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Certain New Chebyshev and Grüss-Type Inequalities for Unified Fractional Integral Operators via an Extended Generalized Mittag-Leffler Function

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Abstract: In this paper, by adopting the classical method of proofs, we establish certain new Chebyshev and Grüss-type inequalities for unified fractional integral operators via an extended generalized Mittag-Leffler function. The main results are more general and include a large number of available classical fractional integral inequalities in the literature. Furthermore, some new fractional integral inequalities similar to the main results can be also obtained by employing the newly introduced generalized fractional integral operators involving the Mittag-Leffler-like function and weighted function. Consequently, their relevance with known inequalities for different kinds of fractional integral operators are pointed out.

Keywords: Chebyshev integral inequality; Grüss-type inequalities; synchronous functions; Mittag-Leffler function; unified fractional integral operators

MSC: 26D10; 26A33; 31A10



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1. Introduction

Let f and g be two continuous and synchronous functions on $[a, b]$, that is, the two continuous functions f and g satisfying $(f(x) - f(y))(g(x) - g(y)) \geq 0$ for $x, y \in [a, b]$. Then the following inequality holds

$$\int_a^b f(x)g(x)dx \geq \int_a^b f(x)dx \int_a^b g(x)dx. \quad (1)$$

The reverse inequality holds always whenever f and g are two continuous and asynchronous functions. The foregoing inequality (1) is called as the well-known Chebyshev integral inequality. Over the last several years, by employing various kinds of fractional integral operators, many researchers have extended the classical inequalities to fractional integral inequalities at home and abroad, we refer the reader to [1–6] and the references quoted therein. For example, using the Saigo fractional integral operators, Khan et al. [7] presented some inequalities for a class of n -decreasing positive functions. With the help of fixed-point theorems and inequalities analysis techniques, Baleanu et al. [8] and Khan et al. [9] investigated the existence results for hybrid fractional differential equation boundary value problems, respectively. At present, there have been a great deal of fractional integral operators and their applications introduced in the books [10,11]. Furthermore, Belarbi and Dahmani [12] used the Riemann-Liouville fractional integrals to present the Chebyshev-type integral inequalities. In other words, if f and g are two synchronous functions on $C[0, +\infty)$, then, for $x > 0$ and $\alpha > 0$, the following two inequalities hold

$$\mathcal{R}^\alpha(fg)(x) \geq \frac{\Gamma(\alpha + 1)}{x^\alpha} \mathcal{R}^\alpha f(x) \mathcal{R}^\alpha g(x), \quad (2)$$

and for $x > 0$ and $\alpha, \beta > 0$,

$$\frac{x^\alpha}{\Gamma(\alpha+1)} \mathcal{R}^\beta(fg)(x) + \frac{x^\beta}{\Gamma(\beta+1)} \mathcal{R}^\alpha(fg)(x) \geq \mathcal{R}^\alpha f(x) \mathcal{R}^\beta g(x) + \mathcal{R}^\beta f(x) \mathcal{R}^\alpha g(x), \quad (3)$$

where \mathcal{R}^α and \mathcal{R}^β denote the Riemann-Liouville fractional integrals of order α and β , respectively. Similar to the inequalities (2) and (3), Ögünmez and Özkan [13], Chinchane and Pachpatte [14], Purohit and Raina [15], Habib et al. [16] and Set et al. [17] investigated the Chebyshev-type inequalities for the Riemann-Liouville fractional q -integral operators, the Hadamard fractional integral operators, the Saigo fractional integral and q -integral operators, and generalized k -fractional conformable integrals, respectively. Here it is easy to see that the Riemann-Liouville fractional integral and q -integral operators can be seen as the special case of the Saigo fractional integral and q -integral operators, respectively.

By applying the Riemann-Liouville fractional integral operators, Dahmani [18] obtained the following weighted fractional Chebyshev-type integral inequalities, which are the extensions of inequalities (2) and (3). Under the same conditions of inequalities (2) and (3), furthermore, let $u, v : [0, \infty) \rightarrow [0, \infty)$ be continuous. Then we have

$$\begin{aligned} \mathcal{R}^\alpha u(x) \mathcal{R}^\alpha(vfg)(x) + \mathcal{R}^\alpha v(x) \mathcal{R}^\alpha(ufg)(x) \\ \geq \mathcal{R}^\alpha(uf)(x) \mathcal{R}^\alpha(vg)(x) + \mathcal{R}^\alpha(vf)(x) \mathcal{R}^\alpha(ug)(x), \end{aligned} \quad (4)$$

and

$$\begin{aligned} \mathcal{R}^\alpha u(x) \mathcal{R}^\beta(vfg)(x) + \mathcal{R}^\beta v(x) \mathcal{R}^\alpha(ufg)(x) \\ \geq \mathcal{R}^\alpha(uf)(x) \mathcal{R}^\beta(vg)(x) + \mathcal{R}^\beta(vf)(x) \mathcal{R}^\alpha(ug)(x) \end{aligned} \quad (5)$$

for $x > 0$ and $\alpha, \beta > 0$. Similar to inequalities (4) and (5), Chinchane and Pachpatte [19,20], Brahim and Taf [21], Yang [22,23] and Liu et al. [24] studied the weighted fractional Chebyshev-type integral inequalities for Hadamard and Saigo fractional integral operators, fractional integral operators with two parameters of deformation q_1 and q_2 , fractional q -integral operators, Saigo fractional integral and q -integral operators, and generalized fractional integral operators involving the Gauss hypergeometric function, respectively.

In the book [25], the following inequality is provided:

$$\begin{aligned} \left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \left(\frac{1}{b-a} \int_a^b f(x)dx \right) \left(\frac{1}{b-a} \int_a^b g(x)dx \right) \right| \\ \leq \frac{1}{4} (\Phi_2 - \Phi_1)(\Psi_2 - \Psi_1), \end{aligned} \quad (6)$$

where f and g are two integrable functions on $[a, b]$ satisfying the following conditions

$$\Phi_1 \leq f(x) \leq \Phi_2 \text{ and } \Psi_1 \leq g(x) \leq \Psi_2, \quad \Phi_1, \Phi_2, \Psi_1, \Psi_2 \in \mathbb{R}, \quad x \in [a, b]. \quad (7)$$

Here inequality (6) is well-known Grüss inequality. It has attracted extensive attention of scholars all over the world. For example, Elezović et al. [26] derived some Grüss type inequalities related to Chebyshev functional under the function spaces L_p with weight function and exponents. Liu and Ngô [27] gave the inequality of Ostrowski-Grüss type on time scales, which unified corresponding continuous, discrete and quantum calculus versions. Dragomir [28] established some sharp Grüss type inequalities for functions with bounded variation and selfadjoint operators in Hilbert space. Furthermore, Dragomir [29] obtained some Grüss type inequalities for the complex integral under various assumptions.

When f, g satisfy the conditions (7), Dragomir [30] proved the following inequality

$$|S(f, g, u)| \leq \frac{1}{4} (\Phi_2 - \Phi_1)(\Psi_2 - \Psi_1) \left(\int_a^b u(x)dx \right)^2, \quad (8)$$

where u and v are two nonnegative continuous functions on $[a, b]$ and the Chebyshev functionals were defined as

$$S(f, g, u) = \frac{1}{2} T(f, g, u, u) \\ = \int_a^b u(x) dx \int_a^b u(x) f(x) g(x) dx - \int_a^b u(x) f(x) dx \int_a^b u(x) g(x) dx, \quad (9)$$

and

$$T(f, g, u, v) = \int_a^b v(x) dx \int_a^b u(x) f(x) g(x) dx + \int_a^b u(x) dx \int_a^b v(x) f(x) g(x) dx \\ - \int_a^b v(x) f(x) dx \int_a^b u(x) g(x) dx - \int_a^b u(x) f(x) dx \int_a^b v(x) g(x) dx. \quad (10)$$

When $f', g' \in L_\infty(a, b)$, Dragomir [30] had the following inequality

$$|S(f, g, u)| \leq \|f'\|_\infty \|g'\|_\infty \left(\int_a^b u(x) dx \int_a^b x^2 u(x) dx - \left(\int_a^b x u(x) dx \right)^2 \right). \quad (11)$$

Furthermore, let f be M - g -Lipschitzian on $[a, b]$, i.e.,

$$|f(x) - f(y)| \leq M |g(x) - g(y)|, \quad M > 0, \quad x, y \in [a, b], \quad (12)$$

then the following inequality holds [30]

$$|S(f, g, u)| \leq M \left(\int_a^b u(x) dx \int_a^b u(x) g^2(x) dx - \left(\int_a^b u(x) g(x) dx \right)^2 \right). \quad (13)$$

Let f and g be L_1 and L_2 -lipschitzian functions on $[a, b]$, respectively; Dragomir [30] provided the inequality

$$|S(f, g, u)| \leq L_1 L_2 \left(\int_a^b u(x) dx \int_a^b x^2 u(x) dx - \left(\int_a^b x u(x) dx \right)^2 \right). \quad (14)$$

Similar to inequality (6), Dahmani et al. [31] and Zhu et al. [32] studied the Grüss type inequality for Riemann-Liouville fractional integral and q -integral operators satisfying the conditions (7), respectively. Similar to inequality (8), Dahmani and Benzidane [33] gave the Riemann-Liouville fractional q -integral inequality satisfying the conditions (7). Dahmani [34] obtained the fractional integral inequalities (11), (13) and (14) for the extended Chebyshev functional (10) based on the Riemann-Liouville fractional integrals. Based on the Riemann-Liouville fractional q -integral and integral operators, Brahim and Taf [21,35] established the fractional q -integral and integral inequalities (11), (13) and (14) for the extended Chebyshev functional (10) with two parameters of deformation q_1 and q_2 , respectively. By using the Saigo fractional integral and q -integral operators, the author obtained the Saigo fractional integral and q -integral inequalities (8), (11), (13) and (14) for the extended Chebyshev functional (10), respectively. Akdemir et al. [36] gave the general variants of Chebyshev type inequalities using the generalized fractional integral operators.

In 2020, Yang et al. [37] obtained the unified fractional generalized Hadamard and Fejér-Hadamard inequalities for m -convex functions containing extended generalized Mittag-Leffler function. In 2021, Zhang et al. [38] investigated some inequalities for unified fractional integral operators via strongly $(\alpha, h - m)$ -convex function. In 2021, Jung et al. [39] studied the refinements of some integral inequalities for unified fractional integral operators. Motivated by the works mentioned earlier, the main aim of this paper is to establish certain new Chebyshev and Grüss-type inequalities for unified fractional integral operators via an extended generalized Mittag-Leffler function by using the classical method of proofs In Section 2. In Section 3, we show that the unified fractional integral operators contains a lot

of existing fractional integral operators. We also introduce two newly generalized fractional integral operators involving the Mittag-Leffler-like function and weighted function. Using the newly introduced generalized fractional integral operators, some new fractional integral inequalities can be also obtained. Furthermore, their relevance with known inequalities for different kinds of fractional integral operators are pointed out. The main results of this paper are more general and include a great number of available classical inequalities in the literature.

2. Unified Fractional Chebyshev and Grüss-Type Integral Inequalities

In this section, we firstly introduce the definitions of the extended generalized Mittag-Leffler function $\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}(t;p)$ and unified fractional integral operator $(\xi \mathcal{F}_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c} \psi)(x;p)$ as follows.

Definition 1 (See [40,41]). Let $\mu, \alpha, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\alpha), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$ and $0 < k \leq \delta + \Re(\mu)$. Then the extended generalized Mittag-Leffler function $\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}(t;p)$ is defined by

$$\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}(t;p) = \sum_{n=0}^{\infty} \frac{B_p(\gamma + nk, c - \gamma)}{B(\gamma, c - \gamma)} \frac{(c)_{nk}}{\Gamma(n\mu + \alpha)} \frac{t^n}{(l)_{n\delta}}, \quad (15)$$

where $\Re(\mu)$ denotes the real part of complex number, Γ and B represent the Gamma and Beta functions, respectively. Here the generalized Pochhammer symbol $(c)_{nk}$ and an extension of the beta function B_p are defined as follows:

$$(c)_{nk} = \frac{\Gamma(c + nk)}{\Gamma(c)} \quad \text{and} \quad B_p(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} e^{-\frac{p}{t(1-t)}} dt, \quad (16)$$

where $\Re(x), \Re(y), \Re(p) > 0$.

Definition 2 (See [40,41]). Let $\psi, \xi : [a, b] \rightarrow \mathbb{R}$, $0 < a < b$, be the functions such that ψ be positive and $\psi \in L_1[a, b]$, and ξ be differentiable and strictly increasing. Also let ϕ be a positive function such that ϕ/x is an increasing on $[a, +\infty)$ and $\omega, \mu, \alpha, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\alpha), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$ and $0 < k \leq \delta + \Re(\mu)$. Then for $x \in [a, b]$, the left and right unified fractional integral operators are defined by

$$(\xi \mathcal{F}_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c} \psi)(x;p) = \int_a^x \frac{\phi(\xi(x) - \xi(t))}{\xi(x) - \xi(t)} \mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\xi(x) - \xi(t))^\mu; p) \psi(t) d(\xi(t)), \quad (17)$$

$$(\xi \mathcal{F}_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c} \psi)(x;p) = \int_x^b \frac{\phi(\xi(t) - \xi(x))}{\xi(t) - \xi(x)} \mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\xi(t) - \xi(x))^\mu; p) \psi(t) d(\xi(t)). \quad (18)$$

Here we can also define the operators $(\xi \mathcal{G}_{\nu,\beta,l,a^+}^{\phi,\gamma,\delta,k,c} \psi)(x;p)$ and $(\xi \mathcal{G}_{\nu,\beta,l,b^-}^{\phi,\gamma,\delta,k,c} \psi)(x;p)$ similar to the operators in (3) and (4). For convenience, in the section, $(\xi \mathcal{F}_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c} \psi)(x;p)$ and $(\xi \mathcal{G}_{\nu,\beta,l,a^+}^{\phi,\gamma,\delta,k,c} \psi)(x;p)$ are abbreviated as $(\mathfrak{F}^\phi \psi)(x)$ and $(\mathfrak{G}^\phi \psi)(x)$, respectively. Let $(fg)(x) = f(x)g(x)$ and $i(t) = t$ denote the identity operator throughout this paper and the kernel function $\mathcal{K}_x^t(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \xi; \phi)$ be defined by

$$\mathcal{K}_x^t(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \xi; \phi) = \frac{\phi(\xi(x) - \xi(t))}{\xi(x) - \xi(t)} \mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\xi(x) - \xi(t))^\mu; p). \quad (19)$$

Remark 1. From the paper [42], the kernel function $\mathcal{K}_x^t(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \xi; \phi)$ given in (19) have the following properties:

- (P1) Let ξ and ϕ/x be increasing functions. Then for $x < z < t$, $x, t \in [a, b]$, the kernel function $\mathcal{K}_x^t(\mathcal{E}_{\mu, \alpha, l}^{\gamma, \delta, k, c}, \xi; \phi)$ satisfies the following inequality $\mathcal{K}_z^x(\mathcal{E}_{\mu, \alpha, l}^{\gamma, \delta, k, c}, \xi; \phi)\xi'(z) \leq \mathcal{K}_t^x(\mathcal{E}_{\mu, \alpha, l}^{\gamma, \delta, k, c}, \xi; \phi)\xi'(z)$. The reverse of the foregoing inequality holds when ξ and ϕ/x are decreasing.
- (P2) Let ξ and ϕ/x be increasing functions. If $\phi(0) = \phi'(0) = 0$, then for $x, t \in [a, b]$, $\mathcal{K}_x^t(\mathcal{E}_{\mu, \alpha, l}^{\gamma, \delta, k, c}, \xi; \phi) \geq 0$.
- (P3) For $\varrho, \sigma \in \mathbb{R}$, $\mathcal{K}_x^t(\mathcal{E}_{\mu, \alpha, l}^{\gamma, \delta, k, c}, \xi; \varrho\phi_1 + \sigma\phi_2) = \varrho\mathcal{K}_x^t(\mathcal{E}_{\mu, \alpha, l}^{\gamma, \delta, k, c}, \xi; \phi_1) + \sigma\mathcal{K}_x^t(\mathcal{E}_{\mu, \alpha, l}^{\gamma, \delta, k, c}, \xi; \phi_2)$.

In this section, we nextly give some new Chebyshev-type integral inequalities for the synchronous functions involving the left unified fractional integral operators.

Lemma 1. Suppose that f, g are two synchronous functions on $[a, b]$ and let u, v be two nonnegative continuous functions on $[a, b]$. Then the following inequality holds

$$(\mathfrak{F}^\phi u)(x)(\mathfrak{F}^\phi vfg)(x) + (\mathfrak{F}^\phi v)(x)(\mathfrak{F}^\phi ufg)(x) \geq (\mathfrak{F}^\phi uf)(x)(\mathfrak{F}^\phi vg)(x) + (\mathfrak{F}^\phi vf)(x)(\mathfrak{F}^\phi ug)(x). \quad (20)$$

Proof. Since f and g are two synchronous functions on $[a, b]$, then for all $\tau > 0$ and $\rho > 0$, we have

$$(f(\tau) - f(\rho))(g(\tau) - g(\rho)) \geq 0. \quad (21)$$

It follows from (21) that we write

$$f(\tau)g(\tau) + f(\rho)g(\rho) \geq f(\tau)g(\rho) + f(\rho)g(\tau). \quad (22)$$

Multiplying both sides of (22) by $v(\tau)\xi'(\tau)\mathcal{K}_x^\tau(\mathcal{E}_{\mu, \alpha, l}^{\gamma, \delta, k, c}, \xi; \phi)$ and integrating the obtained inequality with regard to τ from a to x , we get

$$(\mathfrak{F}^\phi vfg)(x) + f(\rho)g(\rho)(\mathfrak{F}^\phi v)(x) \geq g(\rho)(\mathfrak{F}^\phi vf)(x) + f(\rho)(\mathfrak{F}^\phi vg)(x). \quad (23)$$

Multiplying both sides of (23) by $u(\rho)\xi'(\rho)\mathcal{K}_x^\rho(\mathcal{E}_{\mu, \alpha, l}^{\gamma, \delta, k, c}, \xi; \phi)$ and integrating the obtained inequality with regard to ρ from a to x , we obtain

$$(\mathfrak{F}^\phi u)(x)(\mathfrak{F}^\phi vfg)(x) + (\mathfrak{F}^\phi v)(x)(\mathfrak{F}^\phi ufg)(x) \geq (\mathfrak{F}^\phi uf)(x)(\mathfrak{F}^\phi vg)(x) + (\mathfrak{F}^\phi vf)(x)(\mathfrak{F}^\phi ug)(x), \quad (24)$$

which implies (20). \square

Theorem 1. Suppose that f, g are two synchronous functions on $[a, b]$ and let $\mathfrak{u}, \mathfrak{v}, \mathfrak{w}$ be three nonnegative continuous functions on $[a, b]$. Then the following inequality holds

$$\begin{aligned} & 2(\mathfrak{F}^\phi \mathfrak{u})(x)((\mathfrak{F}^\phi \mathfrak{v})(x)(\mathfrak{F}^\phi \mathfrak{w}fg)(x) + (\mathfrak{F}^\phi \mathfrak{w})(x)(\mathfrak{F}^\phi \mathfrak{v}fg)(x)) \\ & \quad + 2(\mathfrak{F}^\phi \mathfrak{v})(x)(\mathfrak{F}^\phi \mathfrak{w})(x)(\mathfrak{F}^\phi \mathfrak{u}fg)(x) \\ & \geq (\mathfrak{F}^\phi \mathfrak{u})(x)((\mathfrak{F}^\phi \mathfrak{v}f)(x)(\mathfrak{F}^\phi \mathfrak{w}g)(x) + (\mathfrak{F}^\phi \mathfrak{w}f)(x)(\mathfrak{F}^\phi \mathfrak{v}g)(x)) \\ & \quad + (\mathfrak{F}^\phi \mathfrak{v})(x)((\mathfrak{F}^\phi \mathfrak{u}f)(x)(\mathfrak{F}^\phi \mathfrak{w}g)(x) + (\mathfrak{F}^\phi \mathfrak{w}f)(x)(\mathfrak{F}^\phi \mathfrak{u}g)(x)) \\ & \quad + (\mathfrak{F}^\phi \mathfrak{w})(x)((\mathfrak{F}^\phi \mathfrak{u}f)(x)(\mathfrak{F}^\phi \mathfrak{v}g)(x) + (\mathfrak{F}^\phi \mathfrak{v}f)(x)(\mathfrak{F}^\phi \mathfrak{u}g)(x)). \end{aligned} \quad (25)$$

Proof. Putting $u = \mathfrak{v}$, $v = \mathfrak{w}$ and using Lemma 1, we can write

$$(\mathfrak{F}^\phi \mathfrak{w})(x)(\mathfrak{F}^\phi \mathfrak{w}fg)(x) + (\mathfrak{F}^\phi \mathfrak{w})(x)(\mathfrak{F}^\phi \mathfrak{v}fg)(x) \geq (\mathfrak{F}^\phi \mathfrak{v}f)(x)(\mathfrak{F}^\phi \mathfrak{w}g)(x) + (\mathfrak{F}^\phi \mathfrak{w}f)(x)(\mathfrak{F}^\phi \mathfrak{v}g)(x). \quad (26)$$

Multiplying both sides of (26) by $(\mathfrak{F}^\phi \mathfrak{u})(x)$, we obtain

$$\begin{aligned} & (\mathfrak{F}^\phi \mathfrak{u})(x)((\mathfrak{F}^\phi \mathfrak{v})(x)(\mathfrak{F}^\phi \mathfrak{w}fg)(x) + (\mathfrak{F}^\phi \mathfrak{w})(x)(\mathfrak{F}^\phi \mathfrak{v}fg)(x)) \\ & \geq (\mathfrak{F}^\phi \mathfrak{u})(x)((\mathfrak{F}^\phi \mathfrak{v}f)(x)(\mathfrak{F}^\phi \mathfrak{w}g)(x; p) + (\mathfrak{F}^\phi \mathfrak{w}f)(x)(\mathfrak{F}^\phi \mathfrak{v}g)(x)). \end{aligned} \quad (27)$$

Putting $u = \mathfrak{u}$, $v = \mathfrak{w}$ and using Lemma 1, we can write

$$\begin{aligned} & (\mathfrak{F}^\phi \mathfrak{u})(x)(\mathfrak{F}^\phi \mathfrak{w}fg)(x) + (\mathfrak{F}^\phi \mathfrak{w})(x)(\mathfrak{F}^\phi \mathfrak{u}fg)(x) \\ & \geq (\mathfrak{F}^\phi \mathfrak{u}f)(x)(\mathfrak{F}^\phi \mathfrak{w}g)(x) + (\mathfrak{F}^\phi \mathfrak{w}f)(x)(\mathfrak{F}^\phi \mathfrak{u}g)(x). \end{aligned} \quad (28)$$

Multiplying both sides of (28) by $(\mathfrak{F}^\phi \mathfrak{v})(x)$, we obtain

$$\begin{aligned} & (\mathfrak{F}^\phi \mathfrak{v})(x)(\mathfrak{F}^\phi \mathfrak{u})(x)(\mathfrak{F}^\phi \mathfrak{w}fg)(x) + (\mathfrak{F}^\phi \mathfrak{w})(x)(\mathfrak{F}^\phi \mathfrak{u}fg)(x) \\ & \geq (\mathfrak{F}^\phi \mathfrak{v})(x)((\mathfrak{F}^\phi \mathfrak{u}f)(x)(\mathfrak{F}^\phi \mathfrak{w}g)(x) + (\mathfrak{F}^\phi \mathfrak{w}f)(x)(\mathfrak{F}^\phi \mathfrak{u}g)(x)). \end{aligned} \quad (29)$$

With the same arguments as before, we can get

$$\begin{aligned} & (\mathfrak{F}^\phi \mathfrak{v})(x)(\mathfrak{F}^\phi \mathfrak{u})(x)(\mathfrak{F}^\phi \mathfrak{w}fg)(x) + (\mathfrak{F}^\phi \mathfrak{w})(x)(\mathfrak{F}^\phi \mathfrak{u}fg)(x) \\ & \geq (\mathfrak{F}^\phi \mathfrak{v})(x)((\mathfrak{F}^\phi \mathfrak{u}f)(x)(\mathfrak{F}^\phi \mathfrak{w}g)(x) + (\mathfrak{F}^\phi \mathfrak{w}f)(x)(\mathfrak{F}^\phi \mathfrak{u}g)(x)). \end{aligned} \quad (30)$$

The required inequality (25) follows on adding the inequalities (27), (29) and (30). \square

Lemma 2. Suppose that f, g are two synchronous functions on $[a, b]$ and let u, v be two nonnegative continuous functions on $[a, b]$. Then the following inequality holds

$$\begin{aligned} & (\mathfrak{F}^\phi u)(x)(\mathfrak{G}^\phi vfg)(x) + (\mathfrak{F}^\phi v)(x)(\mathfrak{G}^\phi ufg)(x) \\ & \geq (\mathfrak{F}^\phi uf)(x)(\mathfrak{G}^\phi vg)(x) + (\mathfrak{F}^\phi vf)(x)(\mathfrak{G}^\phi ug)(x). \end{aligned} \quad (31)$$

Proof. Multiplying both sides of (22) by $v(\tau)\zeta'(\tau)\mathcal{K}_x^\tau(\mathcal{E}_{\nu, \beta, \iota}^{\gamma, \delta, k, \epsilon}, \zeta; \varphi)$ ($\mathcal{K}_x^\tau(\mathcal{E}_{\nu, \beta, \iota}^{\gamma, \delta, k, \epsilon}, \zeta; \varphi)$ is defined similarly in (19)) and integrating the obtained inequality with regard to τ from a to x , we get

$$(\mathfrak{G}^\phi vfg)(x) + f(\rho)g(\rho)(\mathfrak{G}^\phi v)(x) \geq g(\rho)(\mathfrak{G}^\phi vf)(x) + f(\rho)(\mathfrak{G}^\phi vg)(x). \quad (32)$$

Multiplying both sides of (32) by $u(\rho)\zeta'(\rho)\mathcal{K}_x^\rho(\mathcal{E}_{\mu, \alpha, l}^{\gamma, \delta, k, \epsilon}, \zeta; \phi)$ and integrating the obtained inequality with regard to ρ from a to x , we obtain

$$\begin{aligned} & (\mathfrak{F}^\phi u)(x)(\mathfrak{G}^\phi vfg)(x) + (\mathfrak{F}^\phi v)(x)(\mathfrak{G}^\phi ufg)(x) \\ & \geq (\mathfrak{F}^\phi uf)(x)(\mathfrak{G}^\phi vg)(x) + (\mathfrak{F}^\phi vf)(x)(\mathfrak{G}^\phi ug)(x), \end{aligned} \quad (33)$$

which implies (31). \square

Theorem 2. Suppose that f, g are two synchronous functions on $[a, b]$ and let $\mathfrak{u}, \mathfrak{v}, \mathfrak{w}$ be three nonnegative continuous functions on $[a, b]$. Then the following inequality holds

$$\begin{aligned} & (\mathfrak{F}^\phi \mathfrak{u})(x)((\mathfrak{F}^\phi \mathfrak{w})(x)(\mathfrak{G}^\phi \mathfrak{v}fg)(x) + 2(\mathfrak{F}^\phi \mathfrak{v})(x)(\mathfrak{G}^\phi \mathfrak{w}fg)(x) + (\mathfrak{F}^\phi \mathfrak{v}fg)(x)(\mathfrak{G}^\phi \mathfrak{w})(x)) \\ & \quad + (\mathfrak{F}^\phi \mathfrak{u}fg)(x)((\mathfrak{F}^\phi \mathfrak{v})(x)(\mathfrak{G}^\phi \mathfrak{w})(x) + (\mathfrak{F}^\phi \mathfrak{w})(x)(\mathfrak{G}^\phi \mathfrak{v})(x))) \\ & \geq (\mathfrak{F}^\phi \mathfrak{u})(x)((\mathfrak{F}^\phi \mathfrak{v}f)(x)(\mathfrak{G}^\phi \mathfrak{w}g)(x) + (\mathfrak{F}^\phi \mathfrak{w}f)(x)(\mathfrak{G}^\phi \mathfrak{v}g)(x)) \\ & \quad + (\mathfrak{F}^\phi \mathfrak{v})(x)((\mathfrak{F}^\phi \mathfrak{u}f)(x)(\mathfrak{G}^\phi \mathfrak{w}g)(x) + (\mathfrak{F}^\phi \mathfrak{w}f)(x)(\mathfrak{G}^\phi \mathfrak{u}g)(x)) \\ & \quad + (\mathfrak{F}^\phi \mathfrak{w})(x)((\mathfrak{F}^\phi \mathfrak{u}f)(x)(\mathfrak{G}^\phi \mathfrak{v}g)(x) + (\mathfrak{F}^\phi \mathfrak{v}f)(x)(\mathfrak{G}^\phi \mathfrak{u}g)(x)). \end{aligned} \quad (34)$$

Proof. Putting $u = \mathbb{v}$, $v = \mathbb{w}$ and using Lemma 2, we can write

$$\begin{aligned} (\mathfrak{F}^\phi_{\mathbb{v}})(x)(\mathfrak{G}^\varphi_{\mathbb{w}}fg)(x) + (\mathfrak{F}^\phi_{\mathbb{w}})(x)(\mathfrak{G}^\varphi_{\mathbb{v}}fg)(x) \\ \geq (\mathfrak{F}^\phi_{\mathbb{v}}f)(x)(\mathfrak{G}^\varphi_{\mathbb{w}}g)(x) + (\mathfrak{F}^\phi_{\mathbb{w}}f)(x)(\mathfrak{G}^\varphi_{\mathbb{v}}g)(x). \end{aligned} \quad (35)$$

Multiplying both sides of (35) by $(\mathfrak{F}^\phi_{\mathbb{u}})(x)$, we obtain

$$\begin{aligned} (\mathfrak{F}^\phi_{\mathbb{u}})(x)((\mathfrak{F}^\phi_{\mathbb{v}})(x)(\mathfrak{G}^\varphi_{\mathbb{w}}fg)(x) + (\mathfrak{F}^\phi_{\mathbb{w}})(x)(\mathfrak{G}^\varphi_{\mathbb{v}}fg)(x)) \\ \geq (\mathfrak{F}^\phi_{\mathbb{u}})(x)((\mathfrak{F}^\phi_{\mathbb{v}}f)(x)(\mathfrak{G}^\varphi_{\mathbb{w}}g)(x) + (\mathfrak{F}^\phi_{\mathbb{w}}f)(x)(\mathfrak{G}^\varphi_{\mathbb{v}}g)(x)). \end{aligned} \quad (36)$$

Putting $u = \mathbb{u}$, $v = \mathbb{w}$ and using Lemma 2, we can write

$$\begin{aligned} (\mathfrak{F}^\phi_{\mathbb{u}})(x)(\mathfrak{G}^\varphi_{\mathbb{w}}fg)(x) + (\mathfrak{F}^\phi_{\mathbb{w}})(x)(\mathfrak{G}^\varphi_{\mathbb{u}}fg)(x) \\ \geq (\mathfrak{F}^\phi_{\mathbb{u}}f)(x)(\mathfrak{G}^\varphi_{\mathbb{w}}g)(x) + (\mathfrak{F}^\phi_{\mathbb{w}}f)(x)(\mathfrak{G}^\varphi_{\mathbb{u}}g)(x). \end{aligned} \quad (37)$$

Multiplying both sides of (37) by $(\mathfrak{F}^\phi_{\mathbb{v}})(x)$, we obtain

$$\begin{aligned} (\mathfrak{F}^\phi_{\mathbb{v}})(x)((\mathfrak{F}^\phi_{\mathbb{u}})(x)(\mathfrak{G}^\varphi_{\mathbb{w}}fg)(x) + (\mathfrak{F}^\phi_{\mathbb{w}})(x)(\mathfrak{G}^\varphi_{\mathbb{u}}fg)(x)) \\ \geq (\mathfrak{F}^\phi_{\mathbb{v}})(x)((\mathfrak{F}^\phi_{\mathbb{u}}f)(x)(\mathfrak{G}^\varphi_{\mathbb{w}}g)(x) + (\mathfrak{F}^\phi_{\mathbb{w}}f)(x)(\mathfrak{G}^\varphi_{\mathbb{u}}g)(x)). \end{aligned} \quad (38)$$

With the same arguments as before, we can get

$$\begin{aligned} (\mathfrak{F}^\phi_{\mathbb{w}})(x)((\mathfrak{F}^\phi_{\mathbb{u}})(x)(\mathfrak{G}^\varphi_{\mathbb{v}}fg)(x) + (\mathfrak{F}^\phi_{\mathbb{v}})(x)(\mathfrak{G}^\varphi_{\mathbb{u}}fg)(x)) \\ \geq (\mathfrak{F}^\phi_{\mathbb{w}})(x)((\mathfrak{F}^\phi_{\mathbb{u}}f)(x)(\mathfrak{G}^\varphi_{\mathbb{v}}g)(x) + (\mathfrak{F}^\phi_{\mathbb{v}}f)(x)(\mathfrak{G}^\varphi_{\mathbb{u}}g)(x)). \end{aligned} \quad (39)$$

The required inequality (34) follows on adding the inequalities (36), (38) and (39). \square

Remark 2. The reverse of the inequalities (25) and (34) hold under the following three cases: (I) The functions f and g asynchronous on $[a, b]$. (II) The weight functions \mathbb{u} , \mathbb{v} and \mathbb{w} are negative on $[a, b]$. (III) Two of the weight functions \mathbb{u} , \mathbb{v} and \mathbb{w} are positive and the third one is negative on $[a, b]$.

Remark 3. Let $\varphi = \phi$, $\hat{\gamma} = \gamma$, $\hat{\delta} = \delta$, $\hat{k} = k$, $\hat{c} = c$, $\nu = \mu$, $\beta = \alpha$, $\iota = l$, and $\zeta = \xi$, then Lemma 2 and Theorem 2 reduce to Lemma 1 and Theorem 1, respectively.

Theorem 3. Suppose that f, g are two synchronous functions on $[a, b]$ and let h, u be two nonnegative continuous functions on $[a, b]$. Then the following inequality holds

$$\begin{aligned} (\mathfrak{F}^\phi_{ufgh})(x)(\mathfrak{G}^\varphi_u)(x) + (\mathfrak{G}^\varphi_{ufgh})(x)(\mathfrak{F}^\phi_u)(x) + (\mathfrak{F}^\phi_{ufg})(x)(\mathfrak{G}^\varphi_{uh})(x) \\ + (\mathfrak{G}^\varphi_{ufg})(x)(\mathfrak{F}^\phi_{uh})(x) \geq (\mathfrak{F}^\phi_{ufh})(x)(\mathfrak{G}^\varphi_{ug})(x) + (\mathfrak{G}^\varphi_{ufh})(x)(\mathfrak{F}^\phi_{ug})(x) \\ + (\mathfrak{F}^\phi_{ugh})(x)(\mathfrak{G}^\varphi_{uf})(x) + (\mathfrak{G}^\varphi_{ugh})(x)(\mathfrak{F}^\phi_{uf})(x). \end{aligned} \quad (40)$$

Proof. Since f and g are two synchronous functions on $[a, b]$ and let h and u be two nonnegative continuous functions on $[a, b]$, then for all $\tau > 0$ and $\rho > 0$, we have

$$(f(\tau) - f(\rho))(g(\tau) - g(\rho))(h(\tau) + h(\rho)) \geq 0. \quad (41)$$

Expanding the left hand side of (41) that we write

$$\begin{aligned} h(\tau)f(\tau)g(\tau) + h(\tau)f(\rho)g(\rho) + h(\rho)f(\tau)g(\tau) + h(\rho)f(\rho)g(\rho) \\ \geq h(\tau)f(\tau)g(\rho) + h(\tau)f(\rho)g(\tau) + h(\rho)f(\tau)g(\rho) + h(\rho)f(\rho)g(\tau). \end{aligned} \quad (42)$$

Multiplying both sides of (42) by $u(\tau)\zeta'(\tau)\mathcal{K}_x^\tau(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \zeta; \phi)$ and integrating the obtained inequality with regard to τ from a to x , we get

$$\begin{aligned} & (\mathfrak{F}^\phi u f g h)(x) + f(\rho)g(\rho)(\mathfrak{F}^\phi u h)(x) + h(\rho)(\mathfrak{F}^\phi u f g)(x) + h(\rho)f(\rho)g(\rho)(\mathfrak{F}^\phi u)(x) \\ & \geq g(\rho)(\mathfrak{F}^\phi u f h)(x) + f(\rho)(\mathfrak{F}^\phi u g h)(x) + h(\rho)g(\rho)(\mathfrak{F}^\phi u f)(x) + h(\rho)f(\rho)(\mathfrak{F}^\phi u g)(x). \end{aligned} \quad (43)$$

Multiplying both sides of (43) by $u(\rho)\zeta'(\rho)\mathcal{K}_x^\rho(\mathcal{E}_{\nu,\beta,\iota}^{\hat{\gamma},\hat{\delta},\hat{k},\hat{c}}, \zeta; \varphi)$ and integrating the obtained inequality with regard to ρ from a to x , we obtain

$$\begin{aligned} & (\mathfrak{G}^\varphi u)(x)(\mathfrak{F}^\phi u f g h)(x) + (\mathfrak{G}^\varphi u f g)(x)(\mathfrak{F}^\phi u h)(x) + (\mathfrak{G}^\varphi u h)(x)(\mathfrak{F}^\phi u f g)(x) \\ & + (\mathfrak{G}^\varphi u f g h)(x)(\mathfrak{F}^\phi u)(x) \geq (\mathfrak{G}^\varphi u g)(x)(\mathfrak{F}^\phi u f h)(x) + (\mathfrak{G}^\varphi u f)(x)(\mathfrak{F}^\phi u g h)(x) \\ & + (\mathfrak{G}^\varphi u g h)(x)(\mathfrak{F}^\phi u f)(x) + (\mathfrak{G}^\varphi u f h)(x)(\mathfrak{F}^\phi u g)(x), \end{aligned} \quad (44)$$

which implies (40). \square

Let $\varphi = \phi$, $\hat{\gamma} = \gamma$, $\hat{\delta} = \delta$, $\hat{k} = k$, $\hat{c} = c$, $\nu = \mu$, $\beta = \alpha$, $\iota = l$, and $\zeta = \zeta$, then we have the following corollary.

Corollary 1. Let f, g be two synchronous functions on $[a, b]$ and let h, u be two nonnegative continuous functions on $[a, b]$. Then the following inequality holds

$$\begin{aligned} & (\mathfrak{F}^\phi u)(x)(\mathfrak{F}^\phi u f g h)(x) + (\mathfrak{F}^\phi u h)(x)(\mathfrak{F}^\phi u f g)(x) \\ & \geq (\mathfrak{F}^\phi u f)(x)(\mathfrak{F}^\phi u g h)(x) + (\mathfrak{F}^\phi u g)(x)(\mathfrak{F}^\phi u f h)(x). \end{aligned} \quad (45)$$

Remark 4. The reverse of the inequalities (40) and (45) hold under the following two cases: (I) The functions f and g asynchronous on $[a, b]$. (II) The function h is negative on $[a, b]$.

Theorem 4. Let f, g and h be three monotonic functions on $[a, b]$ satisfying the condition

$$(f(\tau) - f(\rho))(g(\tau) - g(\rho))(h(\tau) - h(\rho)) \geq 0 \text{ for } \tau, \rho \in [a, b], \quad (46)$$

and let u be a nonnegative continuous function on $[a, b]$. Then the following inequality holds

$$\begin{aligned} & (\mathfrak{F}^\phi u f g h)(x)(\mathfrak{G}^\varphi u)(x) - (\mathfrak{G}^\varphi u f g h)(x)(\mathfrak{F}^\phi u)(x) - (\mathfrak{F}^\phi u f g)(x)(\mathfrak{G}^\varphi u h)(x) \\ & + (\mathfrak{G}^\varphi u f g)(x)(\mathfrak{F}^\phi u h)(x) \geq (\mathfrak{F}^\phi u f h)(x)(\mathfrak{G}^\varphi u g)(x) - (\mathfrak{G}^\varphi u f h)(x)(\mathfrak{F}^\phi u g)(x) \\ & + (\mathfrak{F}^\phi u g h)(x)(\mathfrak{G}^\varphi u f)(x) - (\mathfrak{G}^\varphi u g h)(x)(\mathfrak{F}^\phi u f)(x). \end{aligned} \quad (47)$$

Proof. The proof is similar to that given in Theorem 3. \square

Theorem 5. Let f, g be two integrable functions on $[a, b]$ and let u be a nonnegative continuous function on $[a, b]$. Then the following inequality holds

$$(\mathfrak{F}^\phi u f^2)(x)(\mathfrak{G}^\varphi u)(x) + (\mathfrak{F}^\phi u)(x)(\mathfrak{G}^\varphi u g^2)(x) \geq 2(\mathfrak{F}^\phi u f)(x)(\mathfrak{G}^\varphi u g)(x), \quad (48)$$

$$(\mathfrak{F}^\phi u f^2)(x)(\mathfrak{G}^\varphi u g^2)(x) + (\mathfrak{G}^\varphi u f^2)(x)(\mathfrak{F}^\phi u g^2)(x) \geq 2(\mathfrak{F}^\phi u f g)(x)(\mathfrak{G}^\varphi u f g)(x). \quad (49)$$

Proof. Since $(f(\tau) - g(\rho))^2 \geq 0$ for any $\tau, \rho \in [a, b]$, then

$$f^2(\tau) + g^2(\rho) \geq 2f(\tau)g(\rho) \text{ for } \tau, \rho \in [a, b], \quad (50)$$

Multiplying both sides of (50) by $u(\tau)\zeta'(\tau)\mathcal{K}_x^\tau(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \zeta; \phi)u(\rho)\zeta'(\rho)\mathcal{K}_x^\rho(\mathcal{E}_{\nu,\beta,\iota}^{\hat{\gamma},\hat{\delta},\hat{k},\hat{c}}, \zeta; \varphi)$ and integrating the obtained inequality with regard to τ and ρ from a to x , respectively, we get

$$(\mathfrak{F}^\phi u f^2)(x)(\mathfrak{G}^\varphi u)(x) + (\mathfrak{F}^\phi u)(x)(\mathfrak{G}^\varphi u g^2)(x) \geq 2(\mathfrak{F}^\phi u f)(x)(\mathfrak{G}^\varphi u g)(x), \quad (51)$$

which implies (48). Since $(f(\tau)g(\rho) - f(\rho)g(\tau))^2 \geq 0$ for any $\tau, \rho \in [a, b]$, then

$$f^2(\tau)g^2(\rho) + f^2(\rho)g^2(\tau) \geq 2f(\tau)g(\tau)f(\rho)g(\rho) \text{ for } \tau, \rho \in [a, b], \quad (52)$$

Multiplying both sides of (52) by $u(\tau)\zeta'(\tau)\mathcal{K}_x^\tau(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \zeta; \phi)u(\rho)\zeta'(\rho)\mathcal{K}_x^\rho(\mathcal{E}_{\nu,\beta,\iota}^{\hat{\gamma},\hat{\delta},\hat{k},\hat{c}}, \zeta; \varphi)$ and integrating the obtained inequality with regard to τ and ρ from a to x , respectively, we get

$$(\mathfrak{F}^\phi u f^2)(x)(\mathfrak{G}^\varphi u g^2)(x) + (\mathfrak{F}^\phi u g^2)(x)(\mathfrak{G}^\varphi u f^2)(x) \geq 2(\mathfrak{F}^\phi u f g)(x)(\mathfrak{G}^\varphi u f g)(x), \quad (53)$$

which implies (49). \square

Let $\varphi = \phi$, $\hat{\gamma} = \gamma$, $\hat{\delta} = \delta$, $\hat{k} = k$, $\hat{c} = c$, $\nu = \mu$, $\beta = \alpha$, $\iota = l$, and $\zeta = \xi$, then we have the following corollary.

Corollary 2. Let f, g be two integrable functions on $[a, b]$ and let u be a nonnegative continuous function on $[a, b]$. Then the following inequality holds

$$(\mathfrak{F}^\phi u)(x) \left((\mathfrak{F}^\phi u f^2)(x) + (\mathfrak{F}^\phi u g^2)(x) \right) \geq 2(\mathfrak{F}^\phi u f)(x)(\mathfrak{F}^\phi u g)(x), \quad (54)$$

$$(\mathfrak{F}^\phi u g^2)(x)(\mathfrak{F}^\phi u f^2)(x) \geq ((\mathfrak{F}^\phi u f g)(x))^2 \text{ (Cauchy-Schwartz type inequality)}. \quad (55)$$

Theorem 6. Suppose that f, g are two integrable functions satisfying the condition (7) on $[a, b]$ and let u, v be two nonnegative continuous functions on $[a, b]$. Then the following inequality holds

$$\begin{aligned} & |(\mathfrak{F}^\phi u)(x)(\mathfrak{G}^\varphi v f g)(x) + (\mathfrak{G}^\varphi v)(x)(\mathfrak{F}^\phi u f g)(x) - (\mathfrak{F}^\phi u f)(x)(\mathfrak{G}^\varphi v g)(x) - (\mathfrak{G}^\varphi v f)(x)(\mathfrak{F}^\phi u g)(x)| \\ & \leq (\mathfrak{F}^\phi u)(x)(\mathfrak{G}^\varphi v)(x)(\Phi_2 - \Phi_1)(\Psi_2 - \Psi_1). \end{aligned} \quad (56)$$

Proof. Let $H(\tau, \rho)$ be defined by

$$H(\tau, \rho) = (f(\tau) - f(\rho))(g(\tau) - g(\rho)), \quad \forall \tau, \rho \in [a, b]. \quad (57)$$

Multiplying (57) by $u(\tau)\zeta'(\tau)\mathcal{K}_x^\tau(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \xi; \phi)v(\rho)\zeta'(\rho)\mathcal{K}_x^\rho(\mathcal{E}_{\nu,\beta,\iota}^{\hat{\gamma},\hat{\delta},\hat{k},\hat{c}}, \zeta; \varphi)$ and integrating the resulting result with respect to τ and ρ from a to x , respectively, we can get

$$\begin{aligned} & \int_a^x \int_a^x u(\tau)\zeta'(\tau)\mathcal{K}_x^\tau(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \xi; \phi)v(\rho)\zeta'(\rho)\mathcal{K}_x^\rho(\mathcal{E}_{\nu,\beta,\iota}^{\hat{\gamma},\hat{\delta},\hat{k},\hat{c}}, \zeta; \varphi)H(\tau, \rho)d\tau d\rho \\ & = (\mathfrak{F}^\phi u)(x)(\mathfrak{G}^\varphi v f g)(x) + (\mathfrak{G}^\varphi v)(x)(\mathfrak{F}^\phi u f g)(x) - (\mathfrak{F}^\phi u f)(x)(\mathfrak{G}^\varphi v g)(x) - (\mathfrak{G}^\varphi v f)(x)(\mathfrak{F}^\phi u g)(x). \end{aligned} \quad (58)$$

According to the condition (7), we have

$$|H(\tau, \rho)| = |f(\tau) - f(\rho)||g(\tau) - g(\rho)| \leq (\Phi_2 - \Phi_1)(\Psi_2 - \Psi_1), \quad \forall \tau, \rho \in [a, b]. \quad (59)$$

Combining (58) and (59), we obtain that

$$\begin{aligned} & |(\mathfrak{F}^\phi u)(x)(\mathfrak{G}^\varphi v f g)(x) + (\mathfrak{G}^\varphi v)(x)(\mathfrak{F}^\phi u f g)(x) - (\mathfrak{F}^\phi u f)(x)(\mathfrak{G}^\varphi v g)(x) - (\mathfrak{G}^\varphi v f)(x)(\mathfrak{F}^\phi u g)(x)| \\ & \leq \int_a^x \int_a^x u(\tau)\zeta'(\tau)\mathcal{K}_x^\tau(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \xi; \phi)v(\rho)\zeta'(\rho)\mathcal{K}_x^\rho(\mathcal{E}_{\nu,\beta,\iota}^{\hat{\gamma},\hat{\delta},\hat{k},\hat{c}}, \zeta; \varphi)|H(\tau, \rho)|d\tau d\rho \\ & \leq \int_a^x \int_a^x u(\tau)\zeta'(\tau)\mathcal{K}_x^\tau(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \xi; \phi)v(\rho)\zeta'(\rho)\mathcal{K}_x^\rho(\mathcal{E}_{\nu,\beta,\iota}^{\hat{\gamma},\hat{\delta},\hat{k},\hat{c}}, \zeta; \varphi)(\Phi_2 - \Phi_1)(\Psi_2 - \Psi_1)d\tau d\rho \\ & = (\mathfrak{F}^\phi u)(x)(\mathfrak{G}^\varphi v)(x)(\Phi_2 - \Phi_1)(\Psi_2 - \Psi_1). \end{aligned} \quad (60)$$

This ends the proof. \square

Theorem 7. Suppose that f, g are two integrable functions satisfying the condition (12) and let u, v be two nonnegative continuous functions on $[a, b]$. Then the following inequality holds

$$\begin{aligned}
& |(\mathfrak{F}^\phi u)(x)(\mathfrak{G}^\phi vfg)(x) + (\mathfrak{G}^\phi v)(x)(\mathfrak{F}^\phi ufg)(x) - (\mathfrak{F}^\phi uf)(x)(\mathfrak{G}^\phi vg)(x) - (\mathfrak{G}^\phi vf)(x)(\mathfrak{F}^\phi ug)(x)| \\
& \leq M \left((\mathfrak{F}^\phi u)(x)(\mathfrak{G}^\phi vg^2)(x) + (\mathfrak{G}^\phi v)(x)(\mathfrak{F}^\phi ug^2)(x) - 2(\mathfrak{F}^\phi ug)(x)(\mathfrak{G}^\phi vg)(x) \right). \quad (61)
\end{aligned}$$

Proof. From the condition (12), we have

$$|f(\tau) - f(\rho)| \leq M|g(\tau) - g(\rho)|, \quad \forall \tau, \rho \in [a, b]. \quad (62)$$

According to (57) and (62), we obtain

$$|H(\tau, \rho)| = |f(\tau) - f(\rho)||g(\tau) - g(\rho)| \leq M(g(\tau) - g(\rho))^2. \quad (63)$$

Combining (58) and (63), we get that

$$\begin{aligned}
& \int_a^x \int_a^x u(\tau) \xi'(\tau) \mathcal{H}_x^\tau(\mathcal{E}_{\mu, \alpha, l}^{\gamma, \delta, k, c}, \xi; \phi) v(\rho) \zeta'(\rho) \mathcal{H}_x^\rho(\mathcal{E}_{\nu, \beta, l}^{\hat{\gamma}, \hat{\delta}, \hat{k}, \hat{c}}, \zeta; \varphi) |H(\tau, \rho)| d\tau d\rho \\
& \leq \int_a^x \int_a^x u(\tau) \xi'(\tau) \mathcal{H}_x^\tau(\mathcal{E}_{\mu, \alpha, l}^{\gamma, \delta, k, c}, \xi; \phi) v(\rho) \zeta'(\rho) \mathcal{H}_x^\rho(\mathcal{E}_{\nu, \beta, l}^{\hat{\gamma}, \hat{\delta}, \hat{k}, \hat{c}}, \zeta; \varphi) M(g(\tau) - g(\rho))^2 d\tau d\rho \\
& = M \left((\mathfrak{F}^\phi u)(x)(\mathfrak{G}^\phi vg^2)(x) + (\mathfrak{G}^\phi v)(x)(\mathfrak{F}^\phi ug^2)(x) - 2(\mathfrak{F}^\phi ug)(x)(\mathfrak{G}^\phi vg)(x) \right). \quad (64)
\end{aligned}$$

This completes the proof. \square

Theorem 8. Suppose that f, g are two integrable functions satisfying the lipschitzian condition with the constants L_1, L_2 and let u, v be two nonnegative continuous functions on $[a, b]$. Then we have

$$\begin{aligned}
& |(\mathfrak{F}^\phi u)(x)(\mathfrak{G}^\phi vfg)(x) + (\mathfrak{G}^\phi v)(x)(\mathfrak{F}^\phi ufg)(x) - (\mathfrak{F}^\phi uf)(x)(\mathfrak{G}^\phi vg)(x) - (\mathfrak{G}^\phi vf)(x)(\mathfrak{F}^\phi ug)(x)| \\
& \leq L_1 L_2 \left((\mathfrak{F}^\phi u)(x)(\mathfrak{G}^\phi i^2 v)(x) + (\mathfrak{G}^\phi v)(x)(\mathfrak{F}^\phi i^2 u)(x) - 2(\mathfrak{F}^\phi iu)(x)(\mathfrak{G}^\phi iv)(x) \right). \quad (65)
\end{aligned}$$

Proof. From the conditions of Theorem 8, we have

$$|f(\tau) - f(\rho)| \leq L_1 |\tau - \rho| \quad \text{and} \quad |g(\tau) - g(\rho)| \leq L_2 |\tau - \rho|, \quad \tau, \rho \in [a, b], \quad (66)$$

which implies that

$$|H(\tau, \rho)| = |f(\tau) - f(\rho)||g(\tau) - g(\rho)| \leq L_1 L_2 (\tau - \rho)^2. \quad (67)$$

Combining (58) and (67), we get that

$$\begin{aligned}
& \int_a^x \int_a^x u(\tau) \xi'(\tau) \mathcal{H}_x^\tau(\mathcal{E}_{\mu, \alpha, l}^{\gamma, \delta, k, c}, \xi; \phi) v(\rho) \zeta'(\rho) \mathcal{H}_x^\rho(\mathcal{E}_{\nu, \beta, l}^{\hat{\gamma}, \hat{\delta}, \hat{k}, \hat{c}}, \zeta; \varphi) |H(\tau, \rho)| d\tau d\rho \\
& \leq \int_a^x \int_a^x u(\tau) \xi'(\tau) \mathcal{H}_x^\tau(\mathcal{E}_{\mu, \alpha, l}^{\gamma, \delta, k, c}, \xi; \phi) v(\rho) \zeta'(\rho) \mathcal{H}_x^\rho(\mathcal{E}_{\nu, \beta, l}^{\hat{\gamma}, \hat{\delta}, \hat{k}, \hat{c}}, \zeta; \varphi) L_1 L_2 (\tau - \rho)^2 d\tau d\rho \\
& = L_1 L_2 \left((\mathfrak{F}^\phi u)(x)(\mathfrak{G}^\phi i^2 v)(x) + (\mathfrak{G}^\phi v)(x)(\mathfrak{F}^\phi i^2 u)(x) - 2(\mathfrak{F}^\phi iu)(x)(\mathfrak{G}^\phi iv)(x) \right). \quad (68)
\end{aligned}$$

This ends the proof of Theorem 8. \square

Corollary 3. Let f, g be two differentiable functions on $[a, b]$ and let u, v be two nonnegative continuous functions on $[a, b]$. Then the following inequality holds

$$\begin{aligned}
& |(\mathfrak{F}^\phi u)(x)(\mathfrak{G}^\phi vfg)(x) + (\mathfrak{G}^\phi v)(x)(\mathfrak{F}^\phi ufg)(x) - (\mathfrak{F}^\phi uf)(x)(\mathfrak{G}^\phi vg)(x) - (\mathfrak{G}^\phi vf)(x)(\mathfrak{F}^\phi ug)(x)| \\
& \leq \|f'\|_\infty \|g'\|_\infty \left((\mathfrak{F}^\phi u)(x)(\mathfrak{G}^\phi i^2 v)(x) + (\mathfrak{G}^\phi v)(x)(\mathfrak{F}^\phi i^2 u)(x) - 2(\mathfrak{F}^\phi iu)(x)(\mathfrak{G}^\phi iv)(x) \right). \quad (69)
\end{aligned}$$

Proof. We have $f(\tau) - f(\rho) = \int_{\rho}^{\tau} f'(t)dt$ and $g(\tau) - g(\rho) = \int_{\rho}^{\tau} g'(t)dt$. That is, $|f(\tau) - f(\rho)| \leq \|f'\|_{\infty}|\tau - \rho|$, $|g(\tau) - g(\rho)| \leq \|g'\|_{\infty}|\tau - \rho|$, $\tau, \rho \in [0, \infty)$, and the result follows from Theorem 8. This completes the proof. \square

Theorem 9. Suppose that f, g are two integrable functions satisfying $f' \in L^{\mathbb{P}}[a, b]$, $g' \in L^{\mathbb{Q}}[a, b]$ and let u be a nonnegative continuous function on $[a, b]$, $\mathbb{P}, \mathbb{Q}, \mathbb{r} > 1$ with $1/\mathbb{P} + 1/\mathbb{P}' = 1$, $1/\mathbb{Q} + 1/\mathbb{Q}' = 1$ and $1/\mathbb{r} + 1/\mathbb{r}' = 1$. Then the following weighted fractional integral inequality holds

$$2|(\mathfrak{F}^{\phi}u)(x)(\mathfrak{F}^{\phi}ufg)(x) - (\mathfrak{F}^{\phi}uf)(x)(\mathfrak{F}^{\phi}ug)(x)| \leq \|f'\|_{\mathbb{P}}\|g'\|_{\mathbb{Q}} \int_a^x \int_a^x u(\tau)\xi'(\tau)\mathcal{K}_x^{\tau}(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \xi; \phi)u(\rho)\xi'(\rho)\mathcal{K}_x^{\rho}(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \xi; \phi)|\tau - \rho|^{\frac{1}{\mathbb{P}'} + \frac{1}{\mathbb{Q}'}} d\tau d\rho. \quad (70)$$

Proof. Multiplying both sides of (57) by $u(\tau)\xi'(\tau)\mathcal{K}_x^{\tau}(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \xi; \phi)u(\rho)\xi'(\rho)\mathcal{K}_x^{\rho}(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \xi; \phi)$ and integrating the given result with respect to τ and ρ from a to x , we can state that

$$\begin{aligned} \int_a^x \int_a^x u(\tau)\xi'(\tau)\mathcal{K}_x^{\tau}(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \xi; \phi)u(\rho)\xi'(\rho)\mathcal{K}_x^{\rho}(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \xi; \phi)H(\tau, \rho)d\tau d\rho \\ = 2((\mathfrak{F}^{\phi}u)(x)(\mathfrak{F}^{\phi}ufg)(x) - (\mathfrak{F}^{\phi}uf)(x)(\mathfrak{F}^{\phi}ug)(x)). \end{aligned} \quad (71)$$

On the other hand, from (57), we have

$$H(\tau, \rho) = \int_{\rho}^{\tau} \int_{\rho}^{\tau} f'(\theta)g'(\vartheta)d\theta d\vartheta, \quad \forall \tau, \rho \in [a, b]. \quad (72)$$

By employing the Hölder inequality, we obtain

$$|f(\tau) - f(\rho)| \leq |\tau - \rho|^{\frac{1}{\mathbb{P}'}} \left| \int_{\rho}^{\tau} |f'(\theta)|^{\mathbb{P}} d\theta \right|^{\frac{1}{\mathbb{P}}} \text{ and } |g(\tau) - g(\rho)| \leq |\tau - \rho|^{\frac{1}{\mathbb{Q}'}} \left| \int_{\rho}^{\tau} |g'(\theta)|^{\mathbb{Q}} d\theta \right|^{\frac{1}{\mathbb{Q}}}. \quad (73)$$

Combining (72) and (73), we get

$$|H(\tau, \rho)| \leq |\tau - \rho|^{\frac{1}{\mathbb{P}'} + \frac{1}{\mathbb{Q}'}} \left| \int_{\rho}^{\tau} |f'(\theta)|^{\mathbb{P}} d\theta \right|^{\frac{1}{\mathbb{P}}} \left| \int_{\rho}^{\tau} |g'(\theta)|^{\mathbb{Q}} d\theta \right|^{\frac{1}{\mathbb{Q}}}. \quad (74)$$

According to inequalities (71) and (74), we can write

$$\begin{aligned} 2|(\mathfrak{F}^{\phi}u)(x)(\mathfrak{F}^{\phi}ufg)(x) - (\mathfrak{F}^{\phi}uf)(x)(\mathfrak{F}^{\phi}ug)(x)| \leq \int_a^x \int_a^x u(\tau)\xi'(\tau)\mathcal{K}_x^{\tau}(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \xi; \phi) \\ \times u(\rho)\xi'(\rho)\mathcal{K}_x^{\rho}(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \xi; \phi)|\tau - \rho|^{\frac{1}{\mathbb{P}'} + \frac{1}{\mathbb{Q}'}} \left| \int_{\rho}^{\tau} |f'(\theta)|^{\mathbb{P}} d\theta \right|^{\frac{1}{\mathbb{P}}} \left| \int_{\rho}^{\tau} |g'(\theta)|^{\mathbb{Q}} d\theta \right|^{\frac{1}{\mathbb{Q}}} d\tau d\rho. \end{aligned} \quad (75)$$

Applying the double integral Hölder inequality to (75), we obtain

$$\begin{aligned} 2|(\mathfrak{F}^{\phi}u)(x)(\mathfrak{F}^{\phi}ufg)(x) - (\mathfrak{F}^{\phi}uf)(x)(\mathfrak{F}^{\phi}ug)(x)| \\ \leq \left(\int_a^x \int_a^x u(\tau)\xi'(\tau)\mathcal{K}_x^{\tau}(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \xi; \phi)u(\rho)\xi'(\rho)\mathcal{K}_x^{\rho}(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \xi; \phi)|\tau - \rho|^{\frac{1}{\mathbb{P}'} + \frac{1}{\mathbb{Q}'}} \left| \int_{\rho}^{\tau} |f'(\theta)|^{\mathbb{P}} d\theta \right|^{\frac{\mathbb{r}}{\mathbb{P}}} d\tau d\rho \right)^{\frac{1}{\mathbb{r}}} \\ \times \left(\int_a^x \int_a^x u(\tau)\xi'(\tau)\mathcal{K}_x^{\tau}(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \xi; \phi)u(\rho)\xi'(\rho)\mathcal{K}_x^{\rho}(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \xi; \phi)|\tau - \rho|^{\frac{1}{\mathbb{P}'} + \frac{1}{\mathbb{Q}'}} \left| \int_{\rho}^{\tau} |g'(\theta)|^{\mathbb{Q}} d\theta \right|^{\frac{\mathbb{r}'}{\mathbb{Q}}} d\tau d\rho \right)^{\frac{1}{\mathbb{r}'}}. \end{aligned} \quad (76)$$

Using the following properties

$$\left| \int_{\rho}^{\tau} |f'(\theta)|^{\mathbb{P}} d\theta \right|^{\frac{1}{\mathbb{P}}} \leq \|f'\|_{\mathbb{P}}^{\mathbb{P}} \text{ and } \left| \int_{\rho}^{\tau} |g'(\theta)|^{\mathbb{Q}} d\theta \right|^{\frac{1}{\mathbb{Q}}} \leq \|g'\|_{\mathbb{Q}}^{\mathbb{Q}}, \quad (77)$$

then (76) can be rewritten as

$$\begin{aligned}
& 2|(\mathfrak{F}^\phi u)(x)(\mathfrak{F}^\phi ufg)(x) - (\mathfrak{F}^\phi uf)(x)(\mathfrak{F}^\phi ug)(x)| \\
& \leq \left(\|f'\|_{\mathbb{P}} \int_a^x \int_a^x u(\tau) \zeta'(\tau) u(\rho) \zeta'(\rho) \mathcal{K}_x^\tau(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \zeta; \phi) \mathcal{K}_x^\rho(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \zeta; \phi) |\tau - \rho|^{\frac{1}{\mathbb{P}} + \frac{1}{\mathbb{Q}'}} d\tau d\rho \right)^{\frac{1}{\mathbb{P}}} \\
& \quad \times \left(\|g'\|_{\mathbb{Q}} \int_a^x \int_a^x u(\tau) \zeta'(\tau) \mathcal{K}_x^\tau(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \zeta; \phi) u(\rho) \zeta'(\rho) \mathcal{K}_x^\rho(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \zeta; \phi) |\tau - \rho|^{\frac{1}{\mathbb{P}'} + \frac{1}{\mathbb{Q}'}} d\tau d\rho \right)^{\frac{1}{\mathbb{Q}'}} \\
& = \|f'\|_{\mathbb{P}} \|g'\|_{\mathbb{Q}} \times \int_a^x \int_a^x u(\tau) \zeta'(\tau) \mathcal{K}_x^\tau(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \zeta; \phi) u(\rho) \zeta'(\rho) \mathcal{K}_x^\rho(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \zeta; \phi) |\tau - \rho|^{\frac{1}{\mathbb{P}} + \frac{1}{\mathbb{Q}'}} d\tau d\rho, \quad (78)
\end{aligned}$$

which completes the desired proof. \square

Now we follow the proof of Theorems 6 and 9, we can get the following result.

Theorem 10. Suppose that f, g are two integrable functions satisfying $f' \in L^{\mathbb{P}}[a, b], g' \in L^{\mathbb{Q}}[a, b]$ and let u, v be two nonnegative continuous functions on $[a, b]$, $\mathbb{P}, \mathbb{Q}, \mathbb{R} > 1$ with $1/\mathbb{P} + 1/\mathbb{P}' = 1$, $1/\mathbb{Q} + 1/\mathbb{Q}' = 1$ and $1/\mathbb{R} + 1/\mathbb{R}' = 1$. Then the following weighted fractional integral inequality holds

$$\begin{aligned}
& |(\mathfrak{F}^\phi u)(x)(\mathfrak{G}^\varphi vfg)(x) + (\mathfrak{G}^\varphi v)(x)(\mathfrak{F}^\phi ufg)(x) - (\mathfrak{F}^\phi uf)(x)(\mathfrak{G}^\varphi vg)(x) - (\mathfrak{G}^\varphi vf)(x)(\mathfrak{F}^\phi ug)(x)| \\
& \leq \|f'\|_{\mathbb{P}} \|g'\|_{\mathbb{Q}} \int_a^x \int_a^x u(\tau) \zeta'(\tau) \mathcal{K}_x^\tau(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \zeta; \phi) v(\rho) \zeta'(\rho) \mathcal{K}_x^\rho(\mathcal{E}_{\nu,\beta,\iota}^{\hat{\gamma},\hat{\delta},\hat{k},\hat{c}}, \zeta; \varphi) |\tau - \rho|^{\frac{1}{\mathbb{P}'} + \frac{1}{\mathbb{Q}'}} d\tau d\rho. \quad (79)
\end{aligned}$$

Remark 5. Let $\varphi = \phi, \hat{\gamma} = \gamma, \hat{\delta} = \delta, \hat{k} = k, \hat{c} = c, \nu = \mu, \beta = \alpha, \iota = l$, and $\zeta = \zeta$, then Theorem 10 reduces to Theorem 9. Let $\mathbb{P} = \mathbb{Q} = 2$ in Theorem 10, (79) can give the similar result with the inequality (69).

Nextly, we establish some new Grüss-type integral inequalities involving the left unified fractional integral operators.

Lemma 3. Let f be an integrable function satisfying the condition (7) and let u be a continuous function on $[a, b]$. Then we have the following equation

$$\begin{aligned}
& (\mathfrak{F}^\phi u)(x)(\mathfrak{F}^\phi uf^2)(x) - ((\mathfrak{F}^\phi uf)(x))^2 = (\Phi_2(\mathfrak{F}^\phi u)(x) - (\mathfrak{F}^\phi uf)(x))((\mathfrak{F}^\phi uf)(x) - \Phi_1(\mathfrak{F}^\phi u)(x)) \\
& \quad - (\mathfrak{F}^\phi u)(x)(\mathfrak{F}^\phi u(\Phi_2 - f)(f - \Phi_1))(x). \quad (80)
\end{aligned}$$

Proof. Since f is an integrable function satisfying the condition $\Phi_1 \leq f(x) \leq \Phi_2$ for all $x \in [a, b]$. For any $\rho, \tau \in [a, b]$, we have

$$\begin{aligned}
& (\Phi_2 - f(\rho))(f(\tau) - \Phi_1) + (\Phi_2 - f(\tau))(f(\rho) - \Phi_1) - (\Phi_2 - f(\tau))(f(\tau) - \Phi_1) - (\Phi_2 - f(\rho))(f(\rho) - \Phi_1) \\
& = f^2(\tau) + f^2(\rho) - 2f(\rho)f(\tau). \quad (81)
\end{aligned}$$

Multiplying both sides of (81) by $u(\rho)\zeta'(\rho)\mathcal{K}_x^\rho(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \zeta; \phi)$ and integrating the obtained equality with regard to ρ from a to x , we have

$$\begin{aligned}
& (f(\tau) - \Phi_1)(\Phi_2(\mathfrak{F}^\phi u)(x) - (\mathfrak{F}^\phi uf)(x)) + (\Phi_2 - f(\tau))((\mathfrak{F}^\phi uf)(x) - \Phi_1(\mathfrak{F}^\phi u)(x)) - (\Phi_2 - f(\tau))(f(\tau) - \Phi_1)(\mathfrak{F}^\phi u)(x) \\
& \quad - (\mathfrak{F}^\phi u(\Phi_2 - f)(f - \Phi_1))(x) = f^2(\tau)(\mathfrak{F}^\phi u)(x) + (\mathfrak{F}^\phi uf^2)(x) - 2f(\tau)(\mathfrak{F}^\phi uf)(x). \quad (82)
\end{aligned}$$

Multiplying both sides of (82) by $u(\tau)\zeta'(\tau)\mathcal{K}_x^\tau(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \zeta; \phi)$ and integrating the obtained equality with regard to τ from a to x , we have

$$((\mathfrak{F}^\phi uf)(x) - \Phi_1(\mathfrak{F}^\phi u)(x))(\Phi_2(\mathfrak{F}^\phi u)(x) - (\mathfrak{F}^\phi uf)(x)) + (\Phi_2(\mathfrak{F}^\phi u)(x) - (\mathfrak{F}^\phi uf)(x))((\mathfrak{F}^\phi uf)(x) - \Phi_1(\mathfrak{F}^\phi u)(x))$$

$$\begin{aligned}
& -(\mathfrak{F}^\phi u(\Phi_2 - f)(f - \Phi_1))(x)(\mathfrak{F}^\phi u)(x) - (\mathfrak{F}^\phi u)(x)(\mathfrak{F}^\phi u(\Phi_2 - f)(f - \Phi_1))(x) \\
& = (\mathfrak{F}^\phi u f^2)(x)(\mathfrak{F}^\phi u)(x) + (\mathfrak{F}^\phi u)(x)(\mathfrak{F}^\phi u f^2)(x) - 2(\mathfrak{F}^\phi u f)(x)(\mathfrak{F}^\phi u f)(x), \quad (83)
\end{aligned}$$

which gives (80). \square

Theorem 11. Suppose that f, g are two integrable functions satisfying the condition (7) and let u be a nonnegative continuous function on $[a, b]$. Then we have the following inequality

$$|(\mathfrak{F}^\phi u)(x)(\mathfrak{F}^\phi u f g)(x) - (\mathfrak{F}^\phi u f)(x)(\mathfrak{F}^\phi u g)(x)| \leq \frac{1}{4}(\Phi_2 - \Phi_1)(\Psi_2 - \Psi_1)((\mathfrak{F}^\phi u)(x))^2. \quad (84)$$

Proof. Multiplying both sides of (57) by $u(\tau)\xi'(\tau)\mathcal{K}_x^\tau(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \xi; \phi)u(\rho)\xi'(\rho)\mathcal{K}_x^\rho(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \xi; \phi)$ and integrating the resulting identity with respect to τ and ρ from a to x , we can state

$$\begin{aligned}
& \int_a^x \int_a^x u(\tau)\xi'(\tau)\mathcal{K}_x^\tau(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \xi; \phi)u(\rho)\xi'(\rho)\mathcal{K}_x^\rho(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \xi; \phi)H(\tau, \rho)d\tau d\rho \\
& = 2((\mathfrak{F}^\phi u)(x)(\mathfrak{F}^\phi u f g)(x) - (\mathfrak{F}^\phi u f)(x)(\mathfrak{F}^\phi u g)(x)). \quad (85)
\end{aligned}$$

Thanks to the weighted Cauchy-Schwartz integral inequality for double integrals, we can write that

$$\begin{aligned}
& \left(\int_a^x \int_a^x u(\tau)\xi'(\tau)\mathcal{K}_x^\tau(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \xi; \phi)u(\rho)\xi'(\rho)\mathcal{K}_x^\rho(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \xi; \phi)H(\tau, \rho)d\tau d\rho \right)^2 \\
& \leq \left(\int_a^x \int_a^x u(\tau)\xi'(\tau)\mathcal{K}_x^\tau(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \xi; \phi)u(\rho)\xi'(\rho)\mathcal{K}_x^\rho(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \xi; \phi)(f(\tau) - f(\rho))^2 d\tau d\rho \right) \\
& \quad \times \left(\int_a^x \int_a^x u(\tau)\xi'(\tau)\mathcal{K}_x^\tau(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \xi; \phi)u(\rho)\xi'(\rho)\mathcal{K}_x^\rho(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \xi; \phi)(g(\tau) - g(\rho))^2 d\tau d\rho \right) \\
& = 4((\mathfrak{F}^\phi u)(x)(\mathfrak{F}^\phi u f^2)(x) - ((\mathfrak{F}^\phi u f)(x))^2)((\mathfrak{F}^\phi u)(x)(\mathfrak{F}^\phi u g^2)(x) - ((\mathfrak{F}^\phi u g)(x))^2). \quad (86)
\end{aligned}$$

Since $(\Phi_2 - f(\tau))(f(\tau) - \Phi_1) \geq 0$ and $(\Psi_2 - g(\tau))(g(\tau) - \Psi_1) \geq 0$, we have

$$(\mathfrak{F}^\phi u)(x)(\mathfrak{F}^\phi u(\Phi_2 - f)(f - \Phi_1))(x) \geq 0 \quad \text{and} \quad (\mathfrak{F}^\phi u)(x)(\mathfrak{F}^\phi u(\Psi_2 - g)(g - \Psi_1))(x) \geq 0. \quad (87)$$

Thus, from (87) and Lemma 3, we get

$$(\mathfrak{F}^\phi u)(x)(\mathfrak{F}^\phi u f^2)(x) - ((\mathfrak{F}^\phi u f)(x))^2 \leq (\Phi_2(\mathfrak{F}^\phi u)(x) - (\mathfrak{F}^\phi u f)(x))((\mathfrak{F}^\phi u f)(x) - \Phi_1(\mathfrak{F}^\phi u)(x)), \quad (88)$$

$$(\mathfrak{F}^\phi u)(x)(\mathfrak{F}^\phi u g^2)(x) - ((\mathfrak{F}^\phi u g)(x))^2 \leq (\Psi_2(\mathfrak{F}^\phi u)(x) - (\mathfrak{F}^\phi u g)(x))((\mathfrak{F}^\phi u g)(x) - \Psi_1(\mathfrak{F}^\phi u)(x)). \quad (89)$$

Combining (85), (86), (88) and (89), we deduce that

$$\begin{aligned}
& ((\mathfrak{F}^\phi u)(x)(\mathfrak{F}^\phi u f g)(x) - (\mathfrak{F}^\phi u f)(x)(\mathfrak{F}^\phi u g)(x))^2 \leq (\Phi_2(\mathfrak{F}^\phi u)(x) - (\mathfrak{F}^\phi u f)(x))((\mathfrak{F}^\phi u f)(x) - \Phi_1(\mathfrak{F}^\phi u)(x)) \\
& \quad \times (\Psi_2(\mathfrak{F}^\phi u)(x) - (\mathfrak{F}^\phi u g)(x))((\mathfrak{F}^\phi u g)(x) - \Psi_1(\mathfrak{F}^\phi u)(x)). \quad (90)
\end{aligned}$$

Now using the elementary inequality $4xy \leq (x + y)^2$, $x, y \in \mathbb{R}$, we can state that

$$4(\Phi_2(\mathfrak{F}^\phi u)(x) - (\mathfrak{F}^\phi u f)(x))((\mathfrak{F}^\phi u f)(x) - \Phi_1(\mathfrak{F}^\phi u)(x)) \leq ((\Phi_2 - \Phi_1)(\mathfrak{F}^\phi u)(x))^2, \quad (91)$$

$$4(\Psi_2(\mathfrak{F}^\phi u)(x) - (\mathfrak{F}^\phi u g)(x))((\mathfrak{F}^\phi u g)(x) - \Psi_1(\mathfrak{F}^\phi u)(x)) \leq ((\Psi_2 - \Psi_1)(\mathfrak{F}^\phi u)(x))^2. \quad (92)$$

From (90)–(92), we obtain (84). This complete the proof of Theorem 11. \square

Remark 6. Let $\varphi = \phi$, $\hat{\gamma} = \gamma$, $\hat{\delta} = \delta$, $\hat{k} = k$, $\hat{c} = c$, $\nu = \mu$, $\beta = \alpha$, $\iota = l$, and $\zeta = \xi$ in Theorem 6, then inequality (56) can be rewritten as

$$|(\mathfrak{F}^\phi u)(x)(\mathfrak{F}^\phi ufg)(x) - (\mathfrak{F}^\phi uf)(x)(\mathfrak{F}^\phi ug)(x)| \leq \frac{1}{2}(\Phi_2 - \Phi_1)(\Psi_2 - \Psi_1)((\mathfrak{F}^\phi u)(x))^2. \quad (93)$$

From (84) and (93), it is easy to see that (84) is better than (93).

Lemma 4. Let f, g be two integrable functions on $[a, b]$ and let u, v be two nonnegative continuous functions on $[a, b]$. Then the following inequality holds

$$\begin{aligned} & ((\mathfrak{F}^\phi u)(x)(\mathfrak{G}^\varphi vfg)(x) + (\mathfrak{G}^\varphi v)(x)(\mathfrak{F}^\phi ufg)(x) - (\mathfrak{F}^\phi uf)(x)(\mathfrak{G}^\varphi vg)(x) - (\mathfrak{G}^\varphi vf)(x)(\mathfrak{F}^\phi ug)(x))^2 \\ & \leq \left((\mathfrak{F}^\phi u)(x)(\mathfrak{G}^\varphi vf^2)(x) + (\mathfrak{G}^\varphi v)(x)(\mathfrak{F}^\phi uf^2)(x) - 2(\mathfrak{F}^\phi uf)(x)(\mathfrak{G}^\varphi vf)(x) \right) \\ & \quad \times \left((\mathfrak{F}^\phi u)(x)(\mathfrak{G}^\varphi vg^2)(x) + (\mathfrak{G}^\varphi v)(x)(\mathfrak{F}^\phi ug^2)(x) - 2(\mathfrak{F}^\phi ug)(x)(\mathfrak{G}^\varphi vg)(x) \right). \end{aligned} \quad (94)$$

Proof. Thanks to the weighted Cauchy-Schwartz integral inequality for double integrals, it follows from (58) that we obtain (93). \square

Lemma 5. Let f be an integrable function on $[a, b]$ and let u and v be two nonnegative continuous functions on $[a, b]$. Then we have the following equation

$$\begin{aligned} & (\mathfrak{F}^\phi u)(x)(\mathfrak{G}^\varphi vf^2)(x) + (\mathfrak{G}^\varphi v)(x)(\mathfrak{F}^\phi uf^2)(x) - 2(\mathfrak{F}^\phi uf)(x)(\mathfrak{G}^\varphi vf)(x) \\ & = (\Phi_2(\mathfrak{F}^\phi u)(x) - (\mathfrak{F}^\phi uf)(x))((\mathfrak{G}^\varphi vf)(x) - \Phi_1(\mathfrak{G}^\varphi v)(x)) - (\mathfrak{F}^\phi u)(x)(\mathfrak{G}^\varphi v(\Phi_2 - f)(f - \Phi_1))(x) \\ & \quad + ((\mathfrak{F}^\phi uf)(x) - \Phi_1(\mathfrak{F}^\phi u)(x))(\Phi_2(\mathfrak{G}^\varphi v)(x) - (\mathfrak{G}^\varphi vf)(x)) - (\mathfrak{G}^\varphi v)(x)(\mathfrak{F}^\phi u(\Phi_2 - f)(f - \Phi_1))(x). \end{aligned} \quad (95)$$

Proof. Multiplying both sides of (82) by $v(\tau)\zeta'(\tau)\mathcal{H}_x^\tau(\mathcal{E}_{v,\beta,\iota,a^+}^{\hat{\gamma},\hat{\delta},\hat{k},\hat{c}};\zeta;\varphi)$ and integrating the obtained equation with respect to τ from a to y , we have

$$\begin{aligned} & ((\mathfrak{G}^\varphi vf)(x) - \Phi_1(\mathfrak{G}^\varphi v)(x))(\Phi_2(\mathfrak{F}^\phi u)(x) - (\mathfrak{F}^\phi uf)(x)) - (\mathfrak{F}^\phi u)(x)(\mathfrak{G}^\varphi v(\Phi_2 - f)(f - \Phi_1))(x) \\ & \quad + (\Phi_2(\mathfrak{G}^\varphi v)(x) - (\mathfrak{G}^\varphi vf)(x))((\mathfrak{F}^\phi uf)(x) - \Phi_1(\mathfrak{F}^\phi u)(x)) - (\mathfrak{G}^\varphi v)(x)(\mathfrak{F}^\phi u(\Phi_2 - f)(f - \Phi_1))(x) \\ & = (\mathfrak{G}^\varphi vf^2)(x)(\mathfrak{F}^\phi u)(x) + (\mathfrak{G}^\varphi v)(x)(\mathfrak{F}^\phi uf^2)(x) - 2(\mathfrak{G}^\varphi vf)(x)(\mathfrak{F}^\phi uf)(x). \end{aligned} \quad (96)$$

which gives (95) and proves the Lemma 5. \square

Remark 7. Let $\varphi = \phi$, $\hat{\gamma} = \gamma$, $\hat{\delta} = \delta$, $\hat{k} = k$, $\hat{c} = c$, $v = \mu$, $\beta = \alpha$, $\iota = l$, and $\zeta = \xi$, then Lemma 5 reduces to Lemma 3.

Theorem 12. Suppose that f, g are two integrable functions on $[a, b]$ satisfying the condition (7) and let u, v be two nonnegative continuous functions on $[a, b]$. Then we have

$$\begin{aligned} & ((\mathfrak{F}^\phi u)(x)(\mathfrak{G}^\varphi vfg)(x) + (\mathfrak{G}^\varphi v)(x)(\mathfrak{F}^\phi ufg)(x) - (\mathfrak{F}^\phi uf)(x)(\mathfrak{G}^\varphi vg)(x) - (\mathfrak{G}^\varphi vf)(x)(\mathfrak{F}^\phi ug)(x))^2 \\ & \leq ((\Phi_2(\mathfrak{F}^\phi u)(x) - (\mathfrak{F}^\phi uf)(x))((\mathfrak{G}^\varphi vf)(x) - \Phi_1(\mathfrak{G}^\varphi v)(x)) + (\Phi_2(\mathfrak{G}^\varphi v)(x) - (\mathfrak{G}^\varphi vf)(x)) \\ & \quad \times ((\mathfrak{F}^\phi uf)(x) - \Phi_1(\mathfrak{F}^\phi u)(x)))(\Psi_2(\mathfrak{F}^\phi u)(x) - (\mathfrak{F}^\phi ug)(x))((\mathfrak{G}^\varphi vg)(x) - \Psi_1(\mathfrak{G}^\varphi v)(x)) \\ & \quad + (\Psi_2(\mathfrak{G}^\varphi v)(x) - (\mathfrak{G}^\varphi vf)(x))((\mathfrak{F}^\phi ug)(x) - \Psi_1(\mathfrak{F}^\phi u)(x))). \end{aligned} \quad (97)$$

Proof. Since $(\Phi_2 - f(\tau))(f(\tau) - \Phi_1) \geq 0$ and $(\Psi_2 - g(\tau))(g(\tau) - \Psi_1) \geq 0$, we have

$$-(\mathfrak{F}^\phi u)(x)(\mathfrak{G}^\varphi v(\Phi_2 - f)(f - \Phi_1))(x) - (\mathfrak{G}^\varphi v)(x)(\mathfrak{F}^\phi u(\Phi_2 - f)(f - \Phi_1))(x) \leq 0, \quad (98)$$

$$-(\mathfrak{F}^\phi u)(x)(\mathfrak{G}^\varphi v(\Psi_2 - g)(g - \Psi_1))(x) - (\mathfrak{G}^\varphi v)(x)(\mathfrak{F}^\phi u(\Psi_2 - g)(g - \Psi_1))(x) \leq 0. \quad (99)$$

Applying Lemma 5 to f and g , and using (98), (99) and Lemma 4, we obtain (97). \square

Theorem 13. Let f be an integrable function on $[a, b]$ and let u and v be two nonnegative continuous functions on $[a, b]$. Suppose that there exist two integrable functions Φ_1, Φ_2 on $[a, b]$ such that

$$\Phi_1(t) \leq f(t) \leq \Phi_2(t), \quad \forall t \in [a, b]. \quad (100)$$

Then the following inequality holds:

$$(\mathfrak{F}^\phi u \Phi_2)(x)(\mathfrak{G}^\varphi v f)(x) + (\mathfrak{G}^\varphi v \Phi_1)(x)(\mathfrak{F}^\phi u f)(x) \geq (\mathfrak{G}^\varphi v \Phi_1)(x)(\mathfrak{F}^\phi u \Phi_2)(x) + (\mathfrak{F}^\phi u f)(x)(\mathfrak{G}^\varphi v f)(x). \quad (101)$$

Proof. Since $\Phi_1(t) \leq f(t) \leq \Phi_2(t), \forall t \in [a, b]$, we have

$$(\Phi_2(\tau) - f(\tau))(f(\rho) - \Phi_1(\rho)) \geq 0, \quad \forall \tau, \rho \in [a, b]. \quad (102)$$

This implies that

$$\Phi_2(\tau)f(\rho) + \Phi_1(\rho)f(\tau) \geq \Phi_1(\rho)\Phi_2(\tau) + f(\tau)f(\rho), \quad \forall \tau, \rho \in [a, b]. \quad (103)$$

Multiplying (103) by $u(\tau)\xi'(\tau)\mathcal{K}_x^\tau(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \xi; \phi)v(\rho)\zeta'(\rho)\mathcal{K}_y^\rho(\mathcal{E}_{\nu,\beta,\mu}^{\hat{\gamma},\hat{\delta},\hat{k},\hat{c}}, \zeta; \varphi)$ and integrating the given inequality with respect to τ and ρ from a to x and a to y , respectively, we can get

$$(\mathfrak{F}^\phi u \Phi_2)(x)(\mathfrak{G}^\varphi v f)(x) + (\mathfrak{G}^\varphi v \Phi_1)(x)(\mathfrak{F}^\phi u f)(x) \geq (\mathfrak{G}^\varphi v \Phi_1)(x)(\mathfrak{F}^\phi u \Phi_2)(x) + (\mathfrak{F}^\phi u f)(x)(\mathfrak{G}^\varphi v f)(x), \quad (104)$$

which gives (101). \square

Let $\Phi_1(t) = \Phi_1$ and $\Phi_2(t) = \Phi_2$ in Theorem 13, we have the following corollary.

Corollary 4. Let f be an integrable function on $[a, b]$ satisfying the condition (7). Then we have

$$\Phi_2(\mathfrak{F}^\phi u)(x)(\mathfrak{G}^\varphi v f)(x) + \Phi_1(\mathfrak{G}^\varphi v)(x)(\mathfrak{F}^\phi u f)(x) \geq \Phi_1\Phi_2(\mathfrak{G}^\varphi v)(x)(\mathfrak{F}^\phi u)(x) + (\mathfrak{F}^\phi u f)(x)(\mathfrak{G}^\varphi v f)(x). \quad (105)$$

Theorem 14. Let f, g be two integrable functions on $[a, b]$ and let u, v be two nonnegative continuous functions on $[a, b]$. Suppose that there exist two integrable functions $\Phi_1, \Phi_2, \Psi_1, \Psi_2$ on $[a, b]$ such that

$$\Phi_1(t) \leq f(t) \leq \Phi_2(t) \text{ and } \Psi_1(t) \leq g(t) \leq \Psi_2(t), \quad \forall t \in [a, b]. \quad (106)$$

Then the following inequalities hold:

$$(A1) \quad (\mathfrak{F}^\phi u \Phi_2)(x)(\mathfrak{G}^\varphi v g)(x) + (\mathfrak{G}^\varphi v \Psi_1)(x)(\mathfrak{F}^\phi u f)(x) \geq (\mathfrak{G}^\varphi v \Psi_1)(x)(\mathfrak{F}^\phi u \Phi_2)(x) + (\mathfrak{F}^\phi u f)(x)(\mathfrak{G}^\varphi v g)(x).$$

$$(A2) \quad (\mathfrak{F}^\phi u f)(x)(\mathfrak{G}^\varphi v \Psi_2)(x) + (\mathfrak{G}^\varphi v g)(x)(\mathfrak{F}^\phi u \Phi_1)(x) \geq (\mathfrak{G}^\varphi v \Psi_2)(x)(\mathfrak{F}^\phi u \Phi_1)(x) + (\mathfrak{F}^\phi u f)(x)(\mathfrak{G}^\varphi v g)(x).$$

$$(A3) \quad (\mathfrak{F}^\phi u \Phi_2)(x)(\mathfrak{G}^\varphi v \Psi_2)(x) + (\mathfrak{G}^\varphi v g)(x)(\mathfrak{F}^\phi u f)(x) \geq (\mathfrak{G}^\varphi v \Psi_2)(x)(\mathfrak{F}^\phi u f)(x) + (\mathfrak{F}^\phi u \Phi_2)(x)(\mathfrak{G}^\varphi v g)(x).$$

$$(A4) \quad (\mathfrak{F}^\phi u f)(x)(\mathfrak{G}^\varphi v g)(x) + (\mathfrak{G}^\varphi v \Psi_1)(x)(\mathfrak{F}^\phi u \Phi_1)(x) \geq (\mathfrak{G}^\varphi v g)(x)(\mathfrak{F}^\phi u \Phi_1)(x) + (\mathfrak{F}^\phi u f)(x)(\mathfrak{G}^\varphi v \Psi_1)(x).$$

Proof. Since $\Phi_1(t) \leq f(t) \leq \Phi_2(t)$ and $\Psi_1(t) \leq g(t) \leq \Psi_2(t), \forall t \in [a, b]$, we have

$$(\Phi_2(\tau) - f(\tau))(g(\rho) - \Psi_1(\rho)) \geq 0, \quad \forall \tau, \rho \in [a, b]. \quad (107)$$

This implies that

$$\Phi_2(\tau)g(\rho) + \Psi_1(\rho)f(\tau) \geq \Psi_1(\rho)\Phi_2(\tau) + f(\tau)g(\rho), \quad \forall \tau, \rho \in [a, b]. \quad (108)$$

Multiplying (108) by $u(\tau)\xi'(\tau)\mathcal{K}_x^\tau(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \xi; \phi)v(\rho)\zeta'(\rho)\mathcal{K}_y^\rho(\mathcal{E}_{\nu,\beta,\mu}^{\hat{\gamma},\hat{\delta},\hat{k},\hat{c}}, \zeta; \varphi)$ and integrating the resulting result with regard to τ and ρ from a to x and a to y , respectively, we can get

$$(\mathfrak{F}^\phi u \Phi_2)(x)(\mathfrak{G}^\phi v g)(x) + (\mathfrak{G}^\phi v \Psi_1)(x)(\mathfrak{F}^\phi u f)(x) \geq (\mathfrak{G}^\phi v \Psi_1)(x)(\mathfrak{F}^\phi u \Phi_2)(x) + (\mathfrak{F}^\phi u f)(x)(\mathfrak{G}^\phi v g)(x), \quad (109)$$

which gives (A1). To prove (A2)–(A4), we follow the proof of (A1) by letting (A2) $(f(\tau) - \Phi_1(\tau))(\Psi_2(\rho) - g(\rho)) \geq 0, \forall \tau, \rho \in [a, b]$; (A3) $(\Phi_2(\tau) - f(\tau))(\Psi_2(\rho) - g(\rho)) \geq 0, \forall \tau, \rho \in [a, b]$; (A4) $(f(\tau) - \Phi_1(\tau))(g(\rho) - \Psi_1(\rho)) \geq 0, \forall \tau, \rho \in [a, b]$. This completes the proof of Theorem 14. \square

Let $\Phi_1(t) = \Phi_1, \Phi_2(t) = \Phi_2, \Psi_1(t) = \Psi_1$ and $\Psi_2(t) = \Psi_2$ in Theorem 14, we have the following corollary.

Corollary 5. Let f, g be two integrable functions on $[a, b]$ satisfying the condition (7) and let u, v be two nonnegative continuous functions on $[a, b]$. Then the following inequalities hold:

- (B1) $\Phi_2(\mathfrak{F}^\phi u)(x)(\mathfrak{G}^\phi v g)(x) + \Psi_1(\mathfrak{G}^\phi v)(x)(\mathfrak{F}^\phi u f)(x) \geq \Psi_1 \Phi_2(\mathfrak{G}^\phi v)(x)(\mathfrak{F}^\phi u)(x) + (\mathfrak{F}^\phi u f)(x)(\mathfrak{G}^\phi v g)(x).$
- (B2) $\Psi_2(\mathfrak{F}^\phi u f)(x)(\mathfrak{G}^\phi v)(x) + \Phi_1(\mathfrak{G}^\phi v g)(x)(\mathfrak{F}^\phi u)(x) \geq \Phi_1 \Psi_2(\mathfrak{G}^\phi v)(x)(\mathfrak{F}^\phi u)(x) + (\mathfrak{F}^\phi u f)(x)(\mathfrak{G}^\phi v g)(x).$
- (B3) $\Phi_2 \Psi_2(\mathfrak{F}^\phi u)(x)(\mathfrak{G}^\phi v)(x) + (\mathfrak{G}^\phi v g)(x)(\mathfrak{F}^\phi u f)(x) \geq \Psi_2(\mathfrak{G}^\phi v)(x)(\mathfrak{F}^\phi u f)(x) + \Phi_2(\mathfrak{F}^\phi u)(x)(\mathfrak{G}^\phi v g)(x).$
- (B4) $(\mathfrak{F}^\phi u f)(x)(\mathfrak{G}^\phi v g)(x) + \Phi_1 \Psi_1(\mathfrak{G}^\phi v)(x)(\mathfrak{F}^\phi u)(x) \geq \Phi_1(\mathfrak{G}^\phi v g)(x)(\mathfrak{F}^\phi u)(x) + \Psi_1(\mathfrak{F}^\phi u f)(x)(\mathfrak{G}^\phi v)(x).$

Let $\phi(x) = x^\alpha$ and $\psi(t) = (\zeta(t) - \zeta(a))^{\kappa-1}$ in Definition 2 for $\kappa > 0$, then we have the following equation:

$$\begin{aligned} (\mathfrak{F}^\phi \psi)(x) &= \int_a^x \frac{\phi(\zeta(x) - \zeta(t))}{\zeta(x) - \zeta(t)} \mathcal{E}_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(\zeta(x) - \zeta(t))^\mu; p) \psi(t) d(\zeta(t)) \\ &= \int_a^x (\zeta(x) - \zeta(t))^{\alpha-1} \mathcal{E}_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(\zeta(x) - \zeta(t))^\mu; p) (\zeta(t) - \zeta(a))^{\kappa-1} d(\zeta(t)) \\ &\stackrel{u=\zeta(t)}{=} \int_{\zeta(a)}^{\zeta(x)} (\zeta(x) - u)^{\alpha-1} (u - \zeta(a))^\kappa \mathcal{E}_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(\zeta(x) - u)^\mu; p) du \\ &= \int_{\zeta(a)}^{\zeta(x)} (\zeta(x) - u)^{\alpha-1} (u - \zeta(a))^{\kappa-1} \sum_{n=0}^{\infty} \frac{B_p(\gamma + nk, c - \gamma)}{B(\gamma, c - \gamma)} \frac{(c)_{nk}}{\Gamma(n\mu + \alpha)} \frac{\omega^n}{(l)_{n\delta}} (\zeta(x) - u)^{n\mu} du \\ &= \sum_{n=0}^{\infty} \frac{B_p(\gamma + nk, c - \gamma)}{B(\gamma, c - \gamma)} \frac{(c)_{nk}}{\Gamma(n\mu + \alpha)} \frac{\omega^n}{(l)_{n\delta}} \int_{\zeta(a)}^{\zeta(x)} (\zeta(x) - u)^{n\mu + \alpha - 1} (u - \zeta(a))^{\kappa-1} du. \end{aligned} \quad (110)$$

Here we use the substitution $u = \zeta(x) - s(\zeta(x) - \zeta(a))$ to obtain

$$\begin{aligned} \int_{\zeta(a)}^{\zeta(x)} (\zeta(x) - u)^{n\mu + \alpha - 1} (u - \zeta(a))^{\kappa-1} du &= (\zeta(x) - \zeta(a))^{n\mu + \alpha + \kappa - 1} \int_0^1 s^{n\mu + \alpha - 1} (1 - s)^{\kappa-1} ds \\ &= (\zeta(x) - \zeta(a))^{n\mu + \alpha + \kappa - 1} B(n\mu + \alpha, \kappa). \end{aligned} \quad (111)$$

Based on the inequalities (110) and (111), we can get

$$\begin{aligned} (\mathfrak{F}^\phi \psi)(x) &= \sum_{n=0}^{\infty} \frac{B_p(\gamma + nk, c - \gamma)}{B(\gamma, c - \gamma)} \frac{(c)_{nk}}{\Gamma(n\mu + \alpha)} \frac{\omega^n}{(l)_{n\delta}} (\zeta(x) - \zeta(a))^{n\mu + \alpha + \kappa - 1} B(n\mu + \alpha, \kappa) \\ &= (\zeta(x) - \zeta(a))^{\alpha + \kappa - 1} \sum_{n=0}^{\infty} \frac{B_p(\gamma + nk, c - \gamma)}{B(\gamma, c - \gamma)} \frac{(c)_{nk}}{\Gamma(n\mu + \alpha)} \frac{\omega^n (\zeta(x) - \zeta(a))^{n\mu}}{(l)_{n\delta}} \frac{\Gamma(n\mu + \alpha) \Gamma(\kappa)}{\Gamma(n\mu + \alpha + \kappa)} \\ &= \Gamma(\kappa) (\zeta(x) - \zeta(a))^{\alpha + \kappa - 1} \mathcal{E}_{\mu, \alpha + \kappa, l}^{\gamma, \delta, k, c}(\omega(\zeta(x) - \zeta(a))^\mu; p). \end{aligned} \quad (112)$$

Letting $u(t) = v(t) = 1$, $\zeta(x) = \xi(x)$, $\phi(x) = x^\alpha$, $\varphi(x) = x^\beta$, $\psi_1(t) = (\xi(t) - \xi(a))^{\kappa-1}$ and $\psi_2(t) = (\xi(t) - \xi(a))^{\eta-1}$ in Theorems 13 and 14, according to the inequalities (110)–(112), we obtain the following equations:

$$(\mathfrak{F}^\phi \psi_1)(x) = \Gamma(\kappa)(\xi(x) - \xi(a))^{\alpha+\kappa-1} \mathcal{E}_{\mu, \alpha+\kappa, l}^{\gamma, \delta, k, c}(\omega(\xi(x) - \xi(a))^\mu; p), \quad (113)$$

$$(\mathfrak{G}^\varphi \psi_1)(x) = \Gamma(\kappa)(\xi(x) - \xi(a))^{\beta+\kappa-1} \mathcal{E}_{\nu, \beta+\kappa, \mu}^{\gamma, \delta, k, c}(\omega(\xi(x) - \xi(a))^\nu; p), \quad (114)$$

$$(\mathfrak{G}^\varphi \psi_2)(x) = \Gamma(\eta)(\xi(x) - \xi(a))^{\beta+\eta-1} \mathcal{E}_{\nu, \beta+\eta, \mu}^{\gamma, \delta, k, c}(\omega(\xi(x) - \xi(a))^\nu; p). \quad (115)$$

Based on Theorems 13 and 14, we give directly the following corollaries.

Corollary 6. Let f be an integrable function on $[a, b]$ satisfying

$$\Phi_1(\xi(t) - \xi(a))^{\kappa-1} \leq f(t) \leq \Phi_2(\xi(t) - \xi(a))^{\kappa-1} \quad (116)$$

for $\forall t \in [a, b]$, $\kappa > 0$ and $\Psi_1, \Psi_2 \in \mathbb{R}$. Then the following inequality holds:

$$\Phi_2(\mathfrak{F}^\phi \psi_1)(x)(\mathfrak{G}^\varphi f)(x) + \Phi_1(\mathfrak{G}^\varphi \psi_1)(x)(\mathfrak{F}^\phi f)(x) \geq \Phi_1 \Phi_2(\mathfrak{F}^\phi \psi_1)(x)(\mathfrak{G}^\varphi \psi_1)(x) + (\mathfrak{F}^\phi f)(x)(\mathfrak{G}^\varphi f)(x). \quad (117)$$

Corollary 7. Suppose that there exist two integrable functions Φ_1, Φ_2 on $[a, b]$ such that

$$\Phi_1(t) \leq (\xi(t) - \xi(a))^{\kappa-1} \leq \Phi_2(t), \quad \forall t \in [a, b]. \quad (118)$$

Then the following inequality holds:

$$(\mathfrak{F}^\phi \psi_1)(x)(\mathfrak{G}^\varphi \Phi_1)(x) + (\mathfrak{G}^\varphi \psi_1)(x)(\mathfrak{F}^\phi \Phi_2)(x) \geq (\mathfrak{G}^\varphi \Phi_1)(x)(\mathfrak{F}^\phi \Phi_2)(x) + (\mathfrak{F}^\phi \psi_1)(x)(\mathfrak{G}^\varphi \psi_1)(x). \quad (119)$$

Corollary 8. Let f and g be two integrable functions on $[a, b]$ satisfying (116) and

$$\Psi_1(\xi(t) - \xi(a))^{\eta-1} \leq f(t) \leq \Psi_2(\xi(t) - \xi(a))^{\eta-1} \quad (120)$$

for $\forall t \in [a, b]$, $\eta > 0$ and $\Psi_1, \Psi_2 \in \mathbb{R}$. Then the following inequalities hold:

$$(C1) \quad \Phi_2(\mathfrak{F}^\phi \psi_1)(x)(\mathfrak{G}^\varphi g)(x) + \Psi_1(\mathfrak{G}^\varphi \psi_2)(x)(\mathfrak{F}^\phi f)(x) \geq \Psi_1 \Phi_2(\mathfrak{F}^\phi \psi_1)(x)(\mathfrak{G}^\varphi \psi_2)(x) + (\mathfrak{F}^\phi f)(x)(\mathfrak{G}^\varphi g)(x).$$

$$(C2) \quad \Psi_2(\mathfrak{G}^\varphi \psi_2)(x)(\mathfrak{F}^\phi f)(x) + \Phi_1(\mathfrak{F}^\phi \psi_1)(x)(\mathfrak{G}^\varphi g)(x) \geq \Phi_1 \Psi_2(\mathfrak{F}^\phi \psi_1)(x)(\mathfrak{G}^\varphi \psi_2)(x) + (\mathfrak{F}^\phi f)(x)(\mathfrak{G}^\varphi g)(x).$$

$$(C3) \quad \Phi_2 \Psi_2(\mathfrak{F}^\phi \psi_1)(x)(\mathfrak{G}^\varphi \psi_2)(x) + (\mathfrak{G}^\varphi g)(x)(\mathfrak{F}^\phi f)(x) \geq \Psi_2(\mathfrak{G}^\varphi \psi_2)(x)(\mathfrak{F}^\phi f)(x) + \Phi_2(\mathfrak{F}^\phi \psi_1)(x)(\mathfrak{G}^\varphi g)(x).$$

$$(C4) \quad (\mathfrak{F}^\phi f)(x)(\mathfrak{G}^\varphi g)(x) + \Phi_1 \Psi_1(\mathfrak{F}^\phi \psi_1)(x)(\mathfrak{G}^\varphi \psi_2)(x) \geq \Phi_1(\mathfrak{F}^\phi \psi_1)(x)(\mathfrak{G}^\varphi g)(x) + \Psi_1(\mathfrak{G}^\varphi \psi_2)(x)(\mathfrak{F}^\phi f)(x).$$

Corollary 9. Suppose that there exist two integrable functions Φ_1, Φ_2 on $[a, b]$ such that

$$\Phi_1(t) \leq (\xi(t) - \xi(a))^{\kappa-1} \leq \Phi_2(t) \text{ and } \Psi_1(t) \leq (\xi(t) - \xi(a))^{\eta-1} \leq \Psi_2(t) \quad (121)$$

for $\forall t \in [a, b]$. Then the following inequalities hold:

$$(D1) \quad (\mathfrak{G}^\varphi \psi_2)(x)(\mathfrak{F}^\phi \Phi_2)(x) + (\mathfrak{F}^\phi \psi_1)(x)(\mathfrak{G}^\varphi \Psi_1)(x) \geq (\mathfrak{G}^\varphi \Psi_1)(x)(\mathfrak{F}^\phi \Phi_2)(x) + (\mathfrak{F}^\phi \psi_1)(x)(\mathfrak{G}^\varphi \psi_2)(x).$$

$$(D2) \quad (\mathfrak{F}^\phi \psi_1)(x)(\mathfrak{G}^\varphi \Psi_2)(x) + (\mathfrak{G}^\varphi \psi_2)(x)(\mathfrak{F}^\phi \Phi_1)(x) \geq (\mathfrak{G}^\varphi \Psi_2)(x)(\mathfrak{F}^\phi \Phi_1)(x) + (\mathfrak{F}^\phi \psi_1)(x)(\mathfrak{G}^\varphi \psi_2)(x).$$

$$(D3) \quad (\mathfrak{F}^\phi \Phi_2)(x)(\mathfrak{G}^\varphi \Psi_2)(x) + (\mathfrak{F}^\phi \psi_1)(x)(\mathfrak{G}^\varphi \psi_2)(x) \geq (\mathfrak{F}^\phi \psi_1)(x)(\mathfrak{G}^\varphi \Psi_2)(x) + (\mathfrak{G}^\varphi \psi_2)(x)(\mathfrak{F}^\phi \Phi_2)(x).$$

$$(D4) \quad (\mathfrak{F}^\phi \psi_1)(x)(\mathfrak{G}^\varphi \psi_2)(x) + (\mathfrak{G}^\varphi \Psi_1)(x)(\mathfrak{F}^\phi \Phi_1)(x) \geq (\mathfrak{G}^\varphi \psi_2)(x)(\mathfrak{F}^\phi \Phi_1)(x) + (\mathfrak{F}^\phi \psi_1)(x)(\mathfrak{G}^\varphi \Psi_1)(x).$$

Lemma 6. Let f be an integrable function on $[a, b]$ satisfying the condition (100) and let u and v be two nonnegative continuous functions on $[a, b]$. Then the following equation holds

$$(\mathfrak{F}^\phi u)(x)(\mathfrak{G}^\varphi v f^2)(x) + (\mathfrak{G}^\varphi v)(x)(\mathfrak{F}^\phi u f^2)(x) - 2(\mathfrak{F}^\phi u f)(x)(\mathfrak{G}^\varphi v f)(x)$$

$$\begin{aligned}
&= ((\mathfrak{F}^\phi u \Phi_2)(x) - (\mathfrak{F}^\phi u f)(x))((\mathfrak{G}^\phi v f)(x) - (\mathfrak{G}^\phi v \Phi_1)(x)) - (\mathfrak{F}^\phi u(\Phi_1 + \Phi_2))(x)(\mathfrak{G}^\phi v f)(x) \\
&\quad + ((\mathfrak{F}^\phi u f)(x) - (\mathfrak{F}^\phi u \Phi_1)(x))((\mathfrak{G}^\phi v \Phi_2)(x) - (\mathfrak{G}^\phi v f)(x)) - (\mathfrak{G}^\phi v(\Phi_1 + \Phi_2))(x)(\mathfrak{F}^\phi u f)(x) \\
&\quad + (\mathfrak{F}^\phi u)(x)(\mathfrak{G}^\phi v(\Phi_1 + \Phi_2)f)(x) + (\mathfrak{G}^\phi v)(x)(\mathfrak{F}^\phi u(\Phi_1 + \Phi_2)f)(x) + (\mathfrak{F}^\phi u \Phi_1)(x)(\mathfrak{G}^\phi v \Phi_2)(x) \\
&\quad + (\mathfrak{G}^\phi v \Phi_1)(x)(\mathfrak{F}^\phi u \Phi_2)(x) - (\mathfrak{F}^\phi u)(x)(\mathfrak{G}^\phi v(\Phi_2 - f)(f - \Phi_1))(x) - (\mathfrak{F}^\phi u)(x)(\mathfrak{G}^\phi v \Phi_1 \Phi_2)(x) \\
&\quad - (\mathfrak{G}^\phi v)(x)(\mathfrak{F}^\phi u(\Phi_2 - f)(f - \Phi_1))(x) - (\mathfrak{G}^\phi v)(x)(\mathfrak{F}^\phi u \Phi_1 \Phi_2)(x). \quad (122)
\end{aligned}$$

Proof. Since f is an integrable function on $[a, b]$ satisfying the condition (100), for $\forall \tau, \rho \in [a, b]$, we have

$$\begin{aligned}
&(\Phi_2(\rho) - f(\rho))(f(\tau) - \Phi_1(\tau)) + (\Phi_2(\tau) - f(\tau))(f(\rho) - \Phi_1(\rho)) \\
&\quad - (\Phi_2(\tau) - f(\tau))(f(\tau) - \Phi_1(\tau)) - (\Phi_2(\rho) - f(\rho))(f(\rho) - \Phi_1(\rho)) \\
&= f^2(\tau) + f^2(\rho) - 2f(\rho)f(\tau) + (\Phi_1(\rho) + \Phi_2(\rho))f(\tau) + (\Phi_1(\tau) + \Phi_2(\tau))f(\rho) - \Phi_1(\tau)\Phi_2(\rho) - \Phi_1(\rho)\Phi_2(\tau) \\
&\quad - (\Phi_1(\tau) + \Phi_1(\tau)\Phi_2(\tau) - (\Phi_1(\rho) + \Phi_2(\rho))f(\rho) + \Phi_2(\tau))f(\tau) + \Phi_1(\rho)\Phi_2(\rho). \quad (123)
\end{aligned}$$

Multiplying both sides of (123) by $u(\rho)\zeta'(\rho)\mathcal{H}_x^\rho(\mathcal{E}_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \zeta; \phi)$ and integrating the obtained equality with regard to ρ from a to x , we have

$$\begin{aligned}
&(f(\tau) - \Phi_1(\tau))((\mathfrak{F}^\phi u \Phi_2)(x) - (\mathfrak{F}^\phi u f)(x)) + ((\mathfrak{F}^\phi u f)(x) - (\mathfrak{F}^\phi u \Phi_1)(x))(\Phi_2(\tau) - f(\tau)) \\
&\quad - (\Phi_2(\tau) - f(\tau))(f(\tau) - \Phi_1(\tau))(\mathfrak{F}^\phi u)(x) - (\mathfrak{F}^\phi u(\Phi_2 - f)(f - \Phi_1))(x) \\
&= f^2(\tau)(\mathfrak{F}^\phi u)(x) + (\mathfrak{F}^\phi u f^2)(x) - 2f(\tau)(\mathfrak{F}^\phi u f)(x) + f(\tau)(\mathfrak{F}^\phi u(\Phi_1 + \Phi_2))(x) + (\Phi_1(\tau) + \Phi_2(\tau))(\mathfrak{F}^\phi u f)(x) \\
&\quad - \Phi_1(\tau)(\mathfrak{F}^\phi u \Phi_2)(x) - \Phi_2(\tau)(\mathfrak{F}^\phi u \Phi_1)(x) - (\Phi_1(\tau) + \Phi_2(\tau))f(\tau)(\mathfrak{F}^\phi u)(x) - (\mathfrak{F}^\phi u(\Phi_1 + \Phi_2)f)(x) \\
&\quad + \Phi_1(\tau)\Phi_2(\tau)(\mathfrak{F}^\phi u)(x) + (\mathfrak{F}^\phi u \Phi_1 \Phi_2)(x). \quad (124)
\end{aligned}$$

Multiplying both sides of (124) by $v(\tau)\zeta'(\tau)\mathcal{H}_x^\tau(\mathcal{E}_{\nu,\beta,\mu,a^+}^{\gamma,\delta,\hat{k},\hat{c}}, \zeta; \varphi)$ and integrating the established equality with regard to τ from a to y , we have

$$\begin{aligned}
&((\mathfrak{G}^\phi v f)(x) - (\mathfrak{G}^\phi v \Phi_1)(x))((\mathfrak{F}^\phi u \Phi_2)(x) - (\mathfrak{F}^\phi u f)(x)) - (\mathfrak{G}^\phi v(\Phi_2 - f)(f - \Phi_1))(x)(\mathfrak{F}^\phi u)(x) \\
&\quad + ((\mathfrak{G}^\phi v \Phi_2)(x) - (\mathfrak{G}^\phi v f)(x))((\mathfrak{F}^\phi u f)(x) - (\mathfrak{F}^\phi u \Phi_1)(x)) - (\mathfrak{F}^\phi u(\Phi_2 - f)(f - \Phi_1))(x)(\mathfrak{G}^\phi v)(x) \\
&= (\mathfrak{G}^\phi v f^2)(x)(\mathfrak{F}^\phi u)(x) + (\mathfrak{G}^\phi v)(x)(\mathfrak{F}^\phi u f^2)(x) - 2(\mathfrak{G}^\phi v f)(x)(\mathfrak{F}^\phi u f)(x) + (\mathfrak{G}^\phi v f)(x)(\mathfrak{F}^\phi u(\Phi_1 + \Phi_2))(x) \\
&\quad + (\mathfrak{G}^\phi v(\Phi_1 + \Phi_2))(x)(\mathfrak{F}^\phi u f)(x) - (\mathfrak{G}^\phi v(\Phi_1 + \Phi_2)f)(x)(\mathfrak{F}^\phi u)(x) - (\mathfrak{G}^\phi v)(x)(\mathfrak{F}^\phi u(\Phi_1 + \Phi_2)f)(x) \\
&\quad - (\mathfrak{G}^\phi v \Phi_1)(x)(\mathfrak{F}^\phi u \Phi_2)(x) - (\mathfrak{G}^\phi v \Phi_2)(x)(\mathfrak{F}^\phi u \Phi_1)(x) + (\mathfrak{G}^\phi v \Phi_1 \Phi_2)(x)(\mathfrak{F}^\phi u)(x) + (\mathfrak{G}^\phi v)(x)(\mathfrak{F}^\phi u \Phi_1 \Phi_2)(x).
\end{aligned} \quad (125)$$

which gives the required equality (122). \square

Remark 8. Let $\Phi_1(t) = \Phi_1$, $\Phi_2(t) = \Phi_2$, $\Psi_1(t) = \Psi_1$ and $\Psi_2(t) = \Psi_2$, then Lemma 6 reduces to Lemma 5.

Theorem 15. Let f, g be two integrable functions on $[a, b]$ satisfying the condition (106) and let u, v be two nonnegative continuous functions on $[a, b]$. Then the following inequality holds

$$\begin{aligned}
&|(\mathfrak{F}^\phi u)(x)(\mathfrak{G}^\phi v f g)(x) + (\mathfrak{G}^\phi v)(x)(\mathfrak{F}^\phi u f g)(x) - (\mathfrak{F}^\phi u f)(x)(\mathfrak{G}^\phi v g)(x) - (\mathfrak{G}^\phi v f)(x)(\mathfrak{F}^\phi u g)(x)| \\
&\leq \sqrt{\mathcal{S}(f, \Phi_1, \Phi_2)\mathcal{S}(g, \Psi_1, \Psi_2)}, \quad (126)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{S}(\mathbb{X}, \mathcal{Y}_1, \mathcal{Y}_2) &= ((\mathfrak{F}^\phi u \mathcal{Y}_2)(x) - (\mathfrak{F}^\phi u \mathbb{X})(x))((\mathfrak{G}^\phi v \mathbb{X})(x) - (\mathfrak{G}^\phi v \mathcal{Y}_1)(x)) + (\mathfrak{F}^\phi u)(x)(\mathfrak{G}^\phi v(\mathcal{Y}_1 + \mathcal{Y}_2)\mathbb{X})(x) \\
&\quad + ((\mathfrak{F}^\phi u \mathbb{X})(x) - (\mathfrak{F}^\phi u \mathcal{Y}_1)(x))((\mathfrak{G}^\phi v \mathcal{Y}_2)(x) - (\mathfrak{G}^\phi v \mathbb{X})(x)) + (\mathfrak{G}^\phi v)(x)(\mathfrak{F}^\phi u(\mathcal{Y}_1 + \mathcal{Y}_2)\mathbb{X})(x) \\
&\quad - (\mathfrak{F}^\phi u(\mathcal{Y}_1 + \mathcal{Y}_2))(x)(\mathfrak{G}^\phi v \mathbb{X})(x) - (\mathfrak{G}^\phi v(\mathcal{Y}_1 + \mathcal{Y}_2))(x)(\mathfrak{F}^\phi u \mathbb{X})(x) + (\mathfrak{F}^\phi u \mathcal{Y}_1)(x)(\mathfrak{G}^\phi v \mathcal{Y}_2)(x)
\end{aligned}$$

$$+ (\mathfrak{G}^\varphi v \mathcal{Y}_1)(x) (\mathfrak{F}^\phi u \mathcal{Y}_2)(x) - (\mathfrak{F}^\phi u)(x) (\mathfrak{G}^\varphi v \mathcal{Y}_1 \mathcal{Y}_2)(x) - (\mathfrak{G}^\varphi v)(x) (\mathfrak{F}^\phi u \mathcal{Y}_1 \mathcal{Y}_1)(x). \quad (127)$$

Proof. It follows from Lemma 4 that we have

$$\begin{aligned} & ((\mathfrak{F}^\phi u)(x) (\mathfrak{G}^\varphi v f g)(x) + (\mathfrak{G}^\varphi v)(x) (\mathfrak{F}^\phi u f g)(x) - (\mathfrak{F}^\phi u f)(x) (\mathfrak{G}^\varphi v g)(x) - (\mathfrak{G}^\varphi v f)(x) (\mathfrak{F}^\phi u g)(x))^2 \\ & \leq \left((\mathfrak{F}^\phi u)(x) (\mathfrak{G}^\varphi v f^2)(x) + (\mathfrak{G}^\varphi v)(x) (\mathfrak{F}^\phi u f^2)(x) - 2(\mathfrak{F}^\phi u f)(x) (\mathfrak{G}^\varphi v f)(x) \right) \\ & \quad \times \left((\mathfrak{F}^\phi u)(x) (\mathfrak{G}^\varphi v g^2)(x) + (\mathfrak{G}^\varphi v)(x) (\mathfrak{F}^\phi u g^2)(x) - 2(\mathfrak{F}^\phi u g)(x) (\mathfrak{G}^\varphi v g)(x) \right). \end{aligned} \quad (128)$$

Since $(\Phi_2(\tau) - f(\tau))(f(\tau) - \Phi_1(\tau)) \geq 0$ and $(\Psi_2(\tau) - g(\tau))(g(\tau) - \Psi_1(\tau)) \geq 0$, we have

$$-(\mathfrak{F}^\phi u)(x) (\mathfrak{G}^\varphi v (\Phi_2 - f)(f - \Phi_1))(x) - (\mathfrak{G}^\varphi v)(x) (\mathfrak{F}^\phi u (\Phi_2 - f)(f - \Phi_1))(x) \leq 0, \quad (129)$$

$$-(\mathfrak{F}^\phi u)(x) (\mathfrak{G}^\varphi v (\Psi_2 - g)(g - \Psi_1))(x) - (\mathfrak{G}^\varphi v)(x) (\mathfrak{F}^\phi u (\Psi_2 - g)(g - \Psi_1))(x) \leq 0. \quad (130)$$

From Lemma 6, we can get

$$\begin{aligned} & (\mathfrak{F}^\phi u)(x) (\mathfrak{G}^\varphi v f^2)(x) + (\mathfrak{G}^\varphi v)(x) (\mathfrak{F}^\phi u f^2)(x) - 2(\mathfrak{F}^\phi u f)(x) (\mathfrak{G}^\varphi v f)(x) \\ & \leq ((\mathfrak{F}^\phi u \Phi_2)(x) - (\mathfrak{F}^\phi u f)(x)) ((\mathfrak{G}^\varphi v f)(x) - (\mathfrak{G}^\varphi v \Phi_1)(x)) + (\mathfrak{F}^\phi u)(x) (\mathfrak{G}^\varphi v (\Phi_1 + \Phi_2) f)(x) \\ & \quad + ((\mathfrak{F}^\phi u f)(x) - (\mathfrak{F}^\phi u \Phi_1)(x)) ((\mathfrak{G}^\varphi v \Phi_2)(x) - (\mathfrak{G}^\varphi v f)(x)) + (\mathfrak{G}^\varphi v)(x) (\mathfrak{F}^\phi u (\Phi_1 + \Phi_2) f)(x) \\ & \quad - (\mathfrak{F}^\phi u (\Phi_1 + \Phi_2))(x) (\mathfrak{G}^\varphi v f)(x) - (\mathfrak{G}^\varphi v (\Phi_1 + \Phi_2))(x) (\mathfrak{F}^\phi u f)(x) + (\mathfrak{F}^\phi u \Phi_1)(x) (\mathfrak{G}^\varphi v \Phi_2)(x) \\ & \quad + (\mathfrak{G}^\varphi v \Phi_1)(x) (\mathfrak{F}^\phi u \Phi_2)(x) - (\mathfrak{F}^\phi u)(x) (\mathfrak{G}^\varphi v \Phi_1 \Phi_2)(x) - (\mathfrak{G}^\varphi v)(x) (\mathfrak{F}^\phi u \Phi_1 \Phi_1)(x) = \mathcal{S}(f, \Phi_1, \Phi_2), \end{aligned} \quad (131)$$

$$\begin{aligned} & (\mathfrak{F}^\phi u)(x) (\mathfrak{G}^\varphi v g^2)(x) + (\mathfrak{G}^\varphi v)(x) (\mathfrak{F}^\phi u g^2)(x) - 2(\mathfrak{F}^\phi u g)(x) (\mathfrak{G}^\varphi v g)(x) \\ & \leq (\mathfrak{F}^\phi u \Psi_2)(x) - (\mathfrak{F}^\phi u g)(x) ((\mathfrak{G}^\varphi v g)(x) - (\mathfrak{G}^\varphi v \Psi_1)(x)) + (\mathfrak{F}^\phi u)(x) (\mathfrak{G}^\varphi v (\Psi_1 + \Psi_2) g)(x) \\ & \quad + ((\mathfrak{F}^\phi u g)(x) - (\mathfrak{F}^\phi u \Psi_1)(x)) ((\mathfrak{G}^\varphi v \Psi_2)(x) - (\mathfrak{G}^\varphi v g)(x)) + (\mathfrak{G}^\varphi v)(x) (\mathfrak{F}^\phi u (\Phi_1 + \Phi_2) g)(x) \\ & \quad - (\mathfrak{F}^\phi u (\Psi_1 + \Psi_2))(x) (\mathfrak{G}^\varphi v g)(x) - (\mathfrak{G}^\varphi v (\Psi_1 + \Psi_2))(x) (\mathfrak{F}^\phi u g)(x) + (\mathfrak{F}^\phi u \Psi_1)(x) (\mathfrak{G}^\varphi v \Psi_2)(x) \\ & \quad + (\mathfrak{G}^\varphi v \Psi_1)(x) (\mathfrak{F}^\phi u \Psi_2)(x) - (\mathfrak{F}^\phi u)(x) (\mathfrak{G}^\varphi v \Psi_1 \Psi_2)(x) - (\mathfrak{G}^\varphi v)(x) (\mathfrak{F}^\phi u \Psi_1 \Psi_1)(x) = \mathcal{S}(g, \Psi_1, \Psi_2). \end{aligned} \quad (132)$$

Equations (131) and (132) together with inequality (128) yield the required equality (126). \square

Let $u = v$, $\varphi = \phi$, $\hat{\gamma} = \gamma$, $\hat{\delta} = \delta$, $\hat{k} = k$, $\hat{c} = c$, $\nu = \mu$, $\beta = \alpha$, $\iota = l$, and $\zeta = \xi$, we can obtain the following corollary.

Corollary 10. Let f and g be two integrable functions on $[a, b]$ satisfying the condition (106) and let u be a nonnegative continuous function on $[a, b]$. Then the following inequality holds

$$|(\mathfrak{F}^\phi u)(x) (\mathfrak{F}^\phi u f g)(x) - (\mathfrak{F}^\phi u f)(x) (\mathfrak{F}^\phi u g)(x)| \leq \sqrt{\mathcal{S}(f, \Phi_1, \Phi_2) \mathcal{S}(g, \Psi_1, \Psi_2)}, \quad (133)$$

where

$$\begin{aligned} \mathcal{S}(\mathbb{X}, \mathcal{Y}_1, \mathcal{Y}_2) &= ((\mathfrak{F}^\phi u \mathcal{Y}_2)(x) - (\mathfrak{F}^\phi u \mathbb{X})(x)) ((\mathfrak{F}^\phi u \mathbb{X})(x) - (\mathfrak{F}^\phi u \mathcal{Y}_1)(x)) + (\mathfrak{F}^\phi u)(x) (\mathfrak{F}^\phi u (\mathcal{Y}_1 + \mathcal{Y}_2) \mathbb{X})(x) \\ &+ (\mathfrak{F}^\phi u \mathcal{Y}_1)(x) (\mathfrak{F}^\phi u \mathcal{Y}_2)(x) - (\mathfrak{F}^\phi u \mathbb{X})(x) (\mathfrak{F}^\phi u (\mathcal{Y}_1 + \mathcal{Y}_2))(x) - (\mathfrak{F}^\phi u)(x) (\mathfrak{F}^\phi u \mathcal{Y}_1 \mathcal{Y}_1)(x). \end{aligned} \quad (134)$$

Furthermore, let $\phi = x^\alpha$ and $u(x) = 1$ in Corollary 10, we can obtain the following corollary.

Corollary 11. Let f and g be two integrable functions on $[a, b]$ satisfying the condition (106) and let u be a nonnegative continuous function on $[a, b]$. Then the following inequality holds

$$|(\mathfrak{F}^\phi 1)(x) (\mathfrak{F}^\phi f g)(x) - (\mathfrak{F}^\phi f)(x) (\mathfrak{F}^\phi g)(x)| \leq \sqrt{\mathcal{S}(f, \Phi_1, \Phi_2) \mathcal{S}(g, \Psi_1, \Psi_2)}, \quad (135)$$

where $(\mathfrak{F}^\phi \mathcal{F}_{\mu, \alpha, l, a^+}^{\phi, \gamma, \delta, k, c} 1)(x) = (\xi(x) - \xi(a))^\alpha \mathcal{E}_{\mu, \alpha+1, l}^{\gamma, \delta, k, c}(\omega(\xi(x) - \xi(a))^\mu; p)$ and

$$\begin{aligned} \mathcal{S}(\mathbb{X}, \mathcal{Y}_1, \mathcal{Y}_2) = & ((\mathfrak{F}^\phi \mathcal{Y}_2)(x) - (\mathfrak{F}^\phi \mathbb{X})(x))((\mathfrak{F}^\phi \mathbb{X})(x) - (\mathfrak{F}^\phi \mathcal{Y}_1)(x)) + (\mathfrak{F}^\phi 1)(x)(\mathfrak{F}^\phi (\mathcal{Y}_1 + \mathcal{Y}_2)\mathbb{X})(x) \\ & + (\mathfrak{F}^\phi \mathcal{Y}_1)(x)(\mathfrak{F}^\phi \mathcal{Y}_2)(x) - (\mathfrak{F}^\phi \mathbb{X})(x)(\mathfrak{F}^\phi (\mathcal{Y}_1 + \mathcal{Y}_2))(x) - (\mathfrak{F}^\phi 1)(x)(\mathfrak{F}^\phi \mathcal{Y}_1 \mathcal{Y}_1)(x). \end{aligned} \quad (136)$$

3. Special Cases

Let $\xi(x) = x$, $\phi(x) = x^\alpha$ in Definition 2, we define the following fractional integrals.

Definition 3 (See [43]). Let $\omega, \mu, \alpha, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\alpha), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$ and $0 < k \leq \delta + \Re(\mu)$. Furthermore, suppose that $f \in L_1[a, b]$ and $x \in [a, b]$. Then the generalized fractional integral operators are defined respectively as follows:

$$(\mathcal{F}_{\mu, \alpha, l, a^+}^{\gamma, \delta, k, c} \psi)(x; p) = \int_a^x (x-t)^{\alpha-1} \mathcal{E}_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(x-t)^\mu; p) \psi(t) dt, \quad (137)$$

$$(\mathcal{F}_{\mu, \alpha, l, b^-}^{\gamma, \delta, k, c} \psi)(x; p) = \int_x^b (t-x)^{\alpha-1} \mathcal{E}_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(t-x)^\mu; p) \psi(t) dt. \quad (138)$$

Remark 9. Fractional integral operators defined as in (137) and (138) particularly produce several known fractional integral operators corresponding to different settings of $p, \mu, k, \delta, l, c, \gamma$ in generalized Mittag-Leffler function.

- (E1) Let $p = 0$, fractional integral operators in (137) and (138) coincide with the fractional integral operators $(\mathcal{F}_{\mu, \alpha, l, a^+}^{\gamma, \delta, k, c} \psi)(x)$ and $(\mathcal{F}_{\mu, \alpha, l, b^-}^{\gamma, \delta, k, c} \psi)(x)$ defined by Salim and Faraj [44].
- (E2) Let $l = \delta = 1$, fractional integral operators in (137) and (138) reduce to the fractional integral operators $(\mathcal{F}_{\mu, \alpha, a^+}^{\gamma, k, c} \psi)(x; p)$ and $(\mathcal{F}_{\mu, \alpha, b^-}^{\gamma, k, c} \psi)(x; p)$ defined by Rahman et al. [45].
- (E3) Let $p = 0$ and $l = \delta = 1$, fractional integral operators in (137) and (138) coincide with the fractional integral operators $(\mathcal{F}_{\mu, \alpha, a^+}^{\gamma, k, c} \psi)(x)$ and $(\mathcal{F}_{\mu, \alpha, b^-}^{\gamma, k, c} \psi)(x)$ defined by Srivastava and Tomovski [46].
- (E4) Let $p = 0$ and $l = \delta = k = 1$, fractional integral operators in (137) and (138) coincide with the Prabhakar-type fractional integral operators $(\mathcal{F}_{\mu, \alpha, a^+}^{\gamma, c} \psi)(x)$ and $(\mathcal{F}_{\mu, \alpha, b^-}^{\gamma, c} \psi)(x)$ defined by Parbhakar [47].
- (E5) Let $p = \omega = 0$, fractional integral operators in (137) and (138) reduce to the well-known classical left-sided and right-sided Riemann-Liouville fractional integrals $\mathcal{I}_{a^+}^\alpha \psi(x)$ and $\mathcal{I}_{b^-}^\alpha \psi(x)$.

Let $p = \omega = 0$ in Definition 2, we have the following definition.

Definition 4 (See [48]). Let $\psi, \xi : [a, b] \rightarrow \mathbb{R}$, $0 < a < b$, be the functions such that ψ be positive and $\psi \in L_1[a, b]$, and ξ be differentiable and strictly increasing. Also let ϕ be a positive function such that ϕ/x is an increasing on $[a, +\infty)$. Then for $x \in [a, b]$, the left and right fractional integral operators are defined as follows:

$$(\xi \mathcal{F}_{a^+}^\phi \psi)(x) = \int_a^x \frac{\phi(\xi(x) - \xi(t))}{\xi(x) - \xi(t)} \psi(t) d(\xi(t)), \quad (139)$$

$$(\xi \mathcal{F}_{b^-}^\phi \psi)(x) = \int_x^b \frac{\phi(\xi(t) - \xi(x))}{\xi(t) - \xi(x)} \psi(t) d(\xi(t)). \quad (140)$$

Remark 10. The fractional integral operators defined as in (139) and (140) particularly produce large amounts of known fractional integral operators corresponding to different settings of the functions ξ and ϕ .

- (F1) Let $\phi(x) = x^{\alpha/k} / (k\Gamma_k(\alpha))$ for $\alpha, k > 0$, then operators in (139) and (140) reduce to the generalized Riemann-Liouville k -fractional integrals $\mathcal{F}_{\xi, a^+}^{\alpha, k} \psi(x)$ and $\mathcal{F}_{\xi, b^-}^{\alpha, k} \psi(x)$ defined by Kwun et al. [49].

- (F2) Let $\phi(x) = x^\alpha / \Gamma(\alpha)$ for $\alpha > 0$, then operators in (139) and (140) coincide with the generalized Riemann-Liouville fractional integrals $\mathcal{F}_{\xi,a+}^\alpha \psi(x)$ and $\mathcal{F}_{\xi,b-}^\alpha \psi(x)$ defined in [50].
- (F3) Let $\phi(x) = x^\alpha / \Gamma(\alpha)$ and $\xi(x) = x$ for $\alpha > 0$, then operators in (139) and (140) coincide with the well-known Riemann-Liouville fractional integrals $\mathcal{R}_{a+}^\alpha \psi(x)$ and $\mathcal{R}_{b-}^\alpha \psi(x)$ defined in [50].
- (F4) Let $\phi(x) = x^{\alpha/k} / (k\Gamma_k(\alpha))$ and $\xi(x) = x$ for $\alpha, k > 0$, then operators in (139) and (140) reduce to the Riemann-Liouville k -fractional integrals $\mathcal{F}_{a+}^{\alpha,k} \psi(x)$ and $\mathcal{F}_{b-}^{\alpha,k} \psi(x)$ defined by Mubeen and Habibullah et al. [51].
- (F5) Let $\phi(x) = x^\alpha / \Gamma(\alpha)$ and $\xi(x) = x^\rho / \rho$ for $\alpha, \rho > 0$, then operators in (139) and (140) coincide with the Katugampola fractional integrals ${}^\rho \mathcal{F}_{a+}^\alpha \psi(x)$ and ${}^\rho \mathcal{F}_{b-}^\alpha \psi(x)$ defined by Chen and Katugampola [52].
- (F6) Let $\phi(x) = x^\alpha / \Gamma(\alpha)$ and $\xi(x) = x^\rho$ for $\alpha, \rho > 0$, then operators in (139) and (140) coincide with the ρ -Riemann-Liouville fractional integrals ${}^\rho \mathcal{R}_{a+}^\alpha \psi(x)$ and ${}^\rho \mathcal{R}_{b-}^\alpha \psi(x)$ defined by Dragomir [53].
- (F7) Let $\phi(x) = x^\alpha / \Gamma(\alpha)$ and $\xi(x) = -x^{-1}$ for $\alpha > 0$, then operators in (139) and (140) coincide with the Harmonic fractional integrals $\mathcal{J}_{a+}^\alpha \psi(x)$ and $\mathcal{J}_{b-}^\alpha \psi(x)$ defined by Dragomir [53].
- (F8) Let $\phi(x) = x^\alpha / \Gamma(\alpha)$ and $\xi(x) = x^{\tau+\beta} / (\tau + \beta)$ for $\alpha > 0$, $\tau \in \mathbb{R}$, $\beta \in (0, 1]$ with $\tau + \beta \neq 0$, then operators in (139) and (140) coincide with the generalized conformable fractional integrals ${}^\tau \mathcal{F}_{a+}^\alpha \psi(x)$ and ${}^\tau \mathcal{F}_{b-}^\alpha \psi(x)$ defined by Khan and Khan [54].
- (F9) Let $\phi(x) = x^{\alpha/k} / (k\Gamma_k(\alpha))$, $\xi(x) = (x-a)^\beta / \beta$ and $\zeta(x) = -(b-x)^\beta / \beta$ in (139) and (140) for $k, \alpha, \beta > 0$, respectively, then operators in (139) and (140) reduce to the generalized (k, β) -conformable fractional integrals ${}_\beta \mathcal{F}_{a+}^{\alpha,k} \psi(x)$ and ${}_\beta \mathcal{F}_{b-}^{\alpha,k} \psi(x)$ defined by Habib et al. [16].
- (F10) Let $\phi(x) = x^\alpha / \Gamma(\alpha)$, $\xi(x) = (x-a)^\beta / \beta$ and $\zeta(x) = -(b-x)^\beta / \beta$ in (139) and (140) for $k, \alpha, \beta > 0$, then operators in (139) and (140) coincide with the β -conformable fractional integrals ${}_\beta \mathcal{F}_{a+}^\alpha \psi(x)$ and ${}_\beta \mathcal{F}_{b-}^\alpha \psi(x)$ defined by Jarad et al. [55]. respectively,
- (F11) Let $\phi(x) = x^{\alpha/k} / (k\Gamma_k(\alpha))$, and $\xi(x) = x^{1+\beta} / (1+\beta)$ for $\alpha > 0$, $\beta \neq -1$, then operators in (139) and (140) reduce to the (k, β) -Riemann-Liouville fractional integrals ${}_\beta \mathcal{F}_{a+}^{\alpha,k} \psi(x)$ and ${}_\beta \mathcal{F}_{b-}^{\alpha,k} \psi(x)$ defined by Sarikaya et al. [56].
- (F12) Let $\xi(x) = x$, then operators in (139) and (140) reduce to the left and right-sided generalized fractional integral operators $\mathcal{F}_{a+}^\phi \psi(x)$ and $\mathcal{F}_{b-}^\phi \psi(x)$ defined by Sarikaya et al. [57].
- (F13) Let $\phi(x) = x^\alpha / \Gamma(\alpha)$ and $\xi(x) = \ln x$ for $\alpha > 0$, then operators in (139) and (140) coincide with the left and right-sided Hadamard fractional integrals $\mathcal{H}_{a+}^\alpha \psi(x)$ and $\mathcal{H}_{b-}^\alpha \psi(x)$ defined by Kilbas [58].
- (F14) Let $\phi(x) = (x/\alpha) \exp(-(1-\alpha)x/\alpha)$ and $\xi(x) = x$ for $\alpha \in (0, 1)$, then operators in (139) and (140) reduce to the left and right-sided proportional fractional integral operators $\mathfrak{E}_{a+}^\alpha \psi(x)$ and $\mathfrak{E}_{b-}^\alpha \psi(x)$ with exponential kernel defined by Ahmad et al. [59].
- (F15) Let $\phi(x) = (x/\alpha)^\rho \exp(-(1-\alpha)x/\alpha) / \Gamma(\rho)$ and $\xi(x) = x$ for $\alpha \in (0, 1)$, then operators in (139) and (140) coincide with the left and right-sided generalized proportional fractional integral operators $\mathfrak{E}_{a+}^{\alpha,\rho} \psi(x)$ and $\mathfrak{E}_{b-}^{\alpha,\rho} \psi(x)$ defined by Jarad et al. [60].
- (F16) Let $\phi(x) = (x/\alpha)^\rho \exp(-(1-\alpha)x/\alpha) / \Gamma(\rho)$ for $\alpha \in (0, 1)$, then operators in (139) and (140) coincide with the left and right-sided Hilfer generalized proportional fractional integral operator $\mathfrak{E}_{\xi,a+}^{\alpha,\rho} \psi(x)$ and $\mathfrak{E}_{\xi,b-}^{\alpha,\rho} \psi(x)$ defined by Rashid et al. [61].

Definition 5 (See [62]). Let define the following Mittag-Leffler-like function and k -gamma function

$$\mathcal{E}_{\rho,\lambda}^{\sigma,k}(x) = \sum_{n=0}^{\infty} \frac{\sigma(n)}{k\Gamma_k(\rho kn + \lambda)} x^n \quad (\rho, \lambda > 0; |x| < \mathcal{R}), \quad (141)$$

$$\Gamma_k(\alpha) = \int_0^\infty \exp\left(-\frac{t^k}{k}\right) t^{\alpha-1} dt, \quad (142)$$

where the coefficient $\sigma(n)$ for $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is a bounded sequence of positive real number and \mathcal{R} is a positive number. k -gamma function satisfy $\Gamma_k(\alpha + k) = \alpha \Gamma_k(\alpha)$, $\Gamma_k(\alpha) = k^{\alpha/k-1} \Gamma_k(\alpha/k)$, and $\Gamma(\alpha) = \lim_{k \rightarrow 1} \Gamma_k(\alpha)$.

Similar to Definition 2, we introduce the following generalized fractional integrals.

Definition 6. Let $\psi, \xi : [a, b] \rightarrow \mathbb{R}$, $0 < a < b$, be the functions such that ψ be positive and $\psi \in L_1[a, b]$, and ξ be differentiable and strictly increasing. Also let ϕ be a positive function such that ϕ/x is an increasing on $[a, +\infty)$ and $\omega \in \mathbb{R}$, $\rho, \lambda, \mu > 0$. Then for $x \in [a, b]$, the left and right generalized fractional integral operators are defined by

$${}_{\xi}^{\Omega} \mathcal{F}_{\rho, \lambda, a^+}^{\phi, \sigma, k} \psi(x) = \Omega^{-1}(x) \int_a^x \frac{\phi(\xi(x) - \xi(t))}{\xi(x) - \xi(t)} \Omega(t) \mathcal{E}_{\rho, \lambda}^{\sigma, k}(\omega(\xi(x) - \xi(t))^\mu) \psi(t) d(\xi(t)), \quad (143)$$

$${}_{\xi}^{\Omega} \mathcal{F}_{\rho, \lambda, b^-}^{\phi, \sigma, k} \psi(x) = \Omega^{-1}(x) \int_x^b \frac{\phi(\xi(t) - \xi(x))}{\xi(t) - \xi(x)} \Omega(t) \mathcal{E}_{\rho, \lambda}^{\sigma, k}(\omega(\xi(t) - \xi(x))^\mu) \psi(t) d(\xi(t)), \quad (144)$$

where $\Omega(t)$ is a weighted function with $\Omega(t) \neq 0$ for any $t \in [a, b]$. When $\sigma(0) = 1$, $\lambda = \alpha$ and $\omega = 0$, the operators (143) and (144) degenerate into the left and right integral operators defined by

$${}_{\xi}^{\Omega} \mathcal{F}_{k, a^+}^{\phi} \psi(x) = \frac{\Omega^{-1}(x)}{k \Gamma_k(\alpha)} \int_a^x \frac{\phi(\xi(x) - \xi(t))}{\xi(x) - \xi(t)} \Omega(t) \psi(t) d(\xi(t)), \quad (145)$$

$${}_{\xi}^{\Omega} \mathcal{F}_{k, b^-}^{\phi} \psi(x) = \frac{\Omega^{-1}(x)}{k \Gamma_k(\alpha)} \int_x^b \frac{\phi(\xi(t) - \xi(x))}{\xi(t) - \xi(x)} \Omega(t) \psi(t) d(\xi(t)). \quad (146)$$

Remark 11. The fractional integral operators defined as in (143) and (144) particularly produce several known fractional integral operators corresponding to different settings of the functions ξ and ϕ .

- (G1) Let $\Omega(x) = 1$ and $\phi(x) = x^{\lambda/k}$ for $\lambda, k > 0$, then operators in (143) and (144) reduce to the generalized k -fractional integrals ${}_{\xi} \mathcal{F}_{\rho, \lambda, a^+}^{\sigma, k} \psi(x)$ and ${}_{\xi} \mathcal{F}_{\rho, \lambda, b^-}^{\sigma, k} \psi(x)$ defined by Tunc et al. [63] and Butt et al. [64].
- (G2) Let $\Omega(x) = 1$, $\phi(x) = x^\lambda$ for $\lambda > 0$, then operators in (143) and (144) reduce to the generalized fractional integrals ${}_{\xi} \mathcal{F}_{\rho, \lambda, a^+}^{\sigma} \psi(x)$ and ${}_{\xi} \mathcal{F}_{\rho, \lambda, b^-}^{\sigma} \psi(x)$ defined by Tunc et al. [63] and Butt et al. [64].
- (G3) Let $\Omega(x) = 1$, $\phi(x) = x^\lambda$ and $\xi(x) = x$ for $\lambda > 0$, then operators in (143) and (144) reduce to the generalized Riemann-Liouville fractional integrals $\mathcal{F}_{\rho, \lambda, a^+}^{\sigma} \psi(x)$ and $\mathcal{F}_{\rho, \lambda, b^-}^{\sigma} \psi(x)$.
- (G4) Let $\Omega(x) = 1$, $\phi(x) = x^{\lambda/k}$ and $\xi(x) = x$ for $\lambda, k > 0$, then operators in (143) and (144) reduce to the generalized Riemann-Liouville k -fractional integrals $\mathcal{F}_{\rho, \lambda, a^+}^{\sigma, k} \psi(x)$ and $\mathcal{F}_{\rho, \lambda, b^-}^{\sigma, k} \psi(x)$ defined by Tunc et al. [63] and Butt et al. [64].
- (G5) Let $\Omega(x) = 1$, $\phi(x) = x^{\lambda/k}$ and $\xi(x) = x^{1+\beta}/(1+\beta)$ for $\lambda, k > 0$, $\beta \neq -1$, then operators in (143) and (144) reduce to the generalized (k, β) -fractional integrals ${}_{\beta} \mathcal{F}_{\rho, \lambda, a^+}^{\sigma, k} \psi(x)$ and ${}_{\beta} \mathcal{F}_{\rho, \lambda, b^-}^{\sigma, k} \psi(x)$ defined by Tunc et al. [63] and Butt et al. [64].
- (G6) Let $\Omega(x) = 1$, $\phi(x) = x^{\lambda/k}$ and $\xi(x) = \ln x$ for $\lambda, k > 0$, then operators in (143) and (144) reduce to the generalized Hadamard k -fractional integrals $\mathcal{H}_{\rho, \lambda, a^+}^{\sigma, k} \psi(x)$ and $\mathcal{H}_{\rho, \lambda, b^-}^{\sigma, k} \psi(x)$ defined by Tunc et al. [63] and Butt et al. [64].
- (G7) Let $\Omega(x) = 1$, $\phi(x) = x^{\lambda/k}$ and $\xi(x) = x^\alpha/\alpha$ for $\lambda, k, \alpha > 0$, then operators in (143) and (144) reduce to the generalized Katugampola k -fractional integrals $\mathcal{F}_{\rho, \lambda, a^+}^{\sigma, k, \alpha} \psi(x)$ and $\mathcal{F}_{\rho, \lambda, b^-}^{\sigma, k, \alpha} \psi(x)$.

- (G8) Let $\Omega(x) = 1$, $\phi(x) = x^{\lambda/k}$ and $\xi(x) = x^{\tau+\beta}/(\tau + \beta)$ for $\lambda, k > 0$, $\tau \in \mathbb{R}$, $\beta \in (0, 1]$ with $\tau + \beta \neq 0$, then operators in (143) and (144) reduce to the generalized conformable k -fractional integrals ${}_{\beta}^{\tau}\mathcal{F}_{\rho, \lambda, a+}^{\sigma, k}\psi(x)$ and ${}_{\beta}^{\tau}\mathcal{F}_{\rho, \lambda, b-}^{\sigma, k}\psi(x)$.
- (G9) Let $\Omega(x) = 1$, $\phi(x) = x^{\lambda/k}$ and $\xi(x) = -x^{-1}$ for $\lambda, k > 0$, then operators in (143) and (144) reduce to the generalized Harmonic k -fractional integrals $\mathcal{J}_{\rho, \lambda, a+}^{\sigma, k}\psi(x)$ and $\mathcal{J}_{\rho, \lambda, b-}^{\sigma, k}\psi(x)$.
- (G10) Let $\Omega(x) = \alpha = k = 1$, then operators in (145) and (146) degenerate into the operators in (139) and (140), respectively.
- (G11) Let $\phi(x) = x^{\alpha}$ and $k = 1$, then operator in (145) coincides with the left-side fractional integral operator ${}_{\xi}^{\Omega}\mathcal{F}_{a+}^{\alpha}\psi(x)$ defined by Jarad et al. [65].
- (G12) Let $\Omega(x) = x^{\rho\eta}$, $\xi(x) = x^{\rho}$, $\phi(\xi) = x^{-\rho\alpha}\xi^{\alpha}$ and $k = 1$, then operators in (145) and (146) reduce to the Erdélyi-Kober type fractional integral operators ${}_{\rho}^{\eta}\mathcal{F}_{a+}^{\alpha}\psi(x)$ and ${}_{\rho}^{\eta}\mathcal{F}_{b-}^{\alpha}\psi(x)$ given in [50].
- (G13) Let $\Omega(x) = x^{\rho\eta}$, $\xi(x) = x^{\rho}$, $\phi(\xi) = \rho^{\beta}x^{\gamma+\rho\eta}\xi^{\alpha}$ and $k = 1$, then operator in (145) degenerates into the fractional integral operators ${}_{\rho}^{\eta}\mathcal{F}_{a+}^{\alpha, \gamma}\psi(x)$ given in Sousa et al. [66].

Remark 12. Similar to the main results in Section 2, all inequalities containing the left-side generalized fractional integral operator (143) hold all the same. Furthermore, by using the proof methods of main theorems in Section 2, we can obtain all main results containing the fractional integral operators mentioned in Remarks 9–11.

Remark 13. From Remarks 9–11, we can see easily that the left and right-side unified fractional integral operators (17) and (18) as well as generalized fractional integral operators (143) and (144) involve a large number of existing fractional integral operators. Therefore, the main results of this paper can be seen as the generalizations of the existing results in the literature. For example, some specific results are given as follows.

- (H1) Based on Theorems 1 and 2 and (F3) of Remark 10, Theorems 1 and 2 are reduced to the main results given by Dahmani [18] (Theorems 2 and 4).
- (H2) Based on Theorems 1 and 2 and (F13) of Remark 10, Theorems 1 and 2 degenerate into the primary inequalities presented by Chinchane and Pachpatte [14] (Theorems 3.1 and 3.2) and [19] (Theorems 3.2 and 3.4).
- (H3) Based on Theorems 1 and 2 and (F10) of Remark 10, Theorems 1 and 2 are turned into the main results given by Set et al. [17] (Theorems 5 and 6).
- (H4) Based on Theorems 1 and 2 and (F16) of Remark 10, Theorems 1 and 2 are changed into the primary inequalities presented by Zhou et al. [67] (Theorems 3.2 and 3.4).
- (H5) Based on Theorems 3–5 and (F9) of Remark 10, Theorems 3–5 are reduced to the main results given by Habib et al. [16] (Theorems 3.2–3.5).
- (H6) Based on Theorems 3–5 and Definition 3, Theorems 3–5 are turned into the primary inequalities presented by Yewale and Pachpatte [68] (Theorems 3.1–3.3).
- (H7) Based on Theorems 6–8 and (F13) of Remark 10, Theorems 6–8 are changed into the main results given by Taf and Brahim [69] (Theorems 2.1, 2.3 and 2.7).
- (H8) Based on Theorems 6 and 12 and (G13) of Remark 12, Theorems 6 and 9 degenerate into the primary inequalities presented by Sousa et al. [66] (Theorems 1 and 2).
- (H9) Based on Theorems 9 and 10 and (F3) of Remark 10, Theorems 9 and 10 are turned into the main results given by Dahmani et al. [70] (Theorems 3.1 and 3.2) and Dahmani et al. [71] (Theorems 2 and 3), respectively.
- (H10) Based on Theorems 9 and 10 and (F5) of Remark 10, Theorems 9 and 10 are changed into the main results given by Çelik et al. [72] (Theorems 2.1 and 2.2).
- (H11) Based on Theorems 9 and 10 and (F16) of Remark 10, Theorems 9 and 10 are reduced to the primary inequalities presented by Zhou et al. [73] (Theorems 2.1 and 2.2).
- (H12) Based on Theorems 9 and 10 and (G11) of Remark 12, Theorems 9 and 10 are turned into the main results given by Rahman et al. [74] (Theorems 6 and 7).

- (H13) Based on Theorems 9 and 10 and Definition 3, Theorems 9 and 10 are changed into the main results given by Set et al. [75] (Theorems 2.1 and 2.2).
- (H14) Based on Theorems 11 and 12 and (F3) of Remark 10, Theorems 11 and 12 degenerate into the primary inequalities presented by Dahmani et al. [31] (Theorems 3.1 and 3.3).
- (H15) Based on Theorems 11 and 12 and (G1) of Remark 12, Theorems 11 and 12 are turned into the main results given by Butt et al. [64] (Theorems 2 and 3).
- (H16) Based on Theorems 11 and 12 and (F9) of Remark 10, Theorems 11 and 12 are changed into the primary inequalities presented given by Rahman et al. [76] (Theorems 2 and 5).
- (H17) Based on Theorems 13 and 14 and (F3) of Remark 10, Theorems 13 and 14 are reduced to the main results given by Tariboon et al. [77] (Theorems 2 and 5).
- (H18) Based on Theorems 13–15 and (F11) of Remark 10, Theorems 13–15 degenerate into the primary inequalities presented by Kaçar and Yildirim [78] (Theorems 5–7).
- (H19) Based on Theorems 13 and 14 and (F11) of Remark 10, Theorems 13 and 14 are turned into the main results given by Mubeen and Iqbal [79] (Theorems 2.1 and 2.5).
- (H20) Based on Theorems 13 and 1 and (F10) of Remark 10, Theorems 13–15 are changed into the primary inequalities presented by Rahman et al. [80] (Theorems 2.1 and 2.2).
- (H21) Based on Theorems 13–15 and (F9) of Remark 10, Theorems 13–15 are reduced to the main results given by Farid et al. [81] (Theorems 2.1, 2.2 and 2.4).
- (H22) Based on Theorems 13 and 14 and (F15) of Remark 10, Theorems 13 and 14 are turned into the primary inequalities presented by Rashid et al. [82] (Theorems 3.1 and 3.5).
- (H23) Based on Theorems 13–15 and (G13) of Remark 12, Theorems 13–15 are changed into the main results given by Aljaaidi et al. [83] [Theorems 3.1, 3.4 and 3.7].
- (H24) Based on Theorems 13–15 and (F9) of Remark 10, Theorems 13–15 degenerate into the primary inequalities presented by Rashid et al. [84] (Theorems 2.1, 2.5 and 2.10).

4. Conclusions

In this paper, we have investigated the Chebyshev- and Grüss-type inequalities for unified fractional integral operators via an extended generalized Mittag-Leffler function. Then two generalized fractional integral operators involving the Mittag-Leffler-like function and weighted function have been introduced. Using the newly introduced generalized fractional integral operators, some new inequalities similar to the main results can be also presented. Moreover, their relevance with known inequalities for different kinds of fractional integral operators have been demonstrated. Based on main results in this paper, our future research objects are to investigate some other inequalities by using the unified fractional integral operators and generalized fractional integral operators introduced in this paper.

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