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On Certain Integrals Related to Saran's Hypergeometric Function F_K

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Abstract: In the present paper, we establish two Erdélyi-type integrals for Saran's hypergeometric function F_K, which has applications in specific branches of applied physics and statistics (see below). We employ methods based on the k-dimensional fractional integration by parts to obtain our main integral identities. The first integral generalizes Koschmieder's result and the second integral extends one of Erdélyi's classical hypergeometric integral. Some useful special cases and important remarks are also discussed.

Keywords: Erdélyi-type integral; fractional integration by parts; Saran's function

MSC: 33C65; 33C70

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1. Motivation and Objectives

One of the triple hypergeometric functions due to Saran ([1] p. 294, Equation (2.4); see also [2]) is the F_K function which is defined by

$$\begin{aligned} &F_K[\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_3;x,y,z] \\ &:= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_m(\alpha_2)_{n+p}(\beta_1)_{m+p}(\beta_2)_n}{(\gamma_1)_m(\gamma_2)_n(\gamma_3)_p} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \end{aligned} \tag{1}$$

where $(x,y,z) \in \mathbb{D}_K := \{(x,y,z) \in \mathbb{C}^3 : |x| < 1, |y| < 1, |z| < (1-|x|)(1-|y|)\}$. It may be noted that $\mathbb{D}_K$ is a *complete Reinhardt domain* (see [3] p. 104, Definition 2.3.12) since for $(x,y,z) \in \mathbb{D}_K$ and $\mu_j \in \mathbb{C}$ $(|\mu_j| \le 1, j = 1,2,3)$, we have $|\mu_1 x| \le |x| < 1$, $|\mu_2 y| \le |y| < 1$ and

$$\frac{|\mu_3 z|}{(1-|\mu_1 x|)(1-|\mu_2 y|)} \le \frac{|z|}{(1-|x|)(1-|y|)} < 1,$$

thereby implying that $(\mu_1 x, \mu_2 y, \mu_3 z) \in \mathbb{D}_K$.

During the past few decades, the triple hypergeometric function defined by (1) has been studied by many authors, e.g., Abiodun and Sharma [4], Deshpande [5], Exton [6,7], Pandey [8] and Srivastava and Karlsson [9]. Recently, Luo and Raina [2] investigated some new useful properties and also specifically mentioned the importance of this useful function F_K by pointing out its applications in certain applied sciences. For example, Hutchinson [10] used the F_K-function in his work on compound gamma bivariate distribution, and Kol and Shir [11] used this function in their recent study of the propagator seagull diagram. For more details about the applications of this function, one may refer to the paper [2]; see also [12–16].

The aim of the present paper is to obtain further new results related to the work on Erdélyi-type integrals for the F_K function [2]. For a certain class of hypergeometric functions, the Erdélyi-type integral often connects this class of functions (in terms of an integral representation) to similar forms of functions.

A typical Erdélyi-type integral is given by the following (see [17] p. 178, Equation (11); see also [18] p. 476, Equation (1.1)):

$$ {}_2F_1\left[\begin{matrix}\alpha,\beta\\ \gamma\end{matrix};z\right] = \int_0^1 (1-zx)^{-\alpha'}\,{}_2F_1\left[\begin{matrix}\alpha-\alpha',\beta\\ \lambda\end{matrix};zx\right]{}_2F_1\left[\begin{matrix}\alpha',\beta-\lambda\\ \gamma-\lambda\end{matrix};\frac{(1-x)z}{1-xz}\right]\mathrm{d}\mu_{\lambda,\gamma-\lambda}(x), \tag{2} $$

where $\Re(\gamma) > \Re(\lambda) > 0$, ${}_2F_1$ denotes the familiar Gauss hypergeometric function defined by the following (see [19] p. 13):

$$ {}_2F_1\left[\begin{matrix}\alpha,\beta\\ \gamma\end{matrix};z\right] := \sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{z^n}{n!}\ (|z|<1), $$

and $\mu_{\alpha,\beta}(t)$, as a special case of the *Dirichlet measure*, is defined for $\Re(\alpha) > 0$ and $\Re(\beta) > 0$ by (see [20] p. 52, Definition 3.11-1)

$$ \mathrm{d}\mu_{\alpha,\beta}(t) := \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}t^{\alpha-1}(1-t)^{\beta-1}\mathrm{d}t. \tag{3} $$

Equation (2) was first derived by Erdélyi [17] by making use of the fractional integration by parts and was later rediscovered by Joshi and Vyas [21] by using the series manipulation techniques. Additionally, Equation (2) has some important applications. For example, it was used in solving a certain Abel-type integral equation involving the Appell hypergeometric function F_3 in the kernel [22]. For the latest results concerning the Erdélyi-type integral for hypergeometric functions of one variable, the reader may refer to [18]. Hypergeometric functions of several variables have vastly been studied; see, for example, Refs. [9,23–26]. Due to their importance in the theory and applications, it is always useful and interesting to find new Erdélyi-type integrals associated with such classes of hypergeometric functions.

In the present paper, we focus our investigations on the following two considerations.

Firstly, the authors in their Remark 4.5 of Ref. [2] mention that their main integral identity [2] (p. 14, Theorem 4.1), though general in nature, does not contain Koschmieder's formula [2] (p. 17, Equation (55)) as a special case. However, it was also realized that a more general integral identity may perhaps exist that contains Koschmieder's result. We now confirm by aiming to find such a general form of integral identity which contains Koschmieder's formula, for which we present two independent proofs.

Secondly, Sharma and Manocha [25] in their investigation make use of the familiar methods of fractional integration by parts to establish a much involved integral identity for another class of a triple hypergeometric function F_M of Saran ([1] (p. 294, Equation (2.5); see also [9] (p. 42, Equation (5)), which is defined by

$$ F_M[a,a',a';b,b',b;c,c',c';x,y,z] := \sum_{m,n,p=0}^{\infty}\frac{(a)_m(a')_{n+p}(b)_{m+p}(b')_n}{(c)_m(c')_{n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, $$

where $(x,y,z)\in\mathbb{D}_M := \{(x,y,z)\in\mathbb{C}^3 : |x|<1,\ |y|+|z|<1\}$. Here, $\mathbb{D}_M$ is also a complete Reinhardt domain. It was established in [25] (p. 243) that Saran's function F_M possesses the following integral representation:

$$ \begin{aligned} F_M[a,a',a';b,b',b;c,c',c';x,y,z] &= \int_0^1\int_0^1 (1-vy)^{-d}(1-ux-vz)^{-e} \\ &\cdot F_M\left[a-\lambda,a'-\mu,a'-\mu;e,d,e;c-\lambda,c'-\mu,c'-\mu;\frac{(1-u)x}{1-ux-vz},\frac{(1-v)y}{1-vy},\frac{(1-v)z}{1-ux-vz}\right] \end{aligned} $$

$$\cdot F_M\left[a, a', a', b-e, b'-d, b-e; \lambda, \mu, \mu; ux, vy, vz\right] \mathrm{d}\mu_{\lambda, c-\lambda}(u) \mathrm{d}\mu_{\mu, c'-\mu}(v), \tag{4}$$

where $\Re(c) > \Re(\lambda) > 0$ and $\Re(c') > \Re(\mu) > 0$. In particular, Erdélyi's Formula (2) can be obtained by letting $x = z = 0$ in (4) and noting that

$$F_M\left[a, a', a'; b, b', b; c, c', c'; 0, y, 0\right] = {}_2F_1\left[\begin{matrix} a', b' \\ c' \end{matrix}; y\right].$$

Sharma and Manocha's Formula (4) is an interesting result that depicts an important fact that the Saran's function F_M has also the Erdélyi-type integral relation. In this paper, we show, by specializing a very general type of integral identity, that an integral representation of the Erdélyi-type similar to the result (4) holds also for the F_K function defined above by (1).

2. Some Preliminaries

2.1. Properties of Saran's F_K-Function

The F_K-function has the triple integral representation given by ([2] Equation (3))

$$\begin{aligned} &F_K[\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_3; x, y, z] \\ &\quad = C \int_0^1 \int_0^1 \int_0^1 u^{\alpha_1 - 1}(1-u)^{\gamma_1 - \alpha_1 - 1} v^{\beta_2 - 1}(1-v)^{\gamma_2 - \beta_2 - 1} w^{\beta_1 - 1}(1-w)^{\gamma_3 - \beta_1 - 1} \\ &\qquad \cdot (1-ux)^{\alpha_2 - \beta_1}(1 - ux - vy - wz + uvxy)^{-\alpha_2} \mathrm{d}u\mathrm{d}v\mathrm{d}w, \end{aligned} \tag{5}$$

where $\Re(\gamma_1) > \Re(\alpha_1) > 0$, $\Re(\gamma_3) > \Re(\beta_1) > 0$, $\Re(\gamma_2) > \Re(\beta_2) > 0$ and C is given by

$$C := \frac{\Gamma(\gamma_1)\Gamma(\gamma_2)\Gamma(\gamma_3)}{\Gamma(\alpha_1)\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\gamma_1 - \alpha_1)\Gamma(\gamma_2 - \beta_2)\Gamma(\gamma_3 - \beta_1)}. \tag{6}$$

Alternatively, (5) can be written in a compact form as

$$\begin{aligned} &F_K[\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_3; x, y, z] \\ &\quad = \int_0^1 \int_0^1 \int_0^1 (1-ux)^{\alpha_2 - \beta_1}(1 - ux - vy - wz + uvxy)^{-\alpha_2} \\ &\qquad \cdot \mathrm{d}\mu_{\alpha_1, \gamma_1 - \alpha_1}(u) \mathrm{d}\mu_{\beta_2, \gamma_2 - \beta_2}(v) \mathrm{d}\mu_{\beta_1, \gamma_3 - \beta_1}(w). \end{aligned}$$

By using a simple substitution in (5), Saran [1] (p. 299, Equation (4.2)) derived the following transformation which shall be used in our work below.

$$\begin{aligned} &F_K[\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_3; x, y, z] = (1-x)^{-\beta_1}(1-y)^{-\alpha_2} \\ &\quad \cdot F_K\left[\gamma_1 - \alpha_1, \alpha_2, \alpha_2, \beta_1, \gamma_2 - \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_3; \frac{x}{x-1}, \frac{y}{y-1}, \frac{z}{(1-x)(1-y)}\right]. \end{aligned} \tag{7}$$

It is worth mentioning that Pandey [8] reproduced the transformation (7) by using a contour integral representation of F_K.

2.2. Fractional Integration by Parts for Function of Several Variables

For convenience, we define the fractional derivative by the formal relation

$$\frac{\mathrm{d}^\nu w^\mu}{\mathrm{d}w^\nu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\nu+1)} w^{\mu-\nu}. \tag{8}$$

For functions of one variable (i.e., one-dimensional case), the formula of fractional integration by parts can be found, for example, in [27] (p. 112, Equation (2.9.3)), [18] (p. 478, Equation (2.3)) and [2]. However, for functions of several variables, we could not find in the literature a formal theorem giving the fractional integration by parts of functions of several variables, though Chandel [28], Koschmieder [29,30], Manocha [24], Mittal [31], Manocha and Sharma [25] and Luo and Raina [2] obtained results for the functions F_1, F_2, F_3, F_4, F_A, F_D, F_K and F_M by repeatedly using the fractional integration formula for one variable.

Here, for the clarity of presentation, we give a formal version of k-dimensional fractional integration by parts.

Lemma 1 (k-dimensional fractional integration by parts). *Let* $\mathbf{x} := (x_1, \cdots, x_k)$, $\nu := (\nu_1, \cdots, \nu_k)$ *and* $u(\mathbf{x})$ *and* $v(\mathbf{x})$ *be functions of k-variables defined by*

$$u(\mathbf{x}) = \sum_{\mathbf{m}=0}^{\infty} A_{\mathbf{m}} \prod_{j=1}^{k} (x_j - a_j)^{\rho_j + m_j - 1} \quad \textit{and} \quad v(\mathbf{x}) = \sum_{\mathbf{n}=0}^{\infty} B_{\mathbf{n}} \prod_{j=1}^{k} (b_j - x_j)^{\sigma_j + n_j - 1}.$$

Additionally, let

$$\mathcal{D}_{-}^{\nu} := \frac{\partial^{\nu_1}}{\partial (b_1 - x_1)^{\nu_1}} \cdots \frac{\partial^{\nu_k}}{\partial (b_k - x_k)^{\nu_k}} \quad \textit{and} \quad \mathcal{D}_{+}^{\nu} := \frac{\partial^{\nu_1}}{\partial (x_1 - a_1)^{\nu_1}} \cdots \frac{\partial^{\nu_k}}{\partial (x_k - a_k)^{\nu_k}}.$$

Then

$$\int_{\mathbf{a}}^{\mathbf{b}} u(\mathbf{x}) \mathcal{D}_{-}^{\nu} v(\mathbf{x}) \mathrm{d}\mathbf{x} = \int_{\mathbf{a}}^{\mathbf{b}} v(\mathbf{x}) \mathcal{D}_{+}^{\nu} u(\mathbf{x}) \mathrm{d}\mathbf{x}, \tag{9}$$

holds, provided that the integrals exist.

Proof. To prove (9), we note upon using an elementary fractional integral formula and evaluations that

$$\begin{aligned} \mathcal{D}_{-}^{\nu} v(\mathbf{x}) &= \sum_{\mathbf{n}=0}^{\infty} B_{\mathbf{n}} \prod_{j=1}^{k} \frac{\partial^{\nu_j} (b_j - x_j)^{\sigma_j + n_j - 1}}{\partial (b_j - x_j)^{\nu_j}} \\ &= \sum_{\mathbf{n}=0}^{\infty} B_{\mathbf{n}} \prod_{j=1}^{k} \frac{\Gamma(\sigma_j + n_j)}{\Gamma(\sigma_j - \nu_j + n_j)} (b_j - x_j)^{\sigma_j - \nu_j + n_j - 1}, \end{aligned}$$

so that

$$\begin{aligned} &\int_{\mathbf{a}}^{\mathbf{b}} u(\mathbf{x}) \mathcal{D}_{-}^{\nu} v(\mathbf{x}) \mathrm{d}\mathbf{x} \\ &= \sum_{\mathbf{m}=0}^{\infty} \sum_{\mathbf{n}=0}^{\infty} A_{\mathbf{m}} B_{\mathbf{n}} \prod_{j=1}^{k} \frac{\Gamma(\sigma_j + n_j)}{\Gamma(\sigma_j - \nu_j + n_j)} \int_{a_j}^{b_j} (x_j - a_j)^{\rho_j + m_j - 1} (b_j - x_j)^{\sigma_j - \nu_j + n_j - 1} \mathrm{d}x_j \\ &= \sum_{\mathbf{m}=0}^{\infty} \sum_{\mathbf{n}=0}^{\infty} A_{\mathbf{m}} B_{\mathbf{n}} \prod_{j=1}^{k} \frac{\Gamma(\rho_j + m_j)\Gamma(\sigma_j + n_j)}{\Gamma(\sigma_j + \rho_j - \nu_j + m_j + n_j)} (b_j - a_j)^{\rho_j + \sigma_j - \nu_j + m_j + n_j - 1} \\ &= \sum_{\mathbf{n}=0}^{\infty} \sum_{\mathbf{m}=0}^{\infty} B_{\mathbf{n}} A_{\mathbf{m}} \prod_{j=1}^{k} \frac{\Gamma(\rho_j + m_j)}{\Gamma(\rho_j - \nu_j + m_j)} \int_{a_j}^{b_j} (x_j - a_j)^{\sigma_j + n_j - 1} (b_j - x_j)^{\rho_j - \nu_j + m_j - 1} \mathrm{d}x_j \\ &= \int_{\mathbf{a}}^{\mathbf{b}} v(\mathbf{x}) \mathcal{D}_{+}^{\nu} u(\mathbf{x}) \mathrm{d}\mathbf{x}, \end{aligned}$$

provided that $\Re(\sigma_j) > 0$, $\Re(\sigma_j - \nu_j) > 0$, $\Re(\rho_j) > 0$ and $\Re(\rho_j - \nu_j) > 0$ $(j = 1, 2, \cdots, k)$. This proves Lemma 1. □

2.3. Hypergeometric Function of Several Variables

In Section 4 below, we encounter the Srivastava–Daoust hypergeometric function of several variables [9] (p. 37, Equation (21)); see also [26] (p. 64, Equation (18)). For convenience and compactness' sake, we adopt here slightly varied forms of notations for the Srivastava–Daoust function [9].

As usual, let $(a) := (a_1, \cdots, a_A)$ be a A-dimensional row vector, and let $(b) := (b_1, \cdots, b_B)$ be a B-dimensional row vector. Next, let $\theta^{(j)} := (\theta_1^{(j)}, \cdots, \theta_r^{(j)})$ be a r-dimensional row vector, where j is a positive integer and $\theta_\ell^{(j)} \geq 0$ ($\ell = 1, \cdots, r$). Then,

$$\theta := \left(\theta^{(1)}, \cdots, \theta^{(A)}\right) = \left(\theta_1^{(1)}, \cdots, \theta_r^{(1)}, \cdots, \theta_1^{(A)}, \cdots, \theta_r^{(A)}\right).$$

is a Ar-dimensional row vector. Similarly, $\psi^{(j)} := (\psi_1^{(j)}, \cdots, \psi_r^{(j)})$ is also a r-dimensional row vector with $\psi_\ell^{(j)} \geq 0$ ($\ell = 1, \cdots, r$), and

$$\psi := \left(\psi^{(1)}, \cdots, \psi^{(B)}\right) = \left(\psi_1^{(1)}, \cdots, \psi_r^{(1)}, \cdots, \psi_1^{(B)}, \cdots, \psi_r^{(B)}\right).$$

is a Br-dimensional row vector. The Srivastava–Daoust hypergeometric function in r-complex variables can then be defined as

$$F_B^A \left[\begin{matrix} (a) : \theta \\ (b) : \psi \end{matrix}; z_1, \cdots, z_r \right] = \sum_{m_1, \cdots, m_r = 0}^{\infty} \frac{\prod_{j=1}^{A} (a_j)_{m_1\theta_1^{(j)} + \cdots + m_r\theta_r^{(j)}}}{\prod_{j=1}^{B} (b_j)_{m_1\psi_1^{(j)} + \cdots + m_r\psi_r^{(j)}}} \frac{z_1^{m_1}}{m_1!} \cdots \frac{z_r^{m_r}}{m_r!}, \tag{10}$$

where an empty product is interpreted to be 1.

The series (10) contains the usual definition of the Srivastava–Daoust hypergeometric function as a special case. It is easy to see that it also contains many known multivariable hypergeometric functions (e.g., Lauricella's function, Srivastava's triple hypergeometric function, Saran's functions, Kampé de Fériet's function, etc.) as special cases. We demonstrate here through a concrete example the advantage of our definition (10).

When $r = 2$,

$$\begin{aligned} \theta^{(1)} = \cdots = \theta^{(p)} &= (1, 1), \\ \theta^{(p+1)} = \cdots = \theta^{(p+q)} &= (1, 0), \\ \theta^{(p+q+1)} = \cdots = \theta^{(A)} &= (0, 1), \\ \psi^{(1)} = \cdots = \psi^{(\ell)} &= (1, 1), \\ \psi^{(\ell+1)} = \cdots = \psi^{(\ell+m)} &= (1, 0), \\ \psi^{(\ell+m+1)} = \cdots = \psi^{(C)} &= (0, 1), \end{aligned}$$

$A = p + q + k$ and $B = \ell + m + n$, the series in (10) becomes

$$\sum_{m_1, m_2 = 0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_{m_1+m_2} \prod_{j=1}^{q} (a_{p+j})_{m_1} \prod_{j=1}^{k} (a_{p+q+j})_{m_2}}{\prod_{j=1}^{\ell} (b_j)_{m_1+m_2} \prod_{j=1}^{m} (b_{\ell+j})_{m_1} \prod_{j=1}^{n} (b_{\ell+m+j})_{m_2}} \frac{z_1^{m_1}}{m_1!} \frac{z_2^{m_2}}{m_2!},$$

which is the definition of the Kampé de Fériet function $F_{\ell:m;n}^{p:q;k}[x, y]$ (see [9] p. 27, Equation (28)). This example shows that it is not necessary to make a deliberate distinction between factors, such as $(a_i)_{m_1\theta_1^{(i)} + m_2\theta_2^{(i)}}$ and $(a_j)_{m_1\theta_1^{(j)}}$ in the notation of the function since those components of $\theta^{(i)}$ that are zero will determine the form of $(a_i)_{m_1\theta_1^{(i)} + m_2\theta_2^{(i)}}$.

It may be difficult to establish a general theorem about the convergence of the multiple series (10), unless we impose a *positivity* condition on θ and ψ. However, for a specific series, it is always possible to check its convergence by using the methods described in the book of Srivastava and Karlsson [9]. For the convergence conditions of the Srivastava–Daoust hypergeometric function defined in usual way, the interested reader may refer to [32].

3. The First Integral

In this section, we establish a general integral identity that generalizes the Koschmieder's result (see Corollary 1 below). We use the fractional integration by parts to obtain our result and also point out below that the integral identity can also be proved in an alternative way.

For $\max\{\Re(\eta), \Re(\gamma), \Re(\eta+\gamma-\alpha-\beta)\} > 0$, we define the complex measure $\mu_{\alpha,\beta,\gamma,\eta}$ by

$$\mathrm{d}\mu_{\alpha,\beta,\gamma,\eta}(t) := \frac{\Gamma(\eta+\gamma-\alpha)\Gamma(\eta+\gamma-\beta)}{\Gamma(\eta)\Gamma(\gamma)\Gamma(\eta+\gamma-\alpha-\beta)} t^{\eta-1}(1-t)^{\gamma-1} {}_2F_1\left[\begin{matrix}\alpha,\beta\\ \gamma\end{matrix};1-t\right]\mathrm{d}t. \tag{11}$$

By using [33] (p. 821, Equation (7.512.4))

$$\int_0^1 x^{\gamma-1}(1-x)^{\rho-1} {}_2F_1\left[\begin{matrix}\alpha,\beta\\ \gamma\end{matrix};x\right]\mathrm{d}x = \frac{\Gamma(\gamma)\Gamma(\rho)\Gamma(\gamma+\rho-\alpha-\beta)}{\Gamma(\gamma+\rho-\alpha)\Gamma(\gamma+\rho-\beta)}$$

$$(\Re(\gamma) > 0,\ \Re(\rho) > 0,\ \Re(\gamma+\rho-\alpha-\beta) > 0),$$

it can be verified that $\mu_{\alpha,\beta,\gamma,\eta}([0,1]) = 1$. Evidently, the Dirichlet measure defined in (3) is a special case of (11).

Theorem 1. *For* $\max\{\Re(\gamma_j+\lambda_j-\nu_j-\mu_j), \Re(\mu_j), \Re(\nu_j)\} > 0$ *(*$j = 1,2,3$*), we have*

$$\begin{aligned} &F_K[\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_3;x,y,z] \\ &\quad= \int_0^1\int_0^1\int_0^1 F^{(3)}\left[\begin{matrix} -::-;\alpha_2;\beta_1;\alpha_1,\lambda_1;\beta_2,\lambda_2;\lambda_3 \\ -::-;\ -\ ;\ -\ ;\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3\end{matrix};u_1x,u_2y,u_3z\right] \\ &\qquad\cdot\prod_{j=1}^3 \mathrm{d}\mu_{\gamma_j-\nu_j,\lambda_j-\nu_j,\gamma_j+\lambda_j-\nu_j-\mu_j,\mu_j}(u_j), \end{aligned}$$

where $F^{(3)}[x,y,z]$ *denotes Srivastava's general triple hypergeometric function* [9] (p. 44, Equation (14)).

Proof. Let us start with the integral representation [2] (p. 16, Corollary 4.2):

$$\begin{aligned} &F_K[\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_3;x,y,z] \\ &\quad= \int_0^1\int_0^1\int_0^1 F_K[\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\nu_1,\nu_2,\nu_3;u_1x,u_2y,u_3z]\prod_{j=1}^3 \mathrm{d}\mu_{\nu_j,\gamma_j-\nu_j}(u_j), \end{aligned} \tag{12}$$

where $\max\{\Re(\nu_j), \Re(\gamma_j-\nu_j)\} > 0$ ($j = 1,2,3$). Using the fractional derivative Formula (8), we obtain

$$\begin{aligned} &u_1^{\lambda_1-1}u_2^{\lambda_2-1}u_3^{\lambda_3-1}F_K[\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\nu_1,\nu_2,\nu_3;u_1x,u_2y,u_3z] \\ &\quad= A_1\frac{\partial^{\mu_1-\lambda_1}}{\partial u_1^{\mu_1-\lambda_1}}\frac{\partial^{\mu_2-\lambda_2}}{\partial u_2^{\mu_2-\lambda_2}}\frac{\partial^{\mu_3-\lambda_3}}{\partial u_3^{\mu_3-\lambda_3}}\Bigg\{u_1^{\mu_1-1}u_2^{\mu_2-1}u_3^{\mu_3-1} \\ &\qquad\cdot F^{(3)}\left[\begin{matrix} -::-;\alpha_2;\beta_1;\alpha_1,\lambda_1;\beta_2,\lambda_2;\lambda_3 \\ -::-;\ -\ ;\ -\ ;\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3\end{matrix};u_1x,u_2y,u_3z\right]\Bigg\}, \end{aligned} \tag{13}$$

where

$$A_1 := \prod_{j=1}^3 \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)}.$$

Hence, by using (13), the integral (12) can be written as

$$\begin{aligned}
&F_K[\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_3; x, y, z] \\
&= A_2 \int_0^1 \int_0^1 \int_0^1 \prod_{j=1}^{3} u_j^{\nu_j - \lambda_j} (1 - u_j)^{\gamma_j - \nu_j - 1} \frac{\partial^{\mu_1 - \lambda_1}}{\partial u_1^{\mu_1 - \lambda_1}} \frac{\partial^{\mu_2 - \lambda_2}}{\partial u_2^{\mu_2 - \lambda_2}} \frac{\partial^{\mu_3 - \lambda_3}}{\partial u_3^{\mu_3 - \lambda_3}} \Bigg\{ u_1^{\mu_1 - 1} u_2^{\mu_2 - 1} u_3^{\mu_3 - 1} \\
&\quad \cdot F^{(3)} \left[\begin{matrix} - :: -; \alpha_2; \beta_1; \alpha_1, \lambda_1; \beta_2, \lambda_2; \lambda_3 \\ - :: -; \ - \ ; \ - \ ; \nu_1, \mu_1; \nu_2, \mu_2; \nu_3, \mu_3 \end{matrix} ; u_1 x, u_2 y, u_3 z \right] \Bigg\} \mathrm{d}u_1 \mathrm{d}u_2 \mathrm{d}u_3,
\end{aligned}$$

where

$$A_2 := \prod_{j=1}^{3} \frac{\Gamma(\lambda_j)\Gamma(\gamma_j)}{\Gamma(\mu_j)\Gamma(\nu_j)\Gamma(\gamma_j - \nu_j)}. \tag{14}$$

Now making use of the rule of the fractional integration by parts (9), we obtain

$$\begin{aligned}
F_K[\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_3; x, y, z] &= A_2 \int_0^1 \int_0^1 \int_0^1 u_1^{\mu_1 - 1} u_2^{\mu_2 - 1} u_3^{\mu_3 - 1} \\
&\quad \cdot F^{(3)} \left[\begin{matrix} - :: -; \alpha_2; \beta_1; \alpha_1, \lambda_1; \beta_2, \lambda_2; \lambda_3 \\ - :: -; \ - \ ; \ - \ ; \nu_1, \mu_1; \nu_2, \mu_2; \nu_3, \mu_3 \end{matrix} ; u_1 x, u_2 y, u_3 z \right] \\
&\quad \cdot \prod_{j=1}^{3} \frac{\partial^{\mu_j - \lambda_j}}{\partial (1 - u_j)^{\mu_j - \lambda_j}} \left\{ u_j^{\nu_j - \lambda_j} (1 - u_j)^{\gamma_j - \nu_j - 1} \right\} \mathrm{d}u_1 \mathrm{d}u_2 \mathrm{d}u_3,
\end{aligned}$$

where A_2 is given by (14). The result then follows immediately by using the formula [2] (p. 15):

$$\begin{aligned}
&\frac{\partial^{\mu - \lambda}}{\partial (1 - u)^{\mu - \lambda}} \left\{ u^{\nu - \lambda} (1 - u)^{\gamma - \nu - 1} \right\} \\
&\qquad = \frac{\Gamma(\gamma - \nu)}{\Gamma(\gamma - \nu - \mu + \lambda)} (1 - u)^{\gamma - \nu - \mu + \lambda - 1} {}_2F_1 \left[\begin{matrix} \lambda - \nu, \gamma - \nu \\ \gamma - \nu - \mu + \lambda \end{matrix} ; 1 - u \right]
\end{aligned}$$

and the measure defined in (11). □

Remark 1.

(1) *The method of using fractional integration by parts is one way of proving Theorem 1. We can, however, adopt a direct approach to establish the integral identity of Theorem 1. In fact, if we first express the $F^{(3)}$ function as a triple series, interchange the order of integration and summation and then carry out elementary evaluations, we will arrive at the desired result.*

(2) *Let*

$$f(\mathbf{z}) = \sum_{\mathbf{m}=0}^{\infty} A_{\mathbf{m}} \mathbf{z}^{\mathbf{m}} \ (\mathbf{z} \in F) \quad \textit{and} \quad g(\mathbf{z}) = \sum_{\mathbf{m}=0}^{\infty} B_{\mathbf{m}} \mathbf{z}^{\mathbf{m}} \ (\mathbf{z} \in G),$$

where $\mathbf{z}^{\mathbf{m}} := z_1^{m_1} \cdots z_k^{m_k}$, *then the* Hadamard product *(also called the convolution) of* $f(\mathbf{z})$ *and* $g(\mathbf{z})$ *is defined by*

$$(f * g)(\mathbf{z}) = \sum_{\mathbf{m}=0}^{\infty} A_{\mathbf{m}} B_{\mathbf{m}} \mathbf{z}^{\mathbf{m}} \quad (\mathbf{z} \in F * G).$$

It is easy to verify that

$$\begin{aligned}
&F^{(3)} \left[\begin{matrix} - :: -; \alpha_2; \beta_1; \alpha_1, \lambda_1; \beta_2, \lambda_2; \lambda_3 \\ - :: -; \ - \ ; \ - \ ; \nu_1, \mu_1; \nu_2, \mu_2; \nu_3, \mu_3 \end{matrix} ; u_1, u_2, u_3 \right] \\
&\qquad = F_K[\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \mu_1, \mu_2, \mu_3; u_1, u_2, u_3] * \prod_{j=1}^{3} {}_2F_1 \left[\begin{matrix} 1, \lambda_j \\ \nu_j \end{matrix} ; u_j \right].
\end{aligned}$$

In addition, since $\mathbb{D}_K$ *and the polydisc* $\mathbb{D} := \{(u_1, u_2, u_3) \in \mathbb{C}^3 : |u_j| < 1, j = 1, 2, 3\}$ *are complete Reinhardt domains of holomorphy, it implies therefore from* [34] (p. 22, Observation 5.1) *that the region of convergence of* $F^{(3)}$ *is*

$$\mathbb{D}_K * \mathbb{D} = \mathbb{D}_K \cdot \mathbb{D} = \mathbb{D}_K,$$

where $A \cdot B = \{ab : a \in A, b \in B\}$.

Now when $x = 0$, the F_K-function reduces to the function F_2 and the $F^{(3)}$-function reduces to

$$\begin{aligned}\sum_{m_2,m_3=0}^{\infty} &(\alpha_2)_{m_2+m_3} \frac{(\beta_2)_{m_2}(\lambda_2)_{m_2}(\beta_1)_{m_3}(\lambda_3)_{m_3}}{(\nu_2)_{m_2}(\mu_2)_{m_2}(\nu_3)_{m_3}(\mu_3)_{m_3}} \frac{(u_2 y)^{m_2}}{m_2!} \frac{(u_3 z)^{m_3}}{m_3!} \\ &= F^{1:2;2}_{0:2;2}\left[\begin{matrix}\alpha_2 : \beta_2, \lambda_2; \beta_1, \lambda_3 \\ - : \nu_2, \mu_2; \nu_3, \mu_3\end{matrix}; u_2 y, u_3 z\right],\end{aligned}$$

and, consequently, our Theorem 1 yields the following result due to Koschmieder [29] (p. 253, Equation (10.6)).

Corollary 1 (Koschmieder [29])**.** *For* $\max\{\Re(\gamma_j + \lambda_j - \nu_j - \mu_j), \Re(\mu_j), \Re(\nu_j)\} > 0\,(j = 2, 3)$*, we have*

$$\begin{aligned}F_2[\alpha_2, \beta_2, \beta_1; \gamma_2, \gamma_3; y, z] = \int_0^1 \int_0^1 & F^{1:2;2}_{0:2;2}\left[\begin{matrix}\alpha_2 : \beta_2, \lambda_2; \beta_1, \lambda_3 \\ - : \nu_2, \mu_2; \nu_3, \mu_3\end{matrix}; u_2 y, u_3 z\right] \\ &\cdot \prod_{j=2}^{3} \mathrm{d}\mu_{\gamma_j - \nu_j, \lambda_j - \nu_j, \gamma_j + \lambda_j - \nu_j - \mu_j, \mu_j}(u_j). \end{aligned} \tag{15}$$

Remark 2. *Incidentally, after nearly three decades of the publication of Koschmieder's result* (15) *of* [29] (p. 253)*, Mittal in his paper* [31] (p. 104, Equation (14)) *reproduced the same formula by using similar methods as those mentioned above.*

4. The Second Integral

In this section, we establish a new integral for the function F_K defined above by (1) which evidently provides a generalization of the Erdélyi integral (2). The integral we propose to establish here is quite general and is very different from the one we discussed in Section 3 above.

We first need to prove the following two lemmas:

Lemma 2. *Let* $\max\{|u|, |1-u|, |v|, |1-v|, |w|, |1-w|\} < 1$, $\max\{|x|, |y|\} < 1/2$ *and* $|z| < (1 - 2|x|)(1 - 2|y|)$*, then we have*

$$\begin{aligned}&(1-u)^{\eta_1-1}(1-v)^{\mu_2-1}(1-w)^{\mu_1-1}(1-ux)^{\eta_2-\mu_1}(1-ux-vy-(1-w)z+uvxy)^{-\eta_2} \\ &= D_1(1-x)^{-\mu_1}(1-y)^{-\eta_2} \\ &\quad \cdot \frac{\partial^{\lambda_1-\eta_1}}{\partial(1-u)^{\lambda_1-\eta_1}} \frac{\partial^{\lambda_2-\mu_2}}{\partial(1-v)^{\lambda_2-\mu_2}} \frac{\partial^{\lambda_3-\mu_1}}{\partial(1-w)^{\lambda_3-\mu_1}} \Bigg\{(1-u)^{\lambda_1-1}(1-v)^{\lambda_2-1}(1-w)^{\lambda_3-1} \\ &\quad \cdot F_K\left[\eta_1, \eta_2, \eta_2, \mu_1, \mu_2, \mu_1; \lambda_1, \lambda_2, \lambda_3; \frac{(1-u)x}{x-1}, \frac{(1-v)y}{y-1}, \frac{(1-w)z}{(1-x)(1-y)}\right]\Bigg\},\end{aligned} \tag{16}$$

where

$$D_1 := \frac{\Gamma(\eta_1)\Gamma(\mu_1)\Gamma(\mu_2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)}.$$

Proof. Let $\mathcal{R}$ denote the right-hand side of (16). Under the conditions stated with the lemma, it is easy to see that

$$\left|\frac{(1-u)x}{x-1}\right| < \frac{|x|}{1-|x|} < 1, \quad \left|\frac{(1-v)y}{y-1}\right| < \frac{|y|}{1-|y|} < 1 \tag{17}$$

and

$$\begin{aligned}\left|\frac{(1-w)z}{(1-x)(1-y)}\right| &< \frac{|z|}{(1-|x|)(1-|y|)} \\ &< \left(1-\frac{|x|}{1-|x|}\right)\left(1-\frac{|y|}{1-|y|}\right) < \left(1-\left|\frac{(1-u)x}{x-1}\right|\right)\left(1-\left|\frac{(1-v)y}{y-1}\right|\right).\end{aligned}$$

We can therefore express the F_K-function as a triple series and apply the fractional derivative given in (8) term wise to obtain

$$\begin{aligned}\mathcal{R} &= D_1(1-x)^{-\mu_1}(1-y)^{-\eta_2}\sum_{m,n,p=0}^{\infty}\frac{(\eta_1)_m(\eta_2)_{n+p}(\mu_1)_{m+p}(\mu_2)_n}{(\lambda_1)_m(\lambda_2)_n(\lambda_3)_p m!n!p!} \\ &\quad\cdot\left(\frac{x}{x-1}\right)^m\left(\frac{y}{y-1}\right)^n\left(\frac{z}{(1-x)(1-y)}\right)^p\frac{\partial^{\lambda_1-\eta_1}}{\partial(1-u)^{\lambda_1-\eta_1}}\left\{(1-u)^{\lambda_1+m-1}\right\} \\ &\quad\cdot\frac{\partial^{\lambda_2-\mu_2}}{\partial(1-v)^{\lambda_2-\mu_2}}\left\{(1-v)^{\lambda_2+n-1}\right\}\frac{\partial^{\lambda_3-\mu_1}}{\partial(1-w)^{\lambda_3-\mu_1}}\left\{(1-w)^{\lambda_3+p-1}\right\} \\ &= (1-u)^{\eta_1-1}(1-v)^{\mu_2-1}(1-w)^{\mu_1-1}(1-x)^{-\mu_1}(1-y)^{-\eta_2} \\ &\quad\cdot\sum_{m,n,p=0}^{\infty}\frac{(\eta_2)_{n+p}(\mu_1)_{m+p}}{(\mu_1)_p m!n!p!}\left(\frac{(1-u)x}{x-1}\right)^m\left(\frac{(1-v)y}{y-1}\right)^n\left(\frac{(1-w)z}{(1-x)(1-y)}\right)^p.\end{aligned} \tag{18}$$

We first sum the triple series in the right-hand side of (18) over the index-m and next over the index-n under the same condition (17) to obtain

$$\begin{aligned}\mathcal{R} &= (1-u)^{\eta_1-1}(1-v)^{\mu_2-1}(1-w)^{\mu_1-1}(1-ux)^{-\mu_1}(1-y)^{-\eta_2} \\ &\quad\cdot\sum_{n,p=0}^{\infty}\frac{(\eta_2)_{n+p}}{n!p!}\left(\frac{(1-v)y}{y-1}\right)^n\left(\frac{(1-w)z}{(1-ux)(1-y)}\right)^p \\ &= (1-u)^{\eta_1-1}(1-v)^{\mu_2-1}(1-w)^{\mu_1-1}(1-ux)^{-\mu_1}(1-vy)^{-\eta_2} \\ &\quad\cdot\sum_{p=0}^{\infty}\frac{(\eta_2)_p}{p!}\left(\frac{(1-w)z}{(1-ux)(1-vy)}\right)^p.\end{aligned}$$

The last series can also be summed by noting that

$$\left|\frac{(1-w)z}{(1-ux)(1-vy)}\right| < \frac{|z|}{(1-|x|)(1-|y|)} < \frac{|z|}{(1-2|x|)(1-2|y|)} < 1,$$

and thus

$$\begin{aligned}\mathcal{R} &= (1-u)^{\eta_1-1}(1-v)^{\mu_2-1}(1-w)^{\mu_1-1}(1-ux)^{-\mu_1}(1-vy)^{-\eta_2}\left(1-\tfrac{(1-w)z}{(1-ux)(1-vy)}\right)^{-\eta_2} \\ &= (1-u)^{\eta_1-1}(1-v)^{\mu_2-1}(1-w)^{\mu_1-1}(1-ux)^{\eta_2-\mu_1}(1-ux-vy-(1-w)z+uvxy)^{-\eta_2}.\end{aligned}$$

This completes the proof. □

Let us now define a function $\mathcal{Q}_{x,y,z}(u,v,w)$ involving the fractional derivative operators by

$$\mathcal{Q}_{x,y,z}(u,v,w) := \frac{\partial^{\lambda_1-\eta_1}}{\partial u^{\lambda_1-\eta_1}}\frac{\partial^{\lambda_2-\mu_2}}{\partial v^{\lambda_2-\mu_2}}\frac{\partial^{\lambda_3-\mu_1}}{\partial w^{\lambda_3-\mu_1}}\Big\{u^{\alpha_1-1}(1-u)^{\gamma_1-\alpha_1-\eta_1} \cdot v^{\beta_2-1}(1-v)^{\gamma_2-\beta_2-\mu_2}w^{\gamma_3-\beta_1-1}(1-w)^{\beta_1-\mu_1} \cdot (1-ux)^{\alpha_2-\eta_2-\beta_1+\mu_1}(1-ux-vy-(1-w)z+uvxy)^{-\alpha_2+\eta_2}\Big\}.$$

The following lemma gives an explicit evaluation of the function $\mathcal{Q}_{x,y,z}(u,v,w)$.

Lemma 3. *For* $\max\{|u|,|v|,|w|,|1-w|\}<1$ *and* $(x,y,z)\in\mathbb{D}_K$*, we have*

$$\mathcal{Q}_{x,y,z}(u,v,w) = D_2 u^{\alpha_1-\lambda_1+\eta_1-1}(1-u)^{\gamma_1-\lambda_1-\alpha_1}v^{\beta_2-\lambda_2+\mu_2-1}(1-v)^{\gamma_2-\lambda_2-\beta_2} \cdot w^{\gamma_3+\mu_1-\beta_1-\lambda_3-1}(1-w)^{\beta_1-\lambda_3}\mathcal{F}_{x,y,z}(u,v,w), \tag{19}$$

where

$$D_2 := \frac{\Gamma(\alpha_1)\Gamma(\beta_2)\Gamma(\gamma_3-\beta_1)}{\Gamma(\alpha_1-\lambda_1+\eta_1)\Gamma(\beta_2-\lambda_2+\mu_2)\Gamma(\gamma_3+\mu_1-\beta_1-\lambda_3)}$$

and

$$\mathcal{F}_{x,y,z}(u,v,w) := \sum_{m,n,p=0}^{\infty}\frac{(\alpha_2-\eta_2)_{n+p}(\beta_1-\mu_1)_{m+p}(\alpha_1)_m(\beta_2)_n}{(\beta_1-\mu_1)_p(\alpha_1-\lambda_1+\eta_1)_m(\beta_2-\lambda_2+\mu_2)_n}\frac{(ux)^m}{m!}\frac{(vy)^n}{n!} \cdot \frac{[(1-w)z]^p}{p!}\,{}_2F_1\left[\begin{matrix}\gamma_1-\lambda_1+m,\eta_1-\lambda_1\\ \alpha_1-\lambda_1+\eta_1+m\end{matrix};u\right]{}_2F_1\left[\begin{matrix}\gamma_2-\lambda_2+n,\mu_2-\lambda_2\\ \beta_2-\lambda_2+\mu_2+n\end{matrix};v\right] \cdot {}_2F_1\left[\begin{matrix}\gamma_3-\lambda_3+p,\mu_1-\lambda_3\\ \gamma_3+\mu_1-\beta_1-\lambda_3\end{matrix};w\right]. \tag{20}$$

Proof. The key ingredient of the proof is the following expansion:

$$(1-ux)^{\alpha_2-\eta_2-\beta_1+\mu_1}(1-ux-vy-(1-w)z+uvxy)^{-\alpha_2+\eta_2} = \sum_{m,n,p=0}^{\infty}\frac{(\alpha_2-\eta_2)_{n+p}(\beta_1-\mu_1)_{m+p}}{(\beta_1-\mu_1)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\cdot u^m v^n(1-w)^p \tag{21}$$
$$(\max\{|u|,|v|,|1-w|\}<1;\ (x,y,z)\in\mathbb{D}_K).$$

Let $\mathcal{S}$ denote the triple series in (21). By summing it over m, n and p (as in the proof of Lemma 2), we obtain

$$\begin{aligned}\mathcal{S} &= (1-ux)^{-\beta_1+\mu_1}\sum_{n,p=0}^{\infty}\frac{(\alpha_2-\eta_2)_{n+p}}{n!p!}(vy)^n\left(\frac{(1-w)z}{1-ux}\right)^p \\ &= (1-ux)^{-\beta_1+\mu_1}(1-vy)^{-\alpha_2+\eta_2}\sum_{p=0}^{\infty}\frac{(\alpha_2-\eta_2)_p}{p!}\left(\frac{(1-w)z}{(1-ux)(1-vy)}\right)^p \\ &= (1-ux)^{-\beta_1+\mu_1}(1-vy)^{-\alpha_2+\eta_2}\left(1-\frac{(1-w)z}{(1-ux)(1-vy)}\right)^{-\alpha_2+\eta_2} \\ &= (1-ux)^{\alpha_2-\eta_2-\beta_1+\mu_1}(1-ux-vy-(1-w)z+uvxy)^{-\alpha_2+\eta_2}.\end{aligned}$$

In view of the expansion (21), we have

$$\mathcal{F}_{x,y,z}(u,v,w) = \sum_{m,n,p=0}^{\infty} \frac{(\alpha_2-\eta_2)_{n+p}(\beta_1-\mu_1)_{m+p}}{(\beta_1-\mu_1)_p} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}$$
$$\cdot \frac{\partial^{\lambda_1-\eta_1}}{\partial u^{\lambda_1-\eta_1}}\left\{u^{\alpha_1+m-1}(1-u)^{\gamma_1-\alpha_1-\eta_1}\right\}\frac{\partial^{\lambda_2-\mu_2}}{\partial v^{\lambda_2-\mu_2}}\left\{v^{\beta_2+n-1}(1-v)^{\gamma_2-\beta_2-\mu_2}\right\}$$
$$\cdot \frac{\partial^{\lambda_3-\mu_1}}{\partial w^{\lambda_3-\mu_1}}\left\{w^{\gamma_3-\beta_1-1}(1-w)^{\beta_1-\mu_1+p}\right\}. \tag{22}$$

It is easy to verify that

$$\frac{\partial^{\lambda_1-\eta_1}}{\partial u^{\lambda_1-\eta_1}}\left\{u^{\alpha_1+m-1}(1-u)^{\gamma_1-\alpha_1-\eta_1}\right\} = \frac{\Gamma(\alpha_1)}{\Gamma(\alpha_1-\lambda_1+\eta_1)}\frac{(\alpha_1)_m}{(\alpha_1-\lambda_1+\eta_1)_m}$$
$$\cdot u^{\alpha_1-\lambda_1+\eta_1+m-1}(1-u)^{\gamma_1-\lambda_1-\alpha_1}{}_2F_1\left[\begin{matrix}\gamma_1-\lambda_1+m, \eta_1-\lambda_1\\ \alpha_1-\lambda_1+\eta_1+m\end{matrix}; u\right] \quad (|u|<1), \tag{23}$$
$$\frac{\partial^{\lambda_2-\mu_2}}{\partial v^{\lambda_2-\mu_2}}\left\{v^{\beta_2+n-1}(1-v)^{\gamma_2-\beta_2-\mu_2}\right\} = \frac{\Gamma(\beta_2)}{\Gamma(\beta_2-\lambda_2+\mu_2)}\frac{(\beta_2)_n}{(\beta_2-\lambda_2+\mu_2)_n}$$
$$\cdot v^{\beta_2-\lambda_2+\mu_2+n-1}(1-v)^{\gamma_2-\lambda_2-\beta_2}{}_2F_1\left[\begin{matrix}\gamma_2-\lambda_2+n, \mu-\lambda_2\\ \beta_2-\lambda_2+\mu_2+n\end{matrix}; v\right] \quad (|v|<1) \tag{24}$$

and

$$\frac{\partial^{\lambda_3-\mu_1}}{\partial w^{\lambda_3-\mu_1}}\left\{w^{\gamma_3-\beta_1-1}(1-w)^{\beta_1-\mu_1+p}\right\} = \frac{\Gamma(\gamma_3-\beta_1)}{\Gamma(\gamma_3+\mu_1-\beta_1-\lambda_3)}w^{\gamma_3+\mu_1-\beta_1-\lambda_3-1}$$
$$\cdot (1-w)^{\beta_1-\lambda_3+p}{}_2F_1\left[\begin{matrix}\gamma_3-\lambda_3+p, \mu_1-\lambda_3\\ \gamma_3+\mu_1-\beta_1-\lambda_3\end{matrix}; w\right] \quad (|w|<1). \tag{25}$$

Thus, (20) follows by substituting (23), (24) and (25) in (22). □

The following proposition gives an explicit representation of the function $\mathcal{F}_{x,y,z}(u,v,w)$ defined by (20) in terms of the Srivastava–Daoust function (10).

Proposition 1. *For* $\max\{|u|,|v|,|w|,|1-w|\} < 1$ *and* $(x,y,z) \in \mathbb{D}_K$*, the function* $\mathcal{F}_{x,y,z}(u,v,w)$ *defined by* (20) *can be expressed in terms of the Srivastava–Daoust function as follows:*

$$\mathcal{F}_{x,y,z}(u,v,w) = F_7^{10}\left[\begin{matrix}(a):\theta\\(c):\psi\end{matrix}; u,v,w,ux,vy,(1-w)z\right], \tag{26}$$

where (in terms of the symbolic representations as pointed out with the definition (10)*)*

$$(a) = (\gamma_1-\lambda_1,\ \gamma_2-\lambda_2,\ \beta_1-\mu_1,\ \alpha_2-\eta_2,\ \gamma_3-\lambda_3,\ \eta_1-\lambda_1,\ \mu_2-\lambda_2,\ \mu_1-\lambda_3,\ \alpha_1,\ \beta_2),$$
$$(b) = (\alpha_1-\lambda_1+\eta_1,\ \beta_2-\lambda_2+\mu_2,\ \gamma_3+\mu_1-\beta_1-\lambda_3,\ \gamma_1-\lambda_1,\ \gamma_2-\lambda_2,\ \beta_1-\mu_1,\ \gamma_3-\lambda_3),$$

$$\begin{matrix}\theta^{(1)}=\mathbf{e}_1+\mathbf{e}_4, & \theta^{(3)}=\mathbf{e}_4+\mathbf{e}_6, & \theta^{(5)}=\mathbf{e}_3+\mathbf{e}_6, & \theta^{(7)}=\mathbf{e}_2, & \theta^{(9)}=\mathbf{e}_4,\\ \theta^{(2)}=\mathbf{e}_2+\mathbf{e}_5, & \theta^{(4)}=\mathbf{e}_5+\mathbf{e}_6, & \theta^{(6)}=\mathbf{e}_1, & \theta^{(8)}=\mathbf{e}_3, & \theta^{(10)}=\mathbf{e}_5,\end{matrix}$$

$$\begin{matrix}\psi^{(1)}=\mathbf{e}_1+\mathbf{e}_4, & \psi^{(3)}=\mathbf{e}_3, & \psi^{(5)}=\mathbf{e}_5,\\ \psi^{(2)}=\mathbf{e}_2+\mathbf{e}_5, & \psi^{(4)}=\mathbf{e}_4, & \psi^{(6)}=\psi^{(7)}=\mathbf{e}_6.\end{matrix}$$

Here, as usual, $\mathbf{e}_i := (0, \cdots, 0, 1, 0, \cdots, 0)$ *is the* 6*-dimensional unit vector with* 1 *in the i-th component, and* 0 *otherwise.*

Proof. The proof is quite simple. By interpreting each hypergeometric function occurring in (20) defining the function $\mathcal{F}_{x,y,z}(u,v,w)$ in terms of a series, we easily obtain the sextuple series

$$\mathcal{F}_{x,y,z}(u,v,w) = \sum_{m_1,\cdots,m_6=0}^{\infty} \frac{(\gamma_1-\lambda_1)_{m_1+m_4}(\gamma_2-\lambda_2)_{m_2+m_5}(\beta_1-\mu_1)_{m_4+m_6}}{(\alpha_1-\lambda_1+\eta_1)_{m_1+m_4}(\beta_2-\lambda_2+\mu_2)_{m_2+m_5}} \cdot \frac{(\alpha_2-\eta_2)_{m_5+m_6}(\gamma_3-\lambda_3)_{m_3+m_6}(\eta_1-\lambda_1)_{m_1}(\mu_2-\lambda_2)_{m_2}(\mu_1-\lambda_3)_{m_3}}{(\gamma_3+\mu_1-\beta_1-\lambda_3)_{m_3}(\gamma_1-\lambda_1)_{m_4}(\gamma_2-\lambda_2)_{m_5}(\beta_1-\mu_1)_{m_6}} \cdot \frac{(\alpha_1)_{m_4}(\beta_2)_{m_5}}{(\gamma_3-\lambda_3)_{m_6}}\frac{u^{m_1}}{m_1!}\frac{v^{m_2}}{m_2!}\frac{w^{m_3}}{m_3!}\frac{(ux)^{m_4}}{m_4!}\frac{(vy)^{m_5}}{m_5!}\frac{[(1-w)z]^{m_6}}{m_6!}.$$

Thus by using (10), we obtain the desired form (26). □

When $\alpha_2 = \eta_2$ in (20), then after elementary calculations, the function $\mathcal{F}_{x,y,z}(u,v,w)$ reduces to the product of hypergeometric functions and the Kampé de Fériet function and is given by

$$\mathcal{F}_{x,y,z}(u,v,w)\Big|_{\alpha_2=\eta_2} = {}_2F_1\left[\begin{matrix}\gamma_2-\lambda_2,\mu_2-\lambda_2\\ \beta_2-\lambda_2+\mu_2\end{matrix};v\right]{}_2F_1\left[\begin{matrix}\gamma_3-\lambda_3,\mu_1-\lambda_3\\ \gamma_3+\mu_1-\beta_1-\lambda_3\end{matrix};w\right] \cdot F^{1:2;1}_{1:1;0}\left[\begin{matrix}\gamma_1-\lambda_1:\beta_1-\mu_1,\alpha_1;\eta_1-\lambda_1;\\ \alpha_1-\lambda_1+\eta_1:\gamma_1-\lambda_1;-;\end{matrix}ux,u\right]. \tag{27}$$

Additionally, if we further let $\gamma_1 = \alpha_1 + \lambda_1$ in (27), we obtain

$$\mathcal{F}_{x,y,z}(u,v,w)\Big|_{\substack{\alpha_2=\eta_2\\ \gamma_1=\alpha_1+\lambda_1}} = {}_2F_1\left[\begin{matrix}\gamma_2-\lambda_2,\mu_2-\lambda_2\\ \beta_2-\lambda_2+\mu_2\end{matrix};v\right]{}_2F_1\left[\begin{matrix}\gamma_3-\lambda_3,\mu_1-\lambda_3\\ \gamma_3+\mu_1-\beta_1-\lambda_3\end{matrix};w\right] \cdot F_1[\alpha,\beta_1-\mu_1,\eta_1-\lambda_1;\alpha_1-\lambda_1+\eta_1;ux,u],$$

where F_1 denotes the first Appell function [9] (p. 22, Equation (2)).

On the other hand, when $\gamma_1 = \alpha_1 + \eta_1$, $\gamma_2 = \beta_2 + \mu_2$ and $\mu_1 = \lambda_3$ in (20), then $\mathcal{F}_{x,y,z}(u,v,w)$ becomes Saran's F_K-function:

$$\mathcal{F}_{x,y,z}(u,v,w)\Big|_{\substack{\gamma_1=\alpha_1+\eta_1\\ \gamma_2=\beta_2+\mu_2\\ \mu_1=\lambda_3}} = (1-u)^{\lambda_1-\eta_1}(1-v)^{\lambda_2-\mu_2} \cdot F_K\Big[\alpha_1,\alpha_2-\eta_2,\alpha_2-\eta_2,\beta_1-\lambda_3,\beta_2,\beta_1-\lambda_3; \alpha_1-\lambda_1+\eta_1,\beta_2-\lambda_2+\mu_2,\beta_1-\lambda_3;ux,vy,(1-w)z\Big].$$

We now state and prove the next result.

Theorem 2. *Let*

$$\max\{\Re(\alpha_1-\lambda_1+\eta_1),\Re(\gamma_1-\alpha_1)\}>0, \quad \max\{\Re(\beta_2-\lambda_2+\mu_2),\Re(\gamma_2-\beta_2)\}>0,$$

and $\max\{\Re(\gamma_3+\mu_1-\beta_1-\lambda_3),\Re(\beta_1)\}>0$. *Then we have*

$$\begin{aligned} &F_K[\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_3;x,y,z]\\ &= D\int_0^1\int_0^1\int_0^1(1-ux)^{-\mu_1}(1-vy)^{-\eta_2}\mathcal{F}_{x,y,z}(u,v,w)\\ &\cdot F_K\left[\lambda_1-\eta_1,\eta_2,\eta_2,\mu_1,\lambda_2-\mu_2,\mu_1;\lambda_1,\lambda_2,\lambda_3;\frac{(1-u)x}{1-ux},\frac{(1-v)y}{1-vy},\frac{(1-w)z}{(1-ux)(1-vy)}\right]\\ &\cdot \mathrm{d}\mu_{\alpha_1-\lambda_1+\eta_1,\gamma_1-\alpha_1}(u)\mathrm{d}\mu_{\beta_2-\lambda_2+\mu_2,\gamma_2-\beta_2}(v)\mathrm{d}\mu_{\gamma_3+\mu_1-\beta_1-\lambda_3,\beta_1}(w), \end{aligned} \tag{28}$$

where $\mathcal{F}_{x,y,z}(u,v,w)$ *is defined by* (20) *and*

$$D := \frac{\Gamma(\eta_1)\Gamma(\mu_1)\Gamma(\mu_2)}{\Gamma(\gamma_1-\lambda_1+\eta_1)\Gamma(\gamma_2-\lambda_2+\mu_2)\Gamma(\gamma_3-\lambda_3+\mu_1)}\prod_{\ell=1}^{3}\frac{\Gamma(\gamma_\ell)}{\Gamma(\lambda_\ell)}.$$

Proof. We begin with the integral representation obtained by letting $w \to 1-w$ in (5), that is,

$$\begin{aligned} &F_K[\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_3;x,y,z] \\ &:= C\int_0^1\int_0^1\int_0^1 P_{x,y,z}(u,v,w)Q_{x,y,z}(u,v,w)\mathrm{d}u\mathrm{d}v\mathrm{d}w, \end{aligned} \tag{29}$$

where

$$\begin{aligned} P_{x,y,z}(u,v,w) &:= u^{\alpha_1-1}(1-u)^{\gamma_1-\alpha_1-\eta_1}v^{\beta_2-1}(1-v)^{\gamma_2-\beta_2-\mu_2}w^{\gamma_3-\beta_1-1}(1-w)^{\beta_1-\mu_1} \\ &\quad\cdot(1-ux)^{\alpha_2-\eta_2-\beta_1+\mu_1}(1-ux-vy-(1-w)z+uvxy)^{-\alpha_2+\eta_2}, \\ Q_{x,y,z}(u,v,w) &:= (1-u)^{\eta_1-1}(1-v)^{\mu_2-1}(1-w)^{\mu_1-1} \\ &\quad\cdot(1-ux)^{\eta_2-\mu_1}(1-ux-vy-(1-w)z+uvxy)^{-\eta_2}, \end{aligned}$$

and the constant C is invariant and given by (6). It may be noted that the functions $P_{x,y,z}(u,v,w)$ and $Q_{x,y,z}(u,v,w)$ involve four new free parameters η_1, η_2, μ_1 and μ_2.
Next by applying Lemma 2 to (29), we obtain

$$\begin{aligned} &F_K[\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_3;x,y,z] \\ &= CD_1(1-x)^{-\mu_1}(1-y)^{-\eta_2}\int_0^1\int_0^1\int_0^1 P_{x,y,z}(u,v,w) \\ &\cdot\frac{\partial^{\lambda_1-\eta_1}}{\partial(1-u)^{\lambda_1-\eta_1}}\frac{\partial^{\lambda_2-\mu_2}}{\partial(1-v)^{\lambda_2-\mu_2}}\frac{\partial^{\lambda_3-\mu_1}}{\partial(1-w)^{\lambda_3-\mu_1}}\Big\{(1-u)^{\lambda_1-1}(1-v)^{\lambda_2-1}(1-w)^{\lambda_3-1} \\ &\cdot F_K\left[\eta_1,\eta_2,\eta_2,\mu_1,\mu_2,\mu_1;\lambda_1,\lambda_2,\lambda_3;\frac{(1-u)x}{x-1},\frac{(1-v)y}{y-1},\frac{(1-w)z}{(1-x)(1-y)}\right]\Big\}\mathrm{d}u\mathrm{d}v\mathrm{d}w. \end{aligned} \tag{30}$$

Making use of the transformation (7), the above integral (30) can be further expressed as

$$\begin{aligned} F_K[\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_3;x,y,z] &= CD_1\int_0^1\int_0^1\int_0^1 P_{x,y,z}(u,v,w) \\ &\cdot\frac{\partial^{\lambda_1-\eta_1}}{\partial(1-u)^{\lambda_1-\eta_1}}\frac{\partial^{\lambda_2-\mu_2}}{\partial(1-v)^{\lambda_2-\mu_2}}\frac{\partial^{\lambda_3-\mu_1}}{\partial(1-w)^{\lambda_3-\mu_1}}\Big\{(1-u)^{\lambda_1-1}(1-v)^{\lambda_2-1}(1-w)^{\lambda_3-1} \\ &\cdot(1-ux)^{-\mu_1}(1-vy)^{-\eta_2}F_K\Big[\lambda_1-\eta_1,\eta_2,\eta_2,\mu_1,\lambda_2-\mu_2,\mu_1;\lambda_1,\lambda_2,\lambda_3; \\ &\qquad\frac{(1-u)x}{1-ux},\frac{(1-v)y}{1-vy},\frac{(1-w)z}{(1-ux)(1-vy)}\Big]\Big\}\mathrm{d}u\mathrm{d}v\mathrm{d}w. \end{aligned}$$

Applying next the rule of the fractional integration by parts (9), we obtain

$$\begin{aligned} &F_K[\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_3;x,y,z] \\ &= CD_1\int_0^1\int_0^1\int_0^1(1-u)^{\lambda_1-1}(1-v)^{\lambda_2-1}(1-w)^{\lambda_3-1}(1-ux)^{-\mu_1}(1-vy)^{-\eta_2} \\ &\cdot F_K\left[\lambda_1-\eta_1,\eta_2,\eta_2,\mu_1,\lambda_2-\mu_2,\mu_1;\lambda_1,\lambda_2,\lambda_3;\frac{(1-u)x}{1-ux},\frac{(1-v)y}{1-vy},\frac{(1-w)z}{(1-ux)(1-vy)}\right] \\ &\cdot\frac{\partial^{\lambda_1-\eta_1}}{\partial u^{\lambda_1-\eta_1}}\frac{\partial^{\lambda_2-\mu_2}}{\partial v^{\lambda_2-\mu_2}}\frac{\partial^{\lambda_3-\mu_1}}{\partial w^{\lambda_3-\mu_1}}\Big\{P_{x,y,z}(u,v,w)\Big\}\mathrm{d}u\mathrm{d}v\mathrm{d}w. \end{aligned} \tag{31}$$

The fractional derivative of $P_{x,y,z}(u,v,w)$ appearing in the integrand of (31) is equivalent to $\mathcal{Q}_{x,y,z}(u,v,w)$ which can be evaluated by using Lemma 3, and we have, therefore,

$$\begin{aligned}
&F_K[\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_3;x,y,z] \\
&\quad = CD_1D_2\int_0^1\int_0^1\int_0^1(1-ux)^{-\mu_1}(1-vy)^{-\eta_2}\mathcal{F}_{x,y,z}(u,v,w) \\
&\quad \cdot F_K\left[\lambda_1-\eta_1,\eta_2,\eta_2,\mu_1,\lambda_2-\mu_2,\mu_1;\lambda_1,\lambda_2,\lambda_3;\frac{(1-u)x}{1-ux},\frac{(1-v)y}{1-vy},\frac{(1-w)z}{(1-ux)(1-vy)}\right] \\
&\quad \cdot u^{\alpha_1-\lambda_1+\eta_1-1}(1-u)^{\gamma_1-\alpha_1-1}v^{\beta_2-\lambda_2+\mu_2-1}(1-v)^{\gamma_2-\beta_2-1} \\
&\quad \cdot w^{\gamma_3+\mu_1-\beta_1-\lambda_3-1}(1-w)^{\beta_1-1}\mathrm{d}u\mathrm{d}v\mathrm{d}w.
\end{aligned} \tag{32}$$

We finally obtain (28) upon using (32) and (3). This completes the proof. □

The following corollary may be looked upon as the most important and interesting integral relation concerning the F_K-function.

Corollary 2. *Let* $\max\{\Re(\alpha_1-\lambda_1+\eta_1),\Re(\eta_1)\}>0$, $\max\{\Re(\beta_2-\lambda_2+\mu_2),\Re(\mu_2)\}>0$, *and* $\max\{\Re(\gamma_3-\beta_1),\Re(\beta_1)\}>0$. *Then we have*

$$\begin{aligned}
&F_K[\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\alpha_1+\eta_1,\beta_2+\mu_2,\gamma_3;x,y,z] = \int_0^1\int_0^1\int_0^1(1-ux)^{-\lambda_3}(1-vy)^{-\eta_2} \\
&\cdot F_K\left[\alpha_1,\alpha_2-\eta_2,\alpha_2-\eta_2,\beta_1-\lambda_3,\beta_2,\beta_1-\lambda_3;\alpha_1-\lambda_1+\eta_1,\beta_2-\lambda_2+\mu_2,\beta_1-\lambda_3;ux,vy,wz\right] \\
&\cdot F_K\left[\lambda_1-\eta_1,\eta_2,\eta_2,\lambda_3,\lambda_2-\mu_2,\lambda_3;\lambda_1,\lambda_2,\lambda_3;\frac{(1-u)x}{1-ux},\frac{(1-v)y}{1-vy},\frac{wz}{(1-ux)(1-vy)}\right] \\
&\cdot \mathrm{d}\mu_{\alpha_1-\lambda_1+\eta_1,\lambda_1}(u)\mathrm{d}\mu_{\beta_2-\lambda_2+\mu_2,\lambda_2}(v)\mathrm{d}\mu_{\beta_1,\gamma_3-\beta_1}(w).
\end{aligned} \tag{33}$$

Proof. By setting $\gamma_1=\alpha_1+\eta_1$, $\gamma_2=\beta_2+\mu_2$ and $\mu_1=\lambda_3$ in (28) and letting $w\to 1-w$ in the resulting formula, we obtain

$$\begin{aligned}
&F_K[\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\alpha_1+\eta_1,\beta_2+\mu_2,\gamma_3;x,y,z] = D^*\int_0^1\int_0^1\int_0^1(1-ux)^{-\lambda_3}(1-vy)^{-\eta_2} \\
&\cdot F_K\left[\alpha_1,\alpha_2-\eta_2,\alpha_2-\eta_2,\beta_1-\lambda_3,\beta_2,\beta_1-\lambda_3;\alpha_1-\lambda_1+\eta_1,\beta_2-\lambda_2+\mu_2,\beta_1-\lambda_3;ux,vy,wz\right] \\
&\cdot F_K\left[\lambda_1-\eta_1,\eta_2,\eta_2,\lambda_3,\lambda_2-\mu_2,\lambda_3;\lambda_1,\lambda_2,\lambda_3;\frac{(1-u)x}{1-ux},\frac{(1-v)y}{1-vy},\frac{wz}{(1-ux)(1-vy)}\right] \\
&\cdot (1-u)^{\lambda_1-\eta_1}\mathrm{d}\mu_{\alpha_1-\lambda_1+\eta_1,\eta_1}(u)(1-v)^{\lambda_2-\mu_2}\mathrm{d}\mu_{\beta_2-\lambda_2+\mu_2,\mu_2}(v)\mathrm{d}\mu_{\beta_1,\gamma_3-\beta_1}(w),
\end{aligned}$$

where

$$D^* := \frac{\Gamma(\eta_1)\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\alpha_1+\eta_1)\Gamma(\beta_2+\mu_2)}{\Gamma(\alpha_1-\lambda_1+2\eta_1)\Gamma(\beta_2-\lambda_2+2\mu_2)\prod_{\ell=1}^{3}\Gamma(\lambda_\ell)}.$$

Then the Formula (33) follows by noting that

$$(1-u)^{\lambda_1-\eta_1}\mathrm{d}\mu_{\alpha_1-\lambda_1+\eta_1,\eta_1}(u) = \frac{\Gamma(\lambda_1)}{\Gamma(\eta_1)}\frac{\Gamma(\alpha_1-\lambda_1+2\eta_1)}{\Gamma(\alpha_1+\eta_1)}\mathrm{d}\mu_{\alpha_1-\lambda_1+\eta_1,\lambda_1}(u)$$

and

$$(1-v)^{\lambda_2-\mu_2}\mathrm{d}\mu_{\beta_2-\lambda_2+\mu_2,\mu_2}(v) = \frac{\Gamma(\lambda_2)}{\Gamma(\mu_2)}\frac{\Gamma(\beta_2-\lambda_2+2\mu_2)}{\Gamma(\beta_2+\mu_2)}\mathrm{d}\mu_{\beta_2-\lambda_2+\mu_2,\lambda_2}(v).$$

□

Remark 3.

(1) *It may be noticed that the F_K functions involved in the integrand of* (33) *cannot be directly expressed in terms of simpler functions. For the known reducible cases when the F_K-function reduces to ${}_2F_1$, H_4 and F_2, the interested reader may refer to Refs.* [6] (p. 4, equation (4.7)), [35] (p. 220, Equations (3.6) and (3.7)), [2] (p. 2, Equations (5) and (7)) *and* [36] (p. 58, Equation (2.2)).

(2) *If we let $y = z = 0$ in* (33)*, we easily obtain*

$$
{}_2F_1\left[\begin{matrix}\beta_1, \alpha_1\\ \alpha_1+\eta_1\end{matrix}; x\right] = \int_0^1 (1-ux)^{-\lambda_3}\, {}_2F_1\left[\begin{matrix}\beta_1-\lambda_3, \alpha_1\\ \alpha_1-\lambda_1+\eta_1\end{matrix}; ux\right] \cdot {}_2F_1\left[\begin{matrix}\lambda_3, \lambda_1-\eta_1\\ \lambda_1\end{matrix}; \frac{(1-u)x}{1-ux}\right] \mathrm{d}\mu_{\alpha_1-\lambda_1+\eta_1,\eta_1}(u),
$$

which is the known Erdélyi's integral (2)*.*

5. Conclusions

In this paper, we establish two Erdélyi-type integrals for Saran's F_K function defined by (1). Our method is based on the k-dimensional fractional integration by parts, which is an effective tool and can be applied to Saran's other functions. These integrals, especially integral (33), are fascinating in their forms. More importantly, they can bring new insights into the study of multivariable hypergeometric functions. Their potential connection with the Hadamard convolution, as we pointed out in Remark 1, may be a new direction worth exploring in the future. In fact, in the one-dimensional case, some further connections of the Hadamard convolution with the monodromy formula were also noted by Pérez-Marco [37].

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