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Article

Robust Control for Variable-Order Fractional Interval Systems Subject to Actuator Saturation

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Abstract: In this paper, a class of variable-order fractional interval systems (VO-FIS) in which the system matrices are affected by the fractional order is investigated. Firstly, the sufficient conditions for robust stability of a VO-FIS with a unified order range of $\nu(\sigma) \in (0, 2)$ are proposed. Secondly, the stabilization conditions of a VO-FIS subject to actuator saturation are derived in terms of linear matrix inequalities (LMIs). Then, by using the proposed algorithm through an optimization problem, the stability region is estimated. To summarize, the paper gives a stabilization criterion for VO-FIS subject to actuator saturation. Finally, three numerical examples are proposed to verify the effectiveness of our results.

Keywords: variable-order fractional interval systems; linear matrix inequalities; actuator saturation; stability region

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1. Introduction

In recent decades, there has been a continuing growth in the number of studies on the engineering applications of fractional-order systems (FOS) [1–3], and this control system has attracted more and more scholars' attention [4–8]. This is principally because numerous physical systems that have fractional properties in the real world are marked by fractional-order state equations [9,10]. Stability is fundamental and important to all systems, certainly including FOS [11,12]. A basic theorem for the stability of FOSs was first proposed in [13]. Based on these previous works, many methods have been proposed to solve the stability and stabilization problems of FOSs [14–16]. Among them, using LMI to stabilize the FOS is an effective and systematic method. The fractional-order periods belonging to $(0, 1)$ and $(1, 2)$ are two forms of the existing LMI criteria [17–20]. For instance, [12] discussed the equivalent criterion for the stability of FOS with order ν in $(0, 1)$. In [13], the novel LMI-based stability conditions of FOS are proposed with order ν in $(0, 1)$ and $(1, 2)$, respectively. An LMI criterion based on D-stability is proposed in [14], which describes the stability and robust stability of FOS with order ν in $(0, 1)$. In addition, for these two cases, the admissibility and stability of a singular FOS are proposed in [20–23] by LMIs.

Studies on the stability of fractional-order interval systems (FOIS) have attracted much attention [24,25]. However, because of the coupling uncertainty of the system matrix and the fractional order, there are still numerous challenges in the stability of FOIS [26,27]. The main reasons why stability research faces challenges include two aspects. First, the eigenvalues of a system matrix should be restrained in the left half of the complex plane [28–30]. By using the Matignon lemma [13], if the stability region satisfies $|\arg(\lambda_i(A))| > \frac{\pi\nu}{2}, \ i = 1, \cdots, n$, then FOS is stable. It obvious that if the eigenvalue of the system matrix is positive, the stability of the FOS is also guaranteed. Second, the bilinear matrix inequality (BMI)-based stabilization criteria are extremely complex, and it is not easy to design a controller for them [31,32]. Some assumptions for the stability of FOIS have been put forward to obtain a feasible solution, and these certainly lead to the conservatism.

Recently, variable-order fractional systems have seen significant development and become an international hotspot [33,34]. Variable-order fractional systems can describe many complex phenomena, such as complex diffusion in disordered porous media and in highly heterogeneous fractures [35,36]. The main reason is that the variable-order fractional operator in system modeling has a memory of previous appearances [37–40]. However, research on the stability of variable-order fractional systems is still a challenge in view of the complexity and diversity of variable-order fractional operators. In [41], the stability of variable-order fractional systems is examined using the Arzela–Ascoli theorem . As far as the author knows, little detailed or systematic study has ever been conducted on the stability of a system as follows:

$$D^{v(\sigma)}x(t) = Ax(t),$$

where $\sigma \in \Omega$, and Ω is a compact set in $\mathbb{R}$. Therefore, one of the innovations of this paper is that it combines variable-order fractional systems and fractional-order interval systems to study the stability of VO-FIS.

Actuator saturation will reduce the performance of the control system and even lead to instability. The direct method to not cause such performance degradation is to hold back actuator saturation by operating the control systems in the linear region of the actuator [42–48]. In [49–51], the set invariance conditions are established, and the stabilization of normal systems and singular systems under actuator saturation is discussed. The admissibility criteria in [50] are extended to fuzzy singular systems in [52]. Although [53] provides a method to stabilize FOS under input saturation, the algorithm given is to solve the BMI problem, which is difficult to calculate. This paper overcomes this difficulty and proposes an approach to estimate the stability domain directly by LMIs. In addition, many contributions in the two cases of constant order $\nu \in (0,1)$ and $(1,2)$ are available, but unified results about the stability of VO-FIS in the case of $\nu(\sigma) \in (0,2)$ have scarcely been reported. Moreover, the stabilization issue of VO-FIS with order $\nu(\sigma) \in (0,2)$ subject to actuator saturation has not yet been reported. Therefore, it is very necessary to consider this kind of system.

Motivated by the above-mentioned research, the stabilization of VO-FIS is considered. The main contributions are as follows:

(i) In this paper, a new model of VO-FIS with order (0, 2) is proposed, and based on this system, the stabilization criterion in terms of LMIs is given. In addition, the paper considers the actuator saturation of the system, which expands the scope of application to a certain extent.

(ii) The stability region is estimated by solving an optimization problem in terms of LMIs according to the obtained stability conditions. At present, most studies have used algorithms to solve this problem. However, it is difficult to calculate in the simulation process.

(iii) Compared with the existing results in [28–30], our results are less conservative, since the eigenvalues of the system matrix are restrained in the left half of the complex plane of the references mentioned above. However, from the discussion of numerical simulation in the paper, it is easy to see that the eigenvalues of the system matrix of system (2) are restrained in the right half of the plane.

Notations: X^T is the transpose of the matrix X. The symbols $\mathrm{sym}(X)$ and $\mathrm{asym}(X)$ denote the expressions $X + X^T$ and $X - X^T$, respectively, while $\otimes$ stands for the Kronecker product. In addition, $\lambda_i(X), (i = 1, \cdots, n)$ are the eigenvalues of X, and $s_{\nu(\sigma)} = \sin(\nu(\sigma)\frac{\pi}{2})$, $c_{\nu(\sigma)} = \cos(\nu(\sigma)\frac{\pi}{2})$, and $|x|_\infty = \max_s |x_s|$ for $x \in \mathbb{R}^n$, where x_s is the s-th row of x, and $s = 1, 2, \cdots, n$. The symbol co(X) represents the convex hull of a set of X.

2. Problem Formulation And Preliminaries

Consider an FOS subject to actuator saturation as follows:

$$D^{\nu(\sigma)}x(t) = Ax(t) + B\mathrm{sat}(u(t)), \tag{1}$$

where $\sigma \in \Omega$, and Ω is a compact set in $\mathbb{R}$; $x(t) \in \mathbb{R}^n$ is the state of the system; $u(t) \in \mathbb{R}^m$ is the control input; and $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are known constant matrixes. The vector valued standard saturation function sat: $\mathbb{R}^m \to \mathbb{R}^m$ is defined as:

$$\mathrm{sat}(u) = [\mathrm{sat}(u_1)\ \ \mathrm{sat}(u_2)\ \ \cdots\ \ \mathrm{sat}(u_m)]^T,$$

$$\mathrm{sat}(u_s) = \mathrm{sign}(u_s)\min\{|u_s|, 1\},\ \ s = 1, 2, \cdots, m.$$

The Caputo fractional derivative operator $D^{\nu(\sigma)}$ is defined as follows:

$$D^{\nu(\sigma)}f(t) = \frac{1}{\Gamma(n-\nu(\sigma))}\int_0^t (t-\tau)^{n-\nu(\sigma)-1} f^{(n)}(\tau)d\tau,$$

where n is an integer $(n-1 < \nu(\sigma) < n)$, and $\Gamma(\cdot)$ is the gamma function. The descriptions of the uncertain fractional order $\nu(\sigma)$ and uncertain matrix A are as follows:

$$\nu(\sigma) \in [\underline{\nu}, \overline{\nu}],\ \ A = A_0 + k\nu(\sigma) \triangle A \in [\underline{A}(\nu(\sigma)), \overline{A}(\nu(\sigma))],$$

where k is a given constant. Further, to deal with the uncertainty matrix $\triangle A$, the following notations are introduced [26,27]:

$$D \triangleq [\sqrt{\gamma_{11}}e_1^n \cdots \sqrt{\gamma_{1m}}e_1^n \cdots \sqrt{\gamma_{n1}}e_n^n \cdots \sqrt{\gamma_{nm}}e_n^n]_{n\times mn}, E = D^T,$$

$$F \triangleq \mathrm{diag}\{\delta_{11} \cdots \delta_{1m} \cdots \delta_{n1} \cdots \delta_{nm}\}_{mn\times mn},\ \ |\delta_{ij}| \le 1,$$

where γ_{ij} are given constants, and $e_i^n \in \mathbb{R}^n$ represents the i-th column of the identity matrix I_n. Then, system (1) is denoted as follows:

$$D^{v(\sigma)}x(t) = (A_0 + k\nu(\sigma)DFE)x(t) + B\mathrm{sat}(u(t)), \tag{2}$$

By the state feedback controller $u(t) = Kx(t),\ \ K \in \mathbb{R}^{m\times n}$, system (2) is described as

$$D^{\nu(\sigma)}x(t) = (A_0 + k\nu(\sigma)DFE)x(t) + B\mathrm{sat}(Kx(t)). \tag{3}$$

For convenience, replace $x(t)$ with x in the following writing. Considering the relationship between $\mathrm{sat}(Kx)$ and Kx, for the matrices $H \in \mathbb{R}^{n\times n}$, the symmetric polyhedra is defined as

$$\mathcal{L}(H) = \{x \in \mathbb{R}^n : |Hx|_\infty \le 1\}.$$

In order to derive our main results, the following lemmas are adopted.

Lemma 1 ([49]). *Letting $K, H \in \mathbb{R}^{m\times n}$, for $x \in \mathcal{L}(H)$,*

$$\mathrm{sat}(Kx) \in \mathrm{co}\{(\Lambda_p K + \Lambda_p^- H)x : p = 1, 2, \cdots, 2^m\},$$

where Λ_p represents a diagonal matrix whose diagonal elements become 1 or 0, $\Lambda_p^- = I - \Lambda_p$. Therefore, $\mathrm{sat}(Kx)$ *is expressed as*

$$\mathrm{sat}(Kx) = \sum_{p=1}^{2^m} \lambda_p(\Lambda_p K + \Lambda_p^- H)x,$$

in which $\lambda_p \geq 0$, $\sum_{p=1}^{2^m} \lambda_p = 1$. *Then, system (3) is described as*

$$D^{\nu(\sigma)}x(t) = \sum_{p=1}^{2^m} \lambda_p (A + B\Lambda_p K + B\Lambda_p^- H)x(t). \tag{4}$$

Lemma 2 ([53]). *For* $\varepsilon > 0$, *if system (2) is stable, then there is* $\delta > 0$ *subject to, for each initial condition* x_0 *contained in the closed ball* $\mathcal{B}_\delta = \{x \in \mathbb{R}^n : x^T x \leq \delta\}$, *the solution* $x(t, x_0)$ *located in the closed ball.*

$$\mathcal{B}_\varepsilon \triangleq \{x \in \mathbb{R}^n : x^T x \leq \varepsilon, \ \ \varepsilon > 0\}. \tag{5}$$

Lemma 3 ([7]). *An FOS described by* $D^{\nu(\sigma)}x(t) = Ax(t)$ *with order* $\nu(\sigma) \in (1, 2)$ *is stable iff there exists* $X = X^T > 0$, $X \in \mathbb{R}^{n\times n}$ *such that*

$$\mathrm{sym}[\theta \otimes (AX)] < 0,$$

where $\theta = \begin{bmatrix} s_{\nu(\sigma)} & -c_{\nu(\sigma)} \\ c_{\nu(\sigma)} & s_{\nu(\sigma)} \end{bmatrix}$.

Lemma 4 ([19]). *An FOS described by* $D^{\nu(\sigma)}x(t) = Ax(t)$ *processing order* $\nu(\sigma) \in (0, 1)$ *is stable if and only if there is* $X, Y \in \mathbb{R}^{n\times n}$ *subject to*

$$\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} > 0, \tag{6}$$

$$\mathrm{sym}(s_{\nu(\sigma)} X + c_{\nu(\sigma)} Y) < 0.$$

Lemma 5 ([13]). *An FOS described by* $D^{\nu(\sigma)}x(t) = Ax(t)$ *with order* $\nu(\sigma) \in (0, 2)$ *is stable iff* $|\arg(\lambda_i(A))| > \frac{\pi\nu(\sigma)}{2}$, $\ i = 1, \cdots, n$.

Lemma 6. *If system*

$$D^{\overline{\nu}}x(t) = Ax(t), \tag{7}$$

is stable, then system $D^{\nu(\sigma)}x(t) = Ax(t)$ *is stable.*

Proof. Since system (7) is stable, from Lemma 5, it is obtained that

$$|\arg(\lambda_i(A))| > \frac{\pi\overline{\nu}}{2}, \ \ (i = 1, \cdots, n).$$

According to $v(\sigma) \leq \overline{v}$, one has

$$|\arg(\lambda_i(A))| > \frac{\pi\overline{\nu}}{2} > \frac{\pi\nu(\sigma)}{2}, (i = 1, \cdots, n).$$

Therefore, system $D^{\nu(\sigma)}x(t) = Ax(t)$ is stable. □

3. Main Results

The stability and stabilization theorems for VO-FIS with order $\nu(\sigma) \in (0,2)$ are obtained. The following theorem provides a sufficient condition which guarantees that system (2) with $u(t) = 0$ is stable.

Theorem 1. *System (2) with $u(t) = 0$ for the case $0 < \underline{\nu} < \nu(\sigma) < \overline{\nu} < 2$ is robustly stable if there exist $X, Y, \varepsilon_1, \varepsilon_2 \in \mathbb{R}^{n\times n}$, such that (6) holds and*

$$\Theta(A,\overline{\nu},X,Y) = \Omega_1(A,\overline{\nu},X,Y) + \Omega_2(A,\overline{\nu},X,Y)\chi(\overline{\nu}) < 0, \tag{8}$$

where,

$$\Omega_1(A,\overline{\nu},X,Y) = I_2 \otimes \begin{bmatrix} \mathrm{sym}[A_0(s_{\bar{\upsilon}}X + c_{\bar{\upsilon}}Y)] + (\varepsilon_1+\varepsilon_2)k^2\bar{\upsilon}^2DD^T & s_{\bar{\upsilon}}XE^T & -c_{\bar{\upsilon}}YE^T \\ s_{\bar{\upsilon}}EX & -\varepsilon_1 I & 0 \\ c_{\bar{\upsilon}}EY & 0 & -\varepsilon_2 I \end{bmatrix}.$$

$$\Omega_2(A,\overline{\nu},X,Y) = \begin{bmatrix} \gamma_1 & \gamma_2 \\ -\gamma_2 & \gamma_1 \end{bmatrix},$$

$$\gamma_1 = \begin{bmatrix} \mathrm{sym}(-c_{\bar{\upsilon}}A_0Y) - \varepsilon_2k^2\bar{\upsilon}^2DD^T & (1-s_{\bar{\upsilon}})XE^T & c_{\bar{\upsilon}}YE^T \\ (1-s_{\bar{\upsilon}})EX & 0 & 0 \\ -c_{\bar{\upsilon}}EY & 0 & \varepsilon_2 I \end{bmatrix},$$

$$\gamma_2 = \begin{bmatrix} -\mathrm{asym}(c_{\bar{\upsilon}}A_0X) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \chi(\bar{\upsilon}) \triangleq \begin{cases} 0, 0 < \bar{\upsilon} < 1, \\ 1, 1 \le \bar{\upsilon} < 2. \end{cases}$$

Proof. Let us prove this problem in three cases. In the first case, for $\nu(\sigma) \in [\underline{\nu},\overline{\nu}]$ and $\overline{\nu} < 1$, by using the Schur complement, $\chi(\bar{\upsilon}) \triangleq 0$ and $\Theta(A,\overline{\nu},X,Y) = \Omega_1(A,\overline{\nu},X,Y)$ can be obtained from $\overline{\nu} < 1$. Therefore, the following inequality can be obtained:

$$\begin{aligned} &\mathrm{sym}[(A_0 + k\bar{\upsilon}DFE)(s_{\bar{\upsilon}}X + c_{\bar{\upsilon}}Y)] = \mathrm{sym}[A_0(s_{\bar{\upsilon}}X + c_{\bar{\upsilon}}Y)] + \mathrm{sym}[k\overline{\upsilon}DFE(s_{\bar{\upsilon}}X + c_{\bar{\upsilon}}Y)] \\ &\le \mathrm{sym}[A_0(s_{\bar{\upsilon}}X + c_{\bar{\upsilon}}Y)] + (\varepsilon_1+\varepsilon_2)k^2\bar{\upsilon}^2DD^T + {\varepsilon_1}^{-1}{s_{\bar{\upsilon}}}^2(EX)^T(EX) + {\varepsilon_2}^{-1}{s_{\bar{\upsilon}}}^2(EY)^T(EY) \\ &= \begin{bmatrix} \mathrm{sym}[A_0(s_{\bar{\upsilon}}X + c_{\bar{\upsilon}}Y)] + (\varepsilon_1+\varepsilon_2)k^2\bar{\upsilon}^2DD^T & s_{\bar{\upsilon}}XE^T & -c_{\bar{\upsilon}}YE^T \\ s_{\bar{\upsilon}}EX & -\varepsilon_1 I & 0 \\ c_{\bar{\upsilon}}EY & 0 & -\varepsilon_2 I \end{bmatrix} < 0. \end{aligned}$$

Therefore, it follows that

$$\mathrm{sym}[(A_0 + k\overline{\upsilon}DFE)(s_{\overline{\upsilon}}X + c_{\overline{\upsilon}}Y)] < 0.$$

From Lemma 3, it is obvious that $D^{\overline{\upsilon}}x(t) = (A_0 + k\overline{\upsilon}DFE)x(t)$ is stable. Then, from Lemma 5, it follows that system (2) with $u(t) = 0$ is stable.

In the second case, for $\nu(\sigma) \in [\underline{\nu},\overline{\nu}]$ and $1 < \underline{\nu}$, one gets

$$\Theta = \begin{bmatrix} \mathrm{sym}(s_{\bar{\upsilon}}A_0X) + \varepsilon_1k^2\bar{\upsilon}^2DD^2 & XE^T & -\mathrm{asym}(c_{\bar{\upsilon}}A_0X) & 0 \\ EX & -\varepsilon_1 I & 0 & 0 \\ \mathrm{asym}(c_{\bar{\upsilon}}A_0X) & 0 & \mathrm{sym}(s_{\bar{\upsilon}}A_0X) + \varepsilon_1k^2\bar{\upsilon}^2DD^2 & XE^T \\ 0 & 0 & EX & -\varepsilon_1 I \end{bmatrix} < 0.$$

Pre- and post-multiplying Θ by $\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$ and its transpose, respectively, where I is an identity matrix of appropriate dimensions, it is obtained that

$$\Theta = \begin{bmatrix} \mathrm{sym}[\bar{\theta} \otimes (A_0 X)] + I_2 \otimes (\varepsilon_1 k^2 \bar{v}^2 DD^2) & I_2 \otimes (XE^T) \\ I_2 \otimes (EX) & -I_2 \otimes (\varepsilon_1 I) \end{bmatrix} < 0.$$

By using the Schur complement, it implies

$$\begin{aligned} &\mathrm{sym}\{\bar{\theta} \otimes [(A_0 + k\bar{v}DFE)X]\} = \mathrm{sym}[\bar{\theta} \otimes (A_0 X)] + \mathrm{sym}[I_2 \otimes (k\bar{v}DFEX)] \\ &\leq \mathrm{sym}[\bar{\theta} \otimes (A_0 X)] + \varepsilon_1 k^2 \bar{v}^2 DD^T + \varepsilon_1^{-1}(EX)^T(EX) \\ &= \begin{bmatrix} \mathrm{sym}[\bar{\theta} \otimes (A_0 X)] + I_2 \otimes \varepsilon_1 k^2 \bar{v}^2 DD^T & I_2 \otimes (XE^T) \\ I_2 \otimes (EX) & -I_2 \otimes (\varepsilon_1 I) \end{bmatrix} < 0. \end{aligned}$$

Applying Lemma 3, it follows that $D^{\overline{v}} x(t) = (A_0 + k\overline{v}DFE)x(t)$ is stable, which implies that system (2) is stable by Lemma 6.

In the third case, for $\nu(\sigma) \in [\underline{\nu}, \overline{\nu}]$ and $\underline{\nu} < 1 < \overline{\nu}$, when $\nu(\sigma) \in [\underline{\nu}, 1]$, it is similar to the first case, when $\nu(\sigma) \in [1, \overline{\nu}]$, it is similar to the second case. □

Remark 1. *Theorem 1 considers the stability condition when the system has no controller, but in practical application, there are few cases where the system can be stable without the controller. Therefore, we consider the condition of designing the controller to stabilize the system when the system is unstable.*

Theorem 2. *System (2) for the case $0 < \underline{\nu} < \nu(\sigma) < \overline{\nu} < 2$ is robustly stable if there are $X, Y \in \mathbb{R}^{n \times n}$ subject to (6), and*

$$\tilde{\Theta}_p = \tilde{\Omega}_{1k} + \tilde{\Omega}_{2k}\chi(\overline{v}) < 0, \tag{9}$$

$$\begin{bmatrix} \mathrm{sym}(s_{\bar{v}} X + c_{\bar{v}} Y) - \varepsilon I & Z_{2_s}^T \\ Z_{2_s} & 1 \end{bmatrix} + \begin{bmatrix} \mathrm{sym}[(1 - s_{\bar{v}})X - c_{\bar{v}} Y] & 0 \\ 0 & 0 \end{bmatrix} \chi(\bar{v}) \geq 0,$$

where the definition of $\chi(\overline{v})$ is defined in Theorem 1 and

$$\tilde{\Omega}_{1k} = I_2 \otimes \begin{bmatrix} \Phi & s_{\bar{v}} XE^T & -c_{\bar{v}} YE^T \\ s_{\bar{v}} EX & -\varepsilon_1 I & 0 \\ c_{\bar{v}} EY & 0 & -\varepsilon_2 I \end{bmatrix}, \tag{10}$$

where,

$$\Phi = \mathrm{sym}[A_0(s_{\bar{v}} X + c_{\bar{v}} Y)] + B(\Lambda_p Z_1(\bar{v}) + {\Lambda_p}^{-1} Z_2(\bar{v}))] + (\varepsilon_1 + \varepsilon_2) k^2 \bar{v}^2 DD^T,$$

$$\bar{\Omega}_{2k} = \begin{bmatrix} \tilde{\gamma}_{1k} & \tilde{\gamma}_{2k} \\ -\tilde{\gamma}_{2k} & \tilde{\gamma}_{1k} \end{bmatrix},$$

$$\tilde{\gamma}_{1k} = \begin{bmatrix} \mathrm{sym}(-c_{\bar{v}} A_0 Y) - \varepsilon_2 k^2 \bar{v}^2 DD^T & (1 - s_{\bar{v}})XE^T & c_{\bar{v}} YE^T \\ (1 - s_{\bar{v}})EX & 0 & 0 \\ -c_{\bar{v}} EY & 0 & \varepsilon_2 I \end{bmatrix}$$

$$\tilde{\gamma}_{2k} = \begin{bmatrix} \mathrm{asym}\{c_{\bar{v}}[A_0 X + B(\Lambda_p Z_1(\bar{v}) + {\Lambda_p}^{-1} Z_2(\bar{v}))]\} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Z_1(\bar{v}) \triangleq \begin{cases} K(s_{\bar{v}} X + c_{\bar{v}} Y), 0 < \bar{v} < 1, \\ FX, 1 \leq \bar{v} < 2, \end{cases} \quad Z_2(\bar{v}) \triangleq \begin{cases} H(s_{\bar{v}} X + c_{\bar{v}} Y), 0 < \bar{v} < 1, \\ HX, 1 \leq \bar{v} < 2. \end{cases}$$

Then,

$$K = \begin{cases} Z_1(\bar{v})(s_{\bar{v}}X + c_{\bar{v}}Y)^{-1}, 0 < \bar{v} < 1, \\ Z_1(\bar{v})X^{-1}, 1 \leq \bar{v} < 2, \end{cases} \quad H = \begin{cases} Z_2(\bar{v})(s_{\bar{v}}X + c_{\bar{v}}Y)^{-1}, 0 < \bar{v} < 1, \\ Z_2(\bar{v})X^{-1}, 1 \leq \bar{v} < 2. \end{cases}$$

Proof. Let us prove this problem in three cases. In the first case, for $\nu(\sigma) \in [\underline{\nu}, \overline{\nu}]$ and $\overline{\nu} < 1$, by using the Schur complement and $K = Z_1(\overline{\nu})(s_{\overline{\nu}}X + c_{\overline{\nu}}Y)^{-1}, H = Z_2(\overline{\nu})(s_{\overline{\nu}}X + c_{\overline{\nu}}Y)^{-1}$, and by using the Schur complement lemma and the fact that $2X^TY \leq \varepsilon X^TX + \varepsilon^{-1}Y^TY$, the following is deduced:

$$\begin{aligned} &\mathrm{sym}\{[A_0 + k\overline{\nu}DFE + B(\Lambda_p K + \Lambda_p^{-1}H)](s_{\overline{\nu}}X + c_{\overline{\nu}}Y)\} \\ &= s_{\overline{\nu}}\mathrm{sym}(A_0X) + c_{\overline{\nu}}\mathrm{sym}(A_0Y) + s_{\overline{\nu}}\mathrm{sym}(k\overline{\nu}DFEX) + c_{\overline{\nu}}\mathrm{sym}(k\overline{\nu}DFEY) \\ &+ s_{\overline{\nu}}\mathrm{sym}[B(\Lambda_p K + \Lambda_p^{-1}H)X] + c_{\overline{\nu}}\mathrm{sym}[B(\Lambda_p K + \Lambda_p^{-1}H)Y] \\ &= \begin{bmatrix} \mathrm{sym}[A_0 + B(\Lambda_p K + \Lambda_p^{-1}H)] + (\varepsilon_1 + \varepsilon_2)k^2\overline{\nu}^2DD^T & s_{\overline{\nu}}XE^T & -c_{\overline{\nu}}YE^T \\ s_{\overline{\nu}}EX & -\varepsilon_1 I & 0 \\ c_{\overline{\nu}}EY & 0 & -\varepsilon_2 I \end{bmatrix} < 0 \end{aligned}$$

Therefore, it is easy to get

$$\mathrm{sym}\{[A_0 + k\overline{\nu}DFE + B(\Lambda_p K + \Lambda_p^{-1}H)](s_{\overline{\nu}}X + c_{\overline{\nu}}Y)\} < 0. \tag{11}$$

Letting $\bar{A}_p = A_0 + k\overline{\nu}DFE + B(\Lambda_p K + \Lambda_p^{-1}H)$, one has

$$\mathrm{sym}[\bar{A}_p(s_{\overline{\nu}}X + c_{\overline{\nu}}Y)] < 0, \ \ p = 1, 2, \cdots, 2^m.$$

Then,

$$\mathrm{sym}[\lambda_p \sum_{p=1}^{2^m} \bar{A}_p(s_{\overline{\nu}}X + c_{\overline{\nu}}Y)] < 0, \ \ p = 1, 2, \cdots, 2^m.$$

According to Lemmas 1, 4, and 6, system (2) is stable for any $x_0 \in \mathcal{B}_\delta \subset \mathcal{L}(H)$. Next, the stability region is estimated. It can be obtained from (10) that

$$\mathrm{sym}(s_{\overline{\nu}}X + c_{\overline{\nu}}Y) - \varepsilon I - Z_{2_s}^T Z_{2_s} \geq 0.$$

Then, it is obtained that

$$\frac{1}{\varepsilon}(s_{\overline{\nu}}X + c_{\overline{\nu}}Y)^T(s_{\overline{\nu}}X + c_{\overline{\nu}}Y) - Z_{2_s}^T Z_{2_s} \geq 0. \tag{12}$$

By Schur complement, (13) is equivalent to

$$\begin{bmatrix} \frac{1}{\varepsilon}(s_{\overline{\nu}}X + c_{\overline{\nu}}Y)^T(s_{\overline{\nu}}X + c_{\overline{\nu}}Y) & Z_{2_s}^T \\ Z_{2_s} & 1 \end{bmatrix} \geq 0.$$

Pre- and post-multiplying $\mathrm{diag}((s_{\overline{\nu}}X + c_{\overline{\nu}}Y)^{-T}, 1)$ and its transpose, respectively, it follows that

$$\begin{bmatrix} \frac{1}{\varepsilon}I & H_s^T \\ H_s & 1 \end{bmatrix} \geq 0. \tag{13}$$

By (14), one gets

$$\frac{1}{\varepsilon}I \geq H_s^T H_s.$$

Then,

$$\frac{1}{\varepsilon}x^T x \geq x^T H_s^T H_s x.$$

From Lemma 2 and $x \in \mathcal{B}_\varepsilon$, it is not difficult to get

$$1 \geq \frac{1}{\varepsilon} x^T x \geq x^T H_s^T H_s x.$$

that is,

$$x \in \mathcal{L}(H).$$

Therefore, $\mathcal{B}_\varepsilon$ is selected such that $\mathcal{B}_\delta \subset \mathcal{B}_\varepsilon \subset \mathcal{L}(H)$. Moreover, $\mathcal{B}_\varepsilon$ is used to estimate the stability region $\mathcal{L}(H)$.

In the second case, for $\nu(\sigma) \in [\underline{\nu}, \overline{\nu}]$ and $1 < \underline{\nu}$, the proof is similar to $\nu(\sigma) \in [\underline{\nu}, 1]$.

In the third case, for $\nu(\sigma) \in [\underline{\nu}, \overline{\nu}]$ and $\underline{\nu} < 1 < \overline{\nu}$, it is deduced that when $\nu(\sigma) \in [\underline{\nu}, 1]$, it is similar to the first case. When $\nu(\sigma) \in [1, \overline{\nu}]$, it is similar to the second case.

From Theorem 2, the stability domain is estimated through solving the following optimization problem:

$$\begin{array}{c} \max \varepsilon \\ \text{subject to } (6), (8), (9). \end{array} \tag{14}$$

□

4. Numerical Examples

In this section, three examples of stability and robust stabilization for VO-FIS are given.

Example 1. *Consider unforced FOIS (2) where $\nu(\sigma) \in [0.4, 0.8]$, $k = 1$ and*

$$A_0 = \begin{bmatrix} -5 & -1 & 0.3 \\ 2.3 & 0.4 & 3 \\ 2 & -1 & -1 \end{bmatrix}, \ D = \begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0 & 0.1 & 1 \\ 0.2 & 0.1 & -1 \end{bmatrix},$$

$$E = \begin{bmatrix} 0.2 & 0 & 0.2 \\ 0.3 & 0.1 & 1 \\ 0.2 & 1 & -1 \end{bmatrix}, F = \begin{bmatrix} \cos(0.2\pi t) & 0 & 0 \\ 0 & \sin(0.1\pi t) & 0 \\ 0 & 0 & \sin(0.1\pi t) \end{bmatrix}.$$

By using LMI toolbox in MATLAB, a feasible solution of (6) and (8) is obtained, as follows:

$$X = \begin{bmatrix} 3.5915 & -0.9413 & -1.6629 \\ -0.9413 & 0.6735 & 0.3576 \\ -1.6629 & 0.3576 & 1.1486 \end{bmatrix},$$

$$Y = \begin{bmatrix} 0 & 0.0487 & -0.4584 \\ -0.0487 & 0 & 0.4989 \\ 0.4584 & -0.4989 & 0 \end{bmatrix},$$

$$\varepsilon_1 = 0.5003, \quad \varepsilon_2 = 0.1411.$$

Example 2. *Consider FOIS (2) where $\nu(\sigma) \in [0.4, 0.8]$, $k = 1$ and*

$$A_0 = \begin{bmatrix} 3.2 & 2.5 & -7 \\ 2.3 & 1 & 1.2 \\ 1.9 & 0.4 & 2 \end{bmatrix}, \ B = \begin{bmatrix} 4 & 1 & 3 \\ 1 & 1 & 2 \\ 2 & 0.4 & 3 \end{bmatrix},$$

$$D = \begin{bmatrix} 0.2 & 1 & 0 \\ 0 & 2 & 0.1 \\ 1 & 0.2 & 0.2 \end{bmatrix}, \ E = \begin{bmatrix} 0.2 & 0 & 1 \\ 1 & 2 & 0.2 \\ 0 & 0.1 & 0.2 \end{bmatrix},$$

$$F = \begin{bmatrix} \cos(0.2\pi t) & 0 & 0 \\ 0 & \sin(0.1\pi t) & 0 \\ 0 & 0 & \sin(0.1\pi t) \end{bmatrix}.$$

Then, by solving LMI (15), it follows $\varepsilon = 0.0048$ *and*

$$X = \begin{bmatrix} 0.0672 & -0.0317 & -0.0208 \\ -0.0317 & 0.0669 & 0.0581 \\ -0.0208 & 0.0581 & 0.0796 \end{bmatrix},$$

$$Y = \begin{bmatrix} 0 & 0.0579 & 0.0602 \\ -0.0579 & 0 & -0.0150 \\ -0.0602 & 0.0150 & 0 \end{bmatrix},$$

$$K = \begin{bmatrix} 0.6420 & 4.1018 & 5.3102 \\ -2.4674 & -4.3667 & 3.5311 \\ -3.3331 & -2.9345 & -5.5303 \end{bmatrix},$$

$$H = \begin{bmatrix} -1.5496 & 0.2252 & 8.7944 \\ -10.1473 & -8.1212 & 7.9297 \\ -0.6550 & 0.0621 & -8.4099 \end{bmatrix},$$

$$\varepsilon_1 = 0.0184, \quad \varepsilon_2 = 0.013.$$

Example 3. *Consider FOIS (2) where* $\nu(\sigma) \in [0.9, 1.4]$, $k = 1$ *and*

$$A_0 = \begin{bmatrix} 2.1 & 2.1 & 2 \\ 2.4 & 3.2 & 3.2 \\ 3.9 & 1.3 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1 & 3 \\ 1.4 & 1 & 2.2 \\ 2.6 & 3 & -2 \end{bmatrix},$$

$$D = \begin{bmatrix} 0.1 & 0.4 & 0 \\ 0 & 2 & -0.1 \\ 1 & 0.2 & 0.1 \end{bmatrix}, \quad E = \begin{bmatrix} 0.1 & 0 & 1 \\ 0.4 & 2 & 0.2 \\ 0 & -0.1 & 0.1 \end{bmatrix},$$

$$F = \begin{bmatrix} \cos(0.2\pi t) & 0 & 0 \\ 0 & \sin(0.1\pi t) & 0 \\ 0 & 0 & \sin(0.1\pi t) \end{bmatrix}.$$

By solving LMI (15), it follows that $\varepsilon = 0.0203$ *and*

$$X = \begin{bmatrix} 0.0716 & -0.0013 & -0.0585 \\ -0.0013 & 0.0189 & -0.0040 \\ -0.0585 & -0.0040 & 0.0835 \end{bmatrix},$$

$$K = \begin{bmatrix} -166.1826 & -399.1456 & -268.9191 \\ -435.2862 & -326.3777 & -458.7671 \\ -168.9720 & -294.0463 & -80.4705 \end{bmatrix},$$

$$H = \begin{bmatrix} -2.9176 & -11.1581 & -4.4470 \\ -9.3734 & -0.4442 & -10.5791 \\ -1.3668 & -10.4345 & -0.9516 \end{bmatrix}, \quad \varepsilon_1 = 0.0181.$$

For unforced FOIS (2), Figure 1 shows the eigenvalue perturbed region. The purple and green line are boundaries of order $\nu(\sigma) = 0.8$ and $\nu(\sigma) = 0.4$, respectively. The eigenvalue perturbed region is within the stability boundaries. Therefore, system (2) in Example 1 is stable. Compared with the existing results in [28–30], our results are less conservative, since the eigenvalues of system matrix are restrained in the left half of the complex plane of [28–30]. However, from Figure 1, it is easy to see that the eigenvalues of the system matrix of system (2) are restrained in the right half of the plane.

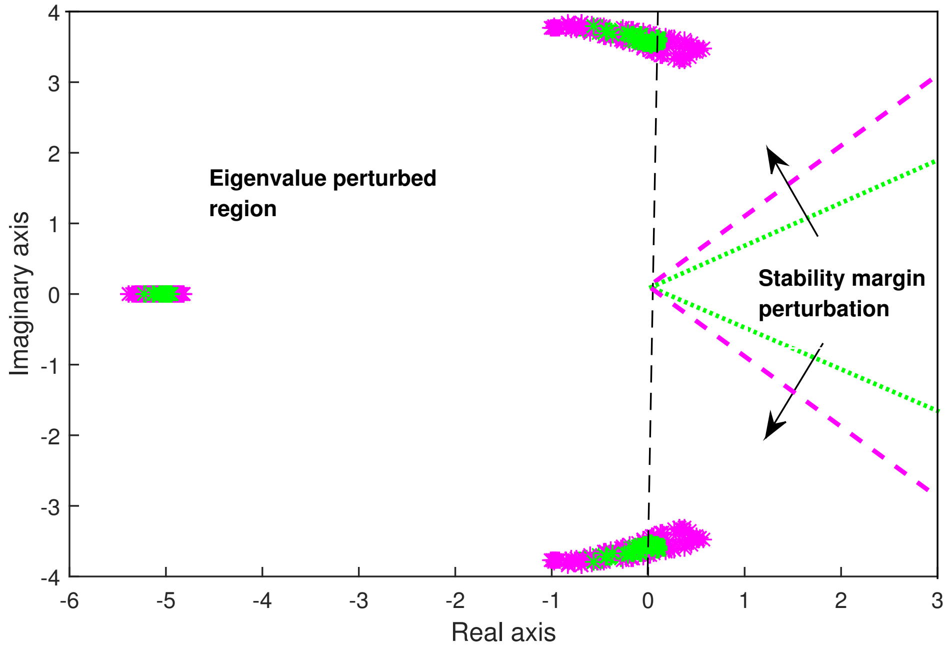

Figure 1. Eigenvalue perturbed region of system (2) in Example 1 with $\nu(\sigma) \in [0.4, 0.8]$.

For system (2) with order $\nu(\sigma) = 0.6$, the stability region $\mathcal{B}_\varepsilon$ is depicted in Figure 2. The simulation result of Example 1 with initial condition $x(0) = [-0.03 \ \ -0.03 \ \ 0.03]^T \in \mathcal{B}_\varepsilon$ is depicted in Figure 3. System (2) is stabilized by the state feedback controller in about 0.2 s.

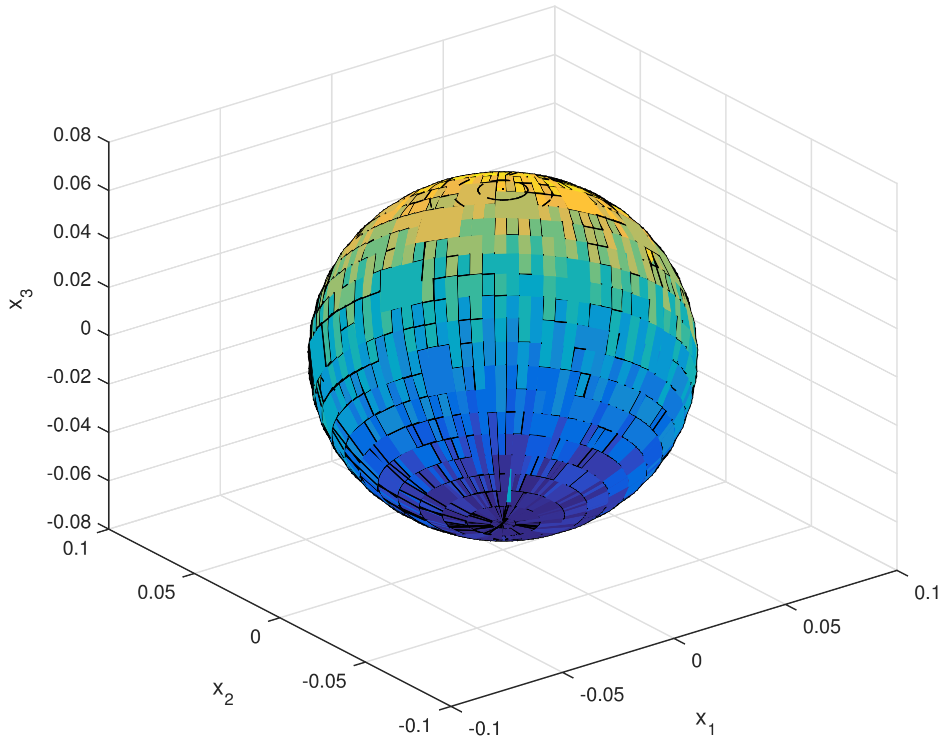

Figure 2. Stability domain $\mathcal{B}_\varepsilon$ in Example 2.

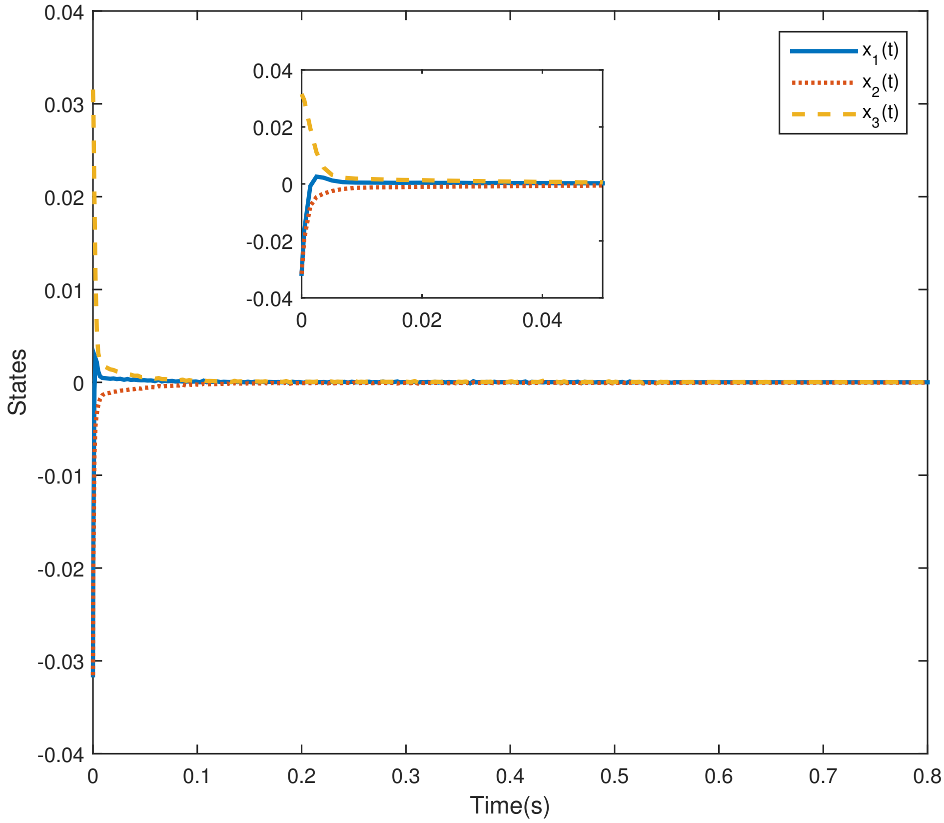

Figure 3. The closed-loop FOS in Example 2.

For FOIS (2) with order $\alpha = 1.2$, the simulation result of Example 2 with the initial condition $x(0) = [-0.01 \quad -0.01 \quad 0.01]^T \in \mathcal{B}_\varepsilon$ is depicted in Figure 4. System (2) is stabilized through the state feedback controller in about 0.4 s.

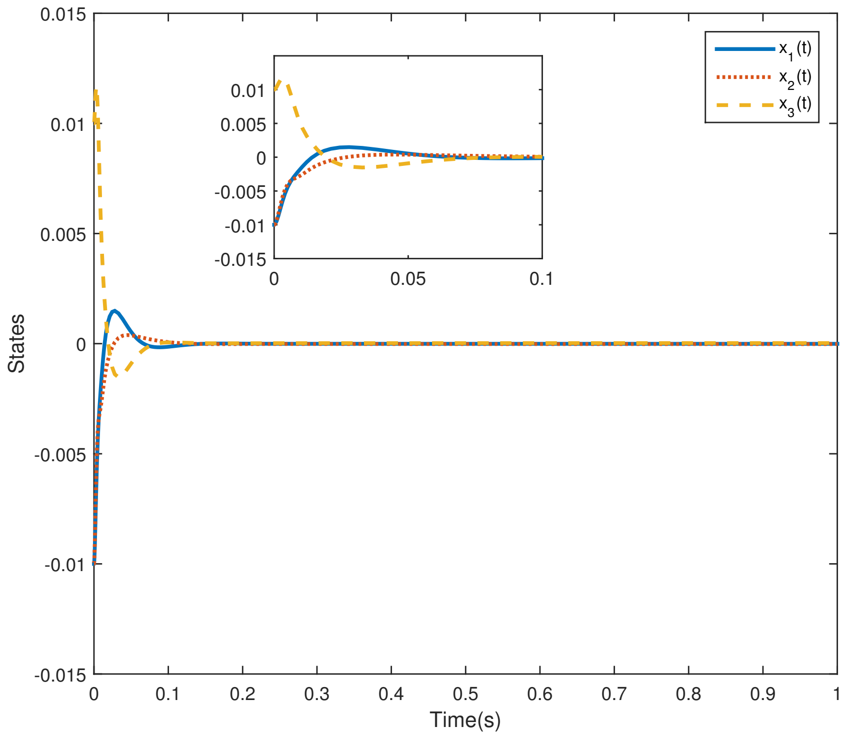

Figure 4. The closed-loop FOS in Example 3.

The eigenvalue perturbed region of FOIS (2) is shown in Figure 5. The purple and green line are edges of order $\nu(\sigma) = 1.4$ and $\nu(\sigma) = 0.9$, respectively. The eigenvalue domain perturbed is within the stability boundaries. Therefore, system (2) is stable.

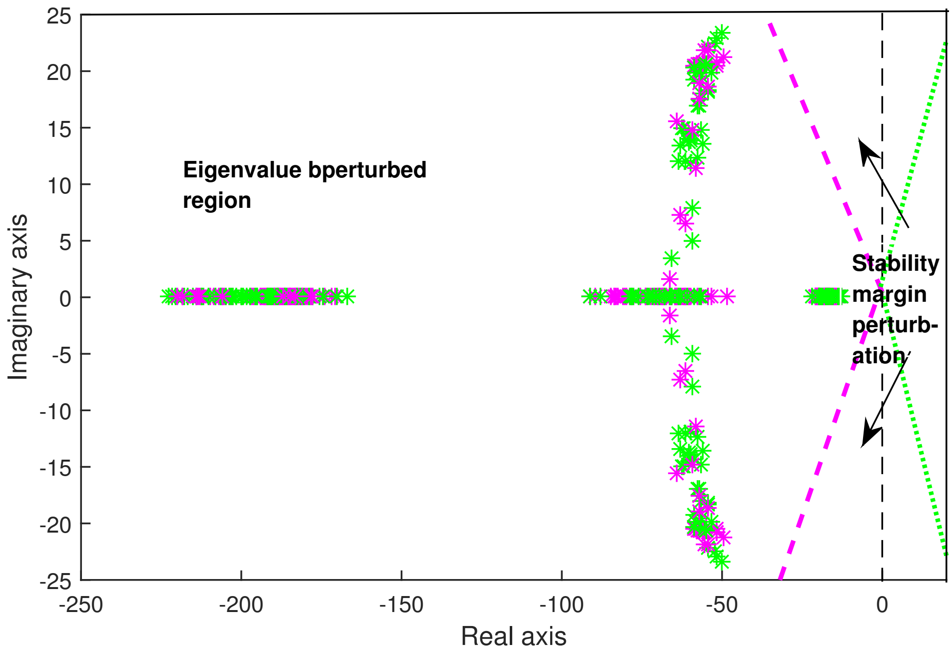

Figure 5. Eigenvalue perturbed region of system (2) in Example 1 with $\nu(\sigma) \in [0.9, 1.4]$.

Remark 2. *In Example 3, the fractional order in [0.9, 1.4] is considered. It is worth emphasizing that the LMI conditions in [28–30] are infeasible, and they fail to solve the robust stability problems. However, in the paper, the LMI conditions in Theorem 2 are feasible. The stability of system (2) is illustrated. Therefore, the whole simulation in this control procedure is explained by its effectiveness.*

Remark 3. *When the fractional order is $\nu(\sigma) \in (0, 1)$, compared to the results in [28,29], the value range of the order of this manuscript is relatively large. For example, when $\nu(\sigma)$ = 0.5, the MATLAB solution result in [28,29] is that the best value of $t > 0$. This means that the solution is invalid, but in this manuscript, the equation has a feasible solution.*

5. Conclusions

This paper's contribution is to propose new sufficient LMI-based conditions for VO-FIS with order $\nu(\sigma) \in (0, 2)$. The robust stabilization for VO-FIS subject to actuator saturation is further discussed. According to the obtained stabilization conditions, the stability domain by solving the optimization problem in terms of LMIs is estimated, since LMIs are calculated directly through the MATLAB toolbox. The next work to be done is to develop the output feedback control technique of FOIS and singular VO-FIS.

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