fractal and fractional

Article

New Exact Solutions of Some Important Nonlinear Fractional Partial Differential Equations with Beta Derivative

Erdogan Mehmet Ozkan

Department of Matematics, Faculty of Science, Davutpasa Campus, Yildiz Technical University, Istanbul 34210, Turkey; mozkan@yildiz.edu.tr

Abstract: In this work, the F-expansion method is used to find exact solutions of the space-time fractional modified Benjamin Bona Mahony equation and the nonlinear time fractional Schrödinger equation with beta derivative. One of the most efficient and significant methods for obtaining new exact solutions to nonlinear equations is this method. With the aid of Maple, more exact solutions defined by the Jacobi elliptic function are obtained. Hyperbolic function solutions and some exact solutions expressed by trigonometric functions are gained in the case of m modulus 1 and 0 limits of the Jacobi elliptic function.

Keywords: Benjamin Bona Mahony equation; Schrödinger equation; F-expansion method; Jacobi elliptic functions; beta derivative

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1. Introduction

Nonlinear partial differential equations (NPDEs) are essential in a variety of domains, including engineering, mathematics, fluid dynamics, and physics. NPDEs have been used to model a variety of real-life challenges. Exact solutions to NPDEs have been obtained using a variety of various and reliable mathematical approaches [1–11]. Fractional calculus is a relatively new field that has gained interest in recent decades. Different physical phenomena, such as viscoelasticity, plasma, solid mechanics, optical fibers, signal processing, electromagnetic waves, fluid dynamics, biomedical sciences, and diffusion processes, are made easily solvable in fractional partial differential equations (FPDEs). Researchers have used numerous ways to acquire exact solutions to FPDEs to make these equations appealing. Many articles have been made recently about obtaining analytical, numerical exact solutions of mathematical problems and some physical phenomena that can be mathematically formed and described using fractional derivatives [12–16]. It seems that these are similar events are often stated in nonlinear fractional partial differential equations. The fractional derivative operator has been defined in many different ways. Some of these are frequently used ones are as follows: Caputo derivative [17], Riemann-Liouville derivative [18], Caputo-Fabrizro [19], Jumarie's modifies Riemann-Liouville derivative [20], Atangana-Baleanu derivative [21]. By aid of these derivative operator, some of the various techniques developed that provide analytical, approximate and exact solution of nonlinear fractional partial differential equations can be listed as sub-equation method [22], the first integral method [23], auxiliary equation method [24], the modified trial equation method [25], the variational iteration method [26], natural transform decomposition method [27].

A new fractional derivative, called conformable derivative, has been described in [28]. Then, using this derivative, exact solution of the time-heat differential equation has been obtained [29]. Moreover Atangana et al. have revealed some definitions, theorems and properties about the conformable derivative [30]. Eventually, a new definition of a fractional derivative called the beta-derivative has been given by Atangana et al. In their paper, the solution of the Hunter-Saxton equation has been obtained using this derivative [31]. In addition, the solutions of the fractional Sharma-Tasso-Oliver, space-time fractional modified Benjamin Bona-Mahony, time fractional Schrödinger equations, which are also given with this derivative, have been obtained by the first integral method [32]. Many aspects of the recommended version have been used to model various physical difficulties and have acted as fractional derivative restrictions. They are not fractional, but they can be regarded an natural extension of the classical derivative. Recently, studies defined with this derivative have gained importance. The interval on which the function is differentiated determines this derivative. It possesses characteristics that the well-known fractional derivatives lack, including the following.

Definition 1. *Suppose that $F(\omega)$ is a function. The beta derivative of $F(\omega)$ is defined by [31]*

$$ {}^{A}D_{\omega}^{\beta}\{F(\omega)\} = \lim_{\gamma \to 0} \frac{F[\omega + \gamma(\omega + \frac{1}{\Gamma(\beta)})^{1-\beta}] - F(\omega)}{\gamma}, \quad \omega > 0, \quad \beta \in (0,1]. \tag{1} $$

There are some important properties for this derivative [32].

1. ${}^{A}D_{\omega}^{\beta}(a_0F(\omega) + a_1G(\omega)) = a_0{}^{A}D_{\omega}^{\beta} + a_1{}^{A}D_{\omega}^{\beta}G(\omega), \forall a_0, a_1 \in \mathbb{R},$
2. ${}^{A}D_{\omega}^{\beta}(c_0) = 0, \forall c_0 \in \mathbb{R},$
3. ${}^{A}D_{\omega}^{\beta}(F(\omega).G(\omega)) = G(\omega){}^{A}D_{\omega}^{\beta}(F(\omega)) + F(\omega){}^{A}D_{\omega}^{\beta}(G(\omega),$
4. ${}^{A}D_{\omega}^{\beta}(F(\omega)/G(\omega)) = \frac{G(\omega){}^{A}D_{\omega}^{\beta}(F(\omega)) - F(\omega){}^{A}D_{\omega}^{\beta}(G(\omega))}{[G(\omega)]^2}, (G \neq 0),$
5. ${}^{A}D_{\omega}^{\beta}(F(G(\omega)) = \left(\omega + \frac{1}{\Gamma(\beta)}\right)^{1-\beta} G'(\omega)F'(G(\omega)),$

where F and G two functions β differentiable and $\beta \in (0,1]$.

The proofs of these properties are given in [30–32].

2. F-Expansion Method

In this section, a detailed explanation of the F-expansion method will be given to obtain the exact solutions of fractional partial differential equations (FPDEs) defined by beta derivative.

Let us consider the space-time FPDE with a beta derivative for a function of two real variables x and t:

$$ H(p, {}^{A}D_t^{\beta}p, {}^{A}D_x^{\beta}p, {}^{A}D_t^{2\beta}p, {}^{A}D_x^{2\beta}p, \ldots) = 0, \quad (0 < \beta \leq 1) \tag{2} $$

1. Firstly, the following travelling wave transformation should be used to transform (2) into an ordinary differential equation

$$ \begin{aligned} p(x,t) &= U(\epsilon), \\ \epsilon &= \frac{k}{\beta}\left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta} - \frac{c}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}, \end{aligned} \tag{3} $$

where k and c arbitrary constant. Substituting (3) into (2), a nonlinear ordinary differential equation (ODE) can be obtain as

$$ N(U, \frac{dU}{d\epsilon}, \frac{d^2U}{d\epsilon^2}, ..) = 0 \tag{4} $$

2. Assume that, the solution $U(\epsilon)$ of (4) can be described as

$$ U(\epsilon) = a_0 + \sum_{i=1}^{M} \left(a_iK^i(\epsilon) + \frac{b_i}{K^i(\epsilon)}\right) \tag{5} $$

where a_0 and a_i, b_i (i = 1, 2,..., M) are constants to be determined, $K(\epsilon)$ is a solution of ODE

$$ [K'(\epsilon)]^2 = f_2[K(\epsilon)]^4 + f_1[K(\epsilon)]^2 + f_0 \tag{6} $$

where f_2, f_1 and f_0 are custom values in the Table 1 [33,34]. M is positive integer which can be determined from Equation (4) as follows where $deg(U(\varepsilon)) = M$ is degree of $U(\varepsilon)$

$$deg\left[\frac{d^qU}{d\varepsilon^q}\right] = M + q,$$
$$deg\left[U^r\left(\frac{d^qU}{d\varepsilon^q}\right)^s\right] = Mr + s(q+M).$$

Table 1. Jacobi elliptic function solutions.

Case	f_0	f_1	f_2	$K(\epsilon)$
1	1	$-(1+m^2)$	m^2	$sn(\epsilon)$ or $cd(\epsilon)$
2	$1-m^2$	$2m^2-1$	$-m^2$	$\mathrm{cn}(\epsilon)$
3	m^2-1	$2-m^2$	-1	$\mathrm{dn}(\epsilon)$
4	m^2	$-(1+m^2)$	1	$\mathrm{ns}(\epsilon)$ or $\mathrm{dc}(\epsilon)$
5	$-m^2$	$2m^2-1$	$1-m^2$	$\mathrm{nc}(\epsilon)$
6	-1	$2-m^2$	$-(1-m^2)$	$\mathrm{nd}(\epsilon)$
7	1	$2-m^2$	$1-m^2$	$\mathrm{sc}(\epsilon)$
8	1	$2m^2-1$	$1-m^2$	$\mathrm{sc}(\epsilon)$
9	$1-m^2$	$2-m^2$	1	$\mathrm{cs}(\epsilon)$
10	$-m^2(1-m^2)$	$2m^2-1$	1	$\mathrm{ds}(\epsilon)$
11	$\frac{1-m^2}{4}$	$\frac{1+m^2}{2}$	$\frac{-1}{4}$	$\mathrm{nc}(\epsilon) \pm \mathrm{sc}(\epsilon)$ or $\frac{cn(\epsilon)}{1 \pm sn(\epsilon)}$
12	$\frac{-(1-m^2)^2}{4}$	$\frac{1+m^2}{2}$	$\frac{-1}{4}$	$\mathrm{mcn}(\epsilon) \pm \mathrm{dn}(\epsilon)$
13	$\frac{1}{4}$	$\frac{1-2m^2}{2}$	$\frac{1}{4}$	$\frac{sn(\epsilon)}{1 \pm cn(\epsilon)}$
14	$\frac{1}{4}$	$\frac{1+m^2}{2}$	$\frac{(1-m^2)^2}{4}$	$\frac{sn(\epsilon)}{cn(\epsilon) \pm dn(\epsilon)}$

3. By substituting (5) with (6) into (4) and collecting the coefficients of $K^j(\epsilon)$ $(j = 0, \pm 1, \pm 2, ...)$, a set of specified algebraic equations consisting of a_0, a_i, b_i $(i = 1, 2, ..., M)$. By solving these algebraic equations, these parameters can be clearly determined.
4. Equation (6) will have Jacobi elliptic function solutions in Table 1. In Table 1, sn(ϵ) = sn(ϵ,m), cd(ϵ) = cd(ϵ,m), cn(ϵ) = cd(ϵ,m), dn(ϵ) = dn(ϵ), ns(ϵ) = ns(ϵ,m), cs(ϵ) = cs(ϵ,m), ds(ϵ) = ds(ϵ,m), sc(ϵ) = sc(ϵ,m), sd(ϵ) = sd(ϵ,m) are the Jacobi elliptic functions with the modulus $0 \le m \le 1$. These functions transform into trigonometric and hyperbolic functions when $m \to 0$ and $m \to 1$ as Table 2 shows [34].

Table 2. Transformation of Jacobian elliptic functions to trigonometric and hyperbolic functions.

$m \to 0$	$m \to 1$
sn(ϵ) = sin(ϵ)	sn(ϵ) = tanh(ϵ)
cd(ϵ) = cos(ϵ)	cn(ϵ) = sech(ϵ)
cn(ϵ) = cos(ϵ)	dn(ϵ) = sech(ϵ)
ns(ϵ) = csc(ϵ)	ns(ϵ) = coth(ϵ)
cs(ϵ) = cot(ϵ)	cs(ϵ) = csch(ϵ)
ds(ϵ) = csc(ϵ)	ds(ϵ) = csch(ϵ)
sc(ϵ) = tan(ϵ)	sc(ϵ) = sinh(ϵ)
sd(ϵ) = sin(ϵ)	sd(ϵ) = sinh(ϵ)
nc(ϵ) = sec(ϵ)	ns(ϵ) = cosh(ϵ)
dn(ϵ) = 1	cd(ϵ) = 1

5. By substituting the parameters found in step3 and the known values in step4 into (5), the solutions of (2) are found.

3. Applications to Fractional Equations with Beta Derivatives

In this section, the exact solutions of the space-time fractional modified Benjamin Bona Mahony equation and the nonlinear time fractional Schrödinger equation with beta-derivative will be investigated by F-expansion method.

Example 1. *Let us consider the space-time fractional modified Benjamin Bona Mahony equation [32,35,36]*

$$ {}^A D_t^{\beta} p(x,t) + {}^A D_x^{\beta} p(x,t) - a p(x,t)^2 \, {}^A D_x^{\beta} p(x,t) + {}^A D_x^{3\beta} p(x,t) = 0, \quad 0 < \beta \leq 1. \tag{7} $$

Let us assume the travelling wave solution of (7) has the form

$$ p(x,t) = U(\epsilon), \quad \epsilon = \frac{k}{\beta}\left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta} - \frac{c}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}. \tag{8} $$

Substituting (8) into (7), we have

$$ -c\frac{dU}{d\epsilon} + k\frac{dU}{d\epsilon} - akU^2\frac{dU}{d\epsilon} + k^3\frac{d^3U}{d\epsilon^3} = 0 \tag{9} $$

Integrating this equation with respect to ϵ and neglecting the constant of integration we obtain

$$ k^3\frac{d^2U}{d\epsilon^2} + (k-c)U - \frac{ak}{3}U^3 = 0 \tag{10} $$

Using the balancing principle between $\frac{d^2U}{d\epsilon^2}$ and $U^3(\epsilon)$ in (10) gives $M = 1$. Thus, from (5), the solution of (10) can be written

$$ U(\epsilon) = a_0 + a_1 K(\epsilon) + \frac{b_1}{K(\epsilon)}, \tag{11} $$

where a_0,a_1,b_1 are constant to be specified and $K(\epsilon)$ fulfills the elliptic Equation (6). Substituting (11) and (6) into (10), sixth order polynomial in $K(\epsilon)$ is found. By equating all coefficients of $K(\epsilon)$ to zero, the following nonlinear system of equations is found.

$$ \begin{aligned} 2k^3 f_2 a_1 - \frac{ak}{3}a_1^3 &= 0 \\ -aka_0a_1^2 &= 0 \\ k^3 f_1 a_1 + (k-c)a_1 - aka_0^2a_1 - aka_1^2b_1 &= 0 \\ (k-c)a_0 - \frac{ak}{3}a_0^3 - 2aka_0a_1b_1 &= 0 \\ k^3 f_1 b_1 + (k-c)b_1 - aka_0b_1 - aka_1b_1^2 &= 0 \\ -aka_0b_1^2 &= 0 \\ 2k^3 f_0 b_1 - \frac{ak}{3}b_1^3 &= 0 \end{aligned} $$

Solving this system with the help of Maple, we can get the following three sets of solutions for unknown coefficients.

set1	*set2*	*set3*
$c = f_1k^3 + k$	$c = f_1k^3 + k \pm 6k^3\sqrt{f_2 f_0}$	$c = f_1k^3 + k$
$a_0 = 0$	$a_0 = 0$	$a_0 = 0$
$a_1 = \pm\sqrt{6\frac{f_2}{a}}k$	$a_1 = \pm\sqrt{6\frac{f_2}{a}}k$	$a_1 = 0$
$b_1 = 0$	$b_1 = \pm\sqrt{6\frac{f_0}{a}}k$	$b_1 = \pm\sqrt{6\frac{f_0}{a}}k$
(*)	(**)	(***)

Substituting (*)–(***) into (11) with (8), we have the following exact solutions for Equation (7):

$$p_{s_1}(x,t) = \pm\sqrt{6\frac{f_2}{a}}kK(\epsilon);\ \epsilon = \frac{k}{\beta}\left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta} - \frac{f_1k^3 + k}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta} \tag{12}$$

$$\begin{aligned} p_{s_2}(x,t) &= \pm\sqrt{6\frac{f_2}{a}}K(\epsilon) \pm \sqrt{6\frac{f_0}{a}}K^{-1}(\epsilon); \\ \epsilon &= \frac{k}{\beta}\left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta} - \frac{k^3f_1 + k \pm 6k^3\sqrt{f_0.f_2}}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta} \end{aligned} \tag{13}$$

$$p_{s_3}(x,t) = \pm\sqrt{6\frac{f_0}{a}}kK^{-1}(\epsilon);\ \epsilon = \frac{k}{\beta}\left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta} - \frac{f_1k^3 + k}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta} \tag{14}$$

Combining (12)–(14) with Tables 1 and 2, the exact solutions of (7) are gained. Let us express some of them below for set1:

case 1. $f_2 = m^2,\ f_1 = -(1+m^2),\ f_0 = 1,\ K(\epsilon) = sn(\epsilon),\ U_1(\epsilon) = \pm\sqrt{6\frac{m^2}{a}}k\, sn(\epsilon);$

$$\epsilon = \frac{k}{\beta}\left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta} - \frac{k^3(-1+m^2) + k}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta} \tag{15}$$

when $m \to 1$, the exact solution of (7) is

$$p_1(x,t) = \pm\sqrt{\frac{6}{a}}k\tanh(\epsilon); \quad \epsilon = \frac{k}{\beta}\left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta} \quad (0 < a) \tag{16}$$

case 2. $f_2 = -m^2,\ f_1 = 2m^2 - 1,\ f_0 = 1 - m^2,\ K(\epsilon) = cn(\epsilon),\ U_2(\epsilon) = \pm\sqrt{6\frac{m^2}{a}}k\, cn(\epsilon);$

$$\epsilon = \frac{k}{\beta}\left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta} - \frac{k^3(2m^2-1) + k}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta} \tag{17}$$

when $m \to 1$, the exact solution of (7) is

$$p_2(x,t) = \pm\sqrt{\frac{-6}{a}}k\,\mathrm{sech}(\epsilon); \quad \epsilon = \frac{k}{\beta}\left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta} - \frac{k^3 + k}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}, \quad (a < 0) \tag{18}$$

In Figure 1, the solitary wave solution $p_2(x,t)$ in (18) with some special parameters are showed.

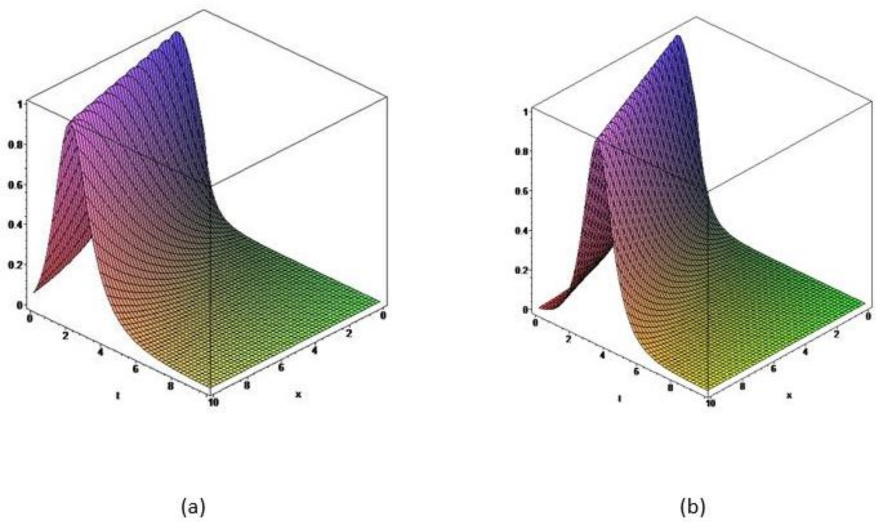


Figure 1. (**a**) The hyperbolic solution of $p_2(x,t)$ when $\beta = 0.5, a = -6, k = 1$. (**b**) The hyperbolic solution of $p_2(x,t)$ when $\beta = 0.75, a = -6, k = 1$.

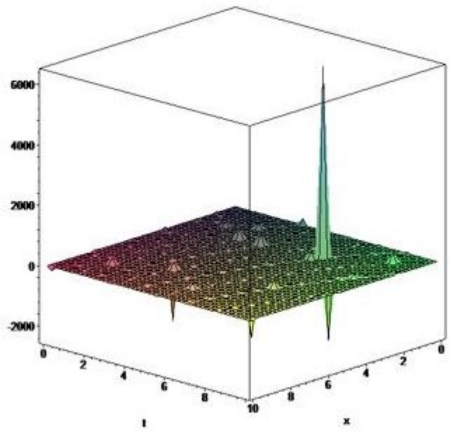
case 5. $f_2 = 1 - m^2,\ f_1 = 2m^2 - 1,\ f_0 = -m^2, K(\epsilon) = nc(\epsilon),\ U_5(\epsilon) = \pm\sqrt{6\frac{1-m^2}{a}}k\, nc(\epsilon);$

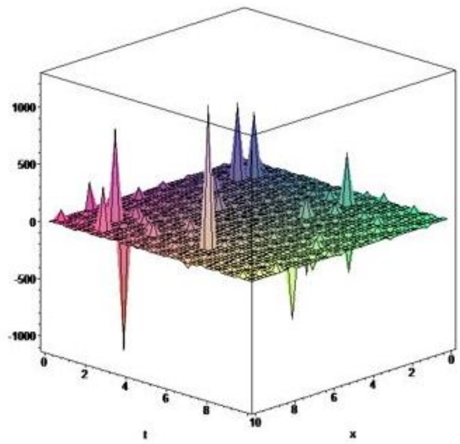
$$\epsilon = \frac{k}{\beta}\left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta} - \frac{k^3(2m^2-1)+k}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta} \tag{19}$$

when $m \to 0$, the exact solution of (7) is

$$p_5(x,t) = \pm\sqrt{\frac{6}{a}}k\sec(\epsilon); \quad \epsilon = \frac{k}{\beta}\left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta} - \frac{-k^3+k}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta} \tag{20}$$

In Figure 2, the periodic wave solution $p_5(x,t)$ in (2) with some special parameters is demonstrated.





(a) (b)

Figure 2. (**a**) The trigonometric solution of $p_5(x,t)$ when $\beta = 0.5, a = 6, k = 2$. (**b**) The trigonometric solution of $p_5(x,t)$ when $\beta = 0.75, a = 6, k = 2$.

case 9. $f_2 = 1,\ f_1 = 2 - m^2,\ f_0 = 1 - m^2,\ K(\epsilon) = cs(\epsilon),\ U_9(\epsilon) = \pm\sqrt{\frac{6}{a}}k\, cs(\epsilon);$

$$\epsilon = \frac{k}{\beta}\left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta} - \frac{k^3(2-m^2)+k}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta} \tag{21}$$

when $m \to 1$, the exact solution of (7) is

$$p_{9a}(x,t) = \pm\sqrt{\frac{6}{a}}k\,\mathrm{csch}(\epsilon); \quad \epsilon = \frac{k}{\beta}\left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta} - \frac{k^3+k}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}, \tag{22}$$

when $m \to 0$, the exact solution of (7) is

$$p_{9b}(x,t) = \pm\sqrt{\frac{6}{a}}k\cot(\epsilon); \quad \epsilon = \frac{k}{\beta}\left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta} - \frac{2k^3+k}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta} \tag{23}$$

The remaining solutions can also be found similarly.

Example 2. *Let us regard the nonlinear time fractional Schrödinger equation [32,37]*

$$i^A D_t^{\beta}\{p(x,t)\} + ap(x,t)_{xx} + b \mid p(x,t) \mid^2 p(x,t) = 0, \quad 0 < \beta \leq 1, \tag{24}$$

where $p(x,t)$ is a complex value function. We get the travelling wave solution of (24) and we apply the transformation as follows:

$$p(x,t) = U(\epsilon).e^{i\varphi}, \quad \epsilon = kx - \frac{2cv}{\beta}\Big(t + \frac{1}{\Gamma(\beta)}\Big)^{\beta}, \quad \varphi = \omega x + \frac{v}{\beta}\Big(t + \frac{1}{\Gamma(\beta)}\Big)^{\beta} \tag{25}$$

where k, c, v and ω are constants. Substituting (25) into (24) we get

$$i\Big(-2cv\frac{dU}{d\epsilon} + 2ak\omega\frac{dU}{d\epsilon}\Big) + ak^2\frac{d^2U}{d\epsilon^2} - (v + a\omega^2)U(\epsilon) + bU^3(\epsilon) = 0 \tag{26}$$

From the imajinary part of (26), we obtain

$$c = \frac{ak\omega}{v} \tag{27}$$

From the real part of (26), by using the balancing principle between $\frac{dU^2}{d\epsilon^2}$ and $U^3(\epsilon)$, we get $M = 1$. Therefore, the solution of

$$aK^2\frac{d^2U}{d\epsilon^2} - (v + aw^2)U(\epsilon) + bU^3(\epsilon) = 0 \tag{28}$$

can be written

$$U(\epsilon) = a_0 + a_1K(\epsilon) + \frac{b_1}{K(\epsilon)}, \tag{29}$$

where a_0, a_1, b_1 are constants to be defined and $K(\epsilon)$ satisfies the elliptic Equation (6). Substituting (29) and (6) into (28), sixth order polynomial in $K(\epsilon)$ is obtained. By equating all coefficients to zero in this polynomial, the following nonlinear system equations is gained.

$$\begin{aligned}
2ak^2a_1f_2 + ba_1^3 &= 0 \\
3a_0a_1^2b &= 0 \\
ak^2f_1a_1 - ak^2(v + a\omega^2)a_1 - 3a_0^2a_1b - 3a_1^2b_1b &= 0 \\
-ak^2(v + aw^2)a_0 + ba_0^3 - 6a_0a_1b_1b &= 0 \\
ak^2f_1b_1 - ak^2(v + a\omega^2)b_1 + 3a_0^2b_1b - +a_1b_1^2b &= 0 \\
3a_0b_1^2b &= 0 \\
2ak^2f_0b_1 + b_1^3b &= 0
\end{aligned}$$

Resolving this system by aid of Maple, we can obtain the following three sets of solutions for unknown coefficients.

set1	*set2*	*set3*
$a_0 = 0$	$a_0 = 0$	$a_0 = 0$
$a_1 = \pm\sqrt{\frac{-2f_2a}{b}}k$	$a_1 = \pm\sqrt{\frac{-2f_2a}{b}}k$	$a_1 = 0$
$b_1 = 0$	$b_1 = \pm\sqrt{\frac{-2f_0a}{b}}k$	$\pm\sqrt{6\frac{f_0a}{b}}k$
$c = c$	$c = c$	$c = c$
$v = -a\omega^2 + f_1$	$v = \pm 6\sqrt{f_0f_2} - a\omega^2 + f_1$	$v = -a\omega^2 + f_0$
(*)	(**)	(***)

Substituting (*)–(***) into (29) with (25), we have the following exact solutions for (24):

$$\begin{aligned}
p_{s_1}(x,t) &= \Big[\pm\sqrt{\frac{-2f_2a}{b}}kK(\epsilon)\Big]e^{i\Big[wx + \frac{-a\omega^2 + f_1}{\beta}\Big(t + \frac{1}{\Gamma(\beta)}\Big)^{\beta}\Big]}; \\
\epsilon &= kx - 2a\frac{k\omega}{\beta}\Big(t + \frac{1}{\Gamma(\beta)}\Big)^{\beta}
\end{aligned} \tag{30}$$

$$p_{s_2}(x,t) = \pm\sqrt{2\frac{f_2 a}{b}} kK(\epsilon) \pm \sqrt{2\frac{f_0 a}{b}} K^{-1}(\epsilon) e^{i\left[wx + \frac{\pm 6\sqrt{\frac{f_2 f_0}{\beta}} - a\omega^2 + f_1}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}\right]};$$
$$\epsilon = kx - \frac{2ak\omega}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta} \tag{31}$$

$$p_{s_3}(x,t) = \pm\sqrt{2\frac{f_0 a}{b}} kK^{-1}(\epsilon) e^{i\left[wx + \frac{-a\omega^2 + f_1}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}\right]};$$
$$\epsilon = kx - \frac{2ak\omega}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta} \tag{32}$$

Combining (30)–(32) with Tables 1 and 2, the exact solutions of (24) are found. Let us define some of them below for set1:

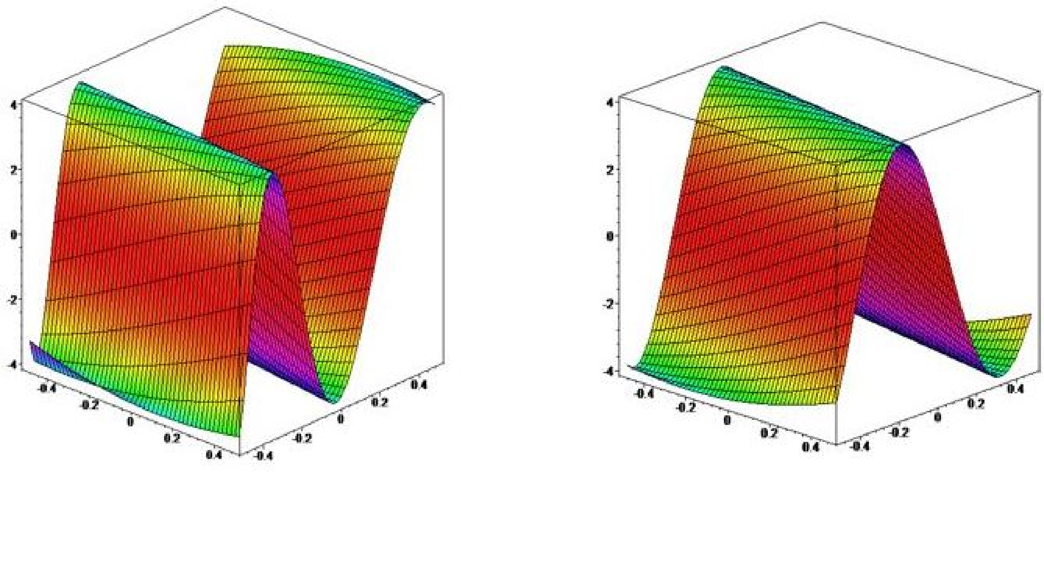
case 1. $f_2 = m^2,\ f_1 = -(1+m^2),\ f_0 = 1,\ K(\epsilon) = sn(\epsilon), U_1(\epsilon) = \pm\sqrt{-2a\frac{m^2}{b}} k\, sn(\epsilon);$

$$\varphi = i\left[wx - \frac{a\omega^2 + 1 + m^2}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}\right] \tag{33}$$

when $m \to 1$, the exact solution of (24) is

$$p_1(x,t) = U_1(\epsilon)e^{i}\varphi = \left[\pm\sqrt{\frac{-2a}{b}} k \tanh(\epsilon)\right] e^{i\left[wx - \frac{a\omega^2 + 2}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}\right]};$$
$$\epsilon = kx - \frac{2ak\omega}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta},\ (ab < 0) \tag{34}$$

In Figure 3, the solitary wave solution $p_1(x,t)$ in (3) with some special parameters is displayed.



(a) (b)

Figure 3. (**a**) The hyperbolic solution of $p_1(x,t)$ when $\beta = 0.5, a = -8, b = 1, k = 1, \omega = 1$. (**b**) The hyperbolic solution of $p_1(x,t)$ when $\beta = 0.75, a = -8, b = 1, k = 1, \omega = 1$.

case 4. $f_2 = 1,\ f_1 = -(1+m^2),\ f_0 = m^2,\ K(\epsilon) = ns(\epsilon),\ U_4(\epsilon) = \left[\pm\sqrt{\frac{-2a}{b}} k\, ns(\epsilon)\right];$

$$\varphi = i\left[wx - \frac{a\omega^2 + 1 + m^2}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}\right] \tag{35}$$

when $m \to 0$, the exact solution of (24) *is*

$$p_{4a}(x,t) = U_{4a}(\epsilon)e^{i}\varphi = \pm\sqrt{\frac{-2a}{b}}k\csc(\epsilon)e^{i\left[wx-\frac{a\omega^2+1}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right]} \tag{36}$$

when $m \to 1$, the exact solution of (24) *is*

$$p_{4b}(x,t) = U_{4b}(\epsilon)e^{i\varphi} = \pm\sqrt{\frac{-2a}{b}}k\coth(\epsilon)e^{i\left[wx-\frac{a\omega^2+2}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right]} \tag{37}$$

where $\epsilon = kx - \frac{2ak\omega}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}, \quad (ab<0)$

case 7. $f_2 = 1-m^2,\ f_1 = 2-m^2,\ f_0 = -1, K(\epsilon) = sc(\epsilon), U_7(\epsilon) = \left[\pm\sqrt{\frac{-2(1-m^2)a}{b}}k\,sc(\epsilon)\right];$

$$\varphi = i\left[wx + \frac{2-a\omega^2-m^2}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right] \tag{38}$$

when $m \to 0$, the exact solution of (24) *is*

$$p_7(x,t) = \pm\sqrt{\tfrac{-2a}{b}}k\tan(\epsilon)e^{i\left[wx+\frac{2-a\omega^2}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right]};\ \epsilon = kx - \frac{2ak\omega}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta},\ (ab<0) \tag{39}$$

In Figure 4, the periodic wave solution $p_7(x,t)$ in (4) *with some special parameters is showed.*

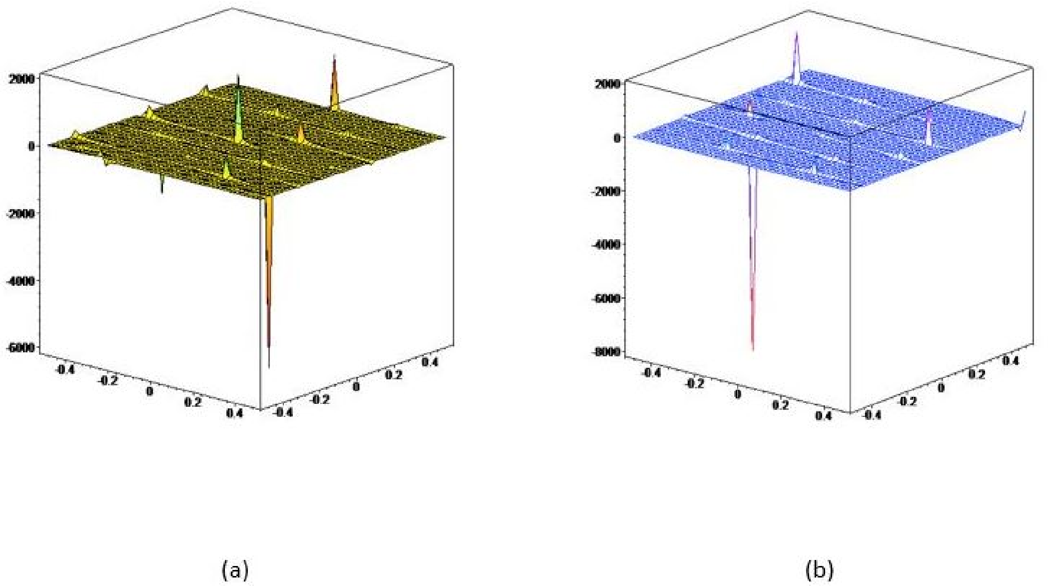


Figure 4. (**a**) The trigonometric solution of $p_1(x,t)$ when $\beta = 0.5, a = -8, b = 1, k = 1, \omega = 1$. (**b**) The trigonometric solution of $p_1(x,t)$ when $\beta = 0.75, a = -8, b = 1, k = 1, \omega = 1$.

The remaining solutions can also be found similarly.

4. Conclusions

In this study, we used the F-expansion method to solve fractional partial differential equations with beta-derivative. We applied this method to find the exact solution of the space-time fractional modified Benjamin-Bona-Mahory equation and the nonlinear time fractional Schrödinger equation.These solutions were validated by symbolic computing system. This study showed that the F-expansion method was an effective way, dependable and powerful to find new exact solutions.

When the results of this method are compared to previous publications, it is clear that they are novel. One of the important features of this method that we use is its diversity compared to other methods [32,35–37]. It is clear that we are able to get this. The findings in this work are valuable for characterizing some nonlinear processes and give good complements to the current literature. Many more nonlinear evolution problems may be solved using this approach. It's worth noting that the suggested approach may be used to solve various nonlinear evolution problems in mathematical physics. The solutions that were obtained in this study could be of significance for the meaning of some concerned physical problems.

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