fractal and fractional

MDPI

Article

Dynamic Analysis and Bifurcation Study on Fractional-Order Tri-Neuron Neural Networks Incorporating Delays

Peiluan Li [1], Jinling Yan [1], Changjin Xu [2,*] and Youlin Shang [1]

[1] School of Mathematics and Statistics, Henan University of Science and Technology, Luoyang 471023, China; 15038522015@163.com (P.L.); haust_yjl@163.com (J.Y.); mathshang@sina.com (Y.S.)
[2] Guizhou Key Laboratory of Economics System Simulation, Guizhou University of Finance and Economics, Guiyang 550004, China
* Correspondence: xcj403@126.com

Abstract: In this manuscript, we principally probe into a class of fractional-order tri-neuron neural networks incorporating delays. Making use of fixed point theorem, we prove the existence and uniqueness of solution to the fractional-order tri-neuron neural networks incorporating delays. By virtue of a suitable function, we prove the uniformly boundedness of the solution to the fractional-order tri-neuron neural networks incorporating delays. With the aid of the stability theory and bifurcation knowledge of fractional-order differential equation, a new delay-independent condition to guarantee the stability and creation of Hopf bifurcation of the fractional-order tri-neuron neural networks incorporating delays is established. Taking advantage of the mixed controller that contains state feedback and parameter perturbation, the stability region and the time of onset of Hopf bifurcation of the fractional-order trineuron neural networks incorporating delays are successfully controlled. Software simulation plots are displayed to illustrate the established key results. The obtained conclusions in this article have important theoretical significance in designing and controlling neural networks.

Keywords: fractional-order tri-neuron neural networks; stability; Hopf bifurcation; Hopf bifurcation control

Citation: Li, P.; Yan, J.; Xu, C.; Shang, Y. Dynamic Analysis and Bifurcation Study on Fractional-Order Tri-Neuron Neural Networks Incorporating Delays. *Fractal Fract.* **2022**, *6*, 161. https://doi.org/10.3390/fractalfract6030161

Academic Editor: Ivanka Stamova

Received: 7 February 2022
Accepted: 8 March 2022
Published: 15 March 2022

1. Introduction

After the classical work of Hopfield [1] on neural networks, many scholars pay much attention to the study of the dynamics of different types of neural networks since neural networks have displayed extraordinary application value in various fields such as associate memory, automatic control, image processing, biological engineering, pattern recognition and so on [2–5]. Generally speaking, delay often occurs in the signal transmission among different neurons due to the finite propagation velocity in neural networks [6]. A natural problem arises: What is the impact of time delay on the dynamical behavior of neural networks? During the past decades, in order to reveal the influence of time delay on various dynamics of neural networks, a great deal of researchers from mathematics and engineering have made great efforts to explore the dynamics of delayed neural networks and lots of valuable fruits have been achieved. For instance, Wang et al. [7] explored the fixed-time synchronization problem of delayed complex-valued BAM neural networks by using pinning control and adaptive pinning control. Gan et al. [8] discussed the anti-synchronization of a class of BAM neural networks involving the Markov scheduling protocol. Syed Ali et al. [9] set up a novel sufficient condition to ensure the global Mittag–Leffler stability of impulsive fractional-order delayed complex-valued BAM neural networks. Chen et al. [10] studied the finite-time stabilization of fractional-order fuzzy quaternion-valued BAM neural networks by applying direct quaternion approach. Making use of basic theories on fractional calculus, inequality skills of fuzzy logic and reduction to absurdity, Chen et al. [10] established the sufficient criteria that guarantee the finite-time stabilization of the fractional-order fuzzy

quaternion-valued BAM neural networks. Cong et al. [11] dealt with the global exponential stability of discrete-time delayed BAM neural networks. By virtue of the mathematical induction method, a new exponential stability criterion for discrete-time delayed BAM neural networks is built. For more specific publications on this aspect, one can see [12–16].

Here we notice that most of the literature (e.g., [1–9,11–16]) only involves integer-order delayed neural networks, not considering the fractional-order cases. Since the introduction of fractional-order derivative by Leibniz and L′Hosptical [17] three hundred years ago, fractional calculus gradually arouse much attention from many scholars. The fractional derivative has more greater advantages than integer-order ones. It can adequately display the global correlation and better description of long-term development process and memory [18]. Recent research shows that fractional calculus has been widely used in many areas such as electrical engineering, wave theory in physics, electronic information, biological technique, control engineering, neural systems and so on [19–21]. Thus, the investigation of the dynamics of fractional-order dynamical systems has attracted much interest from numerous scholars and fruitful results have been achieved. For instance, Zhang et al. [22] explored the multistability of fractional-order competitive neural networks with delays. Li et al. [23] made a detailed analysis on the complete and finite-time synchronization of fractional-order fuzzy neural networks by virtue of nonlinear feedback control. Padmaja and Balasubramaniam [24] set up a new delay and order-dependent passivity criteria for impulsive fractional-order neural networks involving proportional delays. Ke [25] proded into the Mittag–Leffler stability and asymptotic ω-periodicity of fractional-order delayed inertial neural networks. In details, one can see [26–37].

In fractional-order differential equations, time delay is a vital factor that affects the dynamical behavior of systems. In particular, delay-induced Hopf bifurcation is an important topic in fractional-order differential systems. In order to design, optimize and control neural networks to serve humanity, we need to explore the impact of time delay on the dynamics of delayed fractional-order neural networks, especially, we need to investigate the impact of time delay on Hopf bifurcation phenomenon of delayed fractional-order neural networks. By adjusting the value of time delay, we can enlarge or narrow the stability region of neural network systems, then postpone or advance the time of onset of Hopf bifurcation of neural networks. However, many works on delay-induced Hopf bifurcation mainly focus on the integer-order delayed neural networks and there are few works on the fractional-order delayed neural networks. Recently, there has been some literature on delay-induced Hopf bifurcation of fractional-order delayed dynamical systems. For example, Xu et al. [38] established a sufficient condition to ensure the stability and the onset of Hopf bifurcation of fractional-order six-neuron BAM neural networks with multi-delays. Huang et al. [39] explored the Hopf bifurcation of fractional-order quaternion-valued neural networks. Xiao et al. [40] investigated the fractional-order PD control of Hopf bifurcations in fractional-order small-world networks with delays. For more related studies, see Refs. [41–47]. Although some works on Hopf bifurcation of fractional-order delayed neural networks have been carried out, many challenging problems are expected to be solved. Stimulated by the idea, in this present work, we are to deal with the following three aspects: (i) Explore the existence and uniqueness, the boundedness of solution of the involved neural networks with single delay; (ii) Set up a sufficient criterion ensuring the stability and and the creation of Hopf bifurcation of the involved neural networks with single delay; (iii) Make use of a suitable mixed controller that contains state feedback and parameter perturbation to control the time of onset of Hopf bifurcation for the involved neural networks with single delay.

In 2006, Yan [41] investigated the following tri-neuron network system:

$$\begin{cases} \dot{w}_1(t) = -\nu w_1(t) + h(w_1(t)) + h_{12}(w_2(t-\rho)) + h_{13}(w_3(t-\rho)), \\ \dot{w}_2(t) = -\nu w_2(t) + h(w_2(t)) + h_{21}(w_1(t-\rho)) + h_{23}(w_3(t-\rho)), \\ \dot{w}_3(t) = -\nu w_3(t) + h(w_3(t)) + h_{31}(w_1(t-\rho)) + h_{32}(w_2(t-\rho)), \end{cases} \tag{1}$$

where $w_1(t), w_2(t), w_3(t)$ stand for the voltage on the input of the first neuron at time $t(t > 0)$, the voltage on the input of the second neuron at time $t(t > 0)$, the voltage on the input of the third neuron at time $t(t > 0)$, respectively; $\nu > 0$ stands for the rate with which the three neurons will reset its potential to the resetting state in isolation when disconnected from the network and external inputs. $h, h_{12}, h_{13}, h_{21}h_{23}, h_{31}, h_{32}$ are the sigmoidal activation functions of the different neurons. $\rho > 0$ represents the transmission delay of the signal along the axon of the neuron. In details, one can see [41–43]. By analyzing the characteristic equation of system (1) and selecting the time delay as bifurcation parameter, Yan [41] established the delay-independent stability and Hopf bifurcation condition ensuring the stability and the creation of Hopf bifurcation of system (1). In addition, making use of the normal form and center manifold theorem, Yan [41] explored the Hopf bifurcation peculiarities.

On the basis of the analysis above, we will deal with the delay-induced Hopf bifurcation of fractional order neural networks. Inspired by the above viewpoint, we modify model (1) as the following fractional-order form

$$\begin{cases} \dfrac{d^r w_1(t)}{dt^r} = -\nu w_1(t) + h(w_1(t)) + h_{12}(w_2(t-\rho)) + h_{13}(w_3(t-\rho)), \\ \dfrac{d^r w_2(t)}{dt^r} = -\nu w_2(t) + h(w_2(t)) + h_{21}(w_1(t-\rho)) + h_{23}(w_3(t-\rho)), \\ \dfrac{d^r w_3(t)}{dt^r} = -\nu w_3(t) + h(w_3(t)) + h_{31}(w_1(t-\rho)) + h_{32}(w_2(t-\rho)), \end{cases} \tag{2}$$

where $r \in (0, 1]$ is a constant. We give the initial value of system (2) as follows:

$$\begin{cases} w_1(\phi) = w_{1\phi}, \phi \in [-\rho, t_0], \\ w_2(\phi) = w_{2\phi}, \phi \in [-\rho, t_0], \\ w_3(\phi) = w_{1\phi}, \phi \in [-\rho, t_0], \end{cases} \tag{3}$$

where $t_0 > 0$ is a constant. In order to achieve our goal in this paper, we prepare the following necessary assumptions:

$(\mathcal{Q}_1)$ $h, h_{12}, h_{13}, h_{21}, h_{23}, h_{31}, h_{32} \in C^1, h(0) = h_{12}(0) = h_{13}(0) = h_{21}(0) = h_{23}(0) = h_{31}(0) = h_{32}(0) = 0$.

$(\mathcal{Q}_2)$ There exist positive constants $\mathcal{H}, \mathcal{H}_{12}, \mathcal{H}_{13}, \mathcal{H}_{21}, \mathcal{H}_{23}, \mathcal{H}_{31}, \mathcal{H}_{32}$ such that

$$\begin{cases} |h(t_1) - h(t_2)| \le \mathcal{H}|t_1 - t_2|, \\ |h_{12}(t_1) - h_{12}(t_2)| \le \mathcal{H}_{12}|t_1 - t_2|, \\ |h_{13}(t_1) - h_{13}(t_2)| \le \mathcal{H}_{13}|t_1 - t_2|, \\ |h_{21}(t_1) - h_{21}(t_2)| \le \mathcal{H}_{21}|t_1 - t_2|, \\ |h_{23}(t_1) - h_{23}(t_2)| \le \mathcal{H}_{23}|t_1 - t_2|, \\ |h_{31}(t_1) - h_{31}(t_2)| \le \mathcal{H}_{31}|t_1 - t_2|, \\ |h_{32}(t_1) - h_{32}(t_2)| \le \mathcal{H}_{32}|t_1 - t_2|. \end{cases}$$

$(\mathcal{Q}_3)$ There exist positive constants $\mathcal{H}_\star, \mathcal{H}_{12\star}, \mathcal{H}_{13\star}, \mathcal{H}_{21\star}, \mathcal{H}_{23\star}, \mathcal{H}_{31\star}, \mathcal{H}_{32\star}$ such that

$$\begin{cases} |h(t)| \le \mathcal{H}_\star, \\ |h_{12}(t)| \le \mathcal{H}_{12\star}, \\ |h_{13}(t)| \le \mathcal{H}_{13\star}, \\ |h_{21}(t)| \le \mathcal{H}_{21\star}, \\ |h_{23}(t)| \le \mathcal{H}_{23\star}, \\ |h_{31}(t)| \le \mathcal{H}_{31\star}, \\ |h_{32}(t)| \le \mathcal{H}_{32\star}, \end{cases}$$

for all $t \in R$.

We arrange the basic structure of the article as follows. Section 2 lists some related basic theories on fractional-order dynamical system. Section 3 gives the proof existence and uniqueness of solution of system (2). Section 4 shows the proof of boundedness of solution

of system (2). Section 5 establishes the delay-independent sufficient condition to ensure the stability and the creation of Hopf bifurcation of system (2). Section 6 seeks a suitable mixed controller to control the Hopf bifurcation of system (2). Section 7 carries out software simulations to support the derived key conclusions. Section 8 completes this work.

2. Prerequisite Theory

In this segment, we list some related theory on fractional-order systems. Let $R_+ = \{x | x \geq 0, x \in R\}$.

Definition 1 ([44]). *Define the Caputo fractional order derivative as follows:*

$$\mathcal{D}^r \kappa(\varrho) = \frac{1}{\Gamma(l-r)} \int_{\varrho_0}^{\varrho} \frac{\kappa^{(l)}(u)}{(\varrho - u)^{r-l+1}} du,$$

where $\kappa(\varrho) \in ([\varrho_0, \infty), R), \Gamma(\varrho) = \int_0^\infty \varrho^{s-1} e^{-\varrho} d\varrho$, $\varrho \geq \varrho_0$ *and* $l \in Z^+, r \in (r-1, r)$.

The Laplace transform of Caputo-type fractional order derivative is given by

$$\mathcal{L}\{\mathcal{D}^r g(t); s\} = s^r G(s) - \sum_{j=0}^{n-1} s^{r-l-1} g^{(j)}(0), r \in (n-1, n), n \in Z^+,$$

where $G(s) = \mathcal{L}\{g(t)\}$. If $g^{(j)}(0) = 0, j = 1, 2, \cdots, n$, then $\mathcal{L}\{\mathcal{D}^r g(t); s\} = s^r G(s)$.

Definition 2 ([45]). $(w_{1\star}, w_{2\star}, w_{3\star})$ *is an equilibrium point of model (2) provided that*

$$\begin{cases} -\nu w_{1\star} + h(w_{1\star}) + h_{12}(w_{2\star}) + h_{13}(w_{3\star}) = 0, \\ -\nu w_{2\star} + h(w_{2\star}) + h_{21}(w_{1\star}) + h_{23}(w_{3\star}) = 0, \\ -\nu w_{3\star} + h(w_{3\star}) + h_{31}(w_{1\star}) + h_{32}(w_{2\star}) = 0. \end{cases} \tag{4}$$

Lemma 1 ([46]). *Give the fractional-order system as follows:*

$$\frac{d^r u(t)}{dt^r} = g(t, u(t)), u(t_0) = u_{t_0}, t_0 > 0, \tag{5}$$

where $t_0 \in R, r \in (0, 1], g : [t_0, \infty) \times \varphi \to R_+^n, \varphi \subset R_+^n$, *then system (5) has a unique solution defined on* $[t_0, \infty)$ *if* $g(t, u)$ *satisfies the local Lipschitz condition with respect to* u.

Lemma 2 ([47]). *Assume that* $u(t) \in C[t_0, \infty)$ *and satisfies*

$$\begin{cases} D^r u(t) \leq -\zeta_1 u(t) + \zeta_2, \\ u(t_0) = u_{t_0}, \end{cases} \tag{6}$$

where $0 < r < 1, \zeta_1, \zeta_2 \in R, \zeta_1 \neq 0, t_0 \geq 0$. *Then,*

$$u(t) \leq \left(u(t_0) - \frac{\zeta_2}{\zeta_1} \right) E_r[-\zeta_1 (t - t_0)^r] + \frac{\zeta_2}{\zeta_1}. \tag{7}$$

Lemma 3 ([48]). *Consider the following fractional-order model:*

$$\frac{d^r u(t)}{dt^r} = g(t, u(t)), u(0) = u_0, \tag{8}$$

where $r \in (0, 1], g(t, u(t)) : R^+ \times R^m \to R^m$. *Denote* $u_\star$ *the equilibrium point of system (8). Then* $u_\star$ *is locally asymptotically stable provided that every eigenvalue* ϖ *of* $\frac{\partial g(t,u)}{\partial u}|_{u=u_\star}$ *satisfies* $|arg(\varpi)| > \frac{r\pi}{2}$.

Lemma 4 ([49]). *Give the following fractional-order system as follows:*

$$\begin{cases} \dfrac{d^{r_1}\mathcal{P}_1(t)}{dt^{r_1}} = \tau_{11}\mathcal{P}_1(t-\rho_{11}) + \tau_{12}\mathcal{P}_2(t-\rho_{12}) + \cdots + \tau_{1m}\mathcal{P}_m(t-\rho_{1m}), \\ \dfrac{d^{r_2}\mathcal{P}_2(t)}{dt^{r_2}} = \tau_{21}\mathcal{P}_1(t-\rho_{21}) + \tau_{22}\mathcal{P}_2(t-\rho_{22}) + \cdots + \tau_{2m}\mathcal{P}_m(t-\rho_{2m}), \\ \vdots \\ \dfrac{d^{r_m}\mathcal{P}_m(t)}{dt^{r_m}} = \tau_{m1}\mathcal{P}_1(t-\rho_{m1}) + \tau_{n2}\mathcal{P}_2(t-\rho_{m2}) + \cdots + \tau_{mm}\mathcal{P}_m(t-\rho_{mm}), \end{cases} \tag{9}$$

where $r_i \in (0,1)(i = 1, 2, \cdots, m)$. *Set*

$$\Delta(\sigma) = \begin{bmatrix} \sigma^{r_1} - \tau_{11}e^{-\sigma\rho_{11}} & -\tau_{12}e^{-\sigma\rho_{12}} & \cdots & -\tau_{1m}e^{-\sigma\rho_{1m}} \\ -\tau_{21}e^{-\sigma\rho_{12}} & \sigma^{r_2} - \tau_{22}e^{-\sigma\rho_{22}} & \cdots & -\tau_{2m}e^{-\sigma\rho_{2m}} \\ \vdots & \vdots & \ddots & \vdots \\ -\tau_{m1}e^{-\sigma\rho_{m1}} & -\tau_{m2}e^{-\sigma\rho_{m2}} & \cdots & \sigma^{r_m} - \tau_{mm}e^{-\sigma\rho_{mm}}. \end{bmatrix}. \tag{10}$$

Then the equilibrium point of system (9) is asymptotically stable in Lyapunov sense if every root of $\det(\Delta(\sigma)) = 0$ *has negative real parts.*

3. Existence and Uniqueness

Theorem 1. *Denote* $\Psi = \{(w_{1\star}, w_{2\star}, w_{3\star}) \in R^3 : \max\{|w_1|, |w_2|, |w_3|\} < \mathcal{W}\}$, *where* $\mathcal{W} > 0$ *stands for a constant. Then for arbitrary* $\mathcal{W}_{t_0} = (w_{1t_0}, w_{2t_0}, w_{3t_0}) \in \Psi$ *and for all* $t \geq t_0$, *then system (2) with the initial value* $\mathcal{W}_{t_0}$ *has a unique solution* $\mathcal{W}(t) \in \Psi$.

Proof. Let $\mathcal{W} = (w_1, w_2, w_3)$ and $\widetilde{\mathcal{W}} = (\tilde{w}_1, \tilde{w}_2, \tilde{w}_3)$ and $\Pi(\mathcal{W}) = (\Pi_1(\mathcal{W}), \Pi_2(\mathcal{W}), \Pi_3(\mathcal{W}))$, where

$$\begin{cases} \Pi_1(\mathcal{V}) = -\nu w_1(t) + h(w_1(t)) + h_{12}(w_2(t-\rho)) + h_{13}(w_3(t-\rho)), \\ \Pi_2(\mathcal{V}) = -\nu w_2(t) + h(w_2(t)) + h_{21}(w_1(t-\rho)) + h_{23}(w_3(t-\rho)), \\ \Pi_3(\mathcal{V}) = -\nu w_3(t) + h(w_3(t)) + h_{31}(w_1(t-\rho)) + h_{32}(w_2(t-\rho)). \end{cases} \tag{11}$$

$\forall\ \mathcal{W}, \widetilde{\mathcal{W}} \in \Psi$, by virtue of $(\mathcal{Q}_2)$ and $(\mathcal{Q}_3)$, we have

$$\begin{aligned} & ||\Pi(\mathcal{W}) - \Pi(\widetilde{\mathcal{W}})|| \\ = \ & \textstyle\sum_{j=1}^{3} |\Pi_j(\mathcal{W}) - \Pi_j(\widetilde{\mathcal{W}})| \\ = \ & |[-\nu w_1(t) + h(w_1(t)) + h_{12}(w_2(t-\rho)) + h_{13}(w_3(t-\rho))] \\ & -[-\nu \tilde{w}_1(t) + h(\tilde{w}_1(t)) + h_{12}(\tilde{w}_2(t-\rho)) + h_{13}(\tilde{w}_3(t-\rho))]| \\ & +|[-\nu w_2(t) + h(w_2(t)) + h_{21}(w_1(t-\rho)) + h_{23}(w_3(t-\rho))] \\ & -[-\nu \tilde{w}_2(t) + h(\tilde{w}_2(t)) + h_{21}(\tilde{w}_1(t-\rho)) + h_{23}(\tilde{w}_3(t-\rho))]| \\ & +|[-\nu w_3(t) + h(w_3(t)) + h_{31}(w_1(t-\rho)) + h_{32}(w_2(t-\rho))] \\ & -[-\nu \tilde{w}_3(t) + h(\tilde{w}_3(t)) + h_{31}(\tilde{w}_1(t-\rho)) + h_{32}(\tilde{w}_2(t-\rho))]| \\ \leq \ & \nu|w_1(t) - \tilde{w}_1(t)| + \mathcal{H}|w_1(t) - \tilde{w}_1(t)| + \mathcal{H}_{12}|w_2(t-\rho) - \tilde{w}_2(t-\rho)| \\ & +\mathcal{H}_{13}|w_3(t-\rho) - \tilde{w}_3(t-\rho)| + \nu|w_2(t) - \tilde{w}_2(t)| + \mathcal{H}|w_2(t) - \tilde{w}_2(t)| \\ & +\mathcal{H}_{21}|w_1(t-\rho) - \tilde{w}_1(t-\rho)| + \mathcal{H}_{23}|w_3(t-\rho) - \tilde{w}_3(t-\rho)| \\ & +\nu|w_3(t) - \tilde{w}_3(t)| + \mathcal{H}|w_3(t) - \tilde{w}_3(t)| \\ & +\mathcal{H}_{31}|w_1(t-\rho) - \tilde{w}_1(t-\rho)| + \mathcal{H}_{32}|w_2(t-\rho) - \tilde{w}_2(t-\rho)| \\ \leq \ & \mathcal{A}_1|w_1(t) - \tilde{w}_1(t)| + \mathcal{A}_2|w_2(t) - \tilde{w}_2(t)| + \mathcal{A}_3|w_3(t) - \tilde{w}_3(t)|, \end{aligned} \tag{12}$$

where

$$\begin{cases} \mathcal{A}_1 = \nu + \mathcal{H} + \mathcal{H}_{21} + \mathcal{H}_{31}, \\ \mathcal{A}_2 = \nu + \mathcal{H} + \mathcal{H}_{12} + \mathcal{H}_{32}, \\ \mathcal{A}_3 = \nu + \mathcal{H} + \mathcal{H}_{13} + \mathcal{H}_{23}. \end{cases} \tag{13}$$

By (12), we obtain

$$||\Pi(\mathcal{W}) - \Pi(\widetilde{\mathcal{W}})|| \leq \mathcal{A}||\mathcal{W} - \widetilde{\mathcal{W}}|, \tag{14}$$

where

$$\mathcal{A} = \max\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}. \tag{15}$$

By virtue of Lemma 1, we easily know that Theorem 1 is true. The proof completes. □

Remark 1. *According to Lemma 1, we know that $\Pi(\mathcal{W})$ satisfies the local Lipschitz condition with respect to $\mathcal{W}$. By virtue of fixed point theorem, we can conclude that Theorem 1 is correct.*

4. Boundedness

In this segment, we will prove the boundedness of the solution to system (2). Denote $\Psi_+ = \{(w_1, w_2, w_3) \in \Psi : w_1, w_2, w_3 \in R_+\}$.

Theorem 2. *All solutions of system (2) that start with Ψ_+ are uniformly bounded.*

Proof. Define

$$\mathcal{V}(t) = w_1(t) + w_2(t) + w_3(t). \tag{16}$$

It follows from (2) that:

$$\begin{array}{rcl} \frac{d^r \mathcal{V}(t)}{dt^r} & = & \frac{d^r w_1(t)}{dt^r} + \frac{d^r w_2(t)}{dt^r} + \frac{d^r w_3(t)}{dt^r} \\ & = & -\nu w_1(t) + h(w_1(t)) + h_{12}(w_2(t-\rho)) + h_{13}(w_3(t-\rho)) \\ & & -\nu w_2(t) + h(w_2(t)) + h_{21}(w_1(t-\rho)) + h_{23}(w_3(t-\rho)) \\ & & -\nu w_3(t) + h(w_3(t)) + h_{31}(w_1(t-\rho)) + h_{32}(w_2(t-\rho)) \\ & \leq & -\nu w_1(t) + \mathcal{H}_\star + \mathcal{H}_{12\star} + \mathcal{H}_{13\star} \\ & & -\nu w_2(t) + \mathcal{H}_\star + \mathcal{H}_{21\star} + \mathcal{H}_{23\star} \\ & & -\nu w_3(t) + \mathcal{H}_\star + \mathcal{H}_{31\star} + \mathcal{H}_{32\star} \\ & \leq & -\nu \mathcal{V}(t) + \mathcal{H}_0, \end{array} \tag{17}$$

where

$$\mathcal{H}_0 = 3\mathcal{H}_\star + \mathcal{H}_{12\star} + \mathcal{H}_{13\star} + \mathcal{H}_{21\star} + \mathcal{H}_{23\star} + \mathcal{H}_{31\star} + \mathcal{H}_{32\star}. \tag{18}$$

According to Lemma 2, we get

$$\mathcal{V}(t) \leq \left(\mathcal{V}_{t_0} - \frac{\mathcal{H}_0}{\nu}\right) E_r[-\nu(t-t_0)^r] + \frac{\mathcal{H}_0}{\nu} \to \frac{\mathcal{H}_0}{\nu}, t \to \infty. \tag{19}$$

Therefore, all solutions to system (2) which begin with Ψ_+ are uniformly bounded. The proof finishes. □

5. Bifurcation Study

In this segment, we will analyze the stability and the creation of Hopf bifurcation of system (2). By $(\mathcal{Q}_1)$, we can easily know that system (2) has a unique zero equilibrium point $W_0(0,0,0)$. The linear system of model (2) at $W_0(0,0,0)$ is:

$$\begin{cases} \frac{d^r w_1(t)}{dt^r} = -\alpha w_1(t) + a_{12} w_2(t-\rho) + a_{13} w_3(t-\rho), \\ \frac{d^r w_2(t)}{dt^r} = -\alpha w_2(t) + a_{21} w_1(t-\rho) + a_{23} w_3(t-\rho), \\ \frac{d^r w_3(t)}{dt^r} = -\alpha w_3(t) + a_{31} w_1(t-\rho) + a_{32} w_2(t-\rho), \end{cases} \tag{20}$$

where $\alpha = \nu - h^{'}(0), a_{12} = h^{12}(0), a_{13} = h^{13}(0), a_{21} = h^{21}(0), a_{23} = h^{23}(0), a_{31} = h^{31}(0)$, $a_{32} = h^{32}(0)$. The corresponding characteristic equation for (20) is given by:

$$\det \begin{bmatrix} s^r + \alpha & -a_{12}e^{-s\rho} & -a_{13}e^{-s\rho} \\ -a_{21}e^{-s\rho} & s^r + \alpha & -a_{23}e^{-s\rho} \\ -a_{31}e^{-s\rho} & -a_{32}e^{-s\rho} & s^r + \alpha \end{bmatrix} = 0. \tag{21}$$

It follows from (21) that:

$$\mathcal{C}_1(s) + \mathcal{C}_2(s)e^{-2s\rho} + \mathcal{C}_3(s)e^{-3s\rho} = 0, \tag{22}$$

where

$$\begin{cases} \mathcal{C}_1(s) = s^{3r} + \varrho_1 s^{2r} + \varrho_2 s^r + \varrho_3, \\ \mathcal{C}_2(s) = \varrho_4 s^r + \varrho_5, \\ \mathcal{C}_3(s) = \varrho_6, \end{cases} \tag{23}$$

where

$$\begin{cases} \varrho_1 = 3\alpha, \\ \varrho_2 = 3\alpha^2, \\ \varrho_3 = \alpha^3, \\ \varrho_4 = -(a_{12}a_{21} + a_{13}a_{31} + a_{23}a_{32}), \\ \varrho_5 = -\alpha(a_{12}a_{21} + a_{13}a_{31} + a_{23}a_{32}), \\ \varrho_6 = -(a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}). \end{cases} \tag{24}$$

We rewrite (22) as:

$$\mathcal{C}_1(s)e^{s\rho} + \mathcal{C}_2(s)e^{-s\rho} + \varrho_6 e^{-2s\rho} = 0. \tag{25}$$

Let $s = i\theta = \theta\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$ be the root of Equation (25) and let $\mathcal{C}_{lR}(s)$ and $\mathcal{C}_{lI}(s)$ denote the real part and the imaginary part of $\mathcal{C}_l(s)(j = 1, 2)$. Then one has

$$\begin{cases} \mathcal{C}_{lR}(s) = \theta^{3r}\cos\dfrac{3r\pi}{2} + \varrho_1\theta^{2r}\cos r\pi + \varrho_2\theta^r\cos\dfrac{r\pi}{2} + \varrho_3, \\ \mathcal{C}_{lI}(s) = \theta^{3r}\sin\dfrac{3r\pi}{2} + \varrho_1\theta^{2r}\sin r\pi + \varrho_2\theta^r\sin\dfrac{r\pi}{2}, \\ \mathcal{C}_{2R}(s) = \varrho_4\theta^r\cos\dfrac{r\pi}{2} + \varrho_5, \\ \mathcal{C}_{2I}(s) = \varrho_4\theta^r\sin\dfrac{r\pi}{2}. \end{cases} \tag{26}$$

Let

$$\begin{cases} d_1 = \cos\dfrac{3r\pi}{2}, \\ d_2 = \varrho_1\cos r\pi, \\ d_3 = \varrho_2\cos\dfrac{r\pi}{2}, \\ d_4 = \varrho_3, \\ d_5 = \sin\dfrac{3r\pi}{2}, \\ d_6 = \varrho_1\sin r\pi, \\ d_7 = \varrho_2\sin\dfrac{r\pi}{2}, \\ d_8 = \varrho_4\cos\dfrac{r\pi}{2}, \\ d_9 = \varrho_5, \\ d_{10} = \varrho_4\sin\dfrac{r\pi}{2}, \end{cases} \tag{27}$$

then (26) becomes

$$\begin{cases} \mathcal{C}_{lR}(s) = d_1\theta^{3r} + d_2\theta^{2r} + d_3\theta^r + d_4, \\ \mathcal{C}_{lI}(s) = d_5\theta^{3r} + d_6\theta^{2r} + d_7\theta^r, \\ \mathcal{C}_{2R}(s) = d_8\theta^r + d_9, \\ \mathcal{C}_{2I}(s) = d_{10}\theta^r. \end{cases} \tag{28}$$

By virtue of (25) and (28), we have:

$$\begin{array}{l}[(d_1\theta^{3r}+d_2\theta^{2r}+d_3\theta^r+d_4)+i(d_5\theta^{3r}+d_6\theta^{2r}+d_7\theta^r)](\cos\theta\rho+i\sin\theta\rho)\\ +[(d_8\theta^r+d_9)+id_{10}\theta^r](\cos\theta\rho-i\sin\theta\rho)+\varrho_6(\cos 2\theta\rho-i\sin 2\theta\rho)=0.\end{array} \tag{29}$$

It follows from (29) that:

$$\begin{cases}\mathcal{D}_1\cos\theta\rho-\mathcal{D}_2\sin\theta\rho=-\varrho_6\cos 2\theta\rho,\\ \mathcal{D}_3\cos\theta\rho+\mathcal{D}_4\cos\theta\rho=\varrho_6\sin 2\theta\rho,\end{cases} \tag{30}$$

where

$$\begin{cases}\mathcal{D}_1=d_1\theta^{3r}+d_2\theta^{2r}+(d_3+d_8)\theta^r+d_4+d_9,\\ \mathcal{D}_2=d_5\theta^{3r}+d_6\theta^{2r}+(d_7+d_{10})\theta^r,\\ \mathcal{D}_3=d_5\theta^{3r}+d_6\theta^{2r}+(d_7-d_{10})\theta^r,\\ \mathcal{D}_4=d_1\theta^{3r}+d_2\theta^{2r}+(d_3-d_8)\theta^r+d_4-d_9.\end{cases} \tag{31}$$

In view of (30), we get:

$$[\mathcal{D}_1\cos\theta\rho-\mathcal{D}_2\sin\theta\rho]^2+[\mathcal{D}_3\cos\theta\rho+\mathcal{D}_4\cos\theta\rho]^2=\varrho_6^2, \tag{32}$$

which leads to:

$$(\mathcal{D}_1^2+\mathcal{D}_3^2)\cos^2\theta\rho+(\mathcal{D}_2^2+\mathcal{D}_4^2)\sin^2\theta\rho+2(\mathcal{D}_3\mathcal{D}_4-\mathcal{D}_1\mathcal{D}_2)\cos\theta\rho\sin\theta\rho=\varrho_6^2. \tag{33}$$

By (33), we have:

$$(\mathcal{D}_1^2+\mathcal{D}_3^2)\cos^2\theta\rho+(\mathcal{D}_2^2+\mathcal{D}_4^2)\sin^2\theta\rho-\varrho_6^2=-2(\mathcal{D}_3\mathcal{D}_4-\mathcal{D}_1\mathcal{D}_2)\cos\theta\rho\sin\theta\rho, \tag{34}$$

which leads to

$$[(\mathcal{D}_1^2+\mathcal{D}_3^2)\cos^2\theta\rho+(\mathcal{D}_2^2+\mathcal{D}_4^2)\sin^2\theta\rho-\varrho_6^2]^2=4(\mathcal{D}_3\mathcal{D}_4-\mathcal{D}_1\mathcal{D}_2)^2\cos^2\theta\rho\sin^2\theta\rho. \tag{35}$$

It follows from (35) that:

$$[(\mathcal{D}_1^2+\mathcal{D}_3^2)\cos^2\theta\rho+(\mathcal{D}_2^2+\mathcal{D}_4^2)\sin^2\theta\rho-\varrho_6^2]^2=4(\mathcal{D}_3\mathcal{D}_4-\mathcal{D}_1\mathcal{D}_2)^2\cos^2\theta\rho\sin^2\theta\rho. \tag{36}$$

Namely,

$$[(\mathcal{D}_1^2+\mathcal{D}_3^2)\cos^2\theta\rho+(\mathcal{D}_2^2+\mathcal{D}_4^2)\sin^2\theta\rho-\varrho_6^2]^2=4(\mathcal{D}_3\mathcal{D}_4-\mathcal{D}_1\mathcal{D}_2)^2\cos^2\theta\rho\sin^2\theta\rho. \tag{37}$$

Then

$$\mathcal{B}_1\cos^4\theta\rho+\mathcal{B}_2\cos^2\theta\rho+\mathcal{B}_3=0, \tag{38}$$

where

$$\begin{cases}\mathcal{B}_1=(\mathcal{D}_1^2+\mathcal{D}_3^2-\mathcal{D}_2^2-\mathcal{D}_4^2)^2+4(\mathcal{D}_3\mathcal{D}_4-\mathcal{D}_1\mathcal{D}_2)^2,\\ \mathcal{B}_2=2(\mathcal{D}_1^2+\mathcal{D}_3^2)\cos^2\theta\rho+(\mathcal{D}_2^2+\mathcal{D}_4^2)(\mathcal{D}_2^2+\mathcal{D}_4^2-\varrho_6^2),\\ \mathcal{B}_3=(\mathcal{D}_2^2+\mathcal{D}_4^2-\varrho_6^2)^2.\end{cases} \tag{39}$$

Assume that:

$$(\mathcal{Q}_4)\ \mathcal{B}_2<0,\mathcal{B}_2^2\geq 4\mathcal{B}_1\mathcal{B}_3$$

holds, then by (38), we get

$$\cos^2\theta\rho=\frac{-\mathcal{B}_2+\sqrt{\mathcal{B}_2^2-4\mathcal{B}_1\mathcal{B}_3}}{2\mathcal{B}_1}, \tag{40}$$

or

$$\cos^2\theta\rho=\frac{-\mathcal{B}_2-\sqrt{\mathcal{B}_2^2-4\mathcal{B}_1\mathcal{B}_3}}{2\mathcal{B}_1}. \tag{41}$$

According to (40) and (41), we get:

$$\cos\theta\rho = \pm\left(\frac{-\mathcal{B}_2 + \sqrt{\mathcal{B}_2^2 - 4\mathcal{B}_1\mathcal{B}_3}}{2\mathcal{B}_1}\right)^{\frac{1}{2}}, \tag{42}$$

or

$$\cos\theta\rho = \pm\left(\frac{-\mathcal{B}_2 - \sqrt{\mathcal{B}_2^2 - 4\mathcal{B}_1\mathcal{B}_3}}{2\mathcal{B}_1}\right)^{\frac{1}{2}}. \tag{43}$$

Suppose that (38) has twelve real roots $\theta_j (j = 1, 2, \cdots, 6)$. By (42) and (43), we have:

$$\rho_{j1}^l = \frac{1}{\rho_j}\left\{\arccos\left[\left(\frac{-\mathcal{B}_2 + \sqrt{\mathcal{B}_2^2 - 4\mathcal{B}_1\mathcal{B}_3}}{2\mathcal{B}_1}\right)^{\frac{1}{2}}\right] + 2l\pi\right\}, \tag{44}$$

$$\rho_{j2}^l = \frac{1}{\rho_j}\left\{\arccos\left[-\left(\frac{-\mathcal{B}_2 + \sqrt{\mathcal{B}_2^2 - 4\mathcal{B}_1\mathcal{B}_3}}{2\mathcal{B}_1}\right)^{\frac{1}{2}}\right] + 2l\pi\right\}, \tag{45}$$

$$\rho_{j3}^l = \frac{1}{\rho_j}\left\{\arccos\left[\left(\frac{-\mathcal{B}_2 - \sqrt{\mathcal{B}_2^2 - 4\mathcal{B}_1\mathcal{B}_3}}{2\mathcal{B}_1}\right)^{\frac{1}{2}}\right] + 2l\pi\right\}, \tag{46}$$

$$\rho_{j4}^l = \frac{1}{\rho_j}\left\{\arccos\left[-\left(\frac{-\mathcal{B}_2 - \sqrt{\mathcal{B}_2^2 - 4\mathcal{B}_1\mathcal{B}_3}}{2\mathcal{B}_1}\right)^{\frac{1}{2}}\right] + 2l\pi\right\}, \tag{47}$$

where $l = 0, 1, 2, \cdots, j = 1, 2, \cdots, 12$. Let

$$\rho_0 = \min_{k=1,2,3,4}\{\rho_{jk}^0\}, \theta_0 = \theta|_{\rho=\rho_0}. \tag{48}$$

In the sequel, the hypothesis is given:

$(\mathcal{Q}_5)$ $\mathcal{S}_{11}\mathcal{S}_{21} + \mathcal{S}_{12}\mathcal{S}_{22} > 0$, where

$$\begin{cases} \mathcal{S}_{11} = 3r\theta_0^{3r-1}\cos\dfrac{(3r-1)\pi}{2} + 2r\varrho_1\theta_0^{2r-1}\cos\dfrac{(2r-1)\pi}{2} + r\varrho_2\theta_0^{r-1}\cos\dfrac{(r-1)\pi}{2} \\ \quad + r\varrho_4\theta_0^{r-1}\cos\dfrac{(r-1)\pi}{2}\cos 2\theta_0\rho_0 + r\varrho_4\theta_0^{r-1}\sin\dfrac{(r-1)\pi}{2}\sin 2\theta_0\rho_0, \\ \mathcal{S}_{12} = 3r\theta_0^{3r-1}\sin\dfrac{(3r-1)\pi}{2} + 2r\varrho_1\theta_0^{2r-1}\sin\dfrac{(2r-1)\pi}{2} + r\varrho_2\theta_0^{r-1}\sin\dfrac{(r-1)\pi}{2} \\ \quad - r\varrho_4\theta_0^{r-1}\cos\dfrac{(r-1)\pi}{2}\sin 2\theta_0\rho_0 + r\varrho_4\theta_0^{r-1}\sin\dfrac{(r-1)\pi}{2}\cos 2\theta_0\rho_0, \\ \mathcal{S}_{21} = -\rho_0\theta_0^{r+1}\sin\dfrac{r\pi}{2}\cos 2\theta_0\rho_0 + \theta_0\left(\varrho_4\theta_0^r\cos\dfrac{r\pi}{2} + \varrho_5\right)\sin 2\theta_0\rho_0 \\ \quad + 3\rho_0\theta_0\sin 3\theta_0\rho_0, \\ \mathcal{S}_{22} = \rho_0\theta_0^{r+1}\sin\dfrac{r\pi}{2}\sin 2\theta_0\rho_0 + \theta_0\left(\varrho_4\theta_0^r\cos\dfrac{r\pi}{2} + \varrho_5\right)\cos 2\theta_0\rho_0 \\ \quad + 3\rho_0\theta_0\cos 3\theta_0\rho_0. \end{cases} \tag{49}$$

Lemma 5. *Let $s(\rho) = \iota_1(\rho) + i\iota_2(\rho)$ be the root of Equation (22) at $\rho = \rho_0$ such that $\iota_1(\rho_0) = 0, \iota_2(\rho_0) = \varpi_0$, then $Re\left[\frac{ds}{d\rho}\right]\Big|_{\rho=\rho_0,\theta=\theta_0} > 0$.*

Proof. It follows from Equation (22) that:

$$\begin{aligned}&\left(3rs^{3r-1}+2r\varrho_1 s^{2r-1}+r\varrho_2 s^{r-1}\right)\tfrac{ds}{d\rho}+r\varrho_4 s^{r-1}e^{-2s\rho}\tfrac{ds}{d\rho}\\&-2e^{-2s\rho}\left(\tfrac{ds}{d\rho}\rho+s\right)(\varrho_4 s^r+\varrho_5)-3\varrho_6 e^{-3s\rho}\left(\tfrac{ds}{d\rho}\rho+s\right)=0.\end{aligned}\tag{50}$$

Then

$$\left[\frac{ds}{d\rho}\right]^{-1}=\frac{\mathcal{S}_1(s)}{\mathcal{S}_2(s)}-\frac{\rho}{s},\tag{51}$$

where

$$\begin{cases}\mathcal{S}_1(s)=3rs^{3r-1}+2r\varrho_1 s^{2r-1}+r\varrho_2 s^{r-1}+r\varrho_4 s^{r-1}e^{-2s\rho},\\ \mathcal{S}_2(s)=2se^{-2s\rho}(\varrho_4 s^r+\varrho_5)+3s\varrho_6 e^{-3s\rho}.\end{cases}\tag{52}$$

Then

$$\mathrm{Re}\left\{\frac{ds}{d\rho}\right\}\Big|_{\rho=\rho_0,\theta=\theta_0}=\mathrm{Re}\left\{\frac{\mathcal{S}_1(s)}{\mathcal{S}_2(s)}\right\}\Big|_{\rho=\rho_0,\theta=\theta_0}=\frac{\mathcal{S}_{11}\mathcal{S}_{21}+\mathcal{S}_{12}\mathcal{S}_{22}}{\mathcal{S}_{21}^2+\mathcal{S}_{22}^2}.\tag{53}$$

By $(\mathcal{Q}_5)$, one gets:

$$\mathrm{Re}\left\{\left[\frac{ds}{d\rho}\right]^{-1}\right\}\Big|_{\rho=\rho_0,\theta=\theta_0}>0,\tag{54}$$

which completes the proof. □

Now we give the following hypothesis:

$(\mathcal{Q}_6)$ The following inequalities hold:

$$\begin{cases}\mathcal{G}_1=\varrho_1>0,\\ \mathcal{G}_2=\det\begin{bmatrix}\varrho_1 & 1\\ \varrho_3+\varrho_5+\varrho_6 & \varrho_2+\varrho_4\end{bmatrix}>0,\\ \mathcal{G}_3=(\varrho_3+\varrho_5+\varrho_6)\mathcal{G}_2>0.\end{cases}\tag{55}$$

Lemma 6. *If $(\mathcal{Q}_6)$ is satisfied, then the zero equilibrium point of system (2) with $\rho=0$ is locally asymptotically stable.*

Proof. When $\rho=0$, then Equation (22) takes the form:

$$\lambda^3+\varrho_1\lambda^2+(\varrho_2+\varrho_4)\lambda+\varrho_3+\varrho_5+\varrho_6=0.\tag{56}$$

In view of $(\mathcal{Q}_6)$, one knows that all roots λ_j of (54) satisfy $|\arg(\lambda_j)|>\frac{r\pi}{2}(j=1,2,3)$. Thus Lemma 6 is true. The proof ends. □

Based on the analysis above, we can easily derive the following conclusion.

Theorem 3. *Assume that $(\mathcal{Q}_1)$–$(\mathcal{Q}_6)$ are fulfilled, then the zero equilibrium point $W_0(0,0,0)$ of system (2) is locally asymptotically stable if $0\le\rho<\rho_0$ and a Hopf bifurcation arises near $W_0(0,0,0)$ if $\rho=\rho_0$.*

6. Bifurcation Control

In this segment, we will explore the control issue of the Hopf bifurcation of system (2) via mixed controller including state feedback and parameter perturbation. Following the idea of Yuan et al. [50], we establish the following controlled system:

$$\begin{cases}\frac{d^r w_1(t)}{dt^r}=\varepsilon_1[-\nu w_1(t)+h(w_1(t))+h_{12}(w_2(t-\rho))+h_{13}(w_3(t-\rho))]+\varepsilon_2 w_1(t),\\ \frac{d^r w_2(t)}{dt^r}=\varepsilon_1[-\nu w_2(t)+h(w_2(t))+h_{21}(w_1(t-\rho))+h_{23}(w_3(t-\rho))]+\varepsilon_2 w_2(t),\\ \frac{d^r w_3(t)}{dt^r}=\varepsilon_1[-\nu w_3(t)+h(w_3(t))+h_{31}(w_1(t-\rho))+h_{32}(w_2(t-\rho))]+\varepsilon_2 w_3(t),\end{cases}\tag{57}$$

where ε_1 and ε_2 are feedback gain parameters and real constants. It is easy to see that system (57) has a unique equilibrium point $W_0(0,0,0)$. The linear system of model (57) at $W_0(0,0,0)$ is:

$$\begin{cases} \dfrac{d^r w_1(t)}{dt^r} = -\beta w_1(t) + b_{12} w_2(t-\rho) + b_{13} w_3(t-\rho), \\ \dfrac{d^r w_2(t)}{dt^r} = -\beta w_2(t) + b_{21} w_1(t-\rho) + b_{23} w_3(t-\rho), \\ \dfrac{d^r w_3(t)}{dt^r} = -\beta w_3(t) + b_{31} w_1(t-\rho) + b_{32} w_2(t-\rho), \end{cases} \tag{58}$$

where $\beta = \varepsilon_1(\nu - h^{'}(0)), b_{12} = \varepsilon_1 h^{12}(0), b_{13} = \varepsilon_1 h^{13}(0), b_{21} = \varepsilon_1 h^{21}(0), b_{23} = \varepsilon_1 h^{23}(0), b_{31} = \varepsilon_1 h^{31}(0)$,
$b_{32} = \varepsilon_1 h^{32}(0)$. The corresponding characteristic equation for (58) is:

$$\det \begin{bmatrix} s^r + \beta & -b_{12}e^{-s\rho} & -b_{13}e^{-s\rho} \\ -b_{21}e^{-s\rho} & s^r + \beta & -b_{23}e^{-s\rho} \\ -b_{31}e^{-s\rho} & -b_{32}e^{-s\rho} & s^r + \beta \end{bmatrix} = 0. \tag{59}$$

It follows from (58) that

$$\mathcal{H}_1(s) + \mathcal{H}_2(s)e^{-2s\rho} + \mathcal{H}_3(s)e^{-3s\rho} = 0, \tag{60}$$

where

$$\begin{cases} \mathcal{H}_1(s) = s^{3r} + \delta_1 s^{2r} + \delta_2 s^r + \delta_3, \\ \mathcal{H}_2(s) = \delta_4 s^r + \delta_5, \\ \mathcal{H}_3(s) = \delta_6, \end{cases} \tag{61}$$

where

$$\begin{cases} \delta_1 = 3\beta, \\ \delta_2 = 3\beta^2, \\ \delta_3 = \beta^3, \\ \delta_4 = -(b_{12}b_{21} + b_{13}b_{31} + b_{23}b_{32}), \\ \delta_5 = -\alpha(b_{12}b_{21} + b_{13}b_{31} + b_{23}b_{32}), \\ \delta_6 = -(b_{12}b_{23}b_{31} + b_{13}b_{21}b_{32}). \end{cases} \tag{62}$$

We rewrite (60) as:

$$\mathcal{H}_1(s)e^{s\rho} + \mathcal{H}_2(s)e^{-s\rho} + \delta_6 e^{-2s\rho} = 0. \tag{63}$$

Let $s = i\vartheta = \vartheta(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$ be the root of Equation (63) and let $\mathcal{H}_{lR}(s)$ and $\mathcal{H}_{lI}(s)$ denote the real part and the imaginary part of $\mathcal{H}_l(s)(l = 1, 2)$. Then one has

$$\begin{cases} \mathcal{H}_{lR}(s) = \vartheta^{3r}\cos\dfrac{3r\pi}{2} + \delta_1\vartheta^{2r}\cos r\pi + \delta_2\vartheta^r\cos\dfrac{r\pi}{2} + \delta_3, \\ \mathcal{H}_{lI}(s) = \vartheta^{3r}\sin\dfrac{3r\pi}{2} + \delta_1\vartheta^{2r}\sin r\pi + \delta_2\vartheta^r\sin\dfrac{r\pi}{2}, \\ \mathcal{H}_{2R}(s) = \delta_4\vartheta^r\cos\dfrac{r\pi}{2} + \delta_5, \\ \mathcal{H}_{2I}(s) = \delta_4\vartheta^r\sin\dfrac{r\pi}{2}. \end{cases} \tag{64}$$

Let

$$\begin{cases} e_1 = \cos\dfrac{3r\pi}{2}, \\ e_2 = \delta_1 \cos r\pi, \\ e_3 = \delta_2 \cos\dfrac{r\pi}{2}, \\ e_4 = \delta_3, \\ e_5 = \sin\dfrac{3r\pi}{2}, \\ e_6 = \delta_1 \sin r\pi, \\ e_7 = \delta_2 \sin\dfrac{r\pi}{2}, \\ e_8 = \delta_4 \cos\dfrac{r\pi}{2}, \\ e_9 = \delta_5, \\ e_{10} = \delta_4 \sin\dfrac{r\pi}{2}, \end{cases} \tag{65}$$

then (64) becomes

$$\begin{cases} \mathcal{H}_{lR}(s) = e_1\vartheta^{3r} + e_2\vartheta^{2r} + e_3\vartheta^r + e_4, \\ \mathcal{H}_{lI}(s) = e_5\vartheta^{3r} + e_6\vartheta^{2r} + e_7\vartheta^r, \\ \mathcal{H}_{2R}(s) = e_8\vartheta^r + e_9, \\ \mathcal{H}_{2I}(s) = e_{10}\vartheta^r. \end{cases} \tag{66}$$

By virtue of (63) and (66), we have:

$$\begin{aligned} &[(e_1\vartheta^{3r} + e_2\vartheta^{2r} + e_3\vartheta^r + e_4) + i(e_5\vartheta^{3r} + e_6\vartheta^{2r} + e_7\vartheta^r)](\cos\vartheta\rho + i\sin\vartheta\rho) \\ &+[(e_8\vartheta^r + e_9) + ie_{10}\vartheta^r](\cos\vartheta\rho - i\sin\vartheta\rho) + \delta_6(\cos 2\vartheta\rho - i\sin 2\vartheta\rho) = 0. \end{aligned} \tag{67}$$

It follows from (67) that

$$\begin{cases} \mathcal{I}_1\cos\vartheta\rho - \mathcal{I}_2\sin\vartheta\rho = -\delta_6\cos 2\vartheta\rho, \\ \mathcal{I}_3\cos\vartheta\rho + \mathcal{I}_4\cos\vartheta\rho = \delta_6\sin 2\vartheta\rho, \end{cases} \tag{68}$$

where

$$\begin{cases} \mathcal{I}_1 = e_1\vartheta^{3r} + e_2\vartheta^{2r} + (e_3 + e_8)\vartheta^r + e_4 + e_9, \\ \mathcal{I}_2 = e_5\vartheta^{3r} + e_6\vartheta^{2r} + (e_7 + e_{10})\vartheta^r, \\ \mathcal{I}_3 = e_5\vartheta^{3r} + e_6\vartheta^{2r} + (e_7 - e_{10})\vartheta^r, \\ \mathcal{I}_4 = e_1\vartheta^{3r} + e_2\vartheta^{2r} + (e_3 - e_8)\vartheta^r + e_4 - e_9. \end{cases} \tag{69}$$

In view of (69), we get:

$$[\mathcal{I}_1\cos\vartheta\rho - \mathcal{I}_2\sin\vartheta\rho]^2 + [\mathcal{I}_3\cos\vartheta\rho + \mathcal{I}_4\cos\vartheta\rho]^2 = \delta_6^2, \tag{70}$$

which leads to:

$$(\mathcal{I}_1^2 + \mathcal{I}_3^2)\cos^2\vartheta\rho + (\mathcal{I}_2^2 + \mathcal{I}_4^2)\sin^2\vartheta\rho + 2(\mathcal{I}_3\mathcal{I}_4 - \mathcal{I}_1\mathcal{I}_2)\cos\vartheta\rho\sin\vartheta\rho = \delta_6^2. \tag{71}$$

By (71), we have:

$$(\mathcal{I}_1^2 + \mathcal{I}_3^2)\cos^2\vartheta\rho + (\mathcal{I}_2^2 + \mathcal{I}_4^2)\sin^2\vartheta\rho - \delta_6^2 = -2(\mathcal{I}_3\mathcal{I}_4 - \mathcal{I}_1\mathcal{I}_2)\cos\vartheta\rho\sin\vartheta\rho, \tag{72}$$

which leads to:

$$[(\mathcal{I}_1^2 + \mathcal{I}_3^2)\cos^2\vartheta\rho + (\mathcal{I}_2^2 + \mathcal{I}_4^2)\sin^2\vartheta\rho - \delta_6^2]^2 = 4(\mathcal{I}_3\mathcal{I}_4 - \mathcal{I}_1\mathcal{I}_2)^2\cos^2\vartheta\rho\sin^2\vartheta\rho. \tag{73}$$

It follows from (73) that:

$$[(\mathcal{I}_1^2 + \mathcal{I}_3^2)\cos^2\vartheta\rho + (\mathcal{I}_2^2 + \mathcal{I}_4^2)(1 - \cos^2\vartheta\rho) - \delta_6^2]^2 = 4(\mathcal{I}_3\mathcal{I}_4 - \mathcal{I}_1\mathcal{I}_2)^2\cos^2\vartheta\rho(1 - \cos^2\vartheta\rho). \tag{74}$$

Namely,

$$[(\mathcal{I}_1^2+\mathcal{I}_3^2-\mathcal{I}_2^2-\mathcal{I}_4^2)\cos^2\vartheta\rho+\mathcal{I}_2^2+\mathcal{I}_4^2-\delta_6^2]^2=4(\mathcal{I}_3\mathcal{I}_4-\mathcal{I}_1\mathcal{I}_2)^2\cos^2\vartheta\rho(1-\cos^2\vartheta\rho). \tag{75}$$

Then,

$$\mathcal{J}_1\cos^4\vartheta\rho+\mathcal{J}_2\cos^2\vartheta\rho+\mathcal{J}_3=0, \tag{76}$$

where

$$\begin{cases} \mathcal{J}_1=(\mathcal{I}_1^2+\mathcal{I}_3^2-\mathcal{I}_2^2-\mathcal{I}_4^2)^2+4(\mathcal{I}_3\mathcal{I}_4-\mathcal{I}_1\mathcal{I}_2)^2, \\ \mathcal{J}_2=2(\mathcal{I}_1^2+\mathcal{I}_3^2)\cos^2\vartheta\rho+(\mathcal{I}_2^2+\mathcal{I}_4^2)(\mathcal{I}_2^2+\mathcal{I}_4^2-\delta_6^2), \\ \mathcal{J}_3=(\mathcal{I}_2^2+\mathcal{I}_4^2-\delta_6^2)^2. \end{cases} \tag{77}$$

Assume that

$$(\mathcal{Q}_7)\ \mathcal{J}_2<0, \mathcal{J}_2^2\geq 4\mathcal{J}_1\mathcal{J}_3$$

holds, then by (76), we get:

$$\cos^2\vartheta\rho=\frac{-\mathcal{J}_2+\sqrt{\mathcal{J}_2^2-4\mathcal{J}_1\mathcal{J}_3}}{2\mathcal{J}_1}, \tag{78}$$

or

$$\cos^2\vartheta\rho=\frac{-\mathcal{J}_2-\sqrt{\mathcal{J}_2^2-4\mathcal{J}_1\mathcal{J}_3}}{2\mathcal{J}_1}. \tag{79}$$

According to (78) and (79), we can get:

$$\cos\vartheta\rho=\pm\left(\frac{-\mathcal{J}_2+\sqrt{\mathcal{J}_2^2-4\mathcal{J}_1\mathcal{J}_3}}{2\mathcal{J}_1}\right)^{\frac{1}{2}}, \tag{80}$$

or

$$\cos\vartheta\rho=\pm\left(\frac{-\mathcal{J}_2-\sqrt{\mathcal{J}_2^2-4\mathcal{J}_1\mathcal{J}_3}}{2\mathcal{J}_1}\right)^{\frac{1}{2}}. \tag{81}$$

Suppose that (76) has twelve real roots $\vartheta_j(j=1,2,\cdots,6)$. By (80) and (81), we have:

$$\rho_{i1}^l=\frac{1}{\rho_i}\left\{\arccos\left[\left(\frac{-\mathcal{J}_2+\sqrt{\mathcal{J}_2^2-4\mathcal{J}_1\mathcal{J}_3}}{2\mathcal{J}_1}\right)^{\frac{1}{2}}\right]+2l\pi\right\}, \tag{82}$$

$$\rho_{i2}^l=\frac{1}{\rho_i}\left\{\arccos\left[-\left(\frac{-\mathcal{J}_2+\sqrt{\mathcal{J}_2^2-4\mathcal{J}_1\mathcal{J}_3}}{2\mathcal{J}_1}\right)^{\frac{1}{2}}\right]+2l\pi\right\}, \tag{83}$$

$$\rho_{i3}^l=\frac{1}{\rho_i}\left\{\arccos\left[\left(\frac{-\mathcal{J}_2-\sqrt{\mathcal{J}_2^2-4\mathcal{J}_1\mathcal{J}_3}}{2\mathcal{J}_1}\right)^{\frac{1}{2}}\right]+2l\pi\right\}, \tag{84}$$

$$\rho_{i4}^l=\frac{1}{\rho_i}\left\{\arccos\left[-\left(\frac{-\mathcal{J}_2-\sqrt{\mathcal{J}_2^2-4\mathcal{J}_1\mathcal{J}_3}}{2\mathcal{J}_1}\right)^{\frac{1}{2}}\right]+2l\pi\right\}, \tag{85}$$

where $l=0,1,2,\cdots,i=1,2,\cdots,6$. Let

$$\rho_\star=\min_{l=1,2,3,4}\{\rho_{il}^0\}, \vartheta_0=\vartheta|_{\rho=\rho_\star}. \tag{86}$$

In the sequel, the hypothesis is given:

$(\mathcal{Q}_8)$ $\mathcal{N}_{11}\mathcal{N}_{21}+\mathcal{N}_{12}\mathcal{N}_{22}>0$, where

$$\begin{cases} \mathcal{N}_{11}=3r\vartheta_0^{3r-1}\cos\dfrac{(3r-1)\pi}{2}+2r\delta_1\vartheta_0^{2r-1}\cos\dfrac{(2r-1)\pi}{2}+r\delta_2\vartheta_0^{r-1}\cos\dfrac{(r-1)\pi}{2}\\ \quad +r\delta_4\vartheta_0^{r-1}\cos\dfrac{(r-1)\pi}{2}\cos 2\vartheta_0\rho_\star+r\delta_4\vartheta_0^{r-1}\sin\dfrac{(r-1)\pi}{2}\sin 2\vartheta_0\rho_\star,\\ \mathcal{N}_{12}=3r\vartheta_0^{3r-1}\sin\dfrac{(3r-1)\pi}{2}+2r\delta_1\vartheta_0^{2r-1}\sin\dfrac{(2r-1)\pi}{2}+r\delta_2\vartheta_0^{r-1}\sin\dfrac{(r-1)\pi}{2}\\ \quad -r\delta_4\vartheta_0^{r-1}\cos\dfrac{(r-1)\pi}{2}\sin 2\vartheta_0\rho_\star+r\delta_4\vartheta_0^{r-1}\sin\dfrac{(r-1)\pi}{2}\cos 2\vartheta_0\rho_\star,\\ \mathcal{N}_{21}=-\rho_\star\vartheta_0^{r+1}\sin\dfrac{r\pi}{2}\cos 2\vartheta_0\rho_\star+\vartheta_0\Big(\delta_4\vartheta_0^r\cos\dfrac{r\pi}{2}+\delta_5\Big)\sin 2\delta_0\rho_\star\\ \quad +3\rho_\star\vartheta_0\sin 3\vartheta_0\rho_\star,\\ \mathcal{N}_{22}=\rho_\star\vartheta_0^{r+1}\sin\dfrac{r\pi}{2}\sin 2\vartheta_0\rho_\star+\vartheta_0\Big(\delta_4\vartheta_0^r\cos\dfrac{r\pi}{2}+\delta_5\Big)\cos 2\vartheta_0\rho_\star\\ \quad +3\rho_\star\vartheta_0\cos 3\vartheta_0\rho_\star. \end{cases} \tag{87}$$

Lemma 7. *Let $s(\rho)=\varphi_1(\rho)+i\varphi_2(\rho)$ be the root of Equation (60) at $\rho=\rho_\star$ such that $\varphi_1(\rho_\star)=0, \varphi_2(\rho_\star)=\vartheta_0$, then $Re\Big[\frac{ds}{d\rho}\Big]\Big|_{\rho=\rho_\star,\vartheta=\vartheta_0}>0$.*

Proof. It follows from Equation (60) that

$$\begin{aligned} &\big(3rs^{3r-1}+2r\delta_1 s^{2r-1}+r\delta_2 s^{r-1}\big)\tfrac{ds}{d\rho}+r\delta_4 s^{r-1}e^{-2s\rho}\tfrac{ds}{d\rho}\\ &-2e^{-2s\rho}\Big(\tfrac{ds}{d\rho}\rho+s\Big)(\delta_4 s^r+\delta_5)-3\delta_6 e^{-3s\rho}\Big(\tfrac{ds}{d\rho}\rho+s\Big)=0. \end{aligned} \tag{88}$$

Then

$$\left[\frac{ds}{d\rho}\right]^{-1}=\frac{\mathcal{N}_1(s)}{\mathcal{N}_2(s)}-\frac{\rho}{s}, \tag{89}$$

where

$$\begin{cases} \mathcal{N}_1(s)=3rs^{3r-1}+2r\delta_1 s^{2r-1}+r\delta_2 s^{r-1}+r\delta_4 s^{r-1}e^{-2s\rho},\\ \mathcal{N}_2(s)=2se^{-2s\rho}(\delta_4 s^r+\delta_5)+3s\delta_6 e^{-3s\rho}. \end{cases} \tag{90}$$

Then,

$$\mathrm{Re}\left\{\frac{ds}{d\rho}\right\}\Big|_{\rho=\rho_\star,\vartheta=\vartheta_0}=\mathrm{Re}\left\{\frac{\mathcal{N}_1(s)}{\mathcal{N}_2(s)}\right\}\Big|_{\rho=\rho_\star,\vartheta=\vartheta_0}=\frac{\mathcal{N}_{11}\mathcal{N}_{21}+\mathcal{N}_{12}\mathcal{N}_{22}}{\mathcal{N}_{21}^2+\mathcal{N}_{22}^2}. \tag{91}$$

By $(\mathcal{Q}_8)$, one gets

$$\mathrm{Re}\left\{\left[\frac{ds}{d\rho}\right]^{-1}\right\}\Big|_{\rho=\rho_\star,\vartheta=\vartheta_0}>0, \tag{92}$$

which completes the proof. □

Now we give the following hypothesis:

$(\mathcal{Q}_9)$ The following inequalities hold:

$$\begin{cases} \mathcal{F}_1=\delta_1>0,\\ \mathcal{F}_2=\det\begin{bmatrix}\delta_1 & 1\\ \delta_3+\delta_5+\delta_6 & \delta_2+\delta_4\end{bmatrix}>0,\\ \mathcal{F}_3=(\delta_3+\delta_5+\delta_6)\mathcal{F}_2>0. \end{cases} \tag{93}$$

Lemma 8. *If $(\mathcal{Q}_9)$ is satisfied, then the zero equilibrium point of system (57) with $\rho=0$ is locally asymptotically stable.*

Proof. When $\rho = 0$, then Equation (60) takes the form:

$$\lambda^3 + \delta_1\lambda^2 + (\delta_2 + \delta_4)\lambda + \delta_3 + \delta_5 + \delta_6 = 0. \tag{94}$$

In view of $(\mathcal{Q}_9)$, one knows that all roots λ_j of (94) satisfy $|\arg(\lambda_j)| > \frac{r\pi}{2} (j = 1, 2, 3)$. Thus Lemma 8 is true. The proof ends. □

Based on the discussion above, we can easily derive the following conclusion.

Theorem 4. *Assume that $(\mathcal{Q}_1), (\mathcal{Q}_2), (\mathcal{Q}_3), (\mathcal{Q}_7), (\mathcal{Q}_8), (\mathcal{Q}_9)$ are fulfilled, then the zero equilibrium point $W_0(0, 0, 0)$ of system (57) is locally asymptotically stable if $0 \leq \rho < \rho_\star$ and a Hopf bifurcation arises near $W_0(0, 0, 0)$ if $\rho = \rho_\star$.*

Remark 2. *In 2006, Yan [41] investigated the Hopf bifurcation of integer-order delayed tri-neuron neural networks. In this present research, on the basis of Yan [41], we build a class of new fractional-order delayed tri-neuron neural networks. The existence and uniqueness, boundedness of the solution, stability and the onset of Hopf bifurcation, Hopf bifurcation control of fractional-order delayed tri-neuron neural networks is discussed in detail. The research approach of Yan [41] can be be applied to the fractional-order delayed tri-neuron neural networks to establish the key results of this article. Based on this standpoint, we think that our work replenishes the work of Yan [41].*

7. Numerical Simulations

Example 1. *Give the fractional order system as follows:*

$$\begin{cases} \dfrac{d^{0.95}w_1(t)}{dt^r} = -2w_1(t) + \tanh(w_1(t)) + \tanh(w_2(t-\rho)) + \tanh(w_3(t-\rho)), \\ \dfrac{d^{0.95}w_2(t)}{dt^r} = -2w_2(t) + \tanh(w_2(t)) + \tanh(w_1(t-\rho)) + \tanh(w_3(t-\rho)), \\ \dfrac{d^{0.95}w_3(t)}{dt^r} = -2w_3(t) + \tanh(w_3(t)) + \tanh(w_1(t-\rho)) + \tanh(w_2(t-\rho)). \end{cases} \tag{95}$$

Obviously, system (95) owns a unique equilibrium point $W_0(0, 0, 0)$. By virtue of Matlab software, we derive $\theta_0 = 0.9223$ and $\rho_0 \approx 0.6$. Furthermore, we can verify that all the hypotheses $(\mathcal{Q}_1)$–$(\mathcal{Q}_6)$ of Theorem 3 are fulfilled. So the zero equilibrium point $W_0(0, 0, 0)$ of system (95) is locally asymptotically stable if $\rho \in [0, 0.6)$ and system (95) will generates Hopf bifurcation phenomenon near $W_0(0, 0, 0)$ when ρ crosses $\rho_0 \approx 0.6$. Now we select $\rho = 0.5$ is less than $\rho_0 \approx 0.6$, then we get the simulation results that are displayed in Figure 1, which shows that the three variables $w_1(t), w_2(t), w_3(t)$ gradually tend to zero when time t increases. Then we select $\rho = 0.7$ is bigger than $\rho_0 \approx 0.6$, then we get the simulation results that are displayed in Figure 2, which shows that the three variables $w_1(t), w_2(t), w_3(t)$ keep a periodic oscillation (namely, Hopf bifurcation) near $W_0(0, 0, 0)$ when time t increases. The bifurcation figures are given in Figures 3–5. Figure 3 shows the relation between ρ and w_1, Figure 4 shows the relation between ρ and w_2 and Figure 5 shows the relation between ρ and w_3. From Figures 3–5, we can see that the bifurcation value $\rho_0 \approx 0.6$. In addition, the relation of time delay ρ_0 and the amplitude θ_0 is presented in Table 1.

Example 2. *Give the fractional order system as follows:*

$$\begin{cases} \dfrac{d^{0.95}w_1(t)}{dt^r} = 0.6[-2w_1(t) + \tanh(w_1(t)) + h_{12}(w_2(t-\rho)) + \tanh(w_3(t-\rho))] + 0.4w_1(t), \\ \dfrac{d^{0.95}w_2(t)}{dt^r} = 0.6[-2w_2(t) + \tanh(w_2(t)) + h_{21}(w_1(t-\rho)) + \tanh(w_3(t-\rho))] + 0.4w_2(t), \\ \dfrac{d^{0.95}w_3(t)}{dt^r} = 0.6[-2w_3(t) + \tanh(w_3(t)) + h_{31}(w_1(t-\rho)) + \tanh(w_2(t-\rho))] + 0.4w_3(t). \end{cases} \tag{96}$$

Obviously, system (96) owns a unique equilibrium point $W_0(0, 0, 0)$. By virtue of Matlab software, we derive $\vartheta_0 = 1.9070$ and $\rho_\star \approx 0.3$. Furthermore, we can verify that all the hypotheses $(\mathcal{Q}_1), (\mathcal{Q}_2), (\mathcal{Q}_3), (\mathcal{Q}_7), (\mathcal{Q}_8), (\mathcal{Q}_9)$ of Theorem 4 are fulfilled. So the zero equilibrium point

$W_0(0,0,0)$ of system (96) is locally asymptotically stable if $\rho \in [0, 0.3)$ and system (96) will generate the Hopf bifurcation phenomenon near $W_0(0,0,0)$ when ρ crosses $\rho_\star \approx 0.3$. Now, we select $\rho = 0.24$ as less than $\rho_\star \approx 0.6$, then we get the simulation results that are displayed in Figure 6, which show that the three variables $w_1(t), w_2(t), w_3(t)$ gradually tend to zero when time t increases. Then we select $\rho = 0.35$ as bigger than $\rho_\star \approx 0.3$, then we get the simulation results that are displayed in Figure 7, which show that the three variables $w_1(t), w_2(t), w_3(t)$ keep a periodic oscillation (namely, Hopf bifurcation) near $W_0(0,0,0)$ when time t increases. The bifurcation figures are given in Figures 8–10. Figure 8 shows the relation between ρ and w_1, Figure 9 shows the relation between ρ and w_2 and Figure 10 shows the relation between ρ and w_3. From Figures 8–10, we can see that the bifurcation value $\rho_\star \approx 0.3$. In addition, the relation of time delay ρ_0 and the amplitude $\rho_\star$ are presented in Table 2.

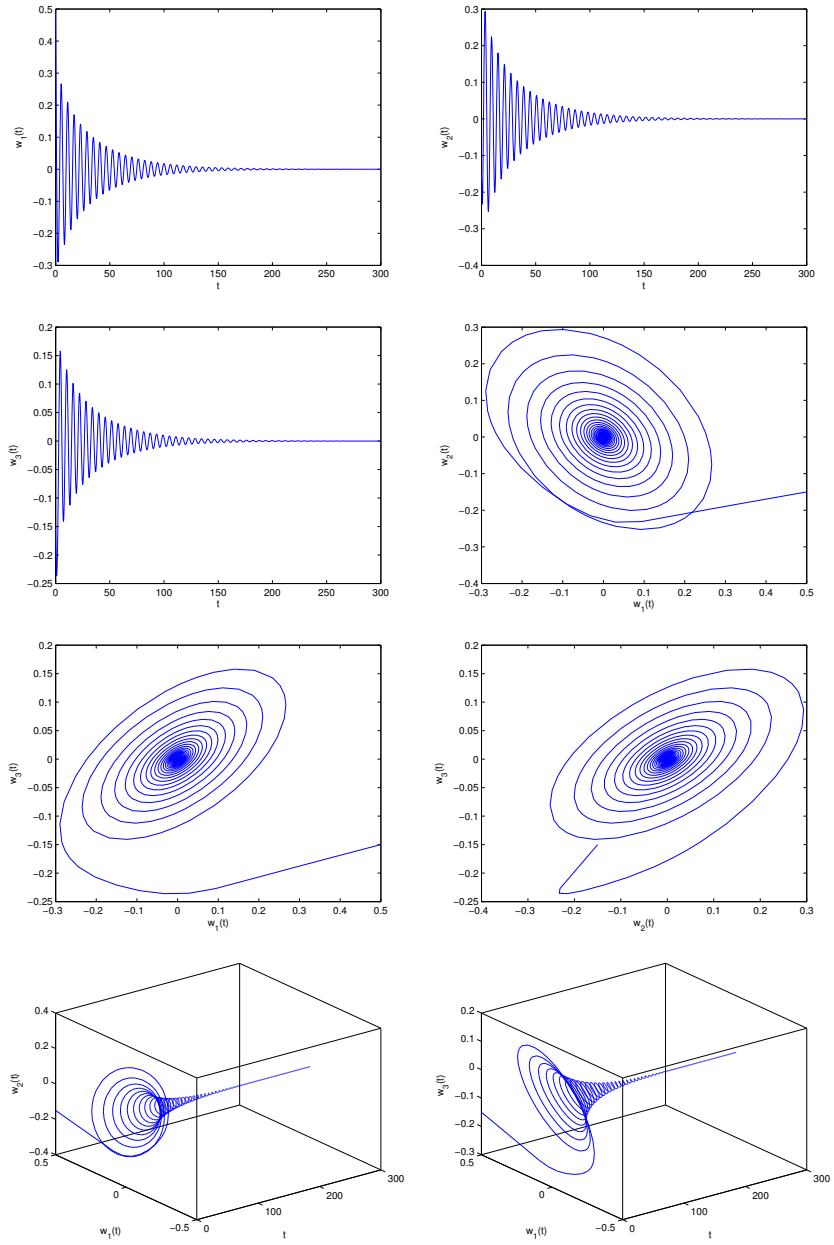

Figure 1. *Cont.*

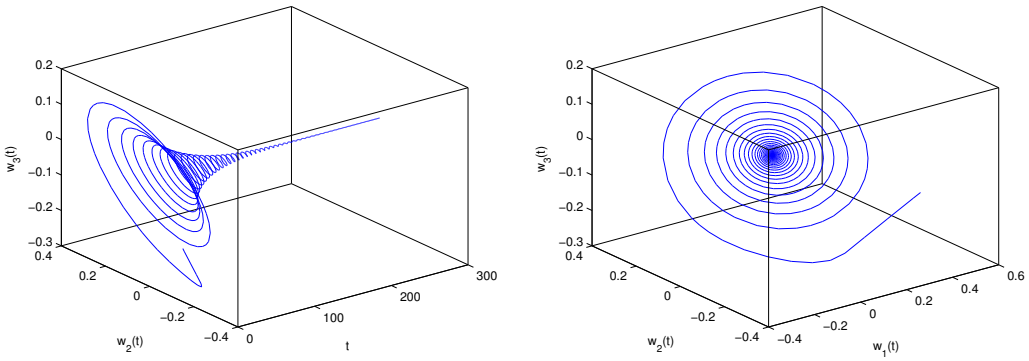

Figure 1. Computer simulation figures of system (95) with $\rho = 0.5 < \rho_0 \approx 0.6$. The three variables $w_1(t) \to 0, w_2(t) \to 0, w_3(t) \to 0$ when time $t \to \infty$.

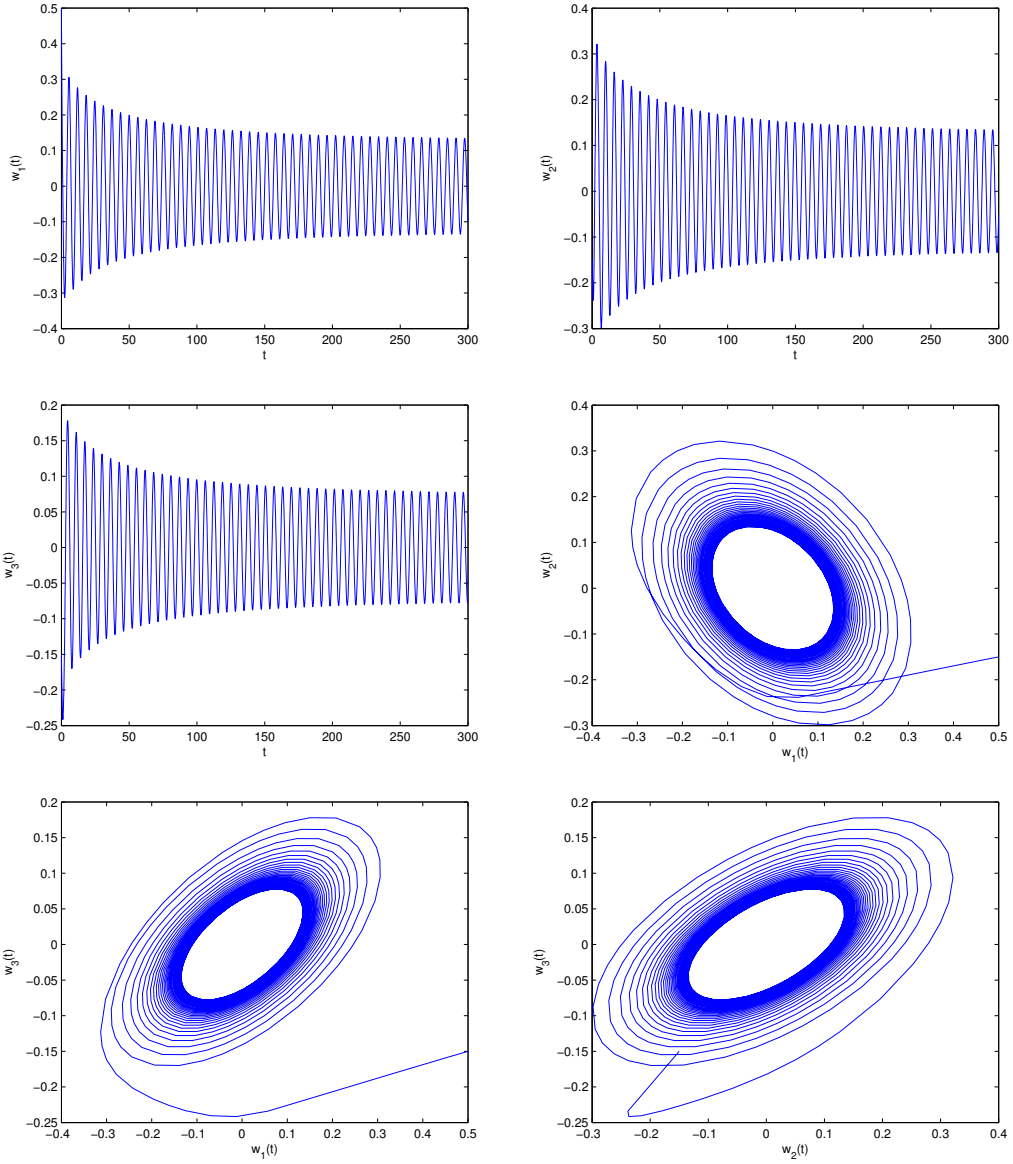

Figure 2. *Cont.*

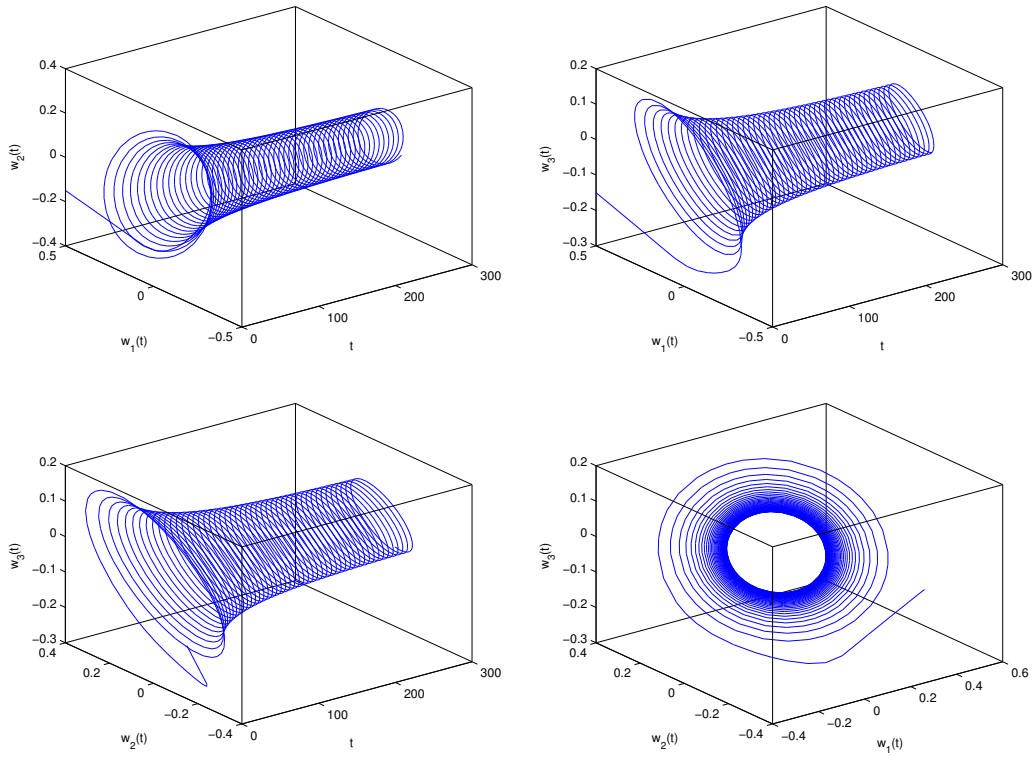

Figure 2. Computer simulation figures of system (95) with $\rho = 0.7 > \rho_0 \approx 0.6$. The Hopf bifurcation arises in the vicinity of $W_0(0, 0, 0)$ when time $t \to \infty$.

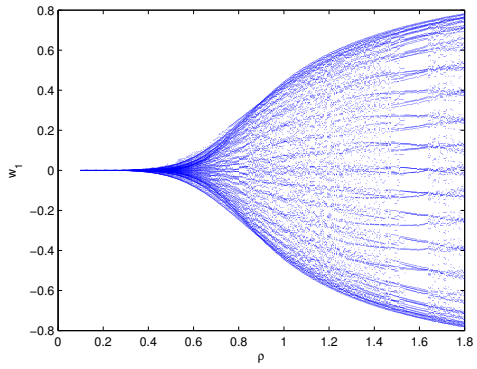

Figure 3. Bifurcation figure of system (95): ρ versus w_1. The bifurcation value $\rho_0 \approx 0.6$.

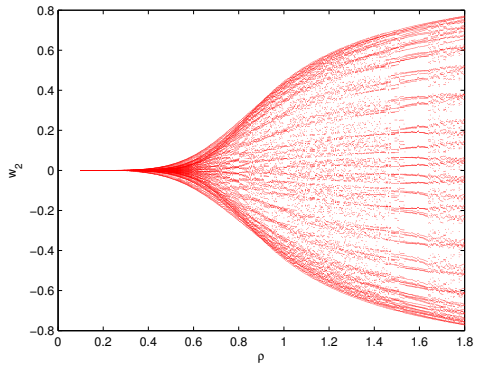

Figure 4. Bifurcation figure of system (95): ρ versus w_2. The bifurcation value $\rho_0 \approx 0.6$.

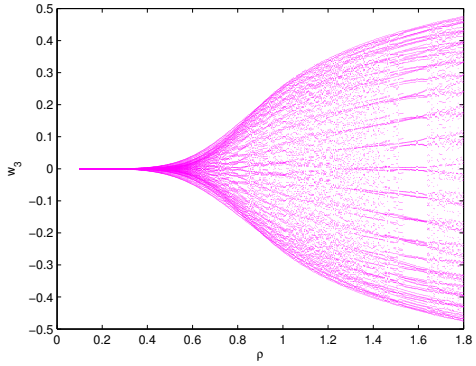

Figure 5. Bifurcation figure of system (95): ρ versus w_3. The bifurcation value $\rho_0 \approx 0.6$.

Table 1. The relation of θ_0 and ρ_0 of system (95).

θ_0	ρ_0
3.6893	0.15
2.3058	0.24
1.4563	0.38
1.2577	0.44
0.9380	0.59
0.9223	0.60
0.7095	0.78
0.6435	0.86
0.5705	0.97

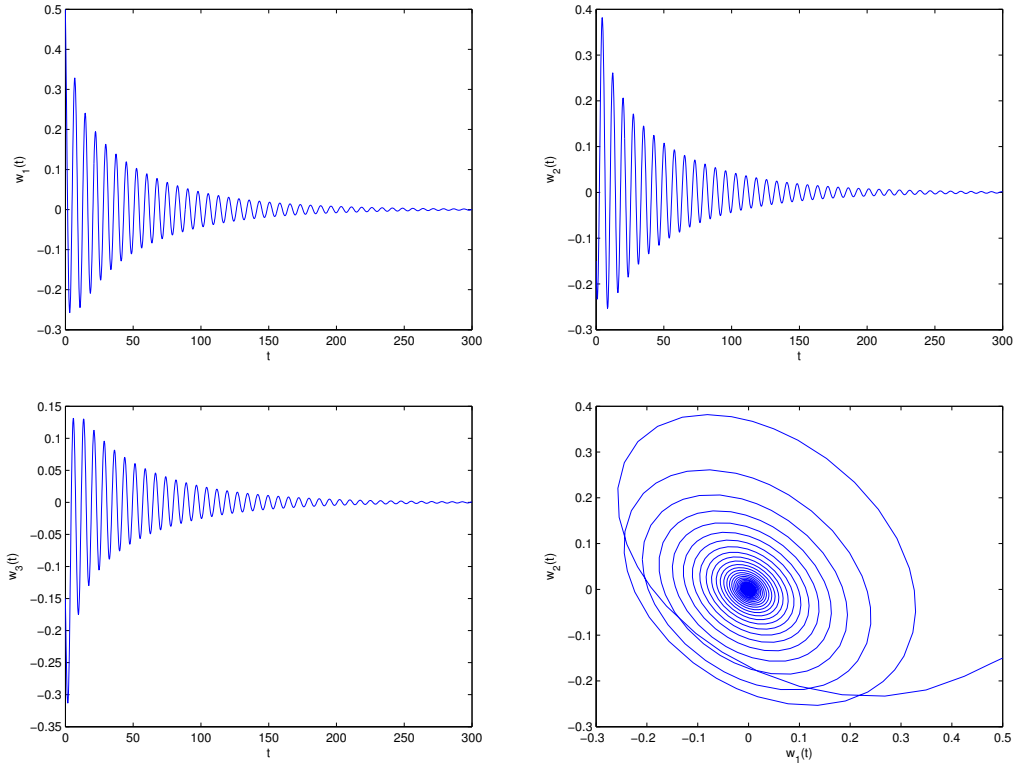

Figure 6. *Cont.*

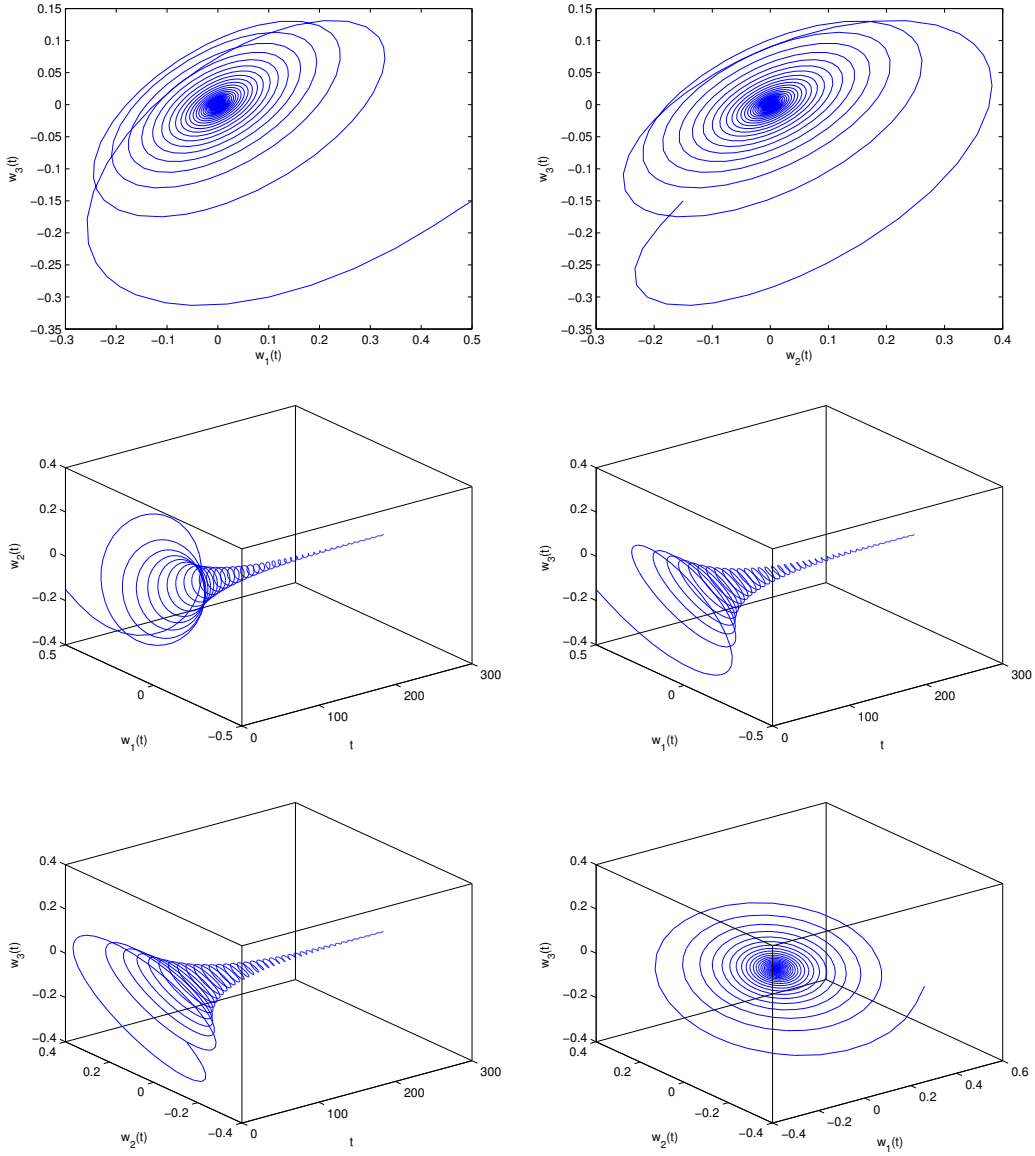

Figure 6. Computer simulation figures of system (96) with $\rho = 0.24 < \rho_\star \approx 0.3$. The three variables $w_1(t) \to 0, w_2(t) \to 0, w_3(t) \to 0$ when time $t \to \infty$.

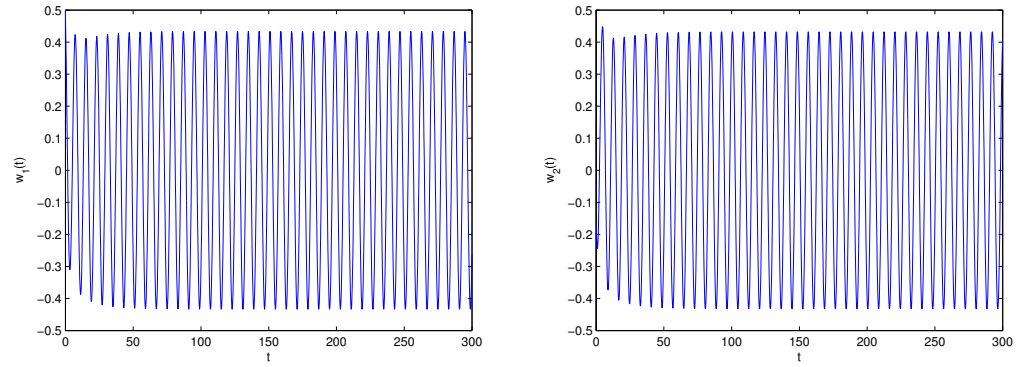

Figure 7. *Cont.*

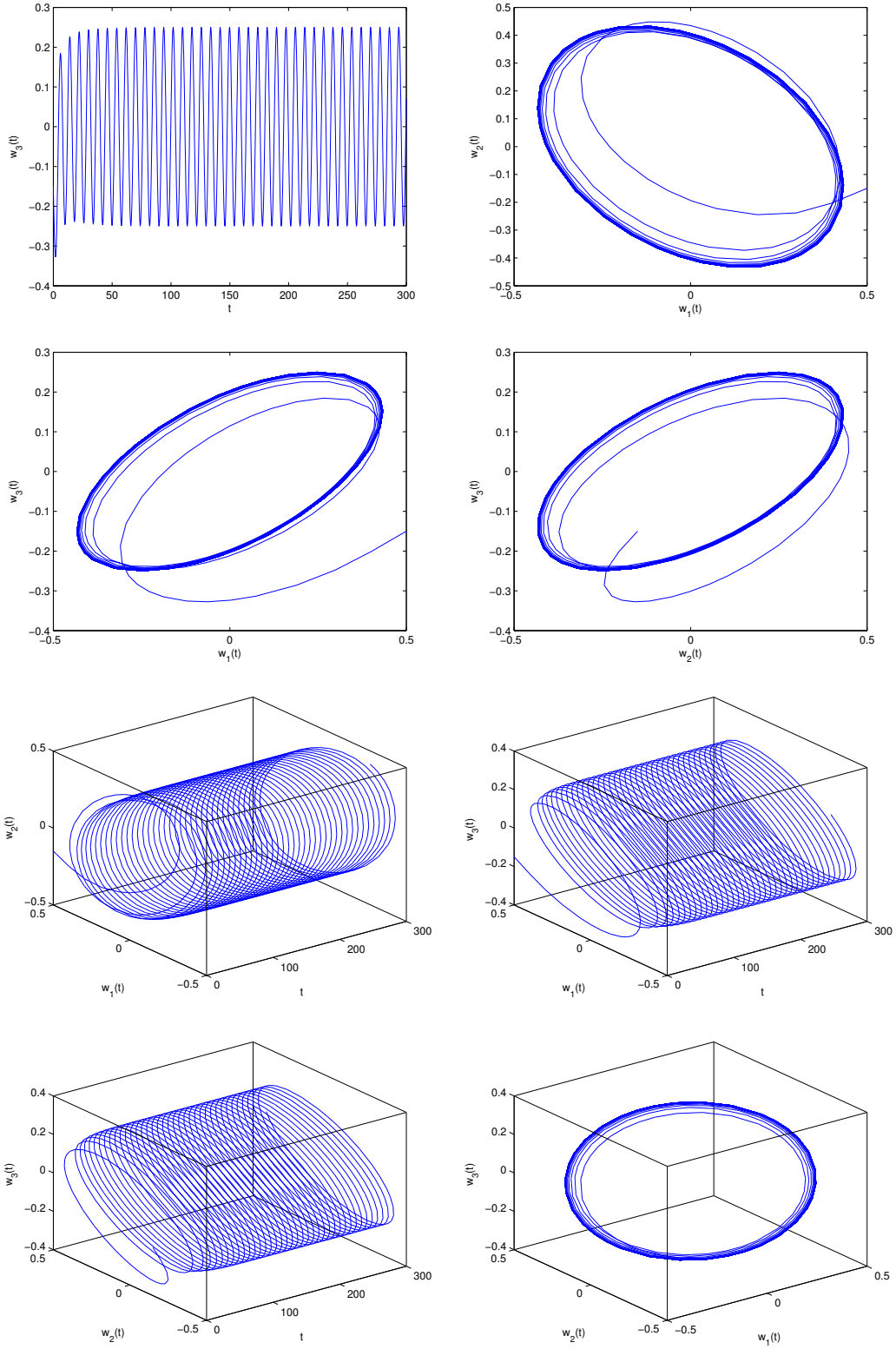

Figure 7. Computer simulation figures of system (96) with $\rho = 0.35 > \rho_\star \approx 0.3$. The Hopf bifurcation arises in the vicinity of $W_0(0, 0, 0)$ when time $t \to \infty$.

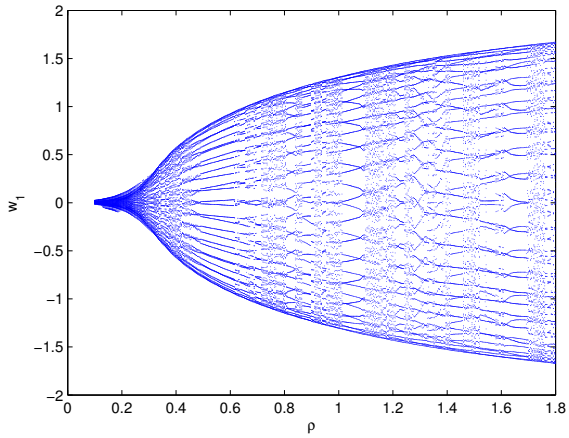

Figure 8. Bifurcation figure of system (96): ρ versus w_1. The bifurcation value $\rho_\star \approx 0.3$.

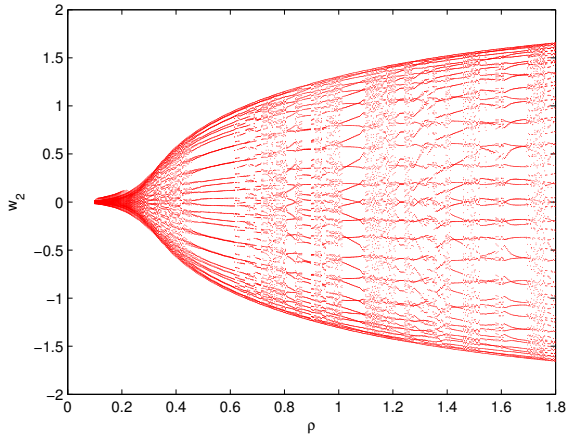

Figure 9. Bifurcation figure of system (96): ρ versus w_2. The bifurcation value $\rho_\star \approx 0.3$.

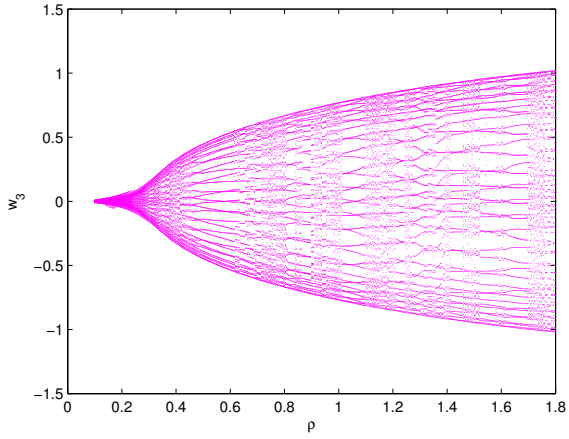

Figure 10. Bifurcation figure of system (96): ρ versus w_3. The bifurcation value $\rho_\star \approx 0.3$.

Table 2. The relation of ϑ_0 and $\rho_\star$ of system (96).

ϑ_0	$\rho_\star$
4.7675	0.12
2.4874	0.23
1.9070	0.30
1.3954	0.41
1.0216	0.56
0.9227	0.62
0.7837	0.73
0.6977	0.82
0.6086	0.94

Remark 3. *In system (95), the bifurcation value* $\rho_0 \approx 0.6$*, in system (96) (namely, the controlled system of system (95)), the bifurcation value* $\rho_0 \approx 0.3$*. we can clearly know that the stability region of system (95) is narrowed and the time of generation of Hopf bifurcation in advance.*

8. Conclusions

The investigation of neural networks has become a very important topic in mathematical and engineering areas. Revealing the intrinsic dynamic characteristics of neural networks has aroused the interest of various scholars. In this article, based on the earlier studies, we built a new fractional-order tri-neuron neural network incorporating delays. The existence and uniqueness, boundedness of the solution of the established fractional-order tri-neuron neural networks are investigated. The delay-independent condition to ensure the stability and the onset of the Hopf bifurcation of the involved fractional-order tri-neuron neural networks is derived. The bifurcation value is determined. By virtue of a mixed controller, which includes state feedback and parameter perturbation, the stability region and the time of onset of the Hopf bifurcation of the considered fractional-order tri-neuron neural networks are adjusted. The research shows that time delay is a crucial parameter which affects the stability region and the time of onset of the Hopf bifurcation of the studied fractional-order tri-neuron neural networks and its controlled neural networks. The derived results have some theoretical value in designing and controlling neural networks. Meanwhile, the exploration idea of this article can also be utilized to study a lot of fractional-order delayed neural networks and some related delayed dynamical systems. In addition, in many neural networks, there exist leakage delays. In this paper, we do not involve the Hopf bifurcation control issue of neural networks with leakage delays. We will focus on this topic in near future.

Author Contributions: Methodology, P.L. and C.X.; Software, C.X.; Formal analysis, P.L. and C.X.; Data curation, Y.S.; writing—original draft preparation; P.L. and C.X.; Writing—review and editing, P.L., J.Y. and C.X. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by National Natural Science Foundation of China (No. 62062018) and and Project of High-level Innovative Talents of Guizhou Province ([2016]5651).

Data Availability Statement: No data were used to support this study.

Acknowledgments: This work is supported by National Natural Science Foundation of China (No. 62062018) and and Project of High-level Innovative Talents of Guizhou Province ([2016]5651). The authors would like to thank the referees and the editor for helpful suggestions incorporated into this paper.

Conflicts of Interest: The authors declare that they have no competing interests.

References

1. Hopfield, J. Neurons with graded response have collective computational properties like those of two-state neurons. *Proc. Natl. Acad. Sci. USA* **1984**, *81*, 3088–3092. [CrossRef] [PubMed]
2. Xu, C.J.; Liao, M.X.; Li, P.L.; Liu, Z.X.; Yuan, S. New results on pseudo almost periodic solutions of quaternion-valued fuzzy cellular neural networks with delays. *Fuzzy Set. Syst.* **2021**, *411*, 25–47. [CrossRef]
3. Thoiyab, N.M.; Muruganantham, P.; Zhu, Q.X.; Gunasekaran, N. Novel results on global stability analysis for multiple time-delayed BAM neural networks under parameter uncertainties. *Chaos Solitons Fractals* **2021**, *152*, 111441. [CrossRef]
4. Cao, Y.; Ramajayam, S.; Sriraman, R.; Samidurai, R. Leakage delay on stabilization of finite-time complex-valued BAM neural network: Decomposition approach. *Neurocomputing* **2021**, *463*, 505–513. [CrossRef]
5. Xu, C.J.; Tang, X.H.; Liao, M.X. Frequency domain analysis for bifurcation in a simplified tri-neuron BAM network model with two delays. *Neural Netw.* **2010**, *23*, 872–880. [CrossRef]
6. Wei, J.J.; Velarde, M.G. Bifurcation analysis and existence of periodic solutions in a simple neural network with delays. *Chaos* **2004**, *14*, 940–953. [CrossRef]
7. Wang, S.Z.; Zhang, Z.Y.; Lin, C.; Chen, J. Fixed-time synchronization for complex-valued BAM neural networks with time-varying delays via pinning control and adaptive pinning control. *Chaos Solitons Fractals* **2021**, *153*, 111583. [CrossRef]
8. Gan, Y.T.; Liu, C.; Peng, H.; Liu, F.; Rao, H.X. Anti-synchronization for periodic BAM neural networks with Markov scheduling protocol. *Neurocomputing* **2020**, *417*, 585–592. [CrossRef]
9. Syed Ali, M.; Narayanan, G.; Shekher, V.; Alsaedi, A.; Ahmad, B. Global Mittag–Leffler stability analysis of impulsive fractional-order complex-valued BAM neural networks with time varying delays. *Commun. Nonlinear Sci. Numer. Simul.* **2020**, *83*, 105088. [CrossRef]
10. Chen, S.L.; Li, H.L.; Kao, Y.G.; Zhang, L.; Hu, C. Finite-time stabilization of fractional-order fuzzy quaternion-valued BAM neural networks via direct quaternion approach. *J. Franklin Inst.* **2021**, *358*, 7650–7673. [CrossRef]
11. Cong, E.Y.; Han, X.; Zhang, X. Global exponential stability analysis of discrete-time BAM neural networks with delays: A mathematical induction approach. *Neurocomputing* **2020**, *379*, 227–235. [CrossRef]
12. Duan, L.Y.; Li, J.M. Fixed-time synchronization of fuzzy neutral-type BAM memristive inertial neural networks with proportional delays. *Inf. Sci.* **2021**, *576*, 522–541. [CrossRef]
13. Yan, M.Y.; Jian, J.G.; Zheng, S. Passivity analysis for uncertain BAM inertial neural networks with time-varying delays. *Neurocomputing* **2021**, *435*, 114–125.
14. Ayachi, M. Measure-pseudo almost periodic dynamical behaviors for BAM neural networks with D operator and hybrid time-varying delays. *Neurocomputing* **2021**. [CrossRef]
15. Kumar, R.; Das, S. Exponential stability of inertial BAM neural network with time-varying impulses and mixed time-varying delays via matrix measure approach. *Commun. Nonlinear Sci. Numer. Simul.* **2020**, *81*, 105016. [CrossRef]
16. Syed Ali, M.; Yogambigai, J.; Saravanan, S.; Elakkia, S. Stochastic stability of neutral-type Markovian-jumping BAM neural networks with time varying delays. *J. Comput. Appl. Math.* **2019**, *349*, 142–156.
17. Bruce, W.J. Colloquium: Fractional calculus view of complexity: A tutorial. *Rev. Modern Phys.* **2014**, *86*, 1169–1186.
18. Shi, J.P.; He, K.; Fang, H. Chaos, Hopf bifurcation and control of a fractional-order delay financial system. *Math. Comput. Simul.* **2022**, *194*, 348–364. [CrossRef]
19. Zhou, B.; Song, Q.K. Stability and Hopf bifurcation analysis of a tri-neuron BAM neural network with distributed delay. *Neurocomputing* **2012**, *82*, 69–83. [CrossRef]
20. Pratap, A.; Raja, R.; Cao, J.; Rihan, F.A.; Seadawy, A.R. Quasi-pinning synchronization and stabilization of fractional order BAM neural networks with delays and discontinuous neuron activations. *Chaos Solitons Fractals* **2020**, *131*, 109491. [CrossRef]
21. Petráš, I. Control of fractional order Chua's system. *J. Elect. Eng.* **2002**, *53*, 219–222.
22. Zhang, F.H.; Huang, T.W.; Wu, Q.J.; Zeng, Z.G. Multistability of delayed fractional-order competitive neural networks. *Neural Netw.* **2021**, *140*, 325–335. [CrossRef] [PubMed]
23. Li, H.L.; Hu, C.; Zhang, L.; Jiang, H.J.; Cao, J.D. Complete and finite-time synchronization of fractional-order fuzzy neural networks via nonlinear feedback control. *Fuzzy Set. Syst.* **2021**. [CrossRef]
24. Padmaja, N.; Balasubramaniam, P. New delay and order-dependent passivity criteria for impulsive fractional-order neural networks with switching parameters and proportional delays. *Neurocomputing* **2021**, *454*, 113–123. [CrossRef]
25. Ke, L. Mittag–Leffler stability and asymptotic ω-periodicity of fractional-order inertial neural networks with time-delays. *Neurocomputing* **2021**, *465*, 53–62. [CrossRef]
26. Du, F.F.; Lu, J.G. Finite-time stability of fractional-order fuzzy cellular neural networks with time delays. *Fuzzy Set. Syst.* **2021**. [CrossRef]
27. Xu, C.J.; Zhang, W.; Liu, Z.X.; Yao, L.Y. Delay-induced periodic oscillation for fractional-order neural networks with mixed delays. *Neurocomputing* **2021**. [CrossRef]
28. Udhayakumar, K.; Rihan, F.A.; Rakkiyappan, R.; Cao, J.D. Fractional-order discontinuous systems with indefinite LKFs: An application to fractional-order neural networks with time delays. *Neural Netw.* **2022**, *145*, 319–330. [CrossRef]
29. Wang, X.H.; Wang, Z.; Xia, J.W. Stability and bifurcation control of a delayed fractional-order eco-epidemiological model with incommensurate orders. *J. Franklin Inst.* **2019**, *356*, 8278–8295. [CrossRef]
30. Li, C.P.; Dao, X.H.; Guo, P. Fractional derivatives in complex planes. *Nonlinear Anal. TMA* **2009**, *71*, 1857–1869. [CrossRef]

31. Guariglia, E. Riemann zeta fractional derivative-functional equation and link with primes. *Adv. Diff. Equat.* **2019**, *261*, 15. [CrossRef]
32. Ortigueira, M.D.; Coito, F. From differences to derivatives. *Frac. Calc. Appl. Anal.* **2004**, *7*, C459–C471.
33. Guariglia, E.; Silvestrov, S. A functional equation for the Riemann zeta fractional derivative. *AIP Conf. Proceed.* **2017**, *1798*, 020063.
34. Wu, G.C.; Lee, E.W.M. Fractional variational iteration method and its application. *Phys. Lett. A* **2010**, *374*, 2506–2509. [CrossRef]
35. Xu, C.J.; Zhang, W.; Aouiti, C.; Liu, Z.X.; Liao, M.X.; Li, P.L. Further investigation on bifurcation and their control of fractional-order BAM neural networks involving four neurons and multiple delays. *Math. Meth. Appl. Sci.* **2021**. [CrossRef]
36. Xu, C.J.; Liu, Z.X.; Aouiti, C.; Li, P.L.; Yao, L.Y.; Yan, J.L. New exploration on bifurcation for fractional-order quaternion-valued neural networks involving leakage delays. *Cogn. Neurodyn.* **2022**. 10.1007/s11571-021-09763-1 [CrossRef]
37. Xu, C.J.; Liao, M.X.; Li, P.L.; Yuan, S. Impact of leakage delay on bifurcation in fractional-order complex-valued neural networks. *Chaos Solitons Fractals* **2021**, *142*, 110535. [CrossRef]
38. Xu, C.J.; Liu, Z.X.; Yao, L.Y.; Aouiti, C. Further exploration on bifurcation of fractional-order six-neuron bi-directional associative memory neural networks with multi-delays. *Appl. Math. Comput.* **2021**, *410*, 126458. [CrossRef]
39. Huang, C.D.; Nie, X.B.; Zhao, X.; Song, Q.K.; Cao, J.D. Novel bifurcation results for a delayed fractional-order quaternion-valued neural network. *Neural Netw.* **2019**, *117*, 67–93. [CrossRef]
40. Xiao, M.; Zheng, W.X.; Lin, J.X.; Jiang, G.P.; Zhao, L.D.; Cao, J.D. Fractional-order PD control at Hopf bifurcations in delayed fractional-order small-world networks. *J. Franklin Inst.* **2017**, *354*, 7643–7667. [CrossRef]
41. Yan, Y.P. Hopf bifurcation and stability for a delayed tri-neuron neural networks. *J. Comput. Appl. Math.* **2006**, *196*, 579–595. [CrossRef]
42. Xu, C.J.; Zhang, W.; Liu, Z.X.; Li, P.L.; Yao, L.Y. Bifurcation study for fractional-order three-layer neural networks involving four time delays. *Cogn. Comput.* **2022**. [CrossRef]
43. Song, Y.L.; Han, M.A.; Wei, J.J. Stability and Hopf bifurcation on a simplified BAM neural network with delays. *Phys. D* **2005**, *200*, 185–204. [CrossRef]
44. Podlubny, I. *Fractional Differential Equations*; Academic Pres: New York, NY, USA, 1999.
45. Bandyopadhyay, B.; Kamal, S. *Stabliization and Control of Fractional Order Systems: A Sliding Mode Approach*; Springer: Heidelberg, Germany, 2015; Volume 317.
46. Li, Y.; Chen, Y.Q.; Podlubny, I. Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag–Leffler stability. *Comput. Math. Appl.* **2009**, *59*, 1810–1821. [CrossRef]
47. Li, H.L.; Zhang, L.; Hu, C.; Jiang, Y.L.; Teng, Z.D. Dynamical analysis of a fractional-order predator-prey model incorporating a prey refuge. *J. Appl. Math. Comput.* **2017**, *54*, 435–449. [CrossRef]
48. Matignon, D. Stability results for fractional differential equations with applications to control processing. *Comput. Eng. Syst. Appl.* **1996**, *2*, 963–968.
49. Deng, W.H.; Li, C.P.; Lü, J.H. Stability analysis of linear fractional differential system with multiple time delays. *Nonlinear Dyn.* **2007**, *48*, 409–416. [CrossRef]
50. Yuan, J.; Zhao, L.Z.; Huang, C.D.; Xiao, M. A novel hybrid control technique for bifurcation in an exponential RED algorithm. *Int. J. Circ. Theory Appl.* **2020**, *48*, 1476–1492. [CrossRef]