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Article

New Fractional Integral Inequalities for Convex Functions Pertaining to Caputo–Fabrizio Operator

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Abstract: In this article, a generalized midpoint-type Hermite–Hadamard inequality and Pachpatte-type inequality via a new fractional integral operator associated with the Caputo–Fabrizio derivative are presented. Furthermore, a new fractional identity for differentiable convex functions of first order is proved. Then, taking this identity into account as an auxiliary result and with the assistance of Hölder, power-mean, Young, and Jensen inequality, some new estimations of the Hermite-Hadamard H-H type inequality as refinements are presented. Applications to special means and trapezoidal quadrature formula are presented to verify the accuracy of the results. Finally, a brief conclusion and future scopes are discussed.

Keywords: Hermite-Hadamard inequality; Caputo-Fabrizio operator; Pachpatte type inequality; Hölder's inequality; Hölder-İşcan inequality

Citation: Sahoo, S.K.; Mohammed, P.O.; Kodamasingh, B.; Tariq, M.; Hamed, Y. S. New Fractional Integral Inequalities for Convex Functions Pertaining to Caputo–Fabrizio Operator. *Fractal Fract.* **2022**, *6*, 171. https://doi.org/10.3390/fractalfract6030171

Academic Editors: Ricardo Almeida and Ahmed I. Zayed

Received: 21 December 2021
Accepted: 11 February 2022
Published: 19 March 2022

1. Introduction

Recently, fractional calculus has been the focus of attention for researchers not only in the field of mathematics but also in fields such as physics [1], nanotechnology [2], medicine [3], bioengineering [4], economy [5,6], fluid mechanics [7], epidemiology [8], and control systems [9]. Fractional calculus is also used in the modeling of diseases [10,11] and solving optimal control problems [12]. In the last decades, fractional calculus has become very popular due to its behavior and wide range of applications in different fields of sciences. Researchers derive new fractional operators and utilize them to solve many real-world problems based on their basic properties. They also employ these new fractional operators to improve several well-known integral inequalities such as H-H inequality [13], Ostrowski inequality [14], Simpson inequality [15], Bullen-type inequality [16], Fejer-type inequality [17], Jensen–Mercer-type inequality [18], and Opial-type inequalities [19]. We suggest interested readers to see the articles [20–27] for a better understanding of developments of fractional integral inequalities.

Caputo and Fabrizio in [28] investigated a new fractional operator, having a nonsingular kernel in its fractional derivatives without the Gamma function. The most satisfying feature of this operator is that if we use Laplace transformation, then any real power can be turned into an integer order. Consequently, this property enables us to get solutions to several related problems. This Caputo–Fabrizio fractional operator is used to investigate many real-life problems such as the modeling of COVID-19 [29], modeling of Hepatitis-B epidemic [30], groundwater flow [31], etc.

In this article, we restrict ourselves to the use of Caputo–Fabrizio fractional operators for integral inequalities. We have established new versions of Hermite–Hadamard and Pachpatte-type inequalities using differintegral of $\frac{\vartheta_1+\vartheta_2}{2}$ type, which provides a new and different direction in the advancement of Caputo–Fabrizio fractional operator with the aid of inequalities.

The classical Hermite–Hadmard inequality is stated as follows (see [32]):

If $\mathcal{G} : \mathbb{X} \subseteq \mathcal{R} \to \mathcal{R}$ is convex in $\mathbb{X}$ for $\vartheta_1, \vartheta_2 \in \mathbb{X}$ and $\vartheta_1 < \vartheta_2$, then

$$\mathcal{G}\left(\frac{\vartheta_1+\vartheta_2}{2}\right) \leq \frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2}\mathcal{G}(x)dx \leq \frac{\mathcal{G}(\vartheta_1)+\mathcal{G}(\vartheta_2)}{2}. \tag{1}$$

The H-H inequality plays an amazing and magnificent role in the literature. Several mathematicians have employed various convexities to improve this inequality. We suggest interested readers to go through the articles [13,33–40] for interesting and different versions of H-H inequality.

The main reason for writing this article is to combine the Caputo–Fabrizio fractional operator with inequalities such as H-H, the Pachpatte type, and the Dragomir Agarwal type for the convex function. The rest of the article is structured as follows. First, we review some fundamental concepts and notions about fractional calculus and integral inequalities. Section 3 deals with presenting inequalities of H-H type and the Pachpatte type employing the Caputo–Fabrizio fractional integral operator for convex functions. We devote Section 4 toward deriving a new integral equality of the Caputo–Fabrizio type. Then, considering this equality, some new estimations of H-H type-related inequalities are discussed. Applications of the results are investigated in Section 5. Finally, in the last Section 6, a brief conclusion is given.

2. Preliminaries

In this section, we recall some known concepts related to our main results.

Definition 1 (see Refs. [41,42])**.** *Let $\mathbb{X}$ be a convex subset of a real vector space $\mathcal{R}$ and let $\mathcal{G} : \mathbb{X} \to \mathcal{R}$ be a function. Then, a function $\mathcal{G}$ is said to be convex, if*

$$\mathcal{G}(\vartheta_1\varrho + (1-\varrho)\vartheta_2) \leq \varrho\mathcal{G}(\vartheta_1) + (1-\varrho)\mathcal{G}(\vartheta_2), \tag{2}$$

holds for all $\vartheta_1, \vartheta_2 \in \mathbb{X}$ and $\varrho \in [0,1]$.

To additionally encourage the conversation of this article, we present the definition of the Riemann–Liouville R-L fractional operator.

Definition 2 (see Ref. [13])**.** *Let $[\vartheta_1, \vartheta_1] \longrightarrow \mathcal{R}$. Then, R-L fractional integrals $\mathrm{I}^{\zeta}_{\vartheta_1^+}\mathcal{G}(z)$ and $\mathrm{I}^{\zeta}_{\vartheta_2^-}\mathcal{G}(z)$ of order $\zeta > 0$ are defined by*

$$\mathrm{I}^{\zeta}_{\vartheta_1^+}\mathcal{G}(z) = \frac{1}{\Gamma(\zeta)}\int_{\vartheta_1}^{z}(z-x)^{\zeta-1}\mathcal{G}(x)dx$$

and

$$\mathrm{I}^{\zeta}_{\vartheta_2^-}\mathcal{G}(z) = \frac{1}{\Gamma(\zeta)}\int_{z}^{\vartheta_2}(x-z)^{\zeta-1}\mathcal{G}(x)dx,$$

where $\Gamma(.)$ is the Gamma function.

In [13], Sarikaya et al. proved the following Hadamard-type inequalities for R-L fractional integrals as follows:

Theorem 1 (see Ref. [13]). *Let $\mathcal{G} : [\vartheta_1, \vartheta_1] \longrightarrow \mathbb{R}$ be a positive mapping with $0 \leq \vartheta_1 \leq \vartheta_2$, $\mathcal{G} \in \mathcal{L}[\vartheta_1, \vartheta_1]$ and $\mathrm{I}^{\zeta}_{\vartheta_1^+}\mathcal{G}(\vartheta_2)$ and $\mathrm{I}^{\zeta}_{\vartheta_2^-}\mathcal{G}(\vartheta_1)$ as fractional operators. If $\mathcal{G}$ is a convex function, then the following inequality for fractional integrals holds:*

$$\mathcal{G}\left(\frac{\vartheta_1+\vartheta_1}{2}\right) \leq \frac{\Gamma(\zeta+1)}{2(\vartheta_2-\vartheta_1)^{\zeta}}\left[\mathrm{I}^{\zeta}_{\vartheta_1^+}\mathcal{G}(\vartheta_2) + \mathrm{I}^{\zeta}_{\vartheta_2^-}\mathcal{G}(\vartheta_1)\right] \leq \frac{\mathcal{G}(\vartheta_1)+\mathcal{G}(\vartheta_2)}{2}.$$

In [43], Sarikaya and Yildirm proved the following mid-point type Hermite–Hadamard inequality for R-L fractional integrals as follows:

Theorem 2 (see Ref. [43]). *Let $\mathcal{G} : [\vartheta_1, \vartheta_1] \longrightarrow \mathbb{R}$ be a positive mapping with $0 \leq \vartheta_1 \leq \vartheta_2$, $\mathcal{G} \in \mathcal{L}[\vartheta_1, \vartheta_1]$, and $\mathrm{I}^{\zeta}_{(\frac{\vartheta_1+\vartheta_1}{2})^+}\mathcal{G}$ and $\mathrm{I}^{\zeta}_{(\frac{\vartheta_1+\vartheta_1}{2})^-}\mathcal{G}$ as fractional operators. If $\mathcal{G}$ is a convex function, then the following inequality for fractional integrals holds:*

$$\mathcal{G}\left(\frac{\vartheta_1+\vartheta_1}{2}\right) \leq \frac{2^{\zeta-1}\Gamma(\zeta+1)}{(\vartheta_2-\vartheta_1)^{\zeta}}\left[\mathrm{I}^{\zeta}_{(\frac{\vartheta_1+\vartheta_1}{2})^+}\mathcal{G}(\vartheta_2) + \mathrm{I}^{\zeta}_{(\frac{\vartheta_1+\vartheta_1}{2})^-}\mathcal{G}(\vartheta_1)\right] \leq \frac{\mathcal{G}(\vartheta_1)+\mathcal{G}(\vartheta_2)}{2}.$$

After these articles, mathematicians started applying different fractional operators to improve integral inequalities of H-H type; for example, see [36,44–46].

To facilitate further discussion on fractional calculus and inequalities, we present the definition of Caputo–Fabrizio fractional operators and some basic notions related to the theory of inequalities.

Definition 3 (see Refs. [28,46–48]). *Let $\mathcal{G} \in H'(\vartheta_1, \vartheta_2), \vartheta_1 < \vartheta_2, \zeta \in [0,1]$, then the notion of left and right Caputo–Fabrizio fractional integrals are defined by,*

$$^{\mathrm{CF}}_{\vartheta_1}\mathrm{I}^{\zeta}\mathcal{G}(\varrho) = \frac{1-\zeta}{\mathrm{B}(\zeta)}\mathcal{G}(\varrho) + \frac{\zeta}{\mathrm{B}(\zeta)}\int_{\vartheta_1}^{\varrho}\mathcal{G}(x)dx,$$

and

$$^{\mathrm{CF}}\mathrm{I}^{\zeta}_{\vartheta_2}\mathcal{G}(\varrho) = \frac{1-\zeta}{\mathrm{B}(\zeta)}\mathcal{G}(\varrho) + \frac{\zeta}{\mathrm{B}(\zeta)}\int_{\varrho}^{\vartheta_2}\mathcal{G}(x)dx.$$

where $\mathrm{B}(\zeta) > 0$ is a normalization function that satisfies $\mathrm{B}(0) = \mathrm{B}(1) = 1$.

Remark 1. *It is worth mentioning that the Caputo–Fabrizio fractional integral in Definition 3 is corresponding to the Caputo–Fabrizio fractional derivative as defined in [28,46–48]. One can see that the exponential kernel in the Caputo–Fabrizio fractional derivative will be convergent if the order ζ is between 0 and 1.*

Definition 4 (Hölder's inequality [49]). *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $\mathcal{F}$ and $\mathcal{G}$ are real functions defined on $[\vartheta_1, \vartheta_2]$ and if $|\mathcal{F}|^p$ and $|\mathcal{G}|^q$ are integrable on $[\vartheta_1, \vartheta_2]$, then the following inequality holds:*

$$\int_{\vartheta_1}^{\vartheta_2}|\mathcal{F}(x)\mathcal{G}(x)|dx \leq \left(\int_{\vartheta_1}^{\vartheta_2}|\mathcal{F}(x)|^p dx\right)^{\frac{1}{p}}\left(\int_{\vartheta_1}^{\vartheta_2}|\mathcal{G}(x)|^q dx\right)^{\frac{1}{q}}. \tag{3}$$

Definition 5 (Power-mean inequality [49]). *Let $q > 1$. If $\mathcal{F}$ and $\mathcal{G}$ are real functions defined on $[\vartheta_1, \vartheta_2]$ and if $|\mathcal{F}|$, $|\mathcal{F}||\mathcal{G}|^q$ are integrable on $[\vartheta_1, \vartheta_2]$, then the following inequality holds:*

$$\int_{\vartheta_1}^{\vartheta_2}|\mathcal{F}(x)\mathcal{G}(x)|dx \leq \left(\int_{\vartheta_1}^{\vartheta_2}|\mathcal{F}(x)|dx\right)^{1-\frac{1}{q}}\left(\int_{\vartheta_1}^{\vartheta_2}|\mathcal{F}(x)||\mathcal{G}(x)|^q dx\right)^{\frac{1}{q}}. \tag{4}$$

Definition 6 (Hölder-İşcan integral inequality [49]). *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $\mathcal{F}$ and $\mathcal{G}$ are real functions defined on $[\vartheta_1, \vartheta_2]$ and if $|\mathcal{F}|^p$ and $|\mathcal{G}|^q$ are integrable on $[\vartheta_1, \vartheta_2]$, then the following inequality holds:*

$$\begin{aligned}\int_{\vartheta_1}^{\vartheta_2} |\mathcal{F}(x)\mathcal{G}(x)|dx \leq & \frac{1}{\vartheta_2 - \vartheta_1}\left(\int_{\vartheta_1}^{\vartheta_2}(\vartheta_2 - x)|\mathcal{F}|^p dx\right)^{\frac{1}{p}}\left(\int_{\vartheta_1}^{\vartheta_2}(\vartheta_2 - x)|\mathcal{G}|^q dx\right)^{\frac{1}{q}} \\ &+ \frac{1}{\vartheta_2 - \vartheta_1}\left(\int_{\vartheta_1}^{\vartheta_2}(x - \vartheta_1)|\mathcal{F}|^p dx\right)^{\frac{1}{p}}\left(\int_{\vartheta_1}^{\vartheta_2}(x - \vartheta_1)|\mathcal{G}|^q dx\right)^{\frac{1}{q}}. \qquad (5)\end{aligned}$$

3. Integral Inequalities via Caputo–Fabrizio Fractional Integral Operator for Convex Functions

Theorem 3. *Let $\mathcal{G} : \mathrm{I} = [\vartheta_1, \vartheta_2] \to \mathcal{R}$ be a convex function on I such that $(\vartheta_1, \vartheta_2) \in \mathrm{I}$ and $\mathcal{G} \in \mathcal{L}[\vartheta_1, \vartheta_2]$. Then, for $\zeta \in [0, 1]$ and $\mathrm{B}(\zeta)$ as the normalizaton function, the following fractional inequality holds:*

$$\begin{aligned}&\mathcal{G}\left(\frac{\vartheta_1 + \vartheta_2}{2}\right) \\ &\quad \leq \frac{\mathrm{B}(\zeta)}{\zeta(\vartheta_2 - \vartheta_1)}\left[{}^{CF}_{\frac{\vartheta_1+\vartheta_2}{2}^-}I^{\zeta}\mathcal{G}(\vartheta_1) + {}^{CF}I^{\zeta}_{\frac{\vartheta_1+\vartheta_2}{2}^+}\mathcal{G}(\vartheta_2) - \frac{1-\zeta}{\mathrm{B}(\zeta)}[\mathcal{G}(\vartheta_1) + \mathcal{G}(\vartheta_2)]\right] \\ &\quad \leq \left[\frac{\mathcal{G}(\vartheta_1) + \mathcal{G}(\vartheta_2)}{2}\right]. \qquad (6)\end{aligned}$$

Proof. Since $\mathcal{G}$ is a convex function on $[\vartheta_1, \vartheta_2]$, we can write

$$\mathcal{G}\left(\frac{\vartheta_1 + \vartheta_2}{2}\right) \leq \frac{1}{\vartheta_2 - \vartheta_1}\int_{\vartheta_1}^{\vartheta_2}\mathcal{G}(x)dx = \frac{1}{\vartheta_2 - \vartheta_1}\left[\int_{\vartheta_1}^{\frac{\vartheta_1+\vartheta_2}{2}}\mathcal{G}(x)dx + \int_{\frac{\vartheta_1+\vartheta_2}{2}}^{\vartheta_2}\mathcal{G}(x)dx\right]. \qquad (7)$$

Multiplying both sides of the above Equation (7) with $\frac{\zeta(\vartheta_2 - \vartheta_1)}{\mathrm{B}(\zeta)}$ and then adding $\frac{1-\zeta}{\mathrm{B}(\zeta)}[\mathcal{G}(\vartheta_1) + \mathcal{G}(\vartheta_2)]$

$$\begin{aligned}&\frac{\zeta(\vartheta_2 - \vartheta_1)}{\mathrm{B}(\zeta)}\mathcal{G}\left(\frac{\vartheta_1 + \vartheta_2}{2}\right) + \frac{1-\zeta}{\mathrm{B}(\zeta)}[\mathcal{G}(\vartheta_1) + \mathcal{G}(\vartheta_2)] \\ &\leq \frac{1-\zeta}{\mathrm{B}(\zeta)}\mathcal{G}(\vartheta_1) + \frac{\zeta}{\mathrm{B}(\zeta)}\int_{\vartheta_1}^{\frac{\vartheta_1+\vartheta_2}{2}}\mathcal{G}(x)dx + \frac{1-\zeta}{\mathrm{B}(\zeta)}\mathcal{G}(\vartheta_2) + \frac{\zeta}{\mathrm{B}(\zeta)}\int_{\frac{\vartheta_1+\vartheta_2}{2}}^{\vartheta_2}\mathcal{G}(x)dx \\ &= \left[{}^{\mathrm{CF}}_{\frac{\vartheta_1+\vartheta_2}{2}^-}\mathrm{I}^{\zeta}\mathcal{G}(\vartheta_1) + {}^{\mathrm{CF}}\mathrm{I}^{\zeta}_{\frac{\vartheta_1+\vartheta_2}{2}^+}\mathcal{G}(\vartheta_2)\right].\end{aligned}$$

Reorganizing the above inequality gives,

$$\begin{aligned}&\mathcal{G}\left(\frac{\vartheta_1 + \vartheta_2}{2}\right) \\ &\quad \leq \frac{\mathrm{B}(\zeta)}{\zeta(\vartheta_2 - \vartheta_1)}\left[{}^{\mathrm{CF}}_{\frac{\vartheta_1+\vartheta_2}{2}^-}\mathrm{I}^{\zeta}\mathcal{G}(\vartheta_1) + {}^{\mathrm{CF}}\mathrm{I}^{\zeta}_{\frac{\vartheta_1+\vartheta_2}{2}^+}\mathcal{G}(\vartheta_2) - \frac{1-\zeta}{\mathrm{B}(\zeta)}[\mathcal{G}(\vartheta_1) + \mathcal{G}(\vartheta_2)]\right]. \qquad (8)\end{aligned}$$

This completes the proof of the first part. For the second inequality, we use

$$\frac{1}{\vartheta_2 - \vartheta_1}\int_{\vartheta_1}^{\vartheta_2}\mathcal{G}(x)dx \leq \frac{\mathcal{G}(\vartheta_1) + \mathcal{G}(\vartheta_2)}{2}.$$

Using the same procedure as above, we have

$$\left[{}^{CF}_{\frac{\vartheta_1+\vartheta_2}{2}-}\mathrm{I}^{\zeta}\mathcal{G}(\vartheta_1) + {}^{CF}\mathrm{I}^{\zeta}_{\frac{\vartheta_1+\vartheta_2}{2}+}\mathcal{G}(\vartheta_2)\right] \leq \left[\frac{\mathcal{G}(\vartheta_1)+\mathcal{G}(\vartheta_2)}{2}\right]\frac{\zeta(\vartheta_2-\vartheta_1)}{\mathrm{B}(\zeta)} + \frac{1-\zeta}{\mathrm{B}(\zeta)}[\mathcal{G}(\vartheta_1)+\mathcal{G}(\vartheta_2)].$$

Reorganizing,

$$\frac{\mathrm{B}(\zeta)}{\zeta(\vartheta_2-\vartheta_1)}\left[{}^{CF}_{\frac{\vartheta_1+\vartheta_2}{2}-}\mathrm{I}^{\zeta}\mathcal{G}(\vartheta_1) + {}^{CF}\mathrm{I}^{\zeta}_{\frac{\vartheta_1+\vartheta_2}{2}+}\mathcal{G}(\vartheta_2) - \frac{1-\zeta}{\mathrm{B}(\zeta)}[\mathcal{G}(\vartheta_1)+\mathcal{G}(\vartheta_2)]\right] \leq \left[\frac{\mathcal{G}(\vartheta_1)+\mathcal{G}(\vartheta_2)}{2}\right]. \quad (9)$$

Reorganizing the above numbered Equations (8) and (9), we have the desired inequality

$$\mathcal{G}\left(\frac{\vartheta_1+\vartheta_2}{2}\right) \leq \frac{\mathrm{B}(\zeta)}{\zeta(\vartheta_2-\vartheta_1)}\left[{}^{CF}_{\frac{\vartheta_1+\vartheta_2}{2}-}\mathrm{I}^{\zeta}\mathcal{G}(\vartheta_1) + {}^{CF}\mathrm{I}^{\zeta}_{\frac{\vartheta_1+\vartheta_2}{2}+}\mathcal{G}(\vartheta_2) - \frac{1-\zeta}{\mathrm{B}(\zeta)}[\mathcal{G}(\vartheta_1)+\mathcal{G}(\vartheta_2)]\right] \leq \left[\frac{\mathcal{G}(\vartheta_1)+\mathcal{G}(\vartheta_2)}{2}\right].$$

This led us to the desired result. □

Pachpatte-Type Inequality: Product of Two Convex Functions

In this section, we present an inequality taking product of two convex functions in the frame of Caputo–Fabrizio fractional operators.

Theorem 4. *Let $\mathcal{G},\mathcal{H} : \mathrm{I} = [\vartheta_1,\vartheta_2] \to \mathcal{R}$ be differentiable functions on* I *such that $(\vartheta_1,\vartheta_2) \in \mathrm{I}$ with $\vartheta_1 < \vartheta_2$ and $\mathcal{GH} \in \mathcal{L}[\vartheta_1,\vartheta_2]$. If $\mathcal{G},\mathcal{H}$ be convex functions, then for $\zeta \in [0,1]$, the following fractional inequality holds:*

$$\frac{2\mathrm{B}(\zeta)}{\zeta(\vartheta_2-\vartheta_1)}\left\{{}^{CF}_{\frac{\vartheta_1+\vartheta_2}{2}-}\mathrm{I}^{\zeta}\mathcal{GH}(\vartheta_1) + {}^{CF}\mathrm{I}^{\zeta}_{\frac{\vartheta_1+\vartheta_2}{2}+}\mathcal{GH}(\vartheta_2) - \frac{1-\zeta}{\mathrm{B}(\zeta)}M(\vartheta_1,\vartheta_2)\right\} \leq \left[\frac{2}{3}M(\vartheta_1,\vartheta_2) + \frac{1}{3}N(\vartheta_1,\vartheta_2)\right], \quad (10)$$

where $\mathrm{B}(\zeta)$ is the normalizaton function, $M(\vartheta_1,\vartheta_2) = [\mathcal{G}(\vartheta_1)\mathcal{H}(\vartheta_1)+\mathcal{G}(\vartheta_2)\mathcal{H}(\vartheta_2)]$ and $N(\vartheta_1,\vartheta_2) = [\mathcal{G}(\vartheta_1)\mathcal{H}(\vartheta_2)+\mathcal{G}(\vartheta_2)\mathcal{H}(\vartheta_1)]$.

Proof. Since $\mathcal{G}$ and $\mathcal{H}$ are convex function on $[\vartheta_1,\vartheta_2]$, $\forall\, \varrho \in [0,1]$, $\vartheta_1,\vartheta_2 \in \mathrm{I}$

$$\mathcal{G}\left(\frac{\varrho}{2}\vartheta_1 + \frac{2-\varrho}{2}\vartheta_2\right) \leq \frac{\varrho}{2}\mathcal{G}(\vartheta_1) + \frac{2-\varrho}{2}\mathcal{G}(\vartheta_2),$$

and

$$\mathcal{H}\left(\frac{\varrho}{2}\vartheta_1 + \frac{2-\varrho}{2}\vartheta_2\right) \leq \frac{\varrho}{2}\mathcal{H}(\vartheta_1) + \frac{2-\varrho}{2}\mathcal{H}(\vartheta_2).$$

Multiplying both the above inequalities side by side, we have

$$\begin{aligned}\mathcal{G}\left(\frac{\varrho}{2}\vartheta_1+\frac{2-\varrho}{2}\vartheta_2\right)\mathcal{H}\left(\frac{\varrho}{2}\vartheta_1+\frac{2-\varrho}{2}\vartheta_2\right)&\\ \leq \frac{\varrho^2}{4}\mathcal{G}(\vartheta_1)\mathcal{H}(\vartheta_1)+\frac{(2-\varrho)^2}{4}\mathcal{G}(\vartheta_2)\mathcal{H}(\vartheta_2)&\\ +\frac{\varrho(2-\varrho)}{4}[\mathcal{G}(\vartheta_1)\mathcal{H}(\vartheta_2)+\mathcal{G}(\vartheta_2)\mathcal{H}(\vartheta_1)].&\end{aligned} \quad (11)$$

In addition,

$$\mathcal{G}\left(\frac{\varrho}{2}\vartheta_2+\frac{2-\varrho}{2}\vartheta_1\right)\leq\frac{\varrho}{2}\mathcal{G}(\vartheta_2)+\frac{2-\varrho}{2}\mathcal{G}(\vartheta_1),$$

and

$$\mathcal{H}\left(\frac{\varrho}{2}\vartheta_2+\frac{2-\varrho}{2}\vartheta_1\right)\leq\frac{\varrho}{2}\mathcal{H}(\vartheta_2)+\frac{2-\varrho}{2}\mathcal{H}(\vartheta_1).$$

Similarly

$$\begin{aligned}\mathcal{G}\left(\frac{\varrho}{2}\vartheta_2+\frac{2-\varrho}{2}\vartheta_1\right)\mathcal{H}\left(\frac{\varrho}{2}\vartheta_2+\frac{2-\varrho}{2}\vartheta_1\right)&\\ \leq \frac{\varrho^2}{4}\mathcal{G}(\vartheta_2)\mathcal{H}(\vartheta_2)+\frac{(2-\varrho)^2}{4}\mathcal{G}(\vartheta_1)\mathcal{H}(\vartheta_1)&\\ +\frac{\varrho(2-\varrho)}{4}[\mathcal{G}(\vartheta_1)\mathcal{H}(\vartheta_2)+\mathcal{G}(\vartheta_2)\mathcal{H}(\vartheta_1)].&\end{aligned} \quad (12)$$

Adding both the inequalities (11) and (12) and then integrating with respect to ϱ over [0,1]

$$\begin{aligned}&\int_0^1\mathcal{G}\left(\frac{\varrho}{2}\vartheta_1+\frac{2-\varrho}{2}\vartheta_2\right)\mathcal{H}\left(\frac{\varrho}{2}\vartheta_1+\frac{2-\varrho}{2}\vartheta_2\right)d\varrho\\ &\qquad+\int_0^1\mathcal{G}\left(\frac{\varrho}{2}\vartheta_2+\frac{2-\varrho}{2}\vartheta_1\right)\mathcal{H}\left(\frac{\varrho}{2}\vartheta_2+\frac{2-\varrho}{2}\vartheta_1\right)d\varrho\\ &\leq\frac{[\mathcal{G}(\vartheta_1)\mathcal{H}(\vartheta_1)+\mathcal{G}(\vartheta_2)\mathcal{H}(\vartheta_2)]}{4}\int_0^1\varrho^2+(2-\varrho)^2d\varrho\\ &\qquad+\frac{[\mathcal{G}(\vartheta_1)\mathcal{H}(\vartheta_2)+\mathcal{G}(\vartheta_2)\mathcal{H}(\vartheta_1)]}{2}\int_0^1\varrho(2-\varrho)d\varrho.\end{aligned}$$

This implies

$$\frac{2}{\vartheta_2-\vartheta_1}\left[\int_{\frac{\vartheta_1+\vartheta_2}{2}}^{\vartheta_2}\mathcal{G}(x)\mathcal{H}(x)dx+\int_{\vartheta_1}^{\frac{\vartheta_1+\vartheta_2}{2}}\mathcal{G}(x)\mathcal{H}(x)dx\right]\leq M(\vartheta_1,\vartheta_2)\left[\frac{2}{3}\right]+N(\vartheta_1,\vartheta_2)\left[\frac{1}{3}\right].$$

Multiplying both sides by $\frac{\zeta(\vartheta_2-\vartheta_1)}{2\mathrm{B}(\zeta)}$ and then adding $\frac{1-\zeta}{\mathrm{B}(\zeta)}[\mathcal{G}(\vartheta_1)\mathcal{H}(\vartheta_1)+\mathcal{G}(\vartheta_2)\mathcal{H}(\vartheta_2)]$

$$\begin{aligned}&\frac{1-\zeta}{2\mathrm{B}(\zeta)}\mathcal{G}(\vartheta_1)\mathcal{H}(\vartheta_1)+\frac{\zeta}{\mathrm{B}(\zeta)}\int_{\vartheta_1}^{\frac{\vartheta_1+\vartheta_2}{2}}\mathcal{G}(x)\mathcal{H}(x)dx\\ &\qquad+\frac{1-\zeta}{2\mathrm{B}(\zeta)}\mathcal{G}(\vartheta_2)\mathcal{H}(\vartheta_2)+\frac{\zeta}{\mathrm{B}(\zeta)}\int_{\frac{\vartheta_1+\vartheta_2}{2}}^{\vartheta_1}\mathcal{G}(x)\mathcal{H}(x)dx\\ &\leq\frac{\zeta(\vartheta_2-\vartheta_1)}{2\mathrm{B}(\zeta)}\left[\frac{2}{3}M(\vartheta_1,\vartheta_2)+\frac{1}{3}N(\vartheta_1,\vartheta_2)\right]+\frac{1-\zeta}{\mathrm{B}(\zeta)}[\mathcal{G}(\vartheta_1)\mathcal{H}(\vartheta_1)+\mathcal{G}(\vartheta_2)\mathcal{H}(\vartheta_2)].\end{aligned}$$

Using the definition Caputo–Fabrizio integral operator, we have

$$\left[{}^{\mathrm{CF}}_{\frac{\vartheta_1+\vartheta_2}{2}-}\mathrm{I}^{\zeta}\mathcal{G}\mathcal{H}(\vartheta_1) + {}^{\mathrm{CF}}\mathrm{I}^{\zeta}_{\frac{\vartheta_1+\vartheta_2}{2}+}\mathcal{G}\mathcal{H}(\vartheta_2)\right]$$
$$\leq \frac{\zeta(\vartheta_2-\vartheta_1)}{2\mathrm{B}(\zeta)}\left[\frac{2}{3}M(\vartheta_1,\vartheta_2)+\frac{1}{3}N(\vartheta_1,\vartheta_2)\right]+\frac{1-\zeta}{\mathrm{B}(\zeta)}[\mathcal{G}(\vartheta_1)\mathcal{H}(\vartheta_1)+\mathcal{G}(\vartheta_2)\mathcal{H}(\vartheta_2)].$$

After suitable rearrangements, we have the desired result

$$\frac{2\mathrm{B}(\zeta)}{\zeta(\vartheta_2-\vartheta_1)}\left\{{}^{\mathrm{CF}}_{\frac{\vartheta_1+\vartheta_2}{2}-}\mathrm{I}^{\zeta}\mathcal{G}\mathcal{H}(\vartheta_1) + {}^{\mathrm{CF}}\mathrm{I}^{\zeta}_{\frac{\vartheta_1+\vartheta_2}{2}+}\mathcal{G}\mathcal{H}(\vartheta_2) - \frac{1-\zeta}{\mathrm{B}(\zeta)}M(\vartheta_1,\vartheta_2)\right\}$$
$$\leq \left[\frac{2}{3}M(\vartheta_1,\vartheta_2)+\frac{1}{3}N(\vartheta_1,\vartheta_2)\right].$$

□

4. Dragomir–Agarwal-Type Inequalities: Refinements of H-H Type Inequalities

This section deals with deriving a new identity for differentiable convex functions that involve Caputo–Fabrizio fractional integral operators. Then, taking this identity into account and with the help of some fundamental integral inequalities such as Hölder inequality, Hölder-İşcan integral inequality, power-mean inequality, Young's inequality, and Jensen inequality, several refinements are presented.

Lemma 1. *Let $\mathcal{G}:\mathrm{I}=[\vartheta_1,\vartheta_2]\to\mathcal{R}$ be a differentiable function on I such that $(\vartheta_1,\vartheta_2)\in\mathrm{I}$ with $\vartheta_1<\vartheta_2$ and $\mathcal{G}\in\mathcal{L}[\vartheta_1,\vartheta_2]$. Then, for $\zeta\in[0,1]$, the following fractional equality holds:*

$$\frac{\mathrm{B}(\zeta)}{\zeta(\vartheta_2-\vartheta_1)}\left[{}^{\mathrm{CF}}_{\frac{\vartheta_1+\vartheta_2}{2}-}\mathrm{I}^{\zeta}\mathcal{G}(\vartheta_1) + {}^{\mathrm{CF}}\mathrm{I}^{\zeta}_{\frac{\vartheta_1+\vartheta_2}{2}+}\mathcal{G}(\vartheta_2)\right]-\mathcal{G}\left(\frac{\vartheta_1+\vartheta_2}{2}\right)$$
$$=\frac{\vartheta_2-\vartheta_1}{4}\left\{\int_0^1\varrho\mathcal{G}'\left(\frac{\varrho}{2}\vartheta_1+\frac{2-\varrho}{2}\vartheta_2\right)d\varrho-\int_0^1\varrho\mathcal{G}'\left(\frac{\varrho}{2}\vartheta_2+\frac{2-\varrho}{2}\vartheta_1\right)d\varrho\right\}$$
$$+\frac{1-\zeta}{\zeta(\vartheta_2-\vartheta_1)}[\mathcal{G}(\vartheta_1)+\mathcal{G}(\vartheta_2)],$$

where $\mathrm{B}(\zeta)$ is the normalizaton function.

Proof. It can easily be verified that

$$\frac{\vartheta_2-\vartheta_1}{4}\left\{\int_0^1\varrho\mathcal{G}'\left(\frac{\varrho}{2}\vartheta_1+\frac{2-\varrho}{2}\vartheta_2\right)d\varrho-\int_0^1\varrho\mathcal{G}'\left(\frac{\varrho}{2}\vartheta_2+\frac{2-\varrho}{2}\vartheta_1\right)d\varrho\right\}$$
$$=\frac{1}{\vartheta_2-\vartheta_1}\int_0^1\mathcal{G}(x)dx-\mathcal{G}\left(\frac{\vartheta_1+\vartheta_2}{2}\right)$$
$$=\frac{1}{\vartheta_2-\vartheta_1}\left\{\int_{\vartheta_1}^{\frac{\vartheta_1+\vartheta_2}{2}}\mathcal{G}(x)dx+\int_{\frac{\vartheta_1+\vartheta_2}{2}}^{\vartheta_2}\mathcal{G}(x)dx\right\}-\mathcal{G}\left(\frac{\vartheta_1+\vartheta_2}{2}\right).$$

Multiplying both sides by $\frac{\zeta(\vartheta_2-\vartheta_1)}{\mathrm{B}(\zeta)}$ and then adding $\frac{1-\zeta}{\mathrm{B}(\zeta)}[\mathcal{G}(\vartheta_1)+\mathcal{G}(\vartheta_2)]$, we have

$$\frac{\zeta(\vartheta_2-\vartheta_1)}{B(\zeta)}\frac{\vartheta_2-\vartheta_1}{4}\left[\int_0^1 \varrho\mathcal{G}'\left(\frac{\varrho}{2}\vartheta_1+\frac{2-\varrho}{2}\vartheta_2\right)d\varrho-\int_0^1 \varrho\mathcal{G}'\left(\frac{\varrho}{2}\vartheta_2+\frac{2-\varrho}{2}\vartheta_1\right)d\varrho\right]$$
$$+\frac{1-\zeta}{B(\zeta)}[\mathcal{G}(\vartheta_1)+\mathcal{G}(\vartheta_2)]$$
$$=\frac{1-\zeta}{B(\zeta)}\mathcal{G}(\vartheta_1)+\frac{\zeta}{B(\zeta)}\int_{\vartheta_1}^{\frac{\vartheta_1+\vartheta_2}{2}}\mathcal{G}(x)dx+\frac{1-\zeta}{B(\zeta)}\mathcal{G}(\vartheta_2)+\frac{\zeta}{B(\zeta)}\int_{\frac{\vartheta_1+\vartheta_2}{2}}^{\vartheta_2}\mathcal{G}(x)dx$$
$$-\frac{\zeta(\vartheta_2-\vartheta_1)}{B(\zeta)}\mathcal{G}\left(\frac{\vartheta_1+\vartheta_2}{2}\right)$$
$$=\left[{}^{CF}_{\frac{\vartheta_1+\vartheta_2}{2}^-}\mathrm{I}^{\zeta}\mathcal{G}(\vartheta_1)+{}^{CF}\,\mathrm{I}^{\zeta}_{\frac{\vartheta_1+\vartheta_2}{2}^+}\mathcal{G}(\vartheta_2)\right]-\frac{\zeta(\vartheta_2-\vartheta_1)}{B(\zeta)}\mathcal{G}\left(\frac{\vartheta_1+\vartheta_2}{2}\right).$$

Consequently, we have

$$\frac{B(\zeta)}{\zeta(\vartheta_2-\vartheta_1)}\left[{}^{CF}_{\frac{\vartheta_1+\vartheta_2}{2}^-}\mathrm{I}^{\zeta}\mathcal{G}(\vartheta_1)+{}^{CF}\,\mathrm{I}^{\zeta}_{\frac{\vartheta_1+\vartheta_2}{2}^+}\mathcal{G}(\vartheta_2)\right]-\mathcal{G}\left(\frac{\vartheta_1+\vartheta_2}{2}\right)$$
$$=\frac{\vartheta_2-\vartheta_1}{2}\left\{\int_0^1 \varrho\mathcal{G}'\left(\frac{\varrho}{2}\vartheta_1+\frac{2-\varrho}{2}\vartheta_2\right)d\varrho-\int_0^1 \varrho\mathcal{G}'\left(\frac{\varrho}{2}\vartheta_2+\frac{2-\varrho}{2}\vartheta_1\right)d\varrho\right\}$$
$$+\frac{1-\zeta}{\zeta(\vartheta_2-\vartheta_1)}[\mathcal{G}(\vartheta_1)+\mathcal{G}(\vartheta_2)].$$

This completes the proof of the desired equality. □

Theorem 5. *Let $\mathcal{G}:\mathrm{I}=[\vartheta_1,\vartheta_2]\to\mathcal{R}$ be a differentiable function on* I *such that $(\vartheta_1,\vartheta_2)\in\mathrm{I}$ with $\vartheta_2>\vartheta_1$ and $\mathcal{G}\in\mathcal{L}[\vartheta_1,\vartheta_2]$. If $|\mathcal{G}'|$ is a convex function, then for $\zeta\in[0,1]$, the following fractional inequality holds:*

$$\left|\frac{B(\zeta)}{\zeta(\vartheta_2-\vartheta_1)}\left[{}^{CF}_{\frac{\vartheta_1+\vartheta_2}{2}^-}\mathrm{I}^{\zeta}\mathcal{G}(\vartheta_1)+{}^{CF}\,\mathrm{I}^{\zeta}_{\frac{\vartheta_1+\vartheta_2}{2}^+}\mathcal{G}(\vartheta_2)\right]-\mathcal{G}\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|$$
$$\leq\frac{\vartheta_2-\vartheta_1}{4}\left\{\frac{|\mathcal{G}'(\vartheta_1)|+|\mathcal{G}'(\vartheta_2)|}{2}\right\}+\frac{1-\zeta}{\zeta(\vartheta_2-\vartheta_1)}[\mathcal{G}(\vartheta_1)+\mathcal{G}(\vartheta_2)],$$

where $B(\zeta)$ *is the normalizaton function.*

Proof. From Lemma 1 and properties of modulus, we have

$$\left|\frac{B(\zeta)}{\zeta(\vartheta_2-\vartheta_1)}\left[{}^{CF}_{\frac{\vartheta_1+\vartheta_2}{2}^-}\mathrm{I}^{\zeta}\mathcal{G}(\vartheta_1)+{}^{CF}\,\mathrm{I}^{\zeta}_{\frac{\vartheta_1+\vartheta_2}{2}^+}\mathcal{G}(\vartheta_2)\right]-\mathcal{G}\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|$$
$$=\frac{\vartheta_2-\vartheta_1}{4}\left\{\int_0^1 \varrho\left|\mathcal{G}'\left(\frac{\varrho}{2}\vartheta_1+\frac{2-\varrho}{2}\vartheta_2\right)\right|d\varrho-\int_0^1 \varrho\left|\mathcal{G}'\left(\frac{\varrho}{2}\vartheta_2+\frac{2-\varrho}{2}\vartheta_1\right)\right|d\varrho\right\}$$
$$+\frac{1-\zeta}{\zeta(\vartheta_2-\vartheta_1)}[\mathcal{G}(\vartheta_1)+\mathcal{G}(\vartheta_2)]. \quad (13)$$

Using convexity of $|\mathcal{G}'|$,

$$\begin{aligned}&\int_0^1 \varrho\left|\mathcal{G}'\left(\frac{\varrho}{2}\vartheta_1+\frac{2-\varrho}{2}\vartheta_2\right)\right|d\varrho\\ &\leq \frac{|\mathcal{G}'(\vartheta_1)|}{2}\int_0^1 \varrho^2 d\varrho + \frac{|\mathcal{G}'(\vartheta_2)|}{2}\int_0^1 (2\varrho-\varrho^2)d\varrho\\ &\leq \frac{|\mathcal{G}'(\vartheta_1)|}{6}+\frac{2|\mathcal{G}'(\vartheta_2)|}{3}=\frac{|\mathcal{G}'(\vartheta_1)|+2|\mathcal{G}'(\vartheta_2)|}{6}.\end{aligned}$$

Analogously,

$$\int_0^1 \varrho\left|\mathcal{G}'\left(\frac{\varrho}{2}\vartheta_1+\frac{2-\varrho}{2}\vartheta_2\right)\right|d\varrho=\frac{2|\mathcal{G}'(\vartheta_1)|+|\mathcal{G}'(\vartheta_2)|}{6}.$$

Using the above computations in Equation (13), we have

$$\begin{aligned}&\left|\frac{\mathrm{B}(\zeta)}{\zeta(\vartheta_2-\vartheta_1)}\left[{}^{\mathrm{CF}}_{\frac{\vartheta_1+\vartheta_2}{2}-}\mathrm{I}^{\zeta}\mathcal{G}(\vartheta_1)+{}^{\mathrm{CF}}\mathrm{I}^{\zeta}_{\frac{\vartheta_1+\vartheta_2}{2}+}\mathcal{G}(\vartheta_2)\right]-\mathcal{G}\left(\frac{\vartheta_1+\vartheta_2}{2}\right)-\frac{1-\zeta}{\zeta(\vartheta_2-\vartheta_1)}[\mathcal{G}(\vartheta_1)+\mathcal{G}(\vartheta_2)]\right|\\ &\leq \frac{\vartheta_2-\vartheta_1}{2}\left\{\frac{3|\mathcal{G}'(\vartheta_1)|+3|\mathcal{G}'(\vartheta_2)|}{6}\right\}-\frac{1-\zeta}{\zeta(\vartheta_2-\vartheta_1)}[\mathcal{G}(\vartheta_1)+\mathcal{G}(\vartheta_2)]\\ &=\frac{\vartheta_2-\vartheta_1}{2}\left\{\frac{|\mathcal{G}'(\vartheta_1)|+|\mathcal{G}'(\vartheta_2)|}{2}\right\}-\frac{1-\zeta}{\zeta(\vartheta_2-\vartheta_1)}[\mathcal{G}(\vartheta_1)+\mathcal{G}(\vartheta_2)].\end{aligned}$$

This led us to the desired inequality. □

Theorem 6. *Let $\mathcal{G}:\mathrm{I}=[\vartheta_1,\vartheta_2]\to\mathcal{R}$ be a convex function on* I *such that $(\vartheta_1,\vartheta_2)\in\mathrm{I}$ with $\vartheta_1<\vartheta_2$ and $\mathcal{G}\in\mathcal{L}[\vartheta_1,\vartheta_2]$. If $|\mathcal{G}'|^q$ is a convex function, then for $\zeta\in[0,1]$, $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$, the following fractional inequality holds:*

$$\begin{aligned}&\left|\frac{\mathrm{B}(\zeta)}{\zeta(\vartheta_2-\vartheta_1)}\left[{}^{\mathrm{CF}}_{\frac{\vartheta_1+\vartheta_2}{2}-}\mathrm{I}^{\zeta}\mathcal{G}(\vartheta_1)+{}^{\mathrm{CF}}\mathrm{I}^{\zeta}_{\frac{\vartheta_1+\vartheta_2}{2}+}\mathcal{G}(\vartheta_2)\right]-\mathcal{G}\left(\frac{\vartheta_1+\vartheta_2}{2}\right)-\frac{1-\zeta}{\zeta(\vartheta_2-\vartheta_1)}[\mathcal{G}(\vartheta_1)+\mathcal{G}(\vartheta_2)]\right|\\ &\leq\frac{\vartheta_2-\vartheta_1}{4}\left\{\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|\mathcal{G}'(\vartheta_1)|^q}{4}+\frac{3|\mathcal{G}'(\vartheta_2)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\mathcal{G}'(\vartheta_1)|^q}{4}+\frac{|\mathcal{G}'(\vartheta_2)|^q}{4}\right)^{\frac{1}{q}}\right]\right\},\end{aligned}$$

where $\mathrm{B}(\zeta)$ *is the normalizaton function.*

Proof. The convexity of $|\mathcal{G}'|^q$, produces

$$\left|\mathcal{G}'\left(\frac{\varrho}{2}\vartheta_1+\frac{2-\varrho}{2}\vartheta_2\right)\right|^q\leq\frac{\varrho}{2}|\mathcal{G}'\vartheta_1|^q+\frac{2-\varrho}{2}|\mathcal{G}'\vartheta_2|^q,$$

and

$$\left|\mathcal{G}'\left(\frac{\varrho}{2}\vartheta_2+\frac{2-\varrho}{2}\vartheta_1\right)\right|^q\leq\frac{\varrho}{2}|\mathcal{G}'\vartheta_2|^q+\frac{2-\varrho}{2}|\mathcal{G}'\vartheta_1|^q.$$

Employing Lemma 1 and the Hölder inequality, we have

$$\left|\frac{\mathrm{B}(\zeta)}{\zeta(\vartheta_2-\vartheta_1)}\left[{}^{\mathrm{CF}}_{\frac{\vartheta_1+\vartheta_2}{2}-}\mathrm{I}^{\zeta}\mathcal{G}(\vartheta_1)+{}^{\mathrm{CF}}\mathrm{I}^{\zeta}_{\frac{\vartheta_1+\vartheta_2}{2}+}\mathcal{G}(\vartheta_2)\right]-\mathcal{G}\left(\frac{\vartheta_1+\vartheta_2}{2}\right)-\frac{1-\zeta}{\zeta(\vartheta_2-\vartheta_1)}[\mathcal{G}(\vartheta_1)+\mathcal{G}(\vartheta_2)]\right|$$
$$\leq\frac{\vartheta_2-\vartheta_1}{4}\left\{\int_0^1\varrho\left|\mathcal{G}'\left(\frac{\varrho}{2}\vartheta_1+\frac{2-\varrho}{2}\vartheta_2\right)\right|d\varrho+\int_0^1\varrho\left|\mathcal{G}'\left(\frac{\varrho}{2}\vartheta_1+\frac{2-\varrho}{2}\vartheta_2\right)\right|d\varrho\right\}$$
$$\leq\frac{\vartheta_2-\vartheta_1}{4}\left\{\left(\int_0^1\varrho^p d\varrho\right)^{\frac{1}{p}}\left(\int_0^1|\mathcal{G}'\left(\frac{\varrho}{2}\vartheta_1+\frac{2-\varrho}{2}\vartheta_2\right)|^q d\varrho\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\varrho^p d\varrho\right)^{\frac{1}{p}}\left(\int_0^1|\mathcal{G}'\left(\frac{\varrho}{2}\vartheta_2+\frac{2-\varrho}{2}\vartheta_1\right)|^q dt\right)^{\frac{1}{q}}\right\}$$
$$\leq\frac{\vartheta_2-\vartheta_1}{4}\left\{\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|\mathcal{G}'(\vartheta_1)|^q}{4}+\frac{3|\mathcal{G}'(\vartheta_2)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\mathcal{G}'(\vartheta_1)|^q}{4}+\frac{|\mathcal{G}'(\vartheta_2)|^q}{4}\right)^{\frac{1}{q}}\right]\right\}.$$

This led us to the desired inequality. □

Theorem 7. *Let $\mathcal{G}:\mathrm{I}=[\vartheta_1,\vartheta_2]\to\mathcal{R}$ be a convex function on* I *such that $(\vartheta_1,\vartheta_2)\in\mathrm{I}$ with $\vartheta_2>\vartheta_1$ and $\mathcal{G}\in\mathcal{L}[\vartheta_1,\vartheta_2]$. If $|\mathcal{G}'|^q$ be a convex function, then for $\zeta\in[0,1]$ and $q>1$, the following fractional inequality holds:*

$$\left|\frac{\mathrm{B}(\zeta)}{\zeta(\vartheta_2-\vartheta_1)}\left[{}^{\mathrm{CF}}_{\frac{\vartheta_1+\vartheta_2}{2}-}\mathrm{I}^{\zeta}\mathcal{G}(\vartheta_1)+{}^{\mathrm{CF}}\mathrm{I}^{\zeta}_{\frac{\vartheta_1+\vartheta_2}{2}+}\mathcal{G}(\vartheta_2)\right]-\mathcal{G}\left(\frac{\vartheta_1+\vartheta_2}{2}\right)-\frac{1-\zeta}{\zeta(\vartheta_2-\vartheta_1)}[\mathcal{G}(\vartheta_1)+\mathcal{G}(\vartheta_2)]\right|$$
$$\leq\frac{\vartheta_2-\vartheta_1}{4}\left\{\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\frac{|\mathcal{G}'(\vartheta_1)|^q}{6}+\frac{2|\mathcal{G}'(\vartheta_2)|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{2|\mathcal{G}'(\vartheta_1)|^q}{6}+\frac{|\mathcal{G}'(\vartheta_2)|^q}{6}\right)^{\frac{1}{q}}\right]\right\},$$

where $\mathrm{B}(\zeta)$ *is the normalizaton function.*

Proof. From Lemma 1 and using the power mean inequality,

$$\left|\frac{\mathrm{B}(\zeta)}{\zeta(\vartheta_2-\vartheta_1)}\left[{}^{\mathrm{CF}}_{\frac{\vartheta_1+\vartheta_2}{2}-}\mathrm{I}^{\zeta}\mathcal{G}(\vartheta_1)+{}^{\mathrm{CF}}\mathrm{I}^{\zeta}_{\frac{\vartheta_1+\vartheta_2}{2}+}\mathcal{G}(\vartheta_2)\right]-\mathcal{G}\left(\frac{\vartheta_1+\vartheta_2}{2}\right)-\frac{1-\zeta}{\zeta(\vartheta_2-\vartheta_1)}[\mathcal{G}(\vartheta_1)+\mathcal{G}(\vartheta_2)]\right|$$
$$\leq\frac{\vartheta_2-\vartheta_1}{4}\left\{\left(\int_0^1\varrho\left|\mathcal{G}'\left(\frac{\varrho}{2}\vartheta_1+\frac{2-\varrho}{2}\vartheta_2\right)\right|^q d\varrho\right)^{\frac{1}{q}}+\left(\int_0^1\varrho\left|\mathcal{G}'\left(\frac{\varrho}{2}\vartheta_1+\frac{2-\varrho}{2}\vartheta_2\right)\right|^q d\varrho\right)^{\frac{1}{q}}\right\}$$
$$\leq\frac{\vartheta_2-\vartheta_1}{4}\left\{\left(\int_0^1\varrho d\varrho\right)^{1-\frac{1}{q}}\left(\int_0^1\varrho\left|\mathcal{G}'\left(\frac{\varrho}{2}\vartheta_1+\frac{2-\varrho}{2}\vartheta_2\right)\right|^q d\varrho\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\varrho d\varrho\right)^{1-\frac{1}{q}}\left(\int_0^1\varrho\left|\mathcal{G}'\left(\frac{\varrho}{2}\vartheta_1+\frac{2-\varrho}{2}\vartheta_2\right)\right|^q d\varrho\right)^{\frac{1}{q}}\right\}$$
$$\leq\frac{\vartheta_2-\vartheta_1}{4}\left\{\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\frac{|\mathcal{G}'(\vartheta_1)|^q}{6}+2\frac{|\mathcal{G}'(\vartheta_2)|^q}{6}\right)^{\frac{1}{q}}+\left(2\frac{|\mathcal{G}'(\vartheta_1)|^q}{6}+\frac{|\mathcal{G}'(\vartheta_2)|^q}{6}\right)^{\frac{1}{q}}\right]\right\}.$$

This led us to the desired inequality. □

Theorem 8. *Let $\mathcal{G}:\mathrm{I}=[\vartheta_1,\vartheta_2]\to\mathcal{R}$ be a differentiable function on* I *such that $(\vartheta_1,\vartheta_2)\in\mathrm{I}$ with $\vartheta_2>\vartheta_1$ and $\mathcal{G}\in\mathcal{L}[\vartheta_1,\vartheta_2]$. If $|\mathcal{G}'|^s$ is a convex function, then, for $\zeta\in[0,1]$ and $\frac{1}{r}+\frac{1}{s}=1$, the following fractional inequality holds:*

$$\left|\frac{\mathrm{B}(\zeta)}{\zeta(\vartheta_2-\vartheta_1)}\left[{}^{\mathrm{CF}}_{\frac{\vartheta_1+\vartheta_2}{2}^-}\mathrm{I}^{\zeta}\mathcal{G}(\vartheta_1)+{}^{\mathrm{CF}}\mathrm{I}^{\zeta}_{\frac{\vartheta_1+\vartheta_2}{2}^+}\mathcal{G}(\vartheta_2)\right]-\mathcal{G}\left(\frac{\vartheta_1+\vartheta_2}{2}\right)-\frac{1-\zeta}{\zeta(\vartheta_2-\vartheta_1)}[\mathcal{G}(\vartheta_1)+\mathcal{G}(\vartheta_2)]\right|$$
$$\leq\frac{\vartheta_2-\vartheta_1}{4}\left[\frac{2}{r(r+1)}+\frac{\left|\mathcal{G}'(\vartheta_1)\right|^s+\left|\mathcal{G}'(\vartheta_2)\right|^s}{s}\right],$$

where $\mathrm{B}(\zeta)$ *is the normalizaton function.*

Proof. From Lemma 1 and Young's inequality

$$\left|\frac{\mathrm{B}(\zeta)}{\zeta(\vartheta_2-\vartheta_1)}\left[{}^{\mathrm{CF}}_{\frac{\vartheta_1+\vartheta_2}{2}^-}\mathrm{I}^{\zeta}\mathcal{G}(\vartheta_1)+{}^{\mathrm{CF}}\mathrm{I}^{\zeta}_{\frac{\vartheta_1+\vartheta_2}{2}^+}\mathcal{G}(\vartheta_2)\right]-\mathcal{G}\left(\frac{\vartheta_1+\vartheta_2}{2}\right)-\frac{1-\zeta}{\zeta(\vartheta_2-\vartheta_1)}[\mathcal{G}(\vartheta_1)+\mathcal{G}(\vartheta_2)]\right|$$
$$\leq\frac{\vartheta_2-\vartheta_1}{4}\left\{\int_0^1\varrho\mathcal{G}'\left(\frac{\varrho}{2}\vartheta_1+\frac{2-\varrho}{2}\vartheta_2\right)d\varrho+\int_0^1\varrho\mathcal{G}'\left(\frac{\varrho}{2}\vartheta_1+\frac{2-\varrho}{2}\vartheta_2\right)d\varrho\right\}.$$

Using Young Inequality

$$xy\leq\frac{1}{r}x^r+\frac{1}{s}y^s.$$

$$\int_0^1\varrho\mathcal{G}'\left(\frac{\varrho}{2}\vartheta_1+\frac{2-\varrho}{2}\vartheta_2\right)dt\leq\frac{1}{r}\int_0^1\varrho^r d\varrho+\frac{1}{s}\int_0^1\left|\mathcal{G}'\left(\frac{\varrho}{2}\vartheta_2+\frac{2-\varrho}{2}\vartheta_1\right)\right|^s d\varrho$$
$$=\frac{1}{r(r+1)}+\frac{1}{s}\left[\frac{\left|\mathcal{G}'(\vartheta_1)\right|^s}{4}+\frac{3\left|\mathcal{G}'(\vartheta_2)\right|^s}{4}\right].$$

Similarly

$$\int_0^1\varrho\mathcal{G}'\left(\frac{\varrho}{2}\vartheta_2+\frac{2-\varrho}{2}\vartheta_1\right)d\varrho\leq\frac{1}{r(r+1)}+\frac{1}{s}\left[\frac{\left|\mathcal{G}'(\vartheta_2)\right|^s}{4}+\frac{3\left|\mathcal{G}'(\vartheta_1)\right|^s}{4}\right].$$

Consequently,

$$\left|\frac{\mathrm{B}(\zeta)}{\zeta(\vartheta_2-\vartheta_1)}\left[{}^{\mathrm{CF}}_{\frac{\vartheta_1+\vartheta_2}{2}^-}\mathrm{I}^{\zeta}\mathcal{G}(\vartheta_1)+{}^{\mathrm{CF}}\mathrm{I}^{\zeta}_{\frac{\vartheta_1+\vartheta_2}{2}^+}\mathcal{G}(\vartheta_2)\right]-\mathcal{G}\left(\frac{\vartheta_1+\vartheta_2}{2}\right)-\frac{1-\zeta}{\zeta(\vartheta_2-\vartheta_1)}[\mathcal{G}(\vartheta_1)+\mathcal{G}(\vartheta_2)]\right|$$
$$\leq\frac{\vartheta_2-\vartheta_1}{4}\left[\frac{2}{r(r+1)}+\frac{1}{s}\left(\left|\mathcal{G}'(\vartheta_1)\right|^s+\left|\mathcal{G}'(\vartheta_2)\right)\right|^s\right]$$
$$=\frac{\vartheta_2-\vartheta_1}{4}\left[\frac{2}{r(r+1)}+\frac{\left|\mathcal{G}'(\vartheta_1)\right|^s+\left|\mathcal{G}'(\vartheta_2)\right|^s}{s}\right].$$

This led us to the desired result. □

Theorem 9. *Let* $\mathcal{G}:\mathrm{I}=[\vartheta_1,\vartheta_2]\to\mathcal{R}$ *be a differentiable function on* I *such that* $(\vartheta_1,\vartheta_2)\in\mathrm{I}$ *with* $\vartheta_1<\vartheta_2$ *and* $\mathcal{G}\in\mathcal{L}[\vartheta_1,\vartheta_2]$. *Then, for* $\zeta\in[0,1]$, *the following fractional inequality holds:*

$$\left|\frac{\mathrm{B}(\zeta)}{\zeta(\vartheta_2-\vartheta_1)}\left[{}^{\mathrm{CF}}_{\frac{\vartheta_1+\vartheta_2}{2}-}\mathrm{I}^{\zeta}\mathcal{G}(\vartheta_1)+{}^{\mathrm{CF}}\mathrm{I}^{\zeta}_{\frac{\vartheta_1+\vartheta_2}{2}+}\mathcal{G}(\vartheta_2)\right]-\mathcal{G}\left(\frac{\vartheta_1+\vartheta_2}{2}\right)-\frac{1-\zeta}{\zeta(\vartheta_2-\vartheta_1)}[\mathcal{G}(\vartheta_1)+\mathcal{G}(\vartheta_2)]\right|$$
$$\leq\frac{\vartheta_2-\vartheta_1}{8}\left\{\left|\mathcal{G}'\left(\frac{\vartheta_1+2\vartheta_2}{3}\right)\right|+\left|\mathcal{G}'\left(\frac{2\vartheta_1+\vartheta_2}{3}\right)\right|\right\},$$

where $\mathrm{B}(\zeta)$ *is the normalizaton function.*

Proof. From Lemma 1 and Jensen inequality,

$$\left|\frac{\mathrm{B}(\zeta)}{\zeta(\vartheta_2-\vartheta_1)}\left[{}^{\mathrm{CF}}_{\frac{\vartheta_1+\vartheta_2}{2}-}\mathrm{I}^{\zeta}\mathcal{G}(\vartheta_1)+{}^{\mathrm{CF}}\mathrm{I}^{\zeta}_{\frac{\vartheta_1+\vartheta_2}{2}+}\mathcal{G}(\vartheta_2)\right]-\mathcal{G}\left(\frac{\vartheta_1+\vartheta_2}{2}\right)-\frac{1-\zeta}{\zeta(\vartheta_2-\vartheta_1)}[\mathcal{G}(\vartheta_1)+\mathcal{G}(\vartheta_2)]\right|$$
$$\leq\frac{\vartheta_2-\vartheta_1}{4}\left\{\int_0^1\varrho\left|\mathcal{G}'\left(\frac{\varrho}{2}\vartheta_1+\frac{2-\varrho}{2}\vartheta_2\right)\right|d\varrho+\int_0^1\varrho\left|\mathcal{G}'\left(\frac{\varrho}{2}\vartheta_1+\frac{2-\varrho}{2}\vartheta_2\right)\right|d\varrho\right\}$$
$$\leq\frac{\vartheta_2-\vartheta_1}{4}\left\{\left(\int_0^1\varrho d\varrho\right)\left|\mathcal{G}'\left(\frac{\int_0^1\varrho\left(\frac{\varrho}{2}\vartheta_1+\frac{2-\varrho}{2}\vartheta_2\right)d\varrho}{\left(\int_0^1\varrho d\varrho\right)}\right)\right|\right.$$
$$\left.+\left(\int_0^1\varrho d\varrho\right)\left|\mathcal{G}'\left(\frac{\int_0^1\varrho\left(\frac{\varrho}{2}\vartheta_2+\frac{2-\varrho}{2}\vartheta_1\right)d\varrho}{\left(\int_0^1\varrho d\varrho\right)}\right)\right|\right\}.$$

Consequently,

$$\left|\frac{\mathrm{B}(\zeta)}{\zeta(\vartheta_2-\vartheta_1)}\left[{}^{\mathrm{CF}}_{\frac{\vartheta_1+\vartheta_2}{2}-}\mathrm{I}^{\zeta}\mathcal{G}(\vartheta_1)+{}^{\mathrm{CF}}\mathrm{I}^{\zeta}_{\frac{\vartheta_1+\vartheta_2}{2}+}\mathcal{G}(\vartheta_2)\right]-\mathcal{G}\left(\frac{\vartheta_1+\vartheta_2}{2}\right)-\frac{1-\zeta}{\zeta(\vartheta_2-\vartheta_1)}[\mathcal{G}(\vartheta_1)+\mathcal{G}(\vartheta_2)]\right|$$
$$\leq\frac{\vartheta_2-\vartheta_1}{4}\left\{\frac{1}{2}\left|\mathcal{G}'\left(\frac{\vartheta_1+2\vartheta_2}{3}\right)\right|+\frac{1}{2}\left|\mathcal{G}'\left(\frac{2\vartheta_1+\vartheta_2}{3}\right)\right|\right\}$$
$$=\frac{\vartheta_2-\vartheta_1}{8}\left\{\left|\mathcal{G}'\left(\frac{\vartheta_1+2\vartheta_2}{3}\right)\right|+\left|\mathcal{G}'\left(\frac{2\vartheta_1+\vartheta_2}{3}\right)\right|\right\}.$$

This led us to the desired inequality. □

Theorem 10. *Let* $\mathcal{G}:\mathrm{I}=[\vartheta_1,\vartheta_2]\to\mathcal{R}$ *be a differentiable function on* I *such that* $(\vartheta_1,\vartheta_2)\in\mathrm{I}$ *with* $\vartheta_2>\vartheta_1$ *and* $\mathcal{G}'\in\mathcal{L}[\vartheta_1,\vartheta_2]$. *If* $|\mathcal{G}'|^q$ *is a convex function, then for* $\zeta\in[0,1]$, $q>1$ *and* $\frac{1}{p}+\frac{1}{q}=1$, *the following fractional inequality holds:*

$$\left|\frac{\mathrm{B}(\zeta)}{\zeta(\vartheta_2-\vartheta_1)}\left[{}^{\mathrm{CF}}_{\frac{\vartheta_1+\vartheta_2}{2}-}\mathrm{I}^{\zeta}\mathcal{G}(\vartheta_1)+{}^{\mathrm{CF}}\mathrm{I}^{\zeta}_{\frac{\vartheta_1+\vartheta_2}{2}+}\mathcal{G}(\vartheta_2)\right]-\mathcal{G}\left(\frac{\vartheta_1+\vartheta_2}{2}\right)-\frac{1-\zeta}{\zeta(\vartheta_2-\vartheta_1)}[\mathcal{G}(\vartheta_1)+\mathcal{G}(\vartheta_2)]\right|$$
$$\leq\frac{\vartheta_2-\vartheta_1}{4}\left\{\left(\frac{1}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{G}'(\vartheta_1)|^q+5|\mathcal{G}'(\vartheta_2)|^q}{12}+\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{G}'(\vartheta_1)|^q}{6}+\frac{|\mathcal{G}'(\vartheta_2)|^q}{3}\right)^{\frac{1}{q}}$$
$$\left.+\left(\frac{1}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{G}'(\vartheta_2)|^q+5|\mathcal{G}'(\vartheta_1)|^q}{12}+\right)^{\frac{1}{q}}+\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{G}'(\vartheta_2)|^q}{6}+\frac{|\mathcal{G}'(\vartheta_1)|^q}{3}\right)^{\frac{1}{q}}\right\},$$

where $\mathrm{B}(\zeta)$ *is the normalizaton function.*

Proof. The convexity of $|\mathcal{G}'|^q$, produces

$$\left|\mathcal{G}'\left(\frac{\varrho}{2}\vartheta_1+\frac{2-\varrho}{2}\vartheta_2\right)\right|^q \le \frac{\varrho}{2}|\mathcal{G}'\vartheta_1|^q+\frac{2-\varrho}{2}|\mathcal{G}'\vartheta_2|^q,$$

and

$$\left|\mathcal{G}'\left(\frac{\varrho}{2}\vartheta_2+\frac{2-\varrho}{2}\vartheta_1\right)\right|^q \le \frac{\varrho}{2}|\mathcal{G}'\vartheta_2|^q+\frac{2-\varrho}{2}|\mathcal{G}'\vartheta_1|^q.$$

Employing Lemma 1 and the Hölder-İşcan integral inequality,

$$\begin{aligned}
&\left|\frac{\mathrm{B}(\zeta)}{\zeta(\vartheta_2-\vartheta_1)}\left[{}^{\mathrm{CF}}_{\frac{\vartheta_1+\vartheta_2}{2}-}\mathrm{I}^{\zeta}\mathcal{G}(\vartheta_1)+{}^{\mathrm{CF}}\mathrm{I}^{\zeta}_{\frac{\vartheta_1+\vartheta_2}{2}+}\mathcal{G}(\vartheta_2)\right]-\mathcal{G}\left(\frac{\vartheta_1+\vartheta_2}{2}\right)-\frac{1-\zeta}{\zeta(\vartheta_2-\vartheta_1)}[\mathcal{G}(\vartheta_1)+\mathcal{G}(\vartheta_2)]\right|\\
&\le \frac{\vartheta_2-\vartheta_1}{4}\left\{\int_0^1 \varrho\left|\mathcal{G}'\left(\frac{\varrho}{2}\vartheta_1+\frac{2-\varrho}{2}\vartheta_2\right)\right|d\varrho+\int_0^1 \varrho\left|\mathcal{G}'\left(\frac{\varrho}{2}\vartheta_1+\frac{2-\varrho}{2}\vartheta_2\right)\right|d\varrho\right\}\\
&\le \frac{\vartheta_2-\vartheta_1}{4}\left\{\left(\int_0^1(1-\varrho)\varrho^p d\varrho\right)^{\frac{1}{p}}\left(\int_0^1(1-\varrho)\left|\mathcal{G}'\left(\frac{\varrho}{2}\vartheta_1+\frac{2-\varrho}{2}\vartheta_2\right)\right|^q d\varrho\right)^{\frac{1}{q}}\right.\\
&+\left(\int_0^1\varrho^{p+1}d\varrho\right)^{\frac{1}{p}}\left(\int_0^1\varrho\left|\mathcal{G}'\left(\frac{\varrho}{2}\vartheta_2+\frac{2-\varrho}{2}\vartheta_1\right)\right|^q dt\right)^{\frac{1}{q}}\\
&+\left(\int_0^1(1-\varrho)\varrho^p d\varrho\right)^{\frac{1}{p}}\left(\int_0^1(1-\varrho)\left|\mathcal{G}'\left(\frac{\varrho}{2}\vartheta_1+\frac{2-\varrho}{2}\vartheta_2\right)\right|^q d\varrho\right)^{\frac{1}{q}}\\
&\left.+\left(\int_0^1\varrho^{p+1}d\varrho\right)^{\frac{1}{p}}\left(\int_0^1\varrho\left|\mathcal{G}'\left(\frac{\varrho}{2}\vartheta_2+\frac{2-\varrho}{2}\vartheta_1\right)\right|^q dt\right)^{\frac{1}{q}}\right\}\\
&=\frac{\vartheta_2-\vartheta_1}{4}\left\{\left(\frac{1}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{G}'(\vartheta_1)|^q+5|\mathcal{G}'(\vartheta_2)|^q}{12}+\right)^{\frac{1}{q}}\right.\\
&+\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{G}'(\vartheta_1)|^q}{6}+\frac{|\mathcal{G}'(\vartheta_2)|^q}{3}\right)^{\frac{1}{q}}+\left(\frac{1}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{G}'(\vartheta_2)|^q+5|\mathcal{G}'(\vartheta_1)|^q}{12}+\right)^{\frac{1}{q}}\\
&\left.+\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{G}'(\vartheta_2)|^q}{6}+\frac{|\mathcal{G}'(\vartheta_1)|^q}{3}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

This led us to the desired inequality. □

5. Trapezoidal Quadrature Formula

Here, we present an application involving error estimation for the trapezoidal quadrature formula by using the results presented in Section 4. Let

$$\mathsf{U}:\vartheta_1=\psi_0<\psi_1<\cdots<\psi_{\ell-1}<\psi_\ell=\vartheta_2$$

be a partition of the closed interval $[\vartheta_1,\vartheta_2]$.

Let us define

$$\mathsf{T}(\mathsf{U},\mathcal{G}):=\sum_{i=0}^{\ell-1}\left(\frac{\mathcal{G}(\psi_i)+\mathcal{G}(\psi_{i+1})}{2}\right)(\psi_{i+1}-\psi_i)$$

and

$$\int_{\vartheta_1}^{\vartheta_2}\mathcal{G}(x)dx:=\mathsf{T}(\mathsf{U},\mathcal{G})+\mathsf{R}(\mathsf{U},\mathcal{G}),$$

where $\mathsf{R}(\mathsf{U},\mathcal{G})$ is the associated remainder term.

From the above notations, we can obtain some new bounds regarding error estimation.

Proposition 1. *Consider a function $\mathcal{G} : [\vartheta_1, \vartheta_2] \to \mathcal{R}$, which is differentiable on $(\vartheta_1, \vartheta_2)$ with $0 < \vartheta_1 < \vartheta_2$. If $\mathcal{G}' \in \mathcal{L}(\vartheta_1, \vartheta_2)$ and $|\mathcal{G}'|$ is a convex function. Then, the following inequality holds:*

$$|\mathsf{R}(\mathsf{U}, \mathcal{G})| \leqq \frac{1}{4}\sum_{i=0}^{\ell-1}\frac{(\psi_{i+1}-\psi_i)^2}{2}\big(|\mathcal{G}'(\psi_i)| + |\mathcal{G}'(\psi_{i+1})|\big).$$

Proof. Applying Theorem 5 on the subinterval $[\psi_i, \psi_{i+1}]$ ($\forall\, i = 0, 1, 2, \cdots, \ell-1$), for $\mathrm{B}(\zeta) = 1$, we have

$$\begin{aligned}&\left|\left(\frac{\mathcal{G}(\psi_i)+\mathcal{G}(\psi_{i+1})}{2}\right)(\psi_{i+1}-r\psi_i) - \int_{\psi_i}^{\psi_{i+1}}\mathcal{G}(x)dx\right| \\ &\leqq \frac{(\psi_{i+1}-\psi_i)^2}{4}\frac{(|\mathcal{G}'(\psi_i)| + |\mathcal{G}'(\psi_{i+1})|)}{2}.\end{aligned}$$

Summing over i from 0 to $\ell - 1$ and using the property of the modulus, we get the desired inequality. □

Proposition 2. *Consider a function $\mathcal{G} : [\vartheta_1, \vartheta_2] \to \mathcal{R}$, which is differentiable on $(\vartheta_1, \vartheta_2)$ with $0 < \vartheta_1 < \vartheta_2$. If $\mathcal{G}' \in \mathcal{L}(\vartheta_1, \vartheta_2)$ and $|\mathcal{G}'|^q$ is a convex function, then for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, the following inequality holds:*

$$\begin{aligned}|\mathsf{R}(\mathsf{U}, \mathcal{G})| \leqq{}& \frac{1}{4}\sum_{i=0}^{\ell-1}(\psi_{i+1}-\psi_i)^2\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \\ &\times\left[\left(\frac{|\mathcal{G}'(\psi_i)|^q}{4} + \frac{3|\mathcal{G}'(\psi_{i+1})|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{3|\mathcal{G}'(\psi_i)|^q}{4} + \frac{|\mathcal{G}'(\psi_{i+1})|^q}{4}\right)^{\frac{1}{q}}\right].\end{aligned}$$

Proof. Applying Theorem 6 on the subinterval $[\psi_i, \psi_{i+1}]$ ($\forall\, i = 0, 1, 2, \cdots, \ell-1$), for $\mathrm{B}(\zeta) = 1$, we have

$$\begin{aligned}&\left|\left(\frac{\mathcal{G}(\psi_i)+\mathcal{G}(\psi_{i+1})}{2}\right)(\psi_{i+1}-r\psi_i) - \int_{\psi_i}^{\psi_{i+1}}\mathcal{G}(x)dx\right| \\ &\leqq \frac{(\psi_{i+1}-\psi_i)^2}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|\mathcal{G}'(\psi_i)|^q}{4} + \frac{3|\mathcal{G}'(\psi_{i+1})|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{3|\mathcal{G}'(\psi_i)|^q}{4} + \frac{|\mathcal{G}'(\psi_{i+1})|^q}{4}\right)^{\frac{1}{q}}\right].\end{aligned}$$

Summing over i from 0 to $\ell - 1$ and using the property of the modulus, the desired inequality is obtained. □

Proposition 3. *Consider a function $\mathcal{G} : [\vartheta_1, \vartheta_2] \to \mathcal{R}$, which is differentiable on $(\vartheta_1, \vartheta_2)$ with $0 < \vartheta_1 < \vartheta_2$. If $\mathcal{G}' \in \mathcal{L}(\vartheta_1, \vartheta_2)$ and $|\mathcal{G}'|$ is a convex function, then the following inequality holds:*

$$|\mathsf{R}(\mathsf{U}, \mathcal{G})| \leqq \frac{1}{8}\sum_{i=0}^{\ell-1}(\psi_{i+1}-\psi_i)^2\left\{\left|\mathcal{G}'\left(\frac{\psi_i+2\psi_{i+1}}{3}\right)\right| + \left|\mathcal{G}'\left(\frac{2\psi_i+\psi_{i+1}}{3}\right)\right|\right\}.$$

Proof. Applying Theorem 9 on the subinterval $[\psi_i, \psi_{i+1}]$ ($\forall\, i = 0, 1, 2, \cdots, \ell-1$), for $\mathrm{B}(\zeta) = 1$, we have

$$\begin{aligned}&\left|\left(\frac{\mathcal{G}(\psi_i)+\mathcal{G}(\psi_{i+1})}{2}\right)(\psi_{i+1}-r\psi_i) - \int_{\psi_i}^{\psi_{i+1}}\mathcal{G}(x)dx\right| \\ &\leqq \frac{(\psi_{i+1}-\psi_i)^2}{8}\left\{\left|\mathcal{G}'\left(\frac{\psi_i+2\psi_{i+1}}{3}\right)\right| + \left|\mathcal{G}'\left(\frac{2\psi_i+\psi_{i+1}}{3}\right)\right|\right\}.\end{aligned}$$

Summing over i from 0 to $\ell - 1$ and using the property of the modulus, the desired inequality is obtained. □

Application to Special Means

Now, we propose some applications to special means of real numbers related to our established results.

1. The arithmetic mean

$$\mathcal{A} = \mathcal{A}(\vartheta_1, \vartheta_2) = \frac{\vartheta_1 + \vartheta_2}{2}, \quad \vartheta_1, \vartheta_2 \in \mathcal{R}.$$

2. The generalized logarithmic mean

$$\mathcal{L}_r^r = \mathcal{L}_r^r(\vartheta_1, \vartheta_2) = \frac{\vartheta_2^{r+1} - \vartheta_1^{r+1}}{(r+1)(\vartheta_2 - \vartheta_1)}.$$

3. The logarithmic mean

$$\mathcal{L}(\vartheta_1, \vartheta_2) = \frac{\vartheta_2 - \vartheta_1}{\ln|\vartheta_2| - \ln|\vartheta_1|}, \quad \vartheta_1 \neq \vartheta_2, \ \vartheta_1\vartheta_2 \neq 0.$$

Proposition 4. *Let* $\vartheta_1, \vartheta_2 \in \mathcal{R}^+, \vartheta_1 < \vartheta_2$, *then*

$$\left| \mathcal{L}_2^2(\vartheta_1, \vartheta_2) - \mathcal{A}^2(\vartheta_1, \vartheta_2) \right| \leq \frac{\vartheta_2 - \vartheta_1}{2} \mathcal{A}(\vartheta_1, \vartheta_2).$$

Proof. In Theorem 5, setting $\mathcal{G}(x) = x^2$ with $\zeta = 1$ and $\mathrm{B}(\zeta) = \mathrm{B}(1) = 1$ completes the proof. □

Proposition 5. *Let* $\vartheta_1, \vartheta_2 \in \mathcal{R}^+, \vartheta_1 < \vartheta_2$, *then*

$$\left| \mathcal{L}^{-1}(\vartheta_1, \vartheta_2) - \mathcal{A}^{-1}(\vartheta_1, \vartheta_2) \right| \leq \frac{(\vartheta_2 - \vartheta_1)}{4} \left[\mathcal{A}(\vartheta_1^{-2}, \vartheta_2^{-2}) \right].$$

In Theorem 5, setting $\mathcal{G}(x) = x^{-1}$ with $\zeta = 1$ and $\mathrm{B}(\zeta) = \mathrm{B}(1) = 1$ completes the proof.

Proposition 6. *Let* $\vartheta_1, \vartheta_2 \in \mathcal{R}^+, \vartheta_1 < \vartheta_2$, *then*

$$\left| \mathcal{L}_n^n(\vartheta_1, \vartheta_2) - \mathcal{A}^n(\vartheta_1, \vartheta_2) \right| \leq \frac{n(\vartheta_2 - \vartheta_1)}{4} \left[\mathcal{A}(\vartheta_1^{n-1}, \vartheta_2^{n-1}) \right].$$

In Theorem 5, setting $\mathcal{G}(x) = x^n$ with $\zeta = 1$ and $\mathrm{B}(\zeta) = \mathrm{B}(1) = 1$ completes the proof.

6. Conclusions

The utilization of different fractional operators in the field of integral inequalities has kept the interest of several mathematicians. The Caputo–Fabrizio fractional operator plays an important role in recent advancements related to differential equations, modeling, and mathematical inequalities. Here, we used a fractional operator that has a nonsingular kernel. Our established new versions of H-H and Pachpatte-type inequalities deals with differeintegrals of $\left(\frac{\vartheta_1+\vartheta_2}{2}\right)$ type, which are new in the literature for Caputo–Fabrizio fractional inequalities. In addition, to provide improvements to H-H-type inequalities, a new identity for differentiable mappings is proved. The accuracy of the results is validated through some application to special means. For future work, we will apply this new fractional operator to improve inequalities of the Ostrowski type, Jensen–Mercer type, and Hermite–Hadmard–Mercer type. We will also try to apply this operator to present Hermite–Hadamard inequality on coordinated convex functions.

Author Contributions: Conceptualization, S.K.S., P.O.M. and B.K.; methodology, B.K. and M.T.; software, S.K.S., P.O.M. and M.T.; validation, P.O.M., B.K. and Y.S.H.; formal analysis, S.K.S., P.O.M., B.K., M.T. and Y.S.H.; investigation, S.K.S. and P.O.M.; writing—original draft preparation, S.K.S., P.O.M. and M.T.; writing—review and editing, S.K.S. and P.O.M.; supervision, B.K. and Y.S.H.; project administration, S.K.S. and P.O.M.; funding acquisition, Y.S.H. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: This work was supported by the Taif University Researchers Supporting Project (No. TURSP 2020/155), Taif University, Taif, Saudi Arabia.

Conflicts of Interest: The authors declare no conflict of interest.

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