fuzzy functions*

$$ \tilde{\mathbf{Z}}_1(x,y)=\psi(x,y)+{}^{RL}_{F}\mathcal{I}^{\alpha}_{0+}z_1(x,y) \tag{17} $$

and

$$ \bar{\mathbf{Z}}_2(x,y)=\varphi(x,y)\ominus{}^{RL}_{F}\mathcal{I}^{\alpha}_{0+}z_2(x,y) \tag{18} $$

are ($\dagger$)-*Caputo gH-differentiable and* ($\ddagger$)-*Caputo gH-differentiable* (*provided they exist*), *respectively. Further,*

$$ {}^{C}_{gH}\mathcal{D}^{\alpha}_{1}\tilde{\mathbf{Z}}_1(x,y)=z_1(x,y) \tag{19} $$

and

$$ {}^{C}_{gH}\mathcal{D}^{\alpha}_{2}\bar{\mathbf{Z}}_2(x,y)=-z_2(x,y). \tag{20} $$

Proof. Applying operator ${}^{C}_{gH}\mathcal{D}^{\alpha}_{1}$ to both sides of (17), based on the definitions of ${}^{C}_{gH}\mathcal{D}^{\alpha}_{1}u(x,y)$ in the special case (i) of Definition 4 for $u\in C^{2,2}_{gH}(J,E_1)$, then it follows from Definition 2.1 of [37] and (6) that

$$ \begin{aligned} &\Big[{}^{C}_{gH}\mathcal{D}^{\alpha}_{1}\tilde{\mathbf{Z}}_1(x,y)\Big]^{\tau} \\ &=\Bigg[{}^{RL}_{F}\mathcal{I}^{1-\alpha}_{0+}\left(\frac{\partial^2(\psi(x,y)^{-}_{\tau}+{}^{RL}_{F}\mathcal{I}^{\alpha}_{0+}\mathbf{z}^{-}_{1_{\tau}}(x,y))}{\partial x\partial y}\right), \\ &\qquad {}^{RL}_{F}\mathcal{I}^{1-\alpha}_{0+}\left(\frac{\partial^2(\psi(x,y)^{+}_{\tau}+{}^{RL}_{F}\mathcal{I}^{\alpha}_{0+}\mathbf{z}^{+}_{1_{\tau}}(x,y))}{\partial x\partial y}\right)\Bigg] \\ &=\left[{}^{RL}_{F}\mathcal{I}^{1-\alpha}_{0+}\left(\frac{\partial^2({}^{RL}_{F}\mathcal{I}^{\alpha}_{0+}\mathbf{z}_1(x,y))}{\partial x\partial y}\right)\right]^{\tau} \\ &=[\mathbf{z}_1(x,y)]^{\tau}. \end{aligned} $$

Moreover,

$$ {}^{C}_{gH}\mathcal{D}^{\alpha}_{1}\tilde{\mathbf{Z}}_1(x,y)=\mathbf{z}_1(x,y). $$

Similarly, employ operator ${}^{C}_{gH}\mathcal{D}^{\alpha}_{2}$ to both sides of (18). Then, based on the special case (ii) of Definition 4, and by Definition 2.1 of [37] and (7), we have

$$ {}^{C}_{gH}\mathcal{D}^{\alpha}_{2}\bar{\mathbf{Z}}_2(x,y)=-\mathbf{z}_2(x,y). $$

This completes the proof. □

Lemma 3. *Let* $\psi(\cdot,\cdot)$ *and* $\varphi(\cdot,\cdot)$ *be the same as in* (11) *and* (12), *separately, let the functions* $f\in C_{\mathcal{J}}(J,E_2,E_1)$ *and* $g\in C_{\mathcal{J}}(J,E_1,E_2)$ *be continuous, and let the functions* $u\in C^{2,2}_{gH}(J,E_1)$

and $v \in C^{2,2}_{gH}(J, E_2)$ *be fuzzy value. Then* (2) *with* (3) *is equivalent to the following nonlinear fractional-order coupled Volterra integro-differential system: For any* $(x, y) \in J$,

$$\begin{cases} u(x,y) = \psi(x,y) + {}^{RL}_{F}\mathcal{I}^{\alpha}_{0+} f(x,y,v(x,y)) \\ v(x,y) = \varphi(x,y) + {}^{RL}_{F}\mathcal{I}^{\beta}_{0+} g(x,y,u(x,y)) \end{cases} \quad \textit{when } k = 1 \tag{21}$$

or

$$\begin{cases} u(x,y) = \psi(x,y) \ominus (-1)\, {}^{RL}_{F}\mathcal{I}^{\alpha}_{0+} f(x,y,v(x,y)) \\ v(x,y) = \varphi(x,y) \ominus (-1)\, {}^{RL}_{F}\mathcal{I}^{\beta}_{0+} g(x,y,u(x,y)) \end{cases} \quad \textit{when } k = 2. \tag{22}$$

Proof. "⇒" Letting $u \in C^{2,2}_{gH}(J, E_1)$ and $v \in C^{2,2}_{gH}(J, E_2)$ satisfy (2) with (3), then one knows that the subsequent proof process of sufficiency is similar to the proof of Lemma 4.1 in [26], and so it is omitted.

"⇐" When $k = 1$, let $(u(x,y), v(x,y))^T$ be a solution of (21), and mark $z(x,y) = f(x,y,v(x,y))$. After applying Caputo fractional differential operator ${}^{C}_{gH}\mathcal{D}^{\alpha}_{1}$ to both sides of the first equation of (21), it follows from (19) that

$${}^{C}_{gH}\mathcal{D}^{\alpha}_{1} u(x,y) = z(x,y),$$

which intends

$${}^{C}_{gH}\mathcal{D}^{\alpha}_{1} u(x,y) = f(x,y,v(x,y)).$$

Furthermore, the first equation of (21) implies that $u(x,0) = \xi_1(x)$, $u(0,y) = \xi_2(y)$. Similar to the second equation of (21), we also obtain

$${}^{C}_{gH}\mathcal{D}^{\beta}_{1} v(x,y) = g(x,y,u(x,y)), \quad v(x,0) = \eta_1(x), v(0,y) = \eta_2(y).$$

Thus, $(u(x,y), v(x,y))^T$ is the solution to (2) with (3).

For $k = 2$, let us employ Caputo fractional differential operator ${}^{C}_{gH}\mathcal{D}^{\alpha}_{2}$ to both sides of the first equation in (22). Then, from (20), one can get

$${}^{C}_{gH}\mathcal{D}^{\alpha}_{2} u(x,y) = z(x,y),$$

i.e., ${}^{C}_{gH}\mathcal{D}^{\alpha}_{2} u(x,y) = f(x,y,v(x,y))$. Additionally, it follows from the first equation of (22) that $u(x,0) = \xi_1(x)$, $u(0,y) = \xi_2(y)$. Further, concerning the second equation of (22), we homogeneously have

$${}^{C}_{gH}\mathcal{D}^{\beta}_{2} v(x,y) = g(x,y,u(x,y)), \quad v(x,0) = \eta_1(x), \quad v(0,y) = \eta_2(y),$$

and the proof of sufficiency is completed. □

Remark 5. *In* [26], *Long et al. only gave the sufficiency, and we expand the existing work and propose sufficiency and necessity of equivalence to* (2) *with* (3) *in Lemma* 3.

For each $\lambda, \tau \in [0,1]$ and any vector $\zeta_1 = \begin{pmatrix} \nu \\ \omega \end{pmatrix}, \zeta_2 = \begin{pmatrix} \tilde{\nu} \\ \tilde{\omega} \end{pmatrix} \in C(J, E_1) \times C(J, E_2)$, let

$$\begin{aligned} \| \zeta_1 \| : &= \max\{\| \nu \|, \| \omega \|\} \\ &= \max\{\rho(\nu, \hat{0}), \rho(\omega, \hat{0})\} \\ &= \max\left\{ \sup_{\substack{(x,y)\in J \\ 0\le\tau\le1}} \max\{|\nu^-_\tau|, |\nu^+_\tau|\}, \sup_{\substack{(x,y)\in J \\ 0\le\tau\le1}} \max\{|\omega^-_\tau|, |\omega^+_\tau|\} \right\}, \end{aligned}$$

where $\hat{0}(x,y)$ is equal to 1 if $x = y = 0$ and is 0 in other cases. Then, from Long et al. [26] and Dong et al. [29], it follows that $(C(J,E_1) \times C(J,E_2), \|\cdot\|)$ is a Banach space. Taking

$$P = \left\{ \begin{pmatrix} \nu \\ \omega \end{pmatrix} \in C(J,E_1) \times C(J,E_2) \middle| \, \nu(x,y),\, \omega(x,y) \geq 0,\, \forall (x,y) \in J \right\},$$

then P is the normal and reproducing cone of $C(J,E_1) \times C(J,E_2)$. The semi-order "$\leq$" in $C(J,E_1) \times C(J,E_2)$ is derived from cone P; that is

$$\zeta_1 \leq \zeta_2 \Longleftrightarrow \zeta_2 - \zeta_1 \in P$$

for $\zeta_1 = \begin{pmatrix} \nu \\ \omega \end{pmatrix}$, $\zeta_2 = \begin{pmatrix} \tilde{\nu} \\ \tilde{\omega} \end{pmatrix} \in C(J,E_1) \times C(J,E_2)$.

In [29], Dong et al. only gave the Gronwall inequality of the form for a single variable function. By Theorem 3.2 of [38] or Lemma 2.3 in [29], we give the following generalization of Gronwall's inequality in the vector form of bivariate function, which plays an important role for obtaining our main results.

Lemma 4. *Let $f \in C_{\mathcal{J}}(J,E_2,E_1)$ and $g \in C_{\mathcal{J}}(J,E_1,E_2)$ satisfy Lipschitz condition* (**LC**) *with coefficients L_1 and L_2 in several; i.e., there exist positive real numbers L_1 and L_2 such that, for all $\theta_1, \theta_2 \in C(J,E_1)$ and any $\gamma_1, \gamma_2 \in C(J,E_2)$,*

$$\begin{cases} d_\infty(f(x,y,\gamma_1), f(x,y,\gamma_2)) \leq L_1 d_\infty(\gamma_1,\gamma_2), \\ d_\infty(g(x,y,\theta_1), g(x,y,\theta_2)) \leq L_2 d_\infty(\theta_1,\theta_2). \end{cases}$$

Assume that Gronwall inequality of the vector form

$$U(x,y) \leq AU(x,y) + H$$

holds, where $U(x,y) = \begin{pmatrix} u_1(x,y) \\ v_1(x,y) \end{pmatrix}$, $H(x,y) = \begin{pmatrix} h_1(x,y) \\ h_2(x,y) \end{pmatrix} \in C(J,E_1) \times C(J,E_2)$, $A = \begin{pmatrix} 0 & L_1 {}^{RL}_{F}\mathcal{I}^{\beta}_{0+} \\ L_2 {}^{RL}_{F}\mathcal{I}^{\alpha}_{0+} & 0 \end{pmatrix}$, and ${}^{RL}_{F}\mathcal{I}^{\alpha}_{0+}$ and ${}^{RL}_{F}\mathcal{I}^{\beta}_{0+}$ represent the fractional integrals of Caputo. In addition, if the following conditions are true:

(**H$_1$**) *Constants $a, b, \alpha, \beta, L_1, L_2 \in (0,1)$,*
(**H$_2$**) $\max\{L_1, L_2\} < \frac{1}{M}$, *where*

$$M = \max\left\{ \frac{a^{\beta_1} b^{\beta_2}}{\Gamma(\beta_1+1)\Gamma(\beta_2+1)}, \frac{a^{\alpha_1} b^{\alpha_2}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)} \right\},$$

then $U(x,y) \leq \sum_{k=0}^{\infty} A^k H$, where $A^{n+1} = A(A^n)$ and $A^0 = I$, the identity matrix.

Proof. Define an operator $\hat{T} : C(J,E_1) \times C(J,E_2) \to C(J,E_1) \times C(J,E_2)$ as

$$(\hat{T}U)(x,y) = AU(x,y) + H.$$

Firstly, we prove that $\hat{T}$ is an increasing operator. In fact, letting $\zeta_1 = \begin{pmatrix} \nu \\ \omega \end{pmatrix} \leq \zeta_2 = \begin{pmatrix} \tilde{\nu} \\ \tilde{\omega} \end{pmatrix}$, that is

$$\begin{cases} \nu(x,y) \leq \tilde{\nu}(x,y), \\ \omega(x,y) \leq \tilde{\omega}(x,y), \end{cases} \qquad \forall (x,y) \in [0,a] \times [0,b],$$

then

$$\begin{aligned}\hat{T}\zeta_2 - \hat{T}\zeta_1 &= A\zeta_2 - A\zeta_1 \\ &= \begin{pmatrix} L_1 {}_F^{RL}\mathcal{I}_{0^+}^{\beta}(\tilde{\omega}-\omega) \\ L_2 {}_F^{RL}\mathcal{I}_{0^+}^{\alpha}(\tilde{\nu}-\nu) \end{pmatrix} \geq \begin{pmatrix} \hat{0} \\ \hat{0} \end{pmatrix}.\end{aligned}$$

Thus, $\hat{T}$ is an increasing operator. Next, that $\|A\| < 1$ shall be shown. Indeed, since

$$\begin{aligned}&\|\zeta_1\| = 1 \\ \Longleftrightarrow &\max\left\{ \sup_{\substack{(x,y)\in J\\ 0\le\tau\le 1}} \max\{|\nu_\tau^-|, |\nu_\tau^+|\}, \sup_{\substack{(x,y)\in J\\ 0\le\tau\le 1}} \max\{|\omega_\tau^-|, |\omega_\tau^+|\} \right\} = 1,\end{aligned}$$

it strings along Definition 2 that

$$\begin{aligned}\|A\| &= \sup_{\|\zeta_1\|=1} \|A\zeta_1\| \\ &\le \sup_{\|\zeta_1\|=1} \max\left\{ L_1 {}_F^{RL}\mathcal{I}_{0^+}^{\beta}, L_2 {}_F^{RL}\mathcal{I}_{0^+}^{\alpha} \right\} \\ &\qquad \times \max\left\{ \sup_{\substack{(x,y)\in J\\ 0\le\tau\le 1}} \max\{|\omega_\tau^-|, |\omega_\tau^+|\}, \sup_{\substack{(x,y)\in J\\ 0\le\tau\le 1}} \max\{|\nu_\tau^-|, |\nu_\tau^+|\} \right\} \\ &\le \max\{L_1, L_2\} \times \sup_{(x,y)\in J} \max\left\{ \frac{a^{\alpha_1} b^{\alpha_2}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}, \frac{a^{\beta_1} b^{\beta_2}}{\Gamma(\beta_1+1)\Gamma(\beta_2+1)} \right\} \\ &< 1,\end{aligned}$$

and so by Theorem 3.2 in [38], one knows that $\hat{T}$ has a unique fixed point U^* and $\lim_{n\to\infty} \hat{T}^n U = U^*$.

At this point, H is taken as the initial value of iteration, which can be obtained through the following calculation:

$$\begin{aligned}U_0 &= H, \\ U_1 &= \hat{T}U_0 = AH + H, \\ U_2 &= \hat{T}U_1 = A^2H + AH + H, \\ &\ \vdots \\ U_n &= \hat{T}U_{n-1} = A^nH + \cdots + AH + H = \sum_{k=0}^{n} A^k H, \\ U^* &= \lim_{n\to\infty} U_n = \sum_{k=0}^{\infty} A^k H.\end{aligned}$$

Hence, it follows that $U(x,y) \le \sum_{k=0}^{\infty} A^k H$ via Lemma 2.3 of [29]. This completes the proof. □

3. Existence and Uniqueness

In this section, using the mathematical inductive method and the Banach fixed point theorem, we prove the existence and uniqueness of two kinds of gH-weak solutions, which are, respectively, called (†)-weak solution and (‡)-weak solution, for (2) with (3). Further, a numerical example is given to verify the results presented in this section.

Theorem 1. *Assume that $f \in C_{\mathcal{J}}(J, E_2, E_1)$ and $g \in C_{\mathcal{J}}(J, E_1, E_2)$ satisfy the Lipschitz condition* (**LC**)*; then* (2) *with* (3) *has a unique* $(\dagger)$*-weak solution defined on J.*

Proof. The proof of Theorem 1 is based on the application of Picard's iteration method. For this, we define two operators $T_1 : C(J, E_1) \to C(J, E_1)$ and $G_1 : C(J, E_2) \to C(J, E_2)$ as

$$T_1(u(x,y)) := \psi(x,y) + {}^{RL}_{F}\mathcal{I}^{\alpha}_{0+} f(x, y, \varphi(x,y) + {}^{RL}_{F}\mathcal{I}^{\beta}_{0+} g(x, y, u(x,y))),$$
$$G_1(v(x,y)) := \varphi(x,y) + {}^{RL}_{F}\mathcal{I}^{\beta}_{0+} g(x, y, \psi(x,y) + {}^{RL}_{F}\mathcal{I}^{\alpha}_{0+} f(x, y, v(x,y))).$$

These imply that T_1 and G_1 concern $v(x,y)$ and $u(x,y)$, respectively. By Lemma 1 (i), now we know that

$$\begin{aligned}
& d_\infty(T_1(u_1), T_1(u_2)) \\
& \leq d_\infty(\psi(x,y), \psi(x,y)) + d_\infty\Big({}^{RL}_{F}\mathcal{I}^{\alpha}_{0+} f(x, y, \varphi(x,y) + {}^{RL}_{F}\mathcal{I}^{\beta}_{0+} g(x, y, u_1(x,y))), \\
& \qquad\qquad {}^{RL}_{F}\mathcal{I}^{\alpha}_{0+} f(x, y, \varphi(x,y) + {}^{RL}_{F}\mathcal{I}^{\beta}_{0+} g(x, y, u_2(x,y)))\Big) \\
& \leq \frac{L_1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1 - 1}(y-t)^{\alpha_2 - 1} \\
& \qquad \times d_\infty\Big({}^{RL}_{F}\mathcal{I}^{\beta}_{0+} g(s, t, u_1(s,t)), {}^{RL}_{F}\mathcal{I}^{\beta}_{0+} g(s, t, u_2(s,t))\Big) dt ds.
\end{aligned}$$

and since

$$\begin{aligned}
& d_\infty\Big({}^{RL}_{F}\mathcal{I}^{\beta}_{0+} g(s, t, u_1(s,t)), {}^{RL}_{F}\mathcal{I}^{\beta}_{0+} g(s, t, u_2(s,t))\Big) \\
& \leq \frac{L_2}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_0^s \int_0^t (s-\mu)^{\beta_1 - 1}(t-\nu)^{\beta_2 - 1} d_\infty(u_1(\mu,\nu), u_2(\mu,\nu)) d\nu d\mu,
\end{aligned}$$

it follows from (16) that

$$\begin{aligned}
& d_\infty(T_1(u_1), T_1(u_2)) \\
& \leq \frac{L_1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \frac{L_2}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_0^x \int_0^y (x-s)^{\alpha_1 - 1}(y-t)^{\alpha_2 - 1} \\
& \qquad \times \left(\int_0^s \int_0^t (s-\mu)^{\beta_1 - 1}(t-\nu)^{\beta_2 - 1} d_\infty(u_1(\mu,\nu), u_2(\mu,\nu)) d\nu d\mu \right) dt ds \\
& \leq \frac{L_1 L_2 \Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(2\beta_1 + \alpha_1)\Gamma(2\beta_2 + \alpha_2)} x^{2\beta_1 + \alpha_1 - 1} y^{2\beta_2 + \alpha_2 - 1} \cdot d_{1-\beta}(u_1, u_2),
\end{aligned}$$

which is equivalent to

$$x^{1-\beta_1} y^{1-\beta_2} d_\infty(T_1(u_1), T_1(u_2)) \leq \frac{L_1 L_2 x^{\beta_1 + \alpha_1} y^{\beta_2 + \alpha_2} \Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(2\beta_1 + \alpha_1)\Gamma(2\beta_2 + \alpha_2)} \cdot d_{1-\beta}(u_1, u_2). \tag{23}$$

Next, we set up the operators for each $n \in \mathbb{N}$,

$$T_1^n(u(x,y)) = T_1(T_1^{n-1}(u(x,y))), \quad G_1^n(v(x,y)) = G_1(G_1^{n-1}(v(x,y))),$$

and by using mathematical induction, prove that the following inequality holds:

$$\begin{aligned}
& d_\infty(T_1^n u_1(x,y), T_1^n u_2(x,y)) \\
& \leq \frac{L_1^n L_2^n x^{(n+1)\beta_1 + n\alpha_1 - 1} y^{(n+1)\beta_2 + n\alpha_2 - 1} \Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma((n+1)\beta_1 + n\alpha_1)\Gamma((n+1)\beta_2 + n\alpha_2)} \cdot d_{1-\beta}(u_1, u_2),
\end{aligned} \tag{24}$$

which signifies that T_1^n is a contraction mapping if n is sufficiently large.

If $n = 1$, then we gain (24) from (23).

When $n = k$, letting (24) also holds, namely,

$$\begin{aligned}&d_\infty\left(T_1^k u_1(x,y)\,,T_1^k u_2(x,y)\right)\\&\le \frac{L_1^k L_2^k x^{(k+1)\beta_1+k\alpha_1-1}y^{(k+1)\beta_2+k\alpha_2-1}\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma((k+1)\beta_1+k\alpha_1)\Gamma((k+1)\beta_2+k\alpha_2)}\cdot d_{1-\beta}(u_1,u_2),\end{aligned}$$

Then we obtain with $n = k+1$,

$$\begin{aligned}&d_\infty(T_1^{k+1}u_1(x,y),T_1^{k+1}u_2(x,y))\\&\le d_\infty(\psi(x,y),\psi(x,y))\\&\quad+d_\infty\Big({}^{RL}_{F}\mathcal{I}^{\alpha}_{0+}f(x,y,\varphi(x,y)+{}^{RL}_{F}\mathcal{I}^{\beta}_{0+}g(x,y,T_1^k u_1(x,y)))\,,\\&\qquad {}^{RL}_{F}\mathcal{I}^{\alpha}_{0+}f(x,y,\varphi(x,y)+{}^{RL}_{F}\mathcal{I}^{\beta}_{0+}g(x,y,T_1^k u_2(x,y)))\Big)\\&\le \frac{L_1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}\\&\qquad\times d_\infty\Big({}^{RL}_{F}\mathcal{I}^{\beta}_{0+}g(s,t,T_1^k u_1(s,t))\,,{}^{RL}_{F}\mathcal{I}^{\beta}_{0+}g(s,t,T_1^k u_2(s,t))\Big)dtds\end{aligned}$$

and because

$$\begin{aligned}&d_\infty\Big({}^{RL}_{F}\mathcal{I}^{\beta}_{0+}g(s,t,T_1^k u_1(s,t))\,,{}^{RL}_{F}\mathcal{I}^{\beta}_{0+}g(s,t,T_1^k u_2(s,t))\Big)\\&\le \frac{L_1^k L_2^k d_{1-\beta}(u_1,u_2)}{\Gamma(\iota_1+1)\Gamma(\iota_2+1)}\int_0^s\int_0^t \mu^{\iota_1}\nu^{\iota_2}(s-\mu)^{\beta_1-1}(t-\nu)^{\beta_2-1}d\nu d\mu,\end{aligned}$$

Here $\iota_i = (k+1)\beta_i + k\alpha_i - 1$ $(i = 1,2)$; one can easily see that

$$\begin{aligned}&d_\infty(T_1^{k+1}u_1(x,y),T_1^{k+1}u_2(x,y))\\&\le \kappa\int_0^x\int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}\\&\qquad\times\left(\int_0^s \mu^{\iota_1}(s-\mu)^{\beta_1-1}d\mu\int_0^t \nu^{\iota_2}(t-\nu)^{\beta_2-1}d\nu\right)dtds\\&=\frac{L_1^{k+1}L_2^{k+1}\Gamma(\beta_1)\Gamma(\beta_2)x^{(k+2)\beta_1+(k+1)\alpha_1-1}y^{(k+2)\beta_2+(k+1)\alpha_2-1}}{\Gamma((k+2)\beta_1+(k+1)\alpha_1)\Gamma((k+2)\beta_2+(k+1)\alpha_2)}\cdot d_{1-\beta}(u_1,u_2),\end{aligned}$$

where $\kappa = \frac{L_1^{k+1}L_2^{k+1}d_{1-\beta}(u_1,u_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\iota_1+1)\Gamma(\iota_2+1)}$. This shows that (24) is also true for $n = k+1$, and we get

$$d_{1-\beta}(T_1^n u_1, T_1^n u_2) \le \frac{L_1^n L_2^n a^{n\beta_1+n\alpha_1}b^{n\beta_2+n\alpha_2}\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma((n+1)\beta_1+n\alpha_1)\Gamma((n+1)\beta_2+n\alpha_2)}\cdot d_{1-\beta}(u_1,u_2)$$

for all $n \in \mathbb{N}$. This in combination with

$$\lim_{n\to\infty}\frac{(L_1L_2a^{\beta_1+\alpha_1}b^{\beta_2+\alpha_2})^n}{\Gamma((n+1)\beta_1+n\alpha_1)\Gamma((n+1)\beta_2+n\alpha_2)} = 0$$

implies that T_1^n is a contraction mapping when n is large enough. By the same deduction, one can also know that G_1^n is a contraction mapping if n is large enough. Hence, there exists a unique $(u, v) \in E_1 \times E_2$ such that the following equations hold:

$$u(x,y) = \psi(x,y) + {}^{RL}_{F}I^{\alpha}_{0+} f(x,y,\varphi(x,y) + {}^{RL}_{F}I^{\beta}_{0+} g(x,y,u(x,y))),$$
$$v(x,y) = \varphi(x,y) + {}^{RL}_{F}I^{\beta}_{0+} g(x,y,\psi(x,y) + {}^{RL}_{F}I^{\alpha}_{0+} f(x,y,v(x,y))),$$

which is the (†)-weak solution of (2) with (3). □

Remark 6. *From Theorem 1, one can know that the existence of* (†)*-weak solutions for* (2) *with* (3) *can be guaranteed by the Lipschitz condition* (**LC**) *alone. Moreover, if we suppose that it is possible to switch to the scales of Banach spaces, as is done in the scientific schools of L.V. Ovsyannikov* [39] *and S.G. Krein and Y.I. Petunin* [40]*, then it is easy to see that one of the methods used in Theorem 1 is similar to that in Ovsyannikov* [39] *and Krein and Petunin* [40]*, but the proof of Theorem 1 must depend on Definitions 2 and 4 and Lemmas 1 and 3, and so the statements proved in this paper cannot turn out to be particular cases of more general theorems proved earlier.*

Below, we will show the existence and uniqueness of the (‡)-weak solution for (2) with (3) by adding the following assumptions for $\widehat{C}^f_{\psi}(J, E_2)$ defined by (13) and $\widehat{C}^g_{\varphi}(J, E_1)$ determined by (14):

(a_1) $\widehat{C}^f_{\psi}(J, E_2) \neq \emptyset$, $\widehat{C}^g_{\varphi}(J, E_1) \neq \emptyset$.

(a_2) If $v(\cdot,\cdot) \in \widehat{C}^f_{\psi}(J, E_2)$, then $V(\cdot,\cdot) \in \widehat{C}^f_{\psi}(J, E_2)$, where

$$V(x,y) = \psi(x,y) \ominus (-1)\, {}^{RL}_{F}\mathcal{I}^{\alpha}_{0+} f(x,y,v(x,y)), \quad \forall (x,y) \in J.$$

When $u(\cdot,\cdot) \in \widehat{C}^g_{\varphi}(J, E_1)$, one has $U(\cdot,\cdot) \in \widehat{C}^g_{\varphi}(J, E_1)$, here

$$U(x,y) = \varphi(x,y) \ominus (-1)\, {}^{RL}_{F}\mathcal{I}^{\beta}_{0+} g(x,y,u(x,y)), \quad \forall (x,y) \in J.$$

Theorem 2. *Assume that $f \in C_{\mathcal{J}}(J, E_2, E_1)$ and $g \in C_{\mathcal{J}}(J, E_1, E_2)$ meet the Lipschitz condition* (**LC**) *and the hypotheses* **(a_1)** *and* **(a_2)** *hold. Then* (2) *with* (3) *has a unique* (‡)*-weak solution.*

Proof. By the hypothesis **(a_1)**, we know that two H-differences $\psi(x,y) \ominus (-1)\, {}^{RL}_{F}\mathcal{I}^{\alpha}_{0+} f(x,y,v(x,y))$ and $\varphi(x,y) \ominus (-1)\, {}^{RL}_{F}\mathcal{I}^{\beta}_{0+} g(x,y,u(x,y))$ exist for all $(x,y) \in J$.

From assumption **(a_2)**, it is reasonable if we define the operators $T_2 : \hat{C}^f_{\psi}(J, E_2) \to \hat{C}^f_{\psi}(J, E_2)$ and $G_2 : \hat{C}^g_{\varphi}(J, E_1) \to \hat{C}^g_{\varphi}(J, E_1)$ as follows

$$G_2(u(x,y)) := \psi(x,y) \ominus (-1)\, {}^{RL}_{F}\mathcal{I}^{\alpha}_{0+} f(x,y,\varphi(x,y) \ominus (-1)\, {}^{RL}_{F}\mathcal{I}^{\beta}_{0+} g(x,y,u(x,y))),$$
$$T_2(v(x,y)) := \varphi(x,y) \ominus (-1)\, {}^{RL}_{F}\mathcal{I}^{\beta}_{0+} g(x,y,\psi(x,y) \ominus (-1)\, {}^{RL}_{F}\mathcal{I}^{\alpha}_{0+} f(x,y,v(x,y))),$$

which indicate that T_2 and G_2 are associated with $u(x,y)$ and $v(x,y)$, respectively. It follows from Lemma 1 (ii) that we have

$$d_\infty(G_2(u_1), G_2(u_2)) \leq \frac{L_1 L_2 \Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(2\beta_1+\alpha_1)\Gamma(2\beta_2+\alpha_2)} x^{2\beta_1+\alpha_1-1} y^{2\beta_2+\alpha_2-1} \cdot d_{1-\beta}(u_1, u_2),$$

which intends

$$x^{1-\beta_1} y^{1-\beta_2} d_\infty(G_2 u_1, G_2 u_2) \leq \frac{L_1 L_2 x^{\beta_1+\alpha_1} y^{\beta_2+\alpha_2} \Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(2\beta_1+\alpha_1)\Gamma(2\beta_2+\alpha_2)} \cdot d_{1-\beta}(u_1, u_2).$$

By the inductive method as the proof of Theorem 1, we get operator sequence $\{G_2^n\}_{n\geq 1}$ established by

$$G_2^n(u(x,y)) = G_2(G_2^{n-1}(u(x,y)))$$

and

$$d_{1-\beta}(G_2^n u_1, G_2^n u_2) \leq \frac{L_1^n L_2^n a^{n\beta_1+n\alpha_1} b^{n\beta_2+n\alpha_2}\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma((n+1)\beta_1+n\alpha_1)\Gamma((n+1)\beta_2+n\alpha_2)} \cdot d_{1-\beta}(u_1,u_2).$$

From

$$\lim_{n\to\infty} \frac{(L_1 L_2 a^{\beta_1+\alpha_1} b^{\beta_2+\alpha_2})^n}{\Gamma((n+1)\beta_1+n\alpha_1)\Gamma((n+1)\beta_2+n\alpha_2)} = 0,$$

it follows that G_2^n is a contraction mapping if n is large enough. Similarly, one can know that T_2^n is also a contraction mapping when n is large enough. Thus, there exists a unique $(u,v) \in E_1 \times E_2$ such that the following equations hold:

$$\begin{aligned}
u(x,y) &= \psi(x,y) \ominus (-1)\, {}_F^{RL}I_{0+}^{\alpha} f(x,y,\varphi(x,y) \ominus (-1)\, {}_F^{RL}I_{0+}^{\beta} g(x,y,u(x,y))),\\
v(x,y) &= \varphi(x,y) \ominus (-1)\, {}_F^{RL}I_{0+}^{\beta} g(x,y,\psi(x,y) \ominus (-1)\, {}_F^{RL}I_{0+}^{\alpha} f(x,y,v(x,y)))
\end{aligned}$$

which is the $(\ddagger)$-weak solution of (2) with (3). □

Based on Example 5.1 of [26], we give the upcoming example, which intuitively and exhaustively demonstrates the existence and uniqueness results of Theorems 1 and 2.

Example 1. *The following coupled system of fuzzy fractional PDEs is considered: For each $(x,y) \in J = [0,a] \times [0,b]$ and $k = 1,2,$*

$$\begin{cases}
{}_{gH}^{C}\mathcal{D}_k^{\alpha} u(x,y) = p(x,y)v(x,y) + q(x,y),\\
{}_{gH}^{C}\mathcal{D}_k^{\beta} v(x,y) = c(x,y)u(x,y) + d(x,y),\\
u(x,0) = u(0,y) = u(0,0) = -2C,\\
v(x,0) = v(0,y) = v(0,0) = 2C,
\end{cases} \tag{25}$$

where $\alpha, \beta \in [0,1] \times [0,1]$, $p(x,y)$, $q(x,y)$, $c(x,y)$ and $d(x,y)$ are polynomial functions, and C is a fuzzy number.

It is easy to see that the functions $f(x,y,v(x,y)) := p(x,y)v(x,y) + b(x,y)$ and $g(x,y,u(x,y)) := q(x,y)u(x,y) + d(x,y)$ in (25) fulfill the Lipschitz condition (**LC**) with constants $L_1 = \max_{(x,y)\in J} |p(x,y)|$ and $L_2 = \max_{(x,y)\in J} |c(x,y)|$, and so (25) exists as a unique $(\dagger)$-weak solution in $C(J,E_1) \times C(J,E_2)$.

For another thing, let us show the existence of the $(\ddagger)$-weak solution for (25). To begin with, choosing $\alpha = \beta = \frac{2}{3}$, $a = 1, b = \frac{1}{2}$, $p(x,y) = -\frac{9}{2[\Gamma(\frac{1}{3})]^2} x^{\frac{1}{3}} y^{\frac{1}{3}}$, $q(x,y) = -\frac{9C}{2[\Gamma(\frac{1}{3})]^2} x^{\frac{4}{3}} y^{\frac{4}{3}}$ and $c(x,y) = \frac{9}{2[\Gamma(\frac{1}{3})]^2} x^{\frac{1}{3}} y^{\frac{1}{3}}$, $d(x,y) = \frac{9C}{2[\Gamma(\frac{1}{3})]^2} x^{\frac{4}{3}} y^{\frac{4}{3}}$, then (25) becomes the following coupled PDE problem:

$$\begin{cases}
{}_{gH}^{C}\mathcal{D}_2^{\frac{2}{3}} u(x,y) = -\dfrac{9}{2[\Gamma(\frac{1}{3})]^2} x^{\frac{1}{3}} y^{\frac{1}{3}} v(x,y) - \dfrac{9C}{2[\Gamma(\frac{1}{3})]^2} x^{\frac{4}{3}} y^{\frac{4}{3}},\\
{}_{gH}^{C}\mathcal{D}_2^{\frac{2}{3}} v(x,y) = \dfrac{9}{2[\Gamma(\frac{1}{3})]^2} x^{\frac{1}{3}} y^{\frac{1}{3}} u(x,y) + \dfrac{9C}{2[\Gamma(\frac{1}{3})]^2} x^{\frac{4}{3}} y^{\frac{4}{3}},\\
u(x,0) = u(0,y) = u(0,0) = -2C,\\
v(x,0) = v(0,y) = v(0,0) = 2C.
\end{cases} \tag{26}$$

One can easily get the Lipschitz coefficients $L_1 = \frac{9}{2\sqrt[3]{2}[\Gamma(\frac{1}{3})]^2}$ and $L_2 = \frac{9}{2\sqrt[3]{2}[\Gamma(\frac{1}{3})]^2}$, $\psi(x,y) = -2C$ and $\varphi(x,y) = 2C$.

In the sequel, by fuzzifying the deterministic solutions according to the Buckley–Feuring strategy due to Long et al. [17] and [26], we find $(\Lambda(x,y,C), \Theta(x,y,C)) = (-2C - Cxy, 2C - Cxy)$, the BF solution (see [10,17]) of (26) to verify the condition **(a_1)** in Theorem 2.

In Example 1, we use Gaussian fuzzy number C with membership function $C(t) = \exp\left(-9(t-c)^2\right)$, where c is a crisp number. The λ-cuts and τ-cuts of C are independently

$$[c_1(\lambda), c_2(\lambda)] = \left[c - \frac{1}{3}\sqrt{\ln\frac{1}{\lambda}}, c + \frac{1}{3}\sqrt{\ln\frac{1}{\lambda}}\right],$$
$$[c_1(\tau), c_2(\tau)] = \left[c - \frac{1}{3}\sqrt{\ln\frac{1}{\tau}}, c + \frac{1}{3}\sqrt{\ln\frac{1}{\tau}}\right],$$

and the continuity of the extended principle shows that the fuzzy solutions of (26) are

$$\begin{aligned}[\Lambda(x,y,C)]^\lambda \quad &= \Bigg[-2\left(c - \frac{1}{3}\sqrt{\ln\frac{1}{\lambda}}\right) - \left(c - \frac{1}{3}\sqrt{\ln\frac{1}{\lambda}}\right)xy, \\ &\qquad -2\left(c + \frac{1}{3}\sqrt{\ln\frac{1}{\lambda}}\right) - \left(c + \frac{1}{3}\sqrt{\ln\frac{1}{\lambda}}\right)xy\Bigg]\end{aligned}$$

and

$$\begin{aligned}[\Theta(x,y,C)]^\tau \quad &= \Bigg[2\left(c - \frac{1}{3}\sqrt{\ln\frac{1}{\tau}}\right) - \left(c - \frac{1}{3}\sqrt{\ln\frac{1}{\tau}}\right)xy, \\ &\qquad 2\left(c + \frac{1}{3}\sqrt{\ln\frac{1}{\tau}}\right) - \left(c + \frac{1}{3}\sqrt{\ln\frac{1}{\tau}}\right)xy\Bigg],\end{aligned}$$

concerning which some λ-cuts and τ-cuts can be simulated; they are shown in Figures 1 and 2. In Figure 1, the top graph represents $[\Theta(x,y,C)]^\tau$ and the bottom graph stands for $[\Lambda(x,y,C)]^\lambda$. The graph on the left shows how the solutions $[\Lambda(x,y,C)]^\lambda$ and $[\Theta(x,y,C)]^\tau$ vary with the independent variables x and λ/τ (i.e., λ or τ) when y is fixed at five constants. The graph on the right shows how the solutions $[\Lambda(x,y,C)]^\lambda$ and $[\Theta(x,y,C)]^\tau$ change with the independent variables y and λ/τ when x is fixed at five constants. Moreover, Figure 2 shows the numerical simulation of the level sets $[\Lambda(x,y,C)]^\lambda$ and $[\Theta(x,y,C)]^\tau$ as a function of λ/τ.

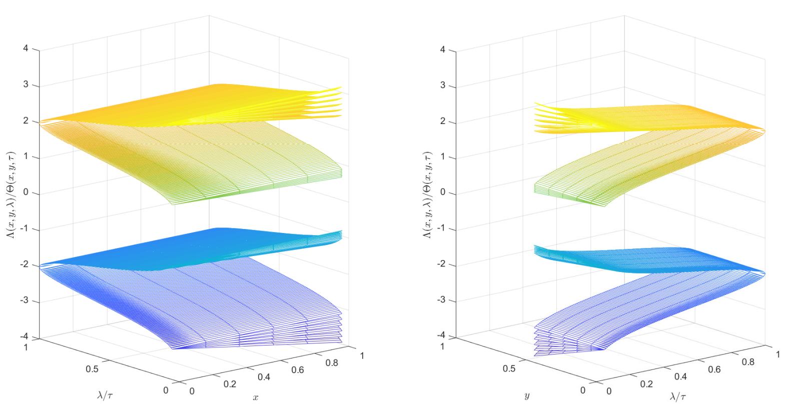

Figure 1. Numerical simulation for fuzzy solutions of (26) with Gaussian fuzzy numbers $[C]^\lambda = \left[1 - \frac{1}{3}\sqrt{\ln\frac{1}{\lambda}}, 1 + \frac{1}{3}\sqrt{\ln\frac{1}{\lambda}}\right]$ and $[C]^\tau = \left[1 - \frac{1}{3}\sqrt{\ln\frac{1}{\tau}}, 1 + \frac{1}{3}\sqrt{\ln\frac{1}{\tau}}\right]$.

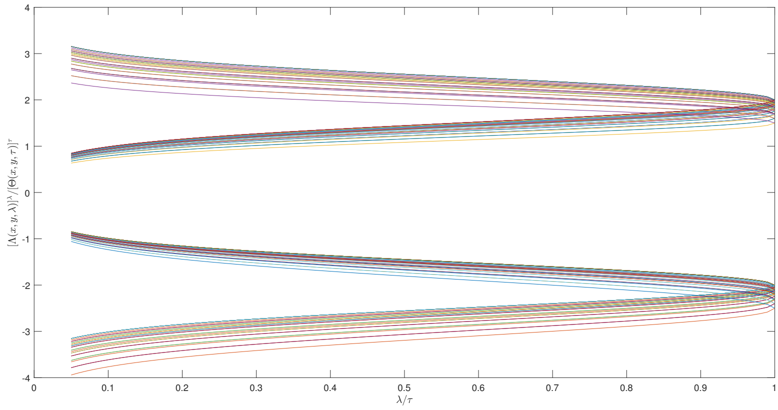

Figure 2. Numerical simulation for level sets of fuzzy solutions of (26) with the Gaussian fuzzy numbers.

From Figures 1 and 2, if the crisp number c in the membership functions of the fuzzy numbers is known, then one can see that the image of the coupling solution and its level set change with the independent variable. However, we cannot get the image when c changes continuously. This is worth improving.

Now we make clear that the condition **(a_2)** in Theorem 2 holds and then prove the existence and uniqueness of the $(\ddagger)$-weak solution for (26).

For briefness, letting $K = \frac{9}{\Gamma(\frac{1}{3})\Gamma(\frac{1}{3})}$, then one has

$$
\begin{aligned}
&[(-1)\,{}_F^{RL}\mathcal{I}_{0+}^{\frac{2}{3}} f(x,y,v(x,y))]^{\lambda} \\
&= \frac{K[v]^{\lambda}}{2\Gamma(\frac{2}{3})\Gamma(\frac{2}{3})} \int_0^x \int_0^y (x-s)^{-\frac{1}{3}}(y-t)^{-\frac{1}{3}} s^{\frac{1}{3}} t^{\frac{1}{3}} ds dt \\
&\quad + \frac{K[C]^{\lambda}}{2\Gamma(\frac{2}{3})\Gamma(\frac{2}{3})} \int_0^x \int_0^y (x-s)^{-\frac{1}{3}}(y-t)^{-\frac{1}{3}} s^{\frac{1}{3}} t^{\frac{1}{3}} ds dt \\
&= \frac{[v]^{\lambda} xy}{2} + \frac{2[C]^{\lambda} x^2 y^2}{9},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&len[(-1)\,{}_F^{RL}\mathcal{I}_{0+}^{\frac{2}{3}} f(x,y,v(x,y))]^{\lambda} \\
&= \frac{2}{3}\sqrt{\ln\frac{1}{\lambda}}\left(xy - \frac{5}{18}x^2y^2\right) \\
&\le \frac{5}{9}\frac{2}{3}\sqrt{\ln\frac{1}{\lambda}} = \frac{10}{27}\sqrt{\ln\frac{1}{\lambda}},
\end{aligned}
$$

and so

$$
len[\psi(x,y)]^{\lambda} \le len[(-1)\,{}_F^{RL}\mathcal{I}_{0+}^{\frac{2}{3}} f(x,y,v(x,y))]^{\lambda}.
$$

Thus, based on Properties 21 of [41], we know that the H-difference $\psi(x,y) \ominus (-1)$ ${}_F^{RL}\mathcal{I}_{0+}^{\frac{2}{3}} f(x,y,v(x,y))$ exists.

From the foregoing proof, it follows that

$$
[\psi(x,y)]^{\lambda} = \left[-2\left(c - \frac{1}{3}\sqrt{\ln\frac{1}{\lambda}}\right), -2\left(c + \frac{1}{3}\sqrt{\ln\frac{1}{\lambda}}\right)\right]
$$

and

$$\begin{aligned}&[(-1)\,{}^{RL}_{F}\mathcal{I}^{\frac{2}{3}}_{0+}f(x,y,v(x,y))]^{\lambda}\\&=\left[\left(xy-\frac{5}{18}x^2y^2\right)\left(c-\frac{1}{3}\sqrt{\ln\frac{1}{\lambda}}\right),\left(xy-\frac{5}{18}x^2y^2\right)\left(c+\frac{1}{3}\sqrt{\ln\frac{1}{\lambda}}\right)\right].\end{aligned}$$

Taking

$$V(x,y)=\psi(x,y)\ominus(-1)\,{}^{RL}_{F}\mathcal{I}^{\frac{2}{3}}_{0+}f(x,y,v(x,y)),$$

then from Example 5.1 in [26] we get

$$\begin{aligned}[V(x,y)]^{\lambda}\quad&=\left[\left(-2-xy+\frac{5}{18}x^2y^2\right)\left(c-\frac{1}{3}\sqrt{\ln\frac{1}{\lambda}}\right),\right.\\&\left.\left(-2-xy+\frac{5}{18}x^2y^2\right)\left(c+\frac{1}{3}\sqrt{\ln\frac{1}{\lambda}}\right)\right],\end{aligned}$$

and

$$len[V(x,y)]^{\lambda}=\frac{2}{3}\sqrt{\ln\frac{1}{\lambda}}\left(-2-xy+\frac{5}{18}x^2y^2\right).$$

Similar to the above steps, one gets

$$[(-1)\,{}^{RL}_{F}\mathcal{I}^{\frac{2}{3}}_{0+}f(x,y,V(x,y))]^{\lambda}=\frac{[V]^{\lambda}}{2}xy+\frac{2[C]^{\lambda}}{9}x^2y^2$$

and

$$\begin{aligned}len[(-1)\,{}^{RL}_{F}\mathcal{I}^{\frac{2}{3}}_{0+}f(x,y,V(x,y))]^{\lambda}\quad&=\frac{2}{3}\sqrt{\ln\frac{1}{\lambda}}\left(-xy-\frac{5}{18}x^2y^2+\frac{5}{36}x^3y^3\right)\\&\le\frac{10}{27}\sqrt{\ln\frac{1}{\lambda}},\end{aligned}$$

which shows that the H-difference

$$\psi(x,y)\ominus(-1)\,{}^{RL}_{F}\mathcal{I}^{\frac{2}{3}}_{0+}f(x,y,V(x,y))$$

exists.

Likewise, we have

$$[(-1)\,{}^{RL}_{F}\mathcal{I}^{\frac{2}{3}}_{0+}g(x,y,u(x,y))]^{\tau}=-\frac{xy}{2}[u]^{\tau}-\frac{2x^2y^2}{9}[C]^{\tau}$$

and

$$len[(-1)\,{}^{RL}_{F}\mathcal{I}^{\frac{2}{3}}_{0+}g(x,y,u(x,y))]^{\tau}\le\frac{22}{54}\sqrt{\ln\frac{1}{\tau}},$$

$$len[\varphi(x,y)]^{\tau}\ge len[(-1)\,{}^{RL}_{F}\mathcal{I}^{\frac{2}{3}}_{0+}g(x,y,u(x,y))]^{\tau}.$$

It follows that $U(x,y)=\varphi(x,y)\ominus(-1)\,{}^{RL}_{F}\mathcal{I}^{\frac{2}{3}}_{0+}g(x,y,u(x,y))$ exists via Properties 21 of [41], and

$$len[(-1)\,{}^{RL}_{F}\mathcal{I}^{\frac{2}{3}}_{0+}g(x,y,U(x,y))]^{\tau}\le\frac{22}{54}\sqrt{\ln\frac{1}{\tau}},$$

That is, the H-difference $\varphi(x,y)\ominus(-1)\,{}^{RL}_{F}\mathcal{I}^{\frac{2}{3}}_{0+}g(x,y,U(x,y))$ exists.

Therefore, in this case, (26) has a unique (‡)-weak solution in $C(J_1,E_1)\times C(J_1,E_2)$.

Remark 7. *From Example 1, one can easily see that due to the "coupling" and the existence of the H-difference, it is more difficult to obtain the existence and uniqueness of the* ($\ddagger$)*-weak solution of* (2) *with* (3)*. This shows that it is challenging and valuable to obtain the results presented in Theorems 1 and 2.*

4. Continuous Dependence and ε-Approximation

In this section, we prove the continuous dependence of two kinds of gH-weak solutions on initial values and ε-approximation solutions of the coupled system for (2) with (3) by using Lemmas 3 and 4 and modifying the initial conditions. Further, we will show that the former is a special case of the latter.

Firstly, we consider whenever solutions are continuous depending on the initial data. Modifying (3), we have the following new coupled system for fuzzy fractional PDEs:

$$\begin{cases} {}^{C}_{gH}\mathcal{D}^{\alpha}_{k}u(x,y) = f(x,y,v(x,y)), \\ {}^{C}_{gH}\mathcal{D}^{\beta}_{k}v(x,y) = g(x,y,u(x,y)), \\ u(x,0) = \xi_{11}(x), \quad v(x,0) = \eta_{11}(x), \\ u(0,y) = \xi_{21}(y), \quad v(0,y) = \eta_{21}(y), \end{cases} \tag{27}$$

or

$$\begin{cases} {}^{C}_{gH}\mathcal{D}^{\alpha}_{k}u(x,y) = f(x,y,v(x,y)), \\ {}^{C}_{gH}\mathcal{D}^{\beta}_{k}v(x,y) = g(x,y,u(x,y)), \\ u(x,0) = \xi_{12}(x), \quad v(x,0) = \eta_{12}(x), \\ u(0,y) = \xi_{22}(y), \quad v(0,y) = \eta_{22}(y), \end{cases} \tag{28}$$

and letting $(u_1(\cdot,\cdot), v_1(\cdot,\cdot))^T$ and $(u_2(\cdot,\cdot), v_2(\cdot,\cdot))^T$ be (ℓ)-weak solutions of (27) and (28) for $\ell = \dagger, \ddagger$, respectively, then we have

$$\begin{cases} u_1(x,y) = \psi_1(x,y) + {}^{RL}_{F}\mathcal{I}^{\alpha}_{0+}f(x,y,v_1(x,y)), \\ v_1(x,y) = \varphi_1(x,y) + {}^{RL}_{F}\mathcal{I}^{\beta}_{0+}g(x,y,u_1(x,y)), \\ u_2(x,y) = \psi_2(x,y) + {}^{RL}_{F}\mathcal{I}^{\alpha}_{0+}f(x,y,v_2(x,y)), \\ v_2(x,y) = \varphi_2(x,y) + {}^{RL}_{F}\mathcal{I}^{\beta}_{0+}g(x,y,u_2(x,y)), \end{cases} \tag{29}$$

or

$$\begin{cases} u_1(x,y) = \psi_1(x,y) \ominus (-1){}^{RL}_{F}\mathcal{I}^{\alpha}_{0+}f(x,y,v_1(x,y)), \\ v_1(x,y) = \varphi_1(x,y) \ominus (-1){}^{RL}_{F}\mathcal{I}^{\beta}_{0+}g(x,y,u_1(x,y)), \\ u_2(x,y) = \psi_2(x,y) \ominus (-1){}^{RL}_{F}\mathcal{I}^{\alpha}_{0+}f(x,y,v_2(x,y)), \\ v_2(x,y) = \varphi_2(x,y) \ominus (-1){}^{RL}_{F}\mathcal{I}^{\beta}_{0+}g(x,y,u_2(x,y)), \end{cases} \tag{30}$$

where

$$\psi_1(x,y) = \xi_{11}(x) + \xi_{21}(y) \ominus \xi_{11}(0), \quad \varphi_1(x,y) = \eta_{11}(x) + \eta_{21}(y) \ominus \eta_{11}(0),$$
$$\psi_2(x,y) = \xi_{12}(x) + \xi_{22}(y) \ominus \xi_{12}(0), \quad \varphi_2(x,y) = \eta_{12}(x) + \eta_{22}(y) \ominus \eta_{12}(0).$$

Theorem 3. *Let* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *in Lemma 4 hold, and* $f \in C_{\mathcal{J}}(J, E_2, E_1)$ *and* $g \in C_{\mathcal{J}}(J, E_1, E_2)$ *satisfy the Lipschitz condition* (**LC**)*. If* $(u_1(\cdot,\cdot), v_1(\cdot,\cdot))^T$ *and* $(u_2(\cdot,\cdot), v_2(\cdot,\cdot))^T$ *are, respectively,*

(ℓ)-weak solutions of (27) and (28) for $\ell = \dagger, \ddagger$, and the corresponding initial values are $(\psi_1, \varphi_1)^T$ and $(\psi_2, \varphi_2)^T$, severally; then the following inequality holds

$$\begin{pmatrix} \rho(u_1,u_2) \\ \rho(v_1,v_2) \end{pmatrix} \le \begin{pmatrix} \rho(\psi_1,\psi_2) \\ \rho(\varphi_1,\varphi_2) \end{pmatrix} + \begin{pmatrix} \frac{a^{\alpha_1}b^{\alpha_2}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)} & 0 \\ 0 & \frac{a^{\beta_1}b^{\beta_2}}{\Gamma(\beta_1+1)\Gamma(\beta_2+1)} \end{pmatrix} \times \sum_{k=0}^{\infty} A^k \begin{pmatrix} L_1\rho(\varphi_1,\varphi_2) \\ L_2\rho(\psi_1,\psi_2) \end{pmatrix}, \tag{31}$$

where A is the same as in Lemma 4.

Proof. Letting $z_1(x,y) = {}^{C}_{gH}\mathcal{D}^{\alpha}_{1}u_1(x,y)$, $z_2(x,y) = {}^{C}_{gH}\mathcal{D}^{\alpha}_{1}u_2(x,y)$, $\omega_1(x,y) = {}^{C}_{gH}\mathcal{D}^{\beta}_{1}v_1(x,y)$ and $\omega_2(x,y) = {}^{C}_{gH}\mathcal{D}^{\beta}_{1}v_2(x,y)$, then without loss of generality, we only consider (29) as follows

$$u_1(x,y) = \psi_1(x,y) + {}^{RL}_{F}\mathcal{I}^{\alpha}_{0+}z_1(x,y), \quad u_2(x,y) = \psi_2(x,y) + {}^{RL}_{F}\mathcal{I}^{\alpha}_{0+}z_2(x,y),$$
$$v_1(x,y) = \varphi_1(x,y) + {}^{RL}_{F}\mathcal{I}^{\beta}_{0+}\omega_1(x,y), \quad v_2(x,y) = \varphi_2(x,y) + {}^{RL}_{F}\mathcal{I}^{\beta}_{0+}\omega_2(x,y)$$

for all $(x,y) \in J$. Thus, (2) with (3) is equivalent to

$$\begin{cases} z_1(x,y) = f(x,y,\varphi_1(x,y) + {}^{RL}_{F}\mathcal{I}^{\beta}_{0+}\omega_1(x,y)), \\ \omega_1(x,y) = g(x,y,\psi_1(x,y) + {}^{RL}_{F}\mathcal{I}^{\alpha}_{0+}z_1(x,y)), \end{cases}$$

and

$$\begin{cases} z_2(x,y) = f(x,y,\varphi_2(x,y) + {}^{RL}_{F}\mathcal{I}^{\beta}_{0+}\omega_2(x,y)), \\ \omega_2(x,y) = g(x,y,\psi_2(x,y) + {}^{RL}_{F}\mathcal{I}^{\alpha}_{0+}z_2(x,y)). \end{cases}$$

It follows from Lemma 1 (i) that

$$\begin{aligned} & d_\infty(z_1(x,y), z_2(x,y)) \\ \le & d_\infty(f(x,y,\varphi_1(x,y) + {}^{RL}_{F}\mathcal{I}^{\beta}_{0+}\omega_1(x,y)), f(x,y,\varphi_2(x,y) + {}^{RL}_{F}\mathcal{I}^{\beta}_{0+}\omega_1(x,y))) \\ & + d_\infty(f(x,y,\varphi_2(x,y) + {}^{RL}_{F}\mathcal{I}^{\beta}_{0+}\omega_1(x,y)), f(x,y,\varphi_2(x,y) + {}^{RL}_{F}\mathcal{I}^{\beta}_{0+}\omega_2(x,y))) \\ \le & L_1 d_\infty(\varphi_1(x,y), \varphi_2(x,y)) + L_1 {}^{RL}_{F}\mathcal{I}^{\beta}_{0+} d_\infty(\omega_1(x,y), \omega_2(x,y)). \end{aligned}$$

Similarly, one has

$$\begin{aligned} & d_\infty(\omega_1(x,y), \omega_2(x,y)) \\ & \le L_2 d_\infty(\psi_1(x,y), \psi_2(x,y)) + L_2 {}^{RL}_{F}\mathcal{I}^{\alpha}_{0+} d_\infty(z_1(x,y), z_2(x,y)). \end{aligned}$$

It can be known from Lemma 4 that

$$\begin{aligned} & \begin{pmatrix} d_\infty(z_1(x,y), z_2(x,y)) \\ d_\infty(\omega_1(x,y), \omega_2(x,y)) \end{pmatrix} \\ \le & \begin{pmatrix} 0 & L_1 {}^{RL}_{F}\mathcal{I}^{\beta}_{0+} \\ L_2 {}^{RL}_{F}\mathcal{I}^{\alpha}_{0+} & 0 \end{pmatrix} \begin{pmatrix} d_\infty(z_1(x,y), z_2(x,y)) \\ d_\infty(\omega_1(x,y), \omega_2(x,y)) \end{pmatrix} \\ & + \begin{pmatrix} L_1 d_\infty(\varphi_1(x,y), \varphi_2(x,y)) \\ L_2 d_\infty(\psi_1(x,y), \psi_2(x,y)) \end{pmatrix} \\ \le & \sum_{k=0}^{\infty} \begin{pmatrix} 0 & L_1 {}^{RL}_{F}\mathcal{I}^{\beta}_{0+} \\ L_2 {}^{RL}_{F}\mathcal{I}^{\alpha}_{0+} & 0 \end{pmatrix}^k \begin{pmatrix} L_1 d_\infty(\varphi_1(x,y), \varphi_2(x,y)) \\ L_2 d_\infty(\psi_1(x,y), \psi_2(x,y)) \end{pmatrix}. \end{aligned}$$

Hence, by Lemmas 1 (i) and 3, we can obtain

$$\begin{aligned}
&\begin{pmatrix} d_\infty(u_1(x,y),u_2(x,y)) \\ d_\infty(v_1(x,y),v_2(x,y)) \end{pmatrix} \\
&\leq \begin{pmatrix} d_\infty(\psi_1(x,y),\psi_2(x,y)) \\ d_\infty(\varphi_1(x,y),\varphi_2(x,y)) \end{pmatrix} + \begin{pmatrix} {}^{RL}_{F}\mathcal{I}^{\alpha}_{0+} & 0 \\ 0 & {}^{RL}_{F}\mathcal{I}^{\beta}_{0+} \end{pmatrix} \\
&\qquad \times \sum_{k=0}^{\infty} A^k \begin{pmatrix} L_1 d_\infty(\varphi_1(x,y),\varphi_2(x,y)) \\ L_2 d_\infty(\psi_1(x,y),\psi_2(x,y)) \end{pmatrix}.
\end{aligned} \tag{32}$$

By using (15) on region J, it follows from (32) that

$$\begin{aligned}
\begin{pmatrix} \rho(u_1,u_2) \\ \rho(v_1,v_2) \end{pmatrix} &\leq \begin{pmatrix} \rho(\psi_1,\psi_2) \\ \rho(\varphi_1,\varphi_2) \end{pmatrix} \begin{pmatrix} {}^{RL}_{F}\mathcal{I}^{\alpha}_{0+} & 0 \\ 0 & {}^{RL}_{F}\mathcal{I}^{\beta}_{0+} \end{pmatrix} \sum_{k=0}^{\infty} A^k \begin{pmatrix} L_1\rho(\varphi_1,\varphi_2) \\ L_2\rho(\psi_1,\psi_2) \end{pmatrix} \\
&\leq \begin{pmatrix} \rho(\psi_1,\psi_2) \\ \rho(\varphi_1,\varphi_2) \end{pmatrix} + \begin{pmatrix} \frac{a^{\alpha_1} b^{\alpha_2}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)} & 0 \\ 0 & \frac{a^{\beta_1} b^{\beta_2}}{\Gamma(\beta_1+1)\Gamma(\beta_2+1)} \end{pmatrix} \\
&\qquad \times \sum_{k=0}^{\infty} A^k \begin{pmatrix} L_1\rho(\varphi_1,\varphi_2) \\ L_2\rho(\psi_1,\psi_2) \end{pmatrix},
\end{aligned}$$

which is (31) and presents continuous dependence of the solution on the initial value for (2) with fuzzy coupling, which can be obtained on the region J. Analogously, one has the same result for (30). This completes the proof. □

In the following, we propose the ε-approximate solution of (27) or (28).

Definition 5. *The function $(u(x,y),v(x,y))^T$ is called the ε-approximate solution of (27) or (28); here $\varepsilon = (\hat{\varepsilon},\bar{\varepsilon})$ if $(u(x,y),v(x,y))^T$ satisfies a coupled system for fuzzy fractional PDEs as follows:*

$$\begin{cases} d_\infty\left({}^{C}_{gH}\mathcal{D}^{\alpha}_{1}u(x,y), f(x,y,v(x,y))\right) \leq \hat{\varepsilon}, \\ d_\infty\left({}^{C}_{gH}\mathcal{D}^{\beta}_{1}v(x,y), g(x,y,u(x,y))\right) \leq \bar{\varepsilon}. \end{cases}$$

Theorem 4. *Suppose that the conditions $(\mathbf{H_1})$ and $(\mathbf{H_2})$ of Lemma 4 hold, and $f \in C_{\mathcal{J}}(J,E_2,E_1)$ and $g \in C_{\mathcal{J}}(J,E_1,E_2)$ satisfy the Lipschitz condition (**LC**), for $i = 1,2$, $(u_i(x,y),v_i(x,y))^T$ is separately the approximate ε_i-solution of (27) or (28), where $\varepsilon_i = (\hat{\varepsilon}_i,\bar{\varepsilon}_i)$, and the corresponding initial value is $(\psi_i,\varphi_i)^T$. Then*

$$\begin{aligned}
\begin{pmatrix} \rho(u_1,u_2) \\ \rho(v_1,v_2) \end{pmatrix} &\leq \begin{pmatrix} \rho(\psi_1,\psi_2) \\ \rho(\varphi_1,\varphi_2) \end{pmatrix} + \begin{pmatrix} \frac{a^{\alpha_1} b^{\alpha_2}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)} & 0 \\ 0 & \frac{a^{\beta_1} b^{\beta_2}}{\Gamma(\beta_1+1)\Gamma(\beta_2+1)} \end{pmatrix} \\
&\qquad \times \sum_{k=0}^{\infty} A^k \begin{pmatrix} L_1\rho(\varphi_1,\varphi_2) + \hat{\varepsilon}_1 + \hat{\varepsilon}_2 \\ L_2\rho(\psi_1,\psi_2) + \bar{\varepsilon}_1 + \bar{\varepsilon}_2 \end{pmatrix},
\end{aligned} \tag{33}$$

where A is the same matrix defined in Lemma 4.

Proof. By Definition 5, it is easy to see that for $i = 1,2$,

$$\begin{cases} d_\infty\left({}^{C}_{gH}\mathcal{D}^{\alpha}_{1}u_i(x,y), f(x,y,v_i(x,y))\right) \leq \hat{\varepsilon}_i, \\ d_\infty\left({}^{C}_{gH}\mathcal{D}^{\beta}_{1}v_i(x,y), g(x,y,u_i(x,y))\right) \leq \bar{\varepsilon}_i. \end{cases} \tag{34}$$

Taking $z_1(x,y) = {}^{C}_{gH}\mathcal{D}^{\alpha}_{1}u_1(x,y)$, $z_2(x,y) = {}^{C}_{gH}\mathcal{D}^{\alpha}_{1}u_2(x,y)$, $\omega_1(x,y) = {}^{C}_{gH}\mathcal{D}^{\beta}_{1}v_1(x,y)$ and $\omega_2(x,y) = {}^{C}_{gH}\mathcal{D}^{\beta}_{1}v_2(x,y)$. In a general way, with regard to (29), one knows that (34) is equivalent to

$$\begin{cases} d_\infty\Big(z_i(x,y), f\Big(x,y,\varphi_i(x,y) + {}^{RL}_{F}\mathcal{I}^{\beta}_{0+}\omega_i(x,y)\Big)\Big) \le \hat{\varepsilon}_i, \\ d_\infty\Big(\omega_i(x,y), g\Big(x,y,\psi_i(x,y) + {}^{RL}_{F}\mathcal{I}^{\alpha}_{0+}z_i(x,y)\Big)\Big) \le \bar{\varepsilon}_i \end{cases}$$

for $i = 1,2$. Since f and g satisfy the Lipschitz conditions (**LC**) , it follows from Lemmas 1 (i), 3 and 4 that

$$\begin{aligned} &\begin{pmatrix} d_\infty(z_1(x,y), z_2(x,y)) \\ d_\infty(\omega_1(x,y), \omega_2(x,y)) \end{pmatrix} \\ &\le \begin{pmatrix} 0 & L_1 {}^{RL}_{F}I^{\beta}_{0+} \\ L_2 {}^{RL}_{F}I^{\alpha}_{0+} & 0 \end{pmatrix} \begin{pmatrix} d_\infty(z_1(x,y), z_2(x,y)) \\ d_\infty(\omega_1(x,y), \omega_2(x,y)) \end{pmatrix} \\ &\qquad + \begin{pmatrix} L_1 d_\infty(\varphi_1(x,y), \varphi_2(x,y)) + \hat{\varepsilon}_1 + \hat{\varepsilon}_2 \\ L_2 d_\infty(\psi_1(x,y), \psi_2(x,y)) + \bar{\varepsilon}_1 + \bar{\varepsilon}_2 \end{pmatrix} \\ &\le \sum_{k=0}^{\infty} A^k \begin{pmatrix} L_1 d_\infty(\varphi_1(x,y), \varphi_2(x,y)) + \hat{\varepsilon}_1 + \hat{\varepsilon}_2 \\ L_2 d_\infty(\psi_1(x,y), \psi_2(x,y)) + \bar{\varepsilon}_1 + \bar{\varepsilon}_2 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} &\begin{pmatrix} d_\infty(u_1(x,y), u_2(x,y)) \\ d_\infty(v_1(x,y), v_2(x,y)) \end{pmatrix} \\ &\le \begin{pmatrix} d_\infty(\psi_1(x,y), \psi_2(x,y)) \\ d_\infty(\varphi_1(x,y), \varphi_2(x,y)) \end{pmatrix} + \begin{pmatrix} {}^{RL}_{F}\mathcal{I}^{\alpha}_{0+} & 0 \\ 0 & {}^{RL}_{F}\mathcal{I}^{\beta}_{0+} \end{pmatrix} \\ &\qquad \times \sum_{k=0}^{\infty} A^k \begin{pmatrix} L_1 d_\infty(\varphi_1(x,y), \varphi_2(x,y)) + \hat{\varepsilon}_1 + \hat{\varepsilon}_2 \\ L_2 d_\infty(\psi_1(x,y), \psi_2(x,y)) + \bar{\varepsilon}_1 + \bar{\varepsilon}_2 \end{pmatrix}. \end{aligned} \tag{35}$$

Taking the upper bound on both sides of (35) on the region J, then by (15), one has

$$\begin{aligned} &\begin{pmatrix} \rho(u_1,u_2) \\ \rho(v_1,v_2) \end{pmatrix} \\ &\le \begin{pmatrix} \rho(\psi_1,\psi_2) \\ \rho(\varphi_1,\varphi_2) \end{pmatrix} \\ &+ \begin{pmatrix} \frac{a^{\alpha_1}b^{\alpha_2}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)} & 0 \\ 0 & \frac{a^{\beta_1}b^{\beta_2}}{\Gamma(\beta_1+1)\Gamma(\beta_2+1)} \end{pmatrix} \times \sum_{k=0}^{\infty} A^k \begin{pmatrix} L_1\rho(\varphi_1,\varphi_2) + \hat{\varepsilon}_1 + \hat{\varepsilon}_2 \\ L_2\rho(\psi_1,\psi_2) + \bar{\varepsilon}_1 + \bar{\varepsilon}_2 \end{pmatrix}, \end{aligned}$$

which means that (33) holds. In a similar way, one can get the same result concerning (30), and so the result presented in Theorem 4 is established. □

Remark 8. *When* $\hat{\varepsilon}_i = \bar{\varepsilon}_i = 0$ *for* $i = 1,2$, (33) *gives* (31) *for the continuous dependence of the solution of* (2) *on the initial value.*

5. Concluding Remarks

In this paper, we studied the following coupled system for fuzzy fractional PDEs under Caputo gH-type derivative of the form

$$\begin{cases} {}^{C}_{gH}\mathcal{D}^{\alpha}_{k}u(x,y) = f(x,y,v(x,y)), \\ {}^{C}_{gH}\mathcal{D}^{\beta}_{k}v(x,y) = g(x,y,u(x,y)), \\ u(x,0) = \xi_1(x), \quad v(x,0) = \eta_1(x), \\ u(0,y) = \xi_2(y), \quad v(0,y) = \eta_2(y) \end{cases} \tag{36}$$

for every $(x,y) \in J = [0,a] \times [0,b]$ and $k = 1,2$. In the sense of the gH-type derivative [26], it is significant to extend the corresponding results of fuzzy fractional PDEs to coupled systems.

Building on some previous work and on the basis of Lipschitz conditions, the main goals and novel work of this paper follow as:

- By using Banach fixed point theorem and mathematical inductive method, we proved existence and uniqueness of two kinds of gH-weak solutions to (36).
- We gave an example for visually embodying the existence and uniqueness theorems and proposed numerical simulations of the (‡) gH-weak solution for (36).
- The equivalence of (36) with a class of nonlinear fractional-order coupled Volterra integro-differential systems was proved, and Gronwall inequality of the vector form was obtained.
- Furthermore, on the basis of the developed Gronwall inequality of the vector form, which is because of the coupling factor in (36), the continuous dependence on the initial data and ε-approximate solution of (36) were established inventively after changing the initial conditions.

As everyone knows that in economics and finance, control and optimization, physics and chemistry, biology and engineering sciences and other practical applications, time delay, impulsive effect or random uncertainty usually occur on account of manual measurement, signal transmission, aging of equipment and so on. Further, implicit equations include explicit equations as special cases. How to find solutions of (36) with time delay terms $x - \lambda$ and $y - \tau$? How to solve the corresponding implicit coupled systems when a function on the right-hand side of (36) contains a Caputo gH-type derivative term? For related work, see, for instance, [3,7,28,29,37] and the references therein. These questions are worth exploring in the next step of this work.

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Abbreviations

The following abbreviations are used in this manuscript:

PDEs	partial differential equations
H-difference	Hukuhara difference
gH-type	generalized Hukuhara type

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