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Some q-Fractional Estimates of Trapezoid like Inequalities Involving Raina's Function

Kamsing Nonlaopon [1], Muhammad Uzair Awan [2,*], Muhammad Zakria Javed [2], Hüseyin Budak [3] and Muhammad Aslam Noor [4]

1 Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand; nkamsi@kku.ac.th
2 Department of Mathematics, Government College University, Faisalabad 38000, Pakistan; zakriajaved071@gmail.com
3 Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce 81620, Turkey; huseyinbudak@duzce.edu.tr
4 Department of Mathematics, COMSATS University Islamabad, Islamabad 45550, Pakistan; noormaslam@gmail.com
* Correspondence: muawan@gcuf.edu.pk

Abstract: In this paper, we derive two new identities involving q-Riemann-Liouville fractional integrals. Using these identities, as auxiliary results, we derive some new q-fractional estimates of trapezoidal-like inequalities, essentially using the class of generalized exponential convex functions.

Keywords: convex; exponential convex; fractional; quantum; inequalities

MSC: 05A30; 26A33; 26A51; 34A08; 26D07; 26D10; 26D15

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1. Introduction

A set $\mathcal{C} \subseteq \mathbb{R}$ is said to be convex, if

$$(1-\nu)x + \nu y \in \mathcal{C}$$

for all $x, y \in \mathcal{C}$ and $\nu \in [0,1]$.

A function $\mathscr{F} : \mathcal{C} \to \mathbb{R}$ is said to be convex, if

$$\mathscr{F}((1-\nu)x + \nu y) \leqslant (1-\nu)\mathscr{F}(x) + \nu\mathscr{F}(y)$$

for all $x, y \in \mathcal{C}$ and $\nu \in [0,1]$.

The classical concepts of convexity are simple but have many applications in different fields of pure and applied sciences. For example, they play a significant role in the theory of optimization, mathematical economics, operations research, etc. In recent years, the classical concepts of convexity have been extended and generalized in different directions using novel and innovative ideas. It has been observed that these new generalizations of classical convexity enjoy some nice properties which classical convexity has. Recently, Cortez et al. [1] presented a new generalization of convexity class as follows:

Definition 1 ([1]). *Let $\rho, \lambda > 0$ and $\sigma = (\sigma(0), \dots, \sigma(k), \dots)$ be a bounded sequence of positive real numbers. A non-empty set $\mathcal{I}$ is said to be generalized convex, if*

$$\varpi_1 + \tau\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2 - \varpi_1) \in \mathcal{I}$$

for all $\varpi_1, \varpi_2 \in \mathcal{I}$ and $\tau \in [0,1]$.

Here, $\mathcal{R}_{\rho,\lambda,\sigma}(z)$ is the Raina's function and is defined as:

$$\mathcal{R}_{\rho,\lambda,\sigma}(z) = \mathcal{R}^{\sigma(0),\sigma(1),\dots}_{\rho,\lambda}(z) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} z^k,$$

where $\rho, \lambda > 0, |z| < R$, $\sigma = \{\sigma(0), \sigma(1), \dots, \sigma(k), \dots\}$ is a bounded sequence of positive real numbers and $\Gamma(\eta) = \int_0^\infty x^{\eta-1}e^{-x}dx$ is the gamma function. For details, see [2].

Cortez et al. [1] also defined the class of generalized convex functions as:

Definition 2 ([1])**.** *Let $\rho, \lambda > 0$ and $\sigma = (\sigma(0), \dots, \sigma(k), \dots)$ be a bounded sequence of positive real numbers. A function $\mathscr{F} : \mathcal{I} \to \mathbb{R}$ is said to be generalized convex, if*

$$\mathscr{F}(\varpi_1 + \tau \mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2 - \varpi_1)) \leqslant (1-\tau)\mathscr{F}(\varpi_1) + \tau \mathscr{F}(\varpi_2)$$

for all $\varpi_1, \varpi_2 \in \mathcal{I}$ and $\tau \in [0,1]$.

Awan et al. [3] introduced the class of exponential convex functions as:

Definition 3 ([3])**.** *A function $\mathscr{F} : \mathcal{C} \to \mathbb{R}$ is said to be exponentially convex, if*

$$\mathscr{F}((1-\nu)x + \nu y) \leqslant (1-\nu)\frac{\mathscr{F}(x)}{\exp(\alpha x)} + \nu \frac{\mathscr{F}(y)}{\exp(\alpha y)},$$

for all $x, y \in \mathcal{C}$ and $\nu \in [0,1]$.

Besides its applications, the theory of convexity has also played a dynamic role in developing the theory of inequalities. A wide class of inequalities is just a direct consequence of the applications of the convexity property of the functions. Hermite–Hadamard's inequality, also known as trapezium-like inequality, is one of the most studied results. It reads as:

Let $\mathscr{F} : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function, then

$$\mathscr{F}\left(\frac{\varpi_1 + \varpi_2}{2}\right) \leqslant \frac{1}{\varpi_2 - \varpi_1} \int_{\varpi_1}^{\varpi_2} \mathscr{F}(x)dx \leqslant \frac{\mathscr{F}(\varpi_1) + \mathscr{F}(\varpi_2)}{2}.$$

For some recent developments related to Hermite–Hadamard's inequality and its applications, see [4].

In recent years, several new techniques have been used to obtain new versions of Hermite–Hadamard's inequality. For instance, Sarikaya et al. [5] utilized the concepts of fractional calculus and obtained the fractional analogues of Hermite–Hadamard's inequality. Alp et al. [6] obtained quantum analogue of Hermite–Hadamard's inequality. Awan et al. [3] obtained a new refinement of Hermite–Hadamard's inequality using the class of exponentially convex functions. Cortez et al. [1] obtained Hermite–Hadamard's inequality using the class of generalized convex functions. Kunt and Aljasem [7] obtained fractional quantum versions of Hermite–Hadamard type of inequalities. Noor et al. [8] obtained some more quantum estimates for Hermite–Hadamard inequalities using the class of convex functions. Sudsutad [9] obtained various new quantum integral inequalities for convex functions. Zhang et al. [10] obtained a new generalized quantum-integral identity and obtained new q-integral inequalities via (α, m)-convexity property of the functions.

The main motivation of this article is to obtain two new identities involving q-Riemann–Liouville fractional integrals. Using these identities as auxiliary results, we derive some new q-fractional estimates of trapezoidal-like inequalities, essentially using the class of generalized exponential convex functions. We hope that the ideas and techniques of this article will inspire interested readers working in this field.

2. Preliminaries

In this section, we recall some previously known concepts and results.
The following concept of q-derivative was introduced and studied in [11].

Definition 4 ([11]). *For a continuous function* $\mathscr{F} : [\varpi_1, \varpi_2] \to \mathbb{R}$ *the* q*-derivative of* $\mathscr{F}$ *at* $x \in [\varpi_1, \varpi_2]$ *is defined as:*

$$ {}_{\varpi_1}D_q\mathscr{F}(x) = \frac{\mathscr{F}(x) - \mathscr{F}(qx + (1-q)\varpi_1)}{(1-q)(x-\varpi_1)}, \quad x \neq \varpi_1. \tag{1}$$

The q-definite integral is defined as:

Definition 5 ([11]). *Let* $\mathscr{F} : [\varpi_1, \varpi_2] \to \mathbb{R}$ *be a continuous function. Then the* q*-definite integral on* $[\varpi_1, \varpi_2]$ *is defined as:*

$$ \int_{\varpi_1}^{x} \mathscr{F}(\nu)\, {}_{\varpi_1}d_q\nu = (1-q)(x-\varpi_1)\sum_{n=0}^{\infty} q^n \mathscr{F}(q^n x + (1-q^n)\varpi_1), \tag{2}$$

for $x \in [\varpi_1, \varpi_2]$.

Interesting additional details of the following concepts can be found in [9,12].

$$ [m]_q = \frac{1-q^m}{1-q}, \quad m \in \mathbb{R}. \tag{3}$$

The q-analogue of power function is defined as, if $\gamma \in \mathbb{R}$, then

$$ (r-m)^{(\gamma)} = r^{\gamma}\prod_{n=0}^{\infty}\frac{r-q^n m}{r-q^{\gamma+n}m}, \quad r \neq 0. \tag{4}$$

The q-gamma function is defined as:

$$ \Gamma_q(\nu) = \frac{(1-q)^{(\nu-1)}}{(1-q)^{\nu-1}}, \quad \nu \in \mathbb{R}/\{0,-1,-2,\dots\}. \tag{5}$$

For any $s, \nu > 0$, the q-beta function is defined as:

$$ B_q(s,\nu) = \int_0^1 u^{(s-1)}(1-qu)^{(\nu-1)} d_q u, \tag{6}$$

and

$$ B_q(s,\nu) = \frac{\Gamma_q(s)\Gamma_q(\nu)}{\Gamma_q(s+\nu)}.$$

The q-Pochhammer symbol is defined as:

$$ (m;q)_0 = 1, \quad \text{and} \quad (m;q)_k = \prod_{n=0}^{k-1}(1-q^n m) \tag{7}$$

for $k \in \mathbb{N} \cup \{\infty\}$.

Theorem 1 ([13]). *Suppose* $\lambda, \mu \in \mathbb{R}$, *then*

$$ \lim_{q\to 1^-}\frac{(q^{\lambda}x;q)}{(q^{\mu}x;q)} = (1-x)^{\mu-\lambda}, \tag{8}$$

uniformly on $\{x \in \mathbb{C} : |x| \leqslant 1\}$, *if* $\mu \geqslant \lambda$, $\lambda + \mu \geqslant 1$, *and uniformly on compact subset of* $\{x \in \mathbb{C} : |x| \leqslant 1, x \neq 1\}$ *for other choices of* μ *and* λ.

The q-shifting operator is defined as:

$$ {}_{\varpi_1}\Phi_q(m) = qm + (1-q)\varpi_1. \tag{9}$$

For any positive integer k, one has:

$$ {}_{\varpi_1}\Phi_q^k(m) = {}_{\varpi_1}\Phi_q^{k-1}({}_{\varpi_1}\Phi_q(m)), \quad {}_{\varpi_1}\Phi_q^0(m) = m. \tag{10}$$

The following properties for q-shifting operator hold:

Theorem 2 ([9,12]). *For any* $r, m \in \mathbb{R}$ *and for all positive integers* k, j, *one has:*

1. ${}_{\varpi_1}\Phi_q^k(m) = {}_{\varpi_1}\Phi_{q^k}(m)$;
2. ${}_{\varpi_1}\Phi_q^k({}_{\varpi_1}\Phi_q^j(m)) = {}_{\varpi_1}\Phi_q^j({}_{\varpi_1}\Phi_q^k(m)) = {}_{\varpi_1}\Phi_q^{j+k}(m)$;
3. ${}_{\varpi_1}\Phi_q(\varpi_1) = \varpi_1$;
4. ${}_{\varpi_1}\Phi_q^k(m) - \varpi_1 = q^k(m - \varpi_1)$;
5. $m - {}_{\varpi_1}\Phi_q^k(m) = (1-q^k)(m-\varpi_1)$;
6. ${}_{\varpi_1}\Phi_q^k(m) = m\, {}_{\frac{\varpi_1}{m}}\Phi_q^k(1)$, *for* $m \neq 0$;
7. ${}_{\varpi_1}\Phi_q(m) - {}_{\varpi_1}\Phi_q^k(r) = q(m - {}_{\varpi_1}\Phi_q^{k-1}(r))$.

The power of q-shifting operator is defined as:

$$ {}_{\varpi_1}(r-m)_q^{(\gamma)} = (r-\varpi_1)^\gamma \prod_{n=0}^{\infty} \frac{r - {}_{\varpi_1}\Phi_q^n(m)}{r - {}_{\varpi_1}\Phi_q^{\gamma+n}(m)}, \quad \gamma \in \mathbb{R}. \tag{11}$$

Theorem 3 ([9,12]). *For any* $\gamma, r, m \in \mathbb{R}$, $r \neq \varpi_1$ *and* $k \in \mathbb{N}$, *one has:*

1. ${}_{\varpi_1}(r-m)_q^{(k)} = (r-\varpi_1)^k \left(\frac{m-\varpi_1}{r-\varpi_1}; q\right)_k$;
2. ${}_{\varpi_1}(r-m)_q^{(\gamma)} = (r-\varpi_1)^\gamma \prod_{n=0}^{\infty} \frac{1-\frac{m-\varpi_1}{r-\varpi_1}q^n}{1-\frac{m-\varpi_1}{r-\varpi_1}q^{n+\gamma}} = (r-\varpi_1)^\gamma \frac{\left(\frac{m-\varpi_1}{r-\varpi_1}; q\right)_\infty}{\left(\frac{m-\varpi_1}{r-\varpi_1}q^\gamma; q\right)_\infty}$;
3. ${}_{\varpi_1}(r - {}_{\varpi_1}\Phi_q^k(r))_q^{(\gamma)} = (r-\varpi_1)^\gamma \frac{(q^k;q)_\infty}{(q^{\gamma+k};q)_\infty}$.

Definition 6 ([9,12]). *Let* $\alpha \geqslant 0$ *and* $\mathscr{F}$ *be a continuous function on* $[\varpi_1, \varpi_2]$. *Then the Riemann–Liouville-type fractional quantum integral is given by* $({}_{\varpi_1}J_q^0\mathscr{F})(\nu) = \mathscr{F}(\nu)$ *and*

$$\begin{aligned}({}_{\varpi_1}J_q^\alpha\mathscr{F})(x) = ({}_{\varpi_1}J_q^\alpha\mathscr{F}(\nu))(x) &= \frac{1}{\Gamma_q(\alpha)} \int_{\varpi_1}^{x} {}_{\varpi_1}(x - {}_{\varpi_1}\Phi_q(\nu))_q^{(\alpha-1)} \mathscr{F}(\nu)\, {}_{\varpi_1}d_q\nu \\ &= \frac{(1-q)(x-\varpi_1)}{\Gamma_q(\alpha)} \sum_{n=0}^{\infty} q^n\, {}_{\varpi_1}(x - {}_{\varpi_1}\Phi_q^{n+1}(x))_q^{(\alpha-1)} \mathscr{F}({}_{\varpi_1}\Phi_q^n(x)),\end{aligned} \tag{12}$$

where $\alpha > 0$ *and* $x \in [\varpi_1, \varpi_2]$.

3. Results and Discussions

In this section, we will discuss our main results. First of all we define the class of generalized exponential convex functions.

Definition 7. *Let* $\rho, \lambda > 0$ *and* $\sigma = (\sigma(0), \ldots, \sigma(k), \ldots)$ *be a bounded sequence of positive real numbers. A function* $\mathscr{F} : \mathcal{I} \to \mathbb{R}$ *is said to be generalized exponential convex, if*

$$\mathscr{F}(\varpi_1 + \tau\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2 - \varpi_1)) \leqslant (1-\tau)\frac{\mathscr{F}(\varpi_1)}{\chi^{\alpha\varpi_1}} + \tau\frac{\mathscr{F}(\varpi_2)}{\chi^{\alpha\varpi_2}}$$

for all $\varpi_1, \varpi_2 \in \mathcal{I}$, $\tau \in [0,1]$ *and* $\chi \geqslant 1$.

Note that if we take $\alpha = 0$ or $\chi = 1$, then the class of generalized exponential convex functions reduces to the class of generalized convex functions introduced and studied in [1]. If we take $\chi = \exp$, then we have the class of exponentially convex functions involving Raina's function. This class is defined as:

Definition 8. *Let* $\rho, \lambda > 0$ *and* $\sigma = (\sigma(0), \ldots, \sigma(k), \ldots)$ *be a bounded sequence of positive real numbers. A function* $\mathscr{F} : \mathcal{I} \to \mathbb{R}$ *is said to be generalized exponential convex, if*

$$\mathscr{F}(\varpi_1 + \tau\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2 - \varpi_1)) \leqslant (1-\tau)\frac{\mathscr{F}(\varpi_1)}{\exp(\alpha\varpi_1)} + \tau\frac{\mathscr{F}(\varpi_2)}{\exp(\alpha\varpi_2)}$$

for all $\varpi_1, \varpi_2 \in \mathcal{I}$ *and* $\tau \in [0,1]$.

Now, we derive our auxiliary results. Before we proceed, for the sake of simplicity, we consider $\Omega = [\varpi_1, \varpi_1 + \mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2 - \varpi_1)]$ and $\Omega^\circ = (\varpi_1, \varpi_1 + \mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2 - \varpi_1))$.

Lemma 1. *Let* $\mathscr{F} : \Omega \to \mathbb{R}$ *be a continuous function and* $\alpha > 0$. *If* ${}_{\varpi_1}D_q\mathscr{F}$ *is* q*-integrable on* Ω°, *then*

$$\begin{aligned}
&\frac{\Gamma_q(\alpha+1)}{\mathcal{R}^{\alpha}_{\rho,\lambda,\sigma}(\varpi_2 - \varpi_1)}\left({}_{\varpi_1}J^{\alpha}_q\mathscr{F}\right)(\varpi_1 + \mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2 - \varpi_1)) - \frac{([\alpha+1]_q - 1)\mathscr{F}(\varpi_1) + \mathscr{F}(\varpi_1 + \mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2 - \varpi_1))}{[\alpha+1]_q} \\
&= \frac{\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2 - \varpi_1)}{[\alpha+1]_q}\int_0^1\left([\alpha+1]_q\, {}_0(1 - {}_0\Phi_q(\nu))^{(\alpha)}_q - 1\right){}_{\varpi_1}D_q\mathscr{F}(\varpi_1 + \nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2 - \varpi_1))\,{}_0d_q\nu.
\end{aligned} \tag{13}$$

Proof. It suffices to show that

$$\begin{aligned}
&\frac{\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2 - \varpi_1)}{[\alpha+1]_q}\int_0^1\left([\alpha+1]_q\, {}_0(1 - {}_0\Phi_q(\nu))^{(\alpha)}_q - 1\right){}_{\varpi_1}D_q\mathscr{F}(\varpi_1 + \nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2 - \varpi_1))\,{}_0d_q\nu \\
&= \mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2 - \varpi_1)\int_0^1 {}_0(1 - {}_0\Phi_q(\nu))^{(\alpha)}_q\, {}_{\varpi_1}D_q\mathscr{F}(\varpi_1 + \nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2 - \varpi_1))\,{}_0d_q\nu \\
&- \frac{\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2 - \varpi_1)}{[\alpha+1]_q}\int_0^1 {}_{\varpi_1}D_q\mathscr{F}(\varpi_1 + \nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2 - \varpi_1))\,{}_0d_q\nu \\
&= \mathcal{S}_1 - \mathcal{S}_2.
\end{aligned} \tag{14}$$

Now,

$$
\begin{aligned}
\mathcal{S}_1 &= \mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\int_0^1 {}_0(1-{}_0\Phi_q(\nu))_q^{(\alpha)}\,{}_{\varpi_1}D_q\mathscr{F}(\varpi_1+\nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))\,{}_0d_q\nu \\
&= \mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\int_0^1 {}_0(1-{}_0\Phi_q(\nu))_q^{(\alpha)}\frac{\mathscr{F}(\varpi_1+\nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))-\mathscr{F}(\varpi_1+q\nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))}{(1-q)\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\nu}\,{}_0d_q\nu \\
&= \frac{1}{1-q}\int_0^1 {}_0(1-{}_0\Phi_q(\nu))_q^{(\alpha)}\frac{\mathscr{F}(\varpi_1+\nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))}{\nu}\,{}_0d_q\nu \\
&\quad -\frac{1}{1-q}\int_0^1 {}_0(1-{}_0\Phi_q(\nu))_q^{(\alpha)}\frac{\mathscr{F}(\varpi_1+q\nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))}{\nu}\,{}_0d_q\nu \\
&= \sum_{n=0}^{\infty} q^n\,{}_0(1-{}_0\Phi_q^{n+1}(1))_q^{(\alpha)}\frac{\mathscr{F}\Big(\varpi_1+{}_0\Phi_q^n(1)\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\Big)}{{}_0\Phi_q^n(1)} \\
&\quad -\sum_{n=0}^{\infty} q^n\,{}_0(1-{}_0\Phi_q^{n+1}(1))_q^{(\alpha)}\frac{\mathscr{F}\Big(\varpi_1+q\,{}_0\Phi_q^n(1)\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\Big)}{{}_0\Phi_q^n(1)} \\
&= \left[\begin{array}{l}\sum_{n=0}^{\infty}\frac{(q^{n+1};q)_\infty}{(q^{\alpha+n+1};q)_\infty}\mathscr{F}\big(\varpi_1+q^n\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\big)\\ -\sum_{n=0}^{\infty}\frac{(q^{n+1};q)_\infty}{(q^{\alpha+n+1};q)_\infty}\mathscr{F}\big(\varpi_1+q^{n+1}\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\big)\end{array}\right] \\
&= \left[\begin{array}{l}\sum_{n=0}^{\infty}(1-q^{\alpha+n})\frac{(q^{n+1};q)_\infty}{(q^{\alpha+n};q)_\infty}\mathscr{F}\big(\varpi_1+q^n\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\big)\\ -\sum_{n=0}^{\infty}(1-q^{n+1})\frac{(q^{n+2};q)_\infty}{(q^{\alpha+n+1};q)_\infty}\mathscr{F}\big(\varpi_1+q^{n+1}\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\big)\end{array}\right] \\
&= \left[\begin{array}{l}\sum_{n=0}^{\infty}\frac{(q^{n+1};q)_\infty}{(q^{\alpha+n};q)_\infty}\mathscr{F}\big(\varpi_1+q^n\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\big)\\ -\sum_{n=0}^{\infty}\frac{(q^{n+2};q)_\infty}{(q^{\alpha+n+1};q)_\infty}\mathscr{F}\big(\varpi_1+q^{n+1}\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\big)\end{array}\right] \\
&= \left[\begin{array}{l}\sum_{n=0}^{\infty}q^{\alpha+n}\frac{(q^{n+1};q)_\infty}{(q^{\alpha+n};q)_\infty}\mathscr{F}\big(\varpi_1+q^n\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\big)\\ -\sum_{n=0}^{\infty}q^{n+2}\frac{(q^{n+2};q)_\infty}{(q^{\alpha+n+1};q)_\infty}\mathscr{F}\big(\varpi_1+q^{n+1}\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\big)\end{array}\right] \\
&= \frac{(q^1;q)_\infty}{(q^\alpha;q)_\infty}\mathscr{F}(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))-\mathscr{F}(\varpi_1) \\
&\quad -\left[\begin{array}{l}\sum_{n=0}^{\infty}q^{\alpha+n}\frac{(q^{n+1};q)_\infty}{(q^{\alpha+n};q)_\infty}\mathscr{F}\big(\varpi_1+q^n\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\big)\\ -\sum_{n=1}^{\infty}q^{n}\frac{(q^{n+1};q)_\infty}{(q^{\alpha+n};q)_\infty}\mathscr{F}\big(\varpi_1+q^{n}\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\big)\end{array}\right] \\
&= \frac{(q^1;q)_\infty}{(q^\alpha;q)_\infty}\mathscr{F}(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))-\mathscr{F}(\varpi_1) \\
&\quad -\left[\begin{array}{l}\sum_{n=0}^{\infty}q^{\alpha+n}\frac{(q^{n+1};q)_\infty}{(q^{\alpha+n};q)_\infty}\mathscr{F}\big(\varpi_1+q^n\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\big)\\ -\sum_{n=0}^{\infty}q^{n}\frac{(q^{n+1};q)_\infty}{(q^{\alpha+n};q)_\infty}\mathscr{F}\big(\varpi_1+q^{n}\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\big)\\ \qquad +\frac{(q^1;q)_\infty}{(q^\alpha;q)_\infty}\mathscr{F}(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))\end{array}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= -\mathscr{F}(\varpi_1) + (1-q^n)\sum_{n=0}^{\infty} q^n \frac{(q^{n+1};q)_\infty}{(q^{\alpha+n};q)_\infty}\mathscr{F}\big(\varpi_1 + q^n \mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\big) \\
&= -\mathscr{F}(\varpi_1) + [\alpha]_q(1-q)\sum_{n=0}^{\infty} q^n \frac{(q^{n+1};q)_\infty}{(q^{\alpha+n};q)_\infty}\mathscr{F}\big(\varpi_1 + q^n \mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\big) \\
&= -\mathscr{F}(\varpi_1) + \frac{[\alpha]_q \Gamma_q(\alpha)}{\mathcal{R}^{\alpha}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)} \\
&\quad\times\left(\frac{(1-q)\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}{\Gamma_q(\alpha)}\sum_{n=0}^{\infty} q^n \zeta^{\alpha-1}(\varpi_2,\varpi_1)\frac{(q^{n+1};q)_\infty}{(q^{\alpha+n};q)_\infty}\mathscr{F}\big(\varpi_1 + q^n \mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\big)\right) \\
&= -\mathscr{F}(\varpi_1) + \frac{\Gamma_q(\alpha+1)}{\mathcal{R}^{\alpha}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)} \\
&\quad\times\left(\frac{(1-q)\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}{\Gamma_q(\alpha)}\sum_{n=0}^{\infty} q^n \zeta^{\alpha-1}(\varpi_2,\varpi_1)\frac{(q^{n+1};q)_\infty}{(q^{(\alpha+1)+(n+1)};q)_\infty}\mathscr{F}\big(\varpi_1 + q^n \mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\big)\right) \\
&= -\mathscr{F}(\varpi_1) + \frac{\Gamma_q(\alpha+1)}{\mathcal{R}^{\alpha}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}\left(\frac{(1-q)\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}{\Gamma_q(\alpha)}\sum_{n=0}^{\infty} q^n\ {}_{\varpi_1}(\varpi_1 + \mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))\right. \\
&\quad\left. -\ {}_{\varpi_1}\Phi_q^{n+1}(\varpi_1 + \mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)))_q^{(\alpha-1)}\mathscr{F}\Big({}_{\varpi_1}\Phi_q^{n}(\varpi_1 + \mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))\Big)\right) \\
&= -\mathscr{F}(\varpi_1) + \frac{\Gamma_q(\alpha+1)}{\mathcal{R}^{\alpha}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)} \\
&\quad\times\left(\frac{1}{\Gamma_q(\alpha)}\int_{\varpi_1}^{\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)} {}_{\varpi_1}(\varpi_1 + \mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1) -\ {}_{\varpi_1}\Phi_q(\nu))_q^{(\alpha-1)}\mathscr{F}(\nu)\ {}_{\varpi_1}d_q\nu\right) \\
&= -\mathscr{F}(\varpi_1) + \frac{\Gamma_q(\alpha+1)}{\mathcal{R}^{\alpha}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}\big({}_{\varpi_1}J_q^{\alpha}\mathscr{F}\big)(\varpi_1 + \mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)).
\end{aligned} \tag{15}
$$

Similarly, we have

$$
\begin{aligned}
\mathcal{S}_2 &= \frac{\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}{[\alpha+1]_q}\int_0^1 {}_{\varpi_1}D_q\mathscr{F}(\varpi_1 + \nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))\ {}_0d_q\nu \\
&= \frac{\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}{[\alpha+1]_q}\int_0^1 \frac{\mathscr{F}(\varpi_1 + \nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)) - \mathscr{F}(\varpi_1 + q\nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))}{(1-q)\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\nu}\ {}_0d_q\nu \\
&= \frac{1}{(1-q)[\alpha+1]_q}\int_0^1 \frac{\mathscr{F}(\varpi_1 + \nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))}{\nu}\ {}_0d_q\nu \\
&\quad - \frac{1}{(1-q)[\alpha+1]_q}\int_0^1 \frac{\mathscr{F}(\varpi_1 + q\nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))}{\nu}\ {}_0d_q\nu \\
&= \frac{1}{[\alpha+1]_q}\left[\sum_{n=0}^{\infty}\mathscr{F}(\varpi_1 + q^n\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)) - \sum_{n=0}^{\infty}\mathscr{F}(\varpi_1 + q^{n+1}\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))\right] \\
&= \frac{\mathscr{F}(\varpi_1 + \mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)) - \mathscr{F}(\varpi_1)}{[\alpha+1]_q}.
\end{aligned} \tag{16}
$$

Using the equalities (15) and (16) in (14), we obtain the desired result. □

Corollary 1. *Under the assumptions of Lemma 1, if we choose $\alpha = 1$, then*

$$\frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}\int_{\varpi_1}^{\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}\mathscr{F}(\nu)\,{}_{\varpi_1}d_q\nu-\frac{q\mathscr{F}(\varpi_1)+\mathscr{F}(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))}{1+q}$$
$$=\frac{q\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}{1+q}\int_0^1(1-(1+q)\nu)\,{}_{\varpi_1}D_q\mathscr{F}(\varpi_1+\nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))\,{}_0d_q\nu. \tag{17}$$

Lemma 2. *Let $\mathscr{F}:\Omega\to\mathbb{R}$ be a continuous function and $\alpha>0$. If ${}_{\varpi_1}D_q\mathscr{F}$ is q-integrable on Ω°, then the following equality holds:*

$$\mathscr{F}\left(\frac{([\alpha+1]_q-1)\varpi_1+(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))}{[\alpha+1]_q}\right)-\frac{\Gamma_q(\alpha+1)}{\mathcal{R}^{\alpha}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}({}_{\varpi_1}J_q^{\alpha}\mathscr{F})(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))$$
$$=\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\left[\begin{array}{l}\int_0^{\frac{1}{[\alpha+1]_q}}\left(1-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}\right){}_{\varpi_1}D_q\mathscr{F}(\varpi_1+\nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))\,{}_0d_q\nu\\+\int_{\frac{1}{[\alpha+1]_q}}^1\left(-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}\right){}_{\varpi_1}D_q\mathscr{F}(\varpi_1+\nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))\,{}_0d_q\nu\end{array}\right]. \tag{18}$$

Proof. Let $S_3=\int_0^{\frac{1}{[\alpha+1]_q}}{}_{\varpi_1}D_q\mathscr{F}(\varpi_1+\nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))\,{}_0d_q\nu$. Then

$$\begin{aligned}S_3&=\int_0^{\frac{1}{[\alpha+1]_q}}{}_{\varpi_1}D_q\mathscr{F}(\varpi_1+\nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))\,{}_0d_q\nu\\
&=\int_0^{\frac{1}{[\alpha+1]_q}}\frac{\mathscr{F}(\varpi_1+\nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))-\mathscr{F}(\varpi_1+q\nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))}{(1-q)\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\nu}\,{}_0d_q\nu\\
&=\frac{1}{(1-q)\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}\int_0^{\frac{1}{[\alpha+1]_q}}\frac{\mathscr{F}(\varpi_1+\nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))}{\nu}\,{}_0d_q\nu\\
&\quad-\frac{1}{(1-q)\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}\int_0^{\frac{1}{[\alpha+1]_q}}\frac{\mathscr{F}(\varpi_1+q\nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))}{\nu}\,{}_0d_q\nu\\
&=\frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)[\alpha+1]_q}\sum_{n=0}^{\infty}q^n\frac{\mathscr{F}\left(\varpi_1+\frac{q^n}{[\alpha+1]_q}\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\right)}{\frac{q^n}{[\alpha+1]_q}}\\
&\quad-\frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)[\alpha+1]_q}\sum_{n=0}^{\infty}q^n\frac{\mathscr{F}\left(\varpi_1+\frac{q^{n+1}}{[\alpha+1]_q}\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\right)}{\frac{q^n}{[\alpha+1]_q}}\\
&=\frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}\left[\sum_{n=0}^{\infty}\mathscr{F}\left(\varpi_1+\frac{q^n}{[\alpha+1]_q}\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\right)-\sum_{n=0}^{\infty}\mathscr{F}\left(\varpi_1+\frac{q^{n+1}}{[\alpha+1]_q}\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\right)\right]\\
&=\frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}\left[\mathscr{F}\left(\frac{([\alpha+1]_q-1)\varpi_1+(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))}{[\alpha+1]_q}\right)-\mathscr{F}(\varpi_1)\right].\end{aligned} \tag{19}$$

By (15), we have

$$\begin{aligned}\mathcal{S}_1 &= \mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\int_0^1 {}_0(1-{}_0\Phi_q(\nu))_q^{(\alpha)}\ {}_{\varpi_1}D_q\mathscr{F}(\varpi_1+\nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))\,{}_0d_q\nu \\ &= -\mathscr{F}(\varpi_1)+\frac{\Gamma_q(\alpha+1)}{\mathcal{R}^{\alpha}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}(\,{}_{\varpi_1}J_q^{\alpha}\mathscr{F})(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)). \end{aligned} \tag{20}$$

Using the equalities (19) and (20), we have

$$\begin{aligned}&\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\left[\begin{array}{l}\int_0^{\frac{1}{[\alpha+1]_q}}\left(1-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}\right){}_{\varpi_1}D_q\mathscr{F}(\varpi_1+\nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))\,{}_0d_q\nu \\ +\int_{\frac{1}{[\alpha+1]_q}}^1 -{}_0(1-\Phi_q(\nu))_q^{(\alpha)}\ {}_{\varpi_1}D_q\mathscr{F}(\varpi_1+\nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))\,{}_0d_q\nu\end{array}\right] \\ &= \mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\left[\begin{array}{l}\int_0^{\frac{1}{[\alpha+1]_q}} {}_{\varpi_1}D_q\mathscr{F}(\varpi_1+\nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))\,{}_0d_q\nu \\ -\int_0^1 {}_0(1-\Phi_q(\nu))_q^{(\alpha)}\ {}_{\varpi_1}D_q\mathscr{F}(\varpi_1+\nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))\,{}_0d_q\nu\end{array}\right] \\ &= \mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\left[\begin{array}{l}\frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}\left[\mathscr{F}\left(\frac{([\alpha+1]_q-1)\varpi_1+(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))}{[\alpha+1]_q}\right)-\mathscr{F}(\varpi_1)\right] \\ -\frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}\left[-\mathscr{F}(\varpi_1)+\frac{\Gamma_q(\alpha+1)}{\mathcal{R}^{\alpha}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}(\,{}_{\varpi_1}J_q^{\alpha}\mathscr{F})(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))\right]\end{array}\right] \\ &= \mathscr{F}\left(\frac{([\alpha+1]_q-1)\varpi_1+(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))}{[\alpha+1]_q}\right)-\frac{\Gamma_q(\alpha+1)}{\mathcal{R}^{\alpha}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}(\,{}_{\varpi_1}J_q^{\alpha}\mathscr{F})(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)).\end{aligned}$$

This completes the proof. □

Corollary 2. *Under the assumptions of Lemma 2, if we take $\alpha = 1$, then the following result holds:*

$$\begin{aligned}&\mathscr{F}\left(\frac{(1+q)\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}{1+q}\right)-\frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}\int_{\varpi_1}^{\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}\mathscr{F}(\nu)\,{}_{\varpi_1}d_q\nu \\ &= q\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\left[\begin{array}{l}\int_0^{\frac{1}{1+q}}\nu\,{}_{\varpi_1}D_q\mathscr{F}(\varpi_1+\nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))\,{}_0d_q\nu \\ +\int_{\frac{1}{1+q}}^1\left(\nu-\frac{1}{q}\right){}_{\varpi_1}D_q\mathscr{F}(\varpi_1+\nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))\,{}_0d_q\nu\end{array}\right].\end{aligned} \tag{21}$$

Theorem 4. *Let $\mathscr{F}:\Omega\to\mathbb{R}$ be a continuous function and $\alpha>0$ and ${}_{\varpi_1}D_q\mathscr{F}$ be q-integrable on Ω°. If $|{}_{\varpi_1}D_q\mathscr{F}|$ is generalized exponential convex on Ω, then*

$$\begin{aligned}&\left|\frac{\Gamma_q(\alpha+1)}{\mathcal{R}^{\alpha}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}(\,{}_{\varpi_1}J_q^{\alpha}\mathscr{F})(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))-\frac{([\alpha+1]_q-1)\mathscr{F}(\varpi_1)+\mathscr{F}(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))}{[\alpha+1]_q}\right| \\ &\leqslant \frac{\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}{[\alpha+1]_q}\left(A_1\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_1)|}{\chi^{\alpha\varpi_1}}+A_2\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)|}{\chi^{\alpha\varpi_2}}\right),\end{aligned} \tag{22}$$

where

$$A_1 = \int_0^1 \left| [\alpha+1]_q\, {}_0(1 - {}_0\Phi_q(\nu))_q^{(\alpha)} - 1 \right| (1-\nu)\, {}_0d_q\nu,$$

and

$$A_2 = \int_0^1 \left| [\alpha+1]_q\, {}_0(1 - {}_0\Phi_q(\nu))_q^{(\alpha)} - 1 \right| \nu\, {}_0d_q\nu.$$

Proof. Using the Lemma (17), property of modulus and the generalized exponential convexity of $|{}_{\varpi_1}D_q\mathscr{F}|$, we have

$$\begin{aligned}
&\left| \frac{\Gamma_q(\alpha+1)}{\mathcal{R}^{\alpha}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}\left({}_{\varpi_1}J_q^{\alpha}\mathscr{F}\right)(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)) - \frac{([\alpha+1]_q-1)\mathscr{F}(\varpi_1)+\mathscr{F}(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))}{[\alpha+1]_q} \right| \\
&\leqslant \frac{\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}{[\alpha+1]_q}\int_0^1 \left| \left([\alpha+1]_q\, {}_0(1 - {}_0\Phi_q(\nu))_q^{(\alpha)} - 1 \right) {}_{\varpi_1}D_q\mathscr{F}(\varpi_1+\nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)) \right| {}_0d_q\nu \\
&\leqslant \frac{\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}{[\alpha+1]_q}\int_0^1 \left| [\alpha+1]_q\, {}_0(1 - {}_0\Phi_q(\nu))_q^{(\alpha)} - 1 \right| \left[(1-\nu)\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_1)|}{\chi^{\alpha\varpi_1}} + \nu\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)|}{\chi^{\alpha\varpi_2}} \right] {}_0d_q\nu \\
&= \frac{\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}{[\alpha+1]_q}\left[\begin{array}{l} \frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_1)|}{\chi^{\alpha\varpi_1}}\int_0^1 \left| [\alpha+1]_q\, {}_0(1 - {}_0\Phi_q(\nu))_q^{(\alpha)} - 1 \right| (1-\nu)\, {}_0d_q\nu \\ + \frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)|}{\chi^{\alpha\varpi_2}}\int_0^1 \left| [\alpha+1]_q\, {}_0(1 - {}_0\Phi_q(\nu))_q^{(\alpha)} - 1 \right| \nu\, {}_0d_q\nu \end{array} \right] \\
&= \frac{\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}{[\alpha+1]_q}\left(\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_1)|}{\chi^{\alpha\varpi_1}}A_1 + \frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)|}{\chi^{\alpha\varpi_2}}A_2 \right),
\end{aligned}$$

which completes the proof. □

Corollary 3. *Under the assumptions of Theorem 4, if we choose $\alpha = 1$, then we have*

$$\begin{aligned}
&\left| \frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)} \int_{\varpi_1}^{\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)} \mathscr{F}(\nu)\, {}_{\varpi_1}d_q\nu - \frac{q\mathscr{F}(\varpi_1)+\mathscr{F}(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))}{1+q} \right| \\
&\leqslant \frac{q^2\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}{(1+q)^4}\left(A_1^*\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_1)|}{\chi^{\alpha\varpi_1}} + A_2^*\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)|}{\chi^{\alpha\varpi_2}} \right),
\end{aligned} \tag{23}$$

where

$$A_1^* = \frac{q+3q^3+2q^4}{1+q+q^2},$$

and

$$A_2^* = \frac{1+4q+q^2}{1+q+q^2}.$$

Theorem 5. *Let $\mathscr{F} : \Omega \to \mathbb{R}$ be a continuous function and $\alpha > 0$ and ${}_{\varpi_1}D_q\mathscr{F}$ be q-integrable on Ω°. If $|{}_{\varpi_1}D_q\mathscr{F}|^r$ is generalized exponential convex on Ω for $r > 1$ and $p^{-1} + r^{-1} = 1$, then the following inequality holds:*

$$\left|\frac{\Gamma_q(\alpha+1)}{\mathcal{R}^{\alpha}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}\left({}_{\varpi_1}J^{\alpha}_q\mathscr{F}\right)(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))-\frac{([\alpha+1]_q-1)\mathscr{F}(\varpi_1)+\mathscr{F}(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))}{[\alpha+1]_q}\right|$$
$$\leqslant\frac{\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}{[\alpha+1]_q}A_3^{\frac{1}{p}}\left(\frac{q\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_1)|^r}{\chi^{\alpha\varpi_1}}+|{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)^r}{1+q}\right)^{\frac{1}{r}},\tag{24}$$

where

$$A_3=\int_0^1\left|[\alpha+1]_{q\,0}(1-{}_0\Phi_q(\nu))^{(\alpha)}_q-1\right|^p{}_0d_q\nu.$$

Proof. Using Lemma (17), Hölder's integral inequality and generalized exponential convexity of $|{}_{\varpi_1}D_q\mathscr{F}|^r$, we have

$$\left|\frac{\Gamma_q(\alpha+1)}{\mathcal{R}^{\alpha}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}\left({}_{\varpi_1}J^{\alpha}_q\mathscr{F}\right)(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))-\frac{([\alpha+1]_q-1)\mathscr{F}(\varpi_1)+\mathscr{F}(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))}{[\alpha+1]_q}\right|$$
$$\leqslant\frac{\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}{[\alpha+1]_q}\int_0^1\left|[\alpha+1]_{q\,0}(1-{}_0\Phi_q(\nu))^{(\alpha)}_q-1\right||{}_{\varpi_1}D_q\mathscr{F}(\varpi_1+\nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))|{}_0d_q\nu$$
$$\leqslant\frac{\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}{[\alpha+1]_q}\left(\int_0^1\left|[\alpha+1]_{q\,0}(1-{}_0\Phi_q(\nu))^{(\alpha)}_q-1\right|^p{}_0d_q\nu\right)^{\frac{1}{p}}\left(\int_0^1|{}_{\varpi_1}D_q\mathscr{F}(\varpi_1+\nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))|^r{}_0d_q\nu\right)^{\frac{1}{r}}$$
$$\leqslant\frac{\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}{[\alpha+1]_q}\left(\int_0^1\left|[\alpha+1]_{q\,0}(1-{}_0\Phi_q(\nu))^{(\alpha)}_q-1\right|^p{}_0d_q\nu\right)^{\frac{1}{p}}$$
$$\times\left(\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_1)|^r}{\chi^{\alpha\varpi_1}}\int_0^1(1-\nu)\,{}_0d_q\nu+\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)|^r}{\chi^{\alpha\varpi_2}}\int_0^1\nu\,{}_0d_q\nu\right)^{\frac{1}{r}}$$
$$=\frac{\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}{[\alpha+1]_q}\left(\int_0^1\left|[\alpha+1]_{q\,0}(1-{}_0\Phi_q(\nu))^{(\alpha)}_q-1\right|^p{}_0d_q\nu\right)^{\frac{1}{p}}\left(\frac{q\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_1)|^r}{\chi^{\alpha\varpi_1}}+\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)|^r}{\chi^{\alpha\varpi_2}}}{1+q}\right)^{\frac{1}{r}},$$

which completes the proof. □

Corollary 4. *Under the assumptions of Theorem 5, if we choose $\alpha=1$, then we have*

$$\left|\frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}\int_{\varpi_1}^{\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}\mathscr{F}(\nu)\,{}_{\varpi_1}d_q\nu-\frac{q\mathscr{F}(\varpi_1)+\mathscr{F}(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))}{1+q}\right|$$
$$\leqslant\frac{q\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}{(1+q)}A_3^{*\frac{1}{p}}\left(\frac{q\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_1)|^r}{\chi^{\alpha\varpi_1}}+|{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)^r}{1+q}\right)^{\frac{1}{r}},\tag{25}$$

where

$$A_3^* = \int_0^1 |1-(1+q)\nu|^p \, {}_0d_q\nu.$$

Theorem 6. *Let $\mathscr{F} : \Omega \to \mathbb{R}$ be a continuous function and $\alpha > 0$ and ${}_{\varpi_1}D_q\mathscr{F}$ be q-integrable on Ω°. If $|{}_{\varpi_1}D_q\mathscr{F}|^r$ is generalized exponential convex on Ω for $r \geqslant 1$, then the following inequality holds:*

$$\left|\frac{\Gamma_q(\alpha+1)}{\mathcal{R}^{\alpha}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}({}_{\varpi_1}J^{\alpha}_q\mathscr{F})(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)) - \frac{([\alpha+1]_q-1)\mathscr{F}(\varpi_1)+\mathscr{F}(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))}{[\alpha+1]_q}\right|$$
$$\leqslant \frac{\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}{[\alpha+1]_q} A_4^{1-\frac{1}{r}}\left(A_1\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_1)|^r}{\chi^{\alpha\varpi_1}} + A_2\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)|^r}{\chi^{\alpha\varpi_2}}\right)^{\frac{1}{r}}, \tag{26}$$

where A_1, A_2 are given in Theorem 4 and A_4 is given as:

$$A_4 = \int_0^1 \left|[\alpha+1]_q\, {}_0(1-{}_0\Phi_q(\nu))^{(\alpha)}_q - 1\right| {}_0d_q\nu.$$

Proof. Using Lemma (17), the power mean integral inequality and generalized exponential convexity of $|{}_{\varpi_1}D_q\mathscr{F}|^r$, we have

$$\left|\frac{\Gamma_q(\alpha+1)}{\mathcal{R}^{\alpha}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}({}_{\varpi_1}J^{\alpha}_q\mathscr{F})(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)) - \frac{([\alpha+1]_q-1)\mathscr{F}(\varpi_1)+\mathscr{F}(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))}{[\alpha+1]_q}\right|$$
$$\leqslant \frac{\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}{[\alpha+1]_q}\int_0^1 \left|[\alpha+1]_q\, {}_0(1-{}_0\Phi_q(\nu))^{(\alpha)}_q - 1\right| |{}_{\varpi_1}D_q\mathscr{F}(\varpi_1+\nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))| \, {}_0d_q\nu$$
$$\leqslant \frac{\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}{[\alpha+1]_q}\left(\int_0^1 \left|[\alpha+1]_q\, {}_0(1-{}_0\Phi_q(\nu))^{(\alpha)}_q - 1\right| {}_0d_q\nu\right)^{1-\frac{1}{r}}$$
$$\times\left(\int_0^1 \left|[\alpha+1]_q\, {}_0(1-{}_0\Phi_q(\nu))^{(\alpha)}_q - 1\right| |{}_{\varpi_1}D_q\mathscr{F}(\varpi_1+\nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))|^r \, {}_0d_q\nu\right)^{\frac{1}{r}}$$
$$\leqslant \frac{\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}{[\alpha+1]_q}\left(\int_0^1 \left|[\alpha+1]_q\, {}_0(1-{}_0\Phi_q(\nu))^{(\alpha)}_q - 1\right| {}_0d_q\nu\right)^{1-\frac{1}{r}}$$
$$\times\left(\int_0^1 \left|[\alpha+1]_q\, {}_0(1-{}_0\Phi_q(\nu))^{(\alpha)}_q - 1\right| \left[\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_1)|^r}{\chi^{\alpha\varpi_1}}(1-\nu) + \frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)|^r}{\chi^{\alpha\varpi_2}}\nu\right] {}_0d_q\nu\right)^{\frac{1}{r}}$$
$$\leqslant \frac{\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}{[\alpha+1]_q}\left(\int_0^1 \left|[\alpha+1]_q\, {}_0(1-{}_0\Phi_q(\nu))^{(\alpha)}_q - 1\right| {}_0d_q\nu\right)^{1-\frac{1}{r}}$$
$$\times\left[\begin{array}{l}\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_1)|^r}{\chi^{\alpha\varpi_1}}\int_0^1 \left|[\alpha+1]_q\, {}_0(1-{}_0\Phi_q(\nu))^{(\alpha)}_q - 1\right|(1-\nu)\, {}_0d_q\nu \\ +\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)|^r}{\chi^{\alpha\varpi_2}}\left|[\alpha+1]_q\, {}_0(1-{}_0\Phi_q(\nu))^{(\alpha)}_q - 1\right|\nu\, {}_0d_q\nu\end{array}\right]^{\frac{1}{r}}$$
$$= \frac{\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}{[\alpha+1]_q} A_4^{1-\frac{1}{r}}\left(A_1\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_1)|^r}{\chi^{\alpha\varpi_1}} + A_3\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)|^r}{\chi^{\alpha\varpi_2}}\right)^{\frac{1}{r}},$$

which completes the proof. □

Corollary 5. *Under the assumptions of Theorem 6, if we choose $\alpha = 1$, then we have*

$$\left|\frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}\int_{\varpi_1}^{\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}\mathscr{F}(\nu)\,{}_{\varpi_1}d_q\nu-\frac{q\mathscr{F}(\varpi_1)+\mathscr{F}(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))}{1+q}\right|$$
$$\leqslant \frac{q\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}{(1+q)}A_4^{*1-\frac{1}{r}}\left(A_1^*\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_1)|^r}{\chi^{\alpha\varpi_1}}+A_2^*\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)|^r}{\chi^{\alpha\varpi_2}}\right)^{\frac{1}{r}}, \tag{27}$$

where A_1^*, A_2^* are already defined in Corollary 3 and

$$A_4^* = \frac{2q+q^2+q^4}{(1+q)^3}.$$

Theorem 7. *Let $\mathscr{F} : \Omega \to \mathbb{R}$ be a continuous function and $\alpha > 0$ and ${}_{\varpi_1}D_q\mathscr{F}$ be q-integrable on Ω°. If $|{}_{\varpi_1}D_q\mathscr{F}|$ is generalized exponential convex on Ω, then*

$$\left|\mathscr{F}\left(\frac{([\alpha+1]_q-1)\varpi_1+(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))}{[\alpha+1]_q}\right)-\frac{\Gamma_q(\alpha+1)}{\mathcal{R}^{\alpha}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}({}_{\varpi_1}J_q^{\alpha}\mathscr{F})(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))\right|$$
$$\leqslant \mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\left[(A_5+A_7)\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_1)|}{\chi^{\alpha\varpi_1}}+(A_6+A_8)\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)|}{\chi^{\alpha\varpi_2}}\right], \tag{28}$$

where

$$A_5 = \int_0^{\frac{1}{[\alpha+1]_q}}\left|1-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}\right|(1-\nu)\,{}_0d_q\nu$$
$$A_6 = \int_0^{\frac{1}{[\alpha+1]_q}}\left|1-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}\right|\nu\,{}_0d_q\nu$$
$$A_7 = \int_{\frac{1}{[\alpha+1]_q}}^1|-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}|(1-\nu)\,{}_0d_q\nu$$
$$A_8 = \int_{\frac{1}{[\alpha+1]_q}}^1|-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}|\nu\,{}_0d_q\nu.$$

Proof. Using Lemma (21) and the generalized exponential convexity of $|{}_{\varpi_1}D_q\mathscr{F}|$, we have

$$\left|\mathscr{F}\left(\frac{([\alpha+1]_q-1)\varpi_1+(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))}{[\alpha+1]_q}\right)-\frac{\Gamma_q(\alpha+1)}{\mathcal{R}^{\alpha}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}({}_{\varpi_1}J^{\alpha}_q\mathscr{F})(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))\right|$$

$$\leqslant \mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\left[\begin{array}{c}\int\limits_0^{\frac{1}{[\alpha+1]_q}}|1-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}||{}_{\varpi_1}D_q\mathscr{F}(\varpi_1+\nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))|{}_0d_q\nu\\+\int\limits_{\frac{1}{[\alpha+1]_q}}^{1}|-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}||{}_{\varpi_1}D_q\mathscr{F}(\varpi_1+\nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))|{}_0d_q\nu\end{array}\right]$$

$$\leqslant \mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\left[\begin{array}{c}\int\limits_0^{\frac{1}{[\alpha+1]_q}}|1-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}|[(1-\nu)\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_1)|}{\chi^{\alpha\varpi_1}}+\nu|{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)]{}_0d_q\nu\\+\int\limits_{\frac{1}{[\alpha+1]_q}}^{1}|-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}|[(1-\nu)|{}_{\varpi_1}D_q\mathscr{F}(\varpi_1)+\nu\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)|}{\chi^{\alpha\varpi_2}}]{}_0d_q\nu\end{array}\right]$$

$$=\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\left[\begin{array}{c}\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_1)|}{\chi^{\alpha\varpi_1}}\left[\int\limits_0^{\frac{1}{[\alpha+1]_q}}|1-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}|(1-\nu)\,{}_0d_q\nu+\int\limits_{\frac{1}{[\alpha+1]_q}}^{1}|-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}|(1-\nu)\,{}_0d_q\nu\right]\\+\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)|}{\chi^{\alpha\varpi_2}}\left[\int\limits_0^{\frac{1}{[\alpha+1]_q}}|1-{}_0(1-\Phi_q(\nu)_q^{(\alpha)})|\nu\,{}_0d_q\nu+\int\limits_{\frac{1}{[\alpha+1]_q}}^{1}|-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}|\nu\,{}_0d_q\nu\right]\end{array}\right].$$

This completes the proof. □

Corollary 6. *Under the assumptions of Theorem 7, if we set $\alpha=1$, then we have the following inequality*

$$\left|\mathscr{F}\left(\frac{(1+q)\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}{1+q}\right)-\frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}\int\limits_{\varpi_1}^{\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}\mathscr{F}(\nu)\,{}_{\varpi_1}d_q\nu\right|$$
$$\leqslant\frac{q\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}{(1+q+q^2)(1+q)^4}\left[3\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_1)|}{\chi^{\alpha\varpi_1}}+(2q^2+2q-1)\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)|}{\chi^{\alpha\varpi_2}}\right].$$

Theorem 8. *Let $\mathscr{F}:\Omega\to\mathbb{R}$ be a continuous function and $\alpha>0$ and ${}_{\varpi_1}D_q\mathscr{F}$ be q-integrable on Ω°. If $|{}_{\varpi_1}D_q\mathscr{F}|^r$ is generalized exponential convex on Ω, then the following inequality holds for $p^{-1}+r^{-1}=1$:*

$$\left|\mathscr{F}\left(\frac{([\alpha+1]_q-1)\varpi_1+(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))}{[\alpha+1]_q}\right)-\frac{\Gamma_q(\alpha+1)}{\mathcal{R}^{\alpha}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}({}_{\varpi_1}J^{\alpha}_q\mathscr{F})(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))\right|$$
$$\leqslant\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\left[\begin{array}{c}A_9^{\frac{1}{p}}\left(\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_1)|^r}{\chi^{\alpha\varpi_1}}\left(\frac{(1+q)[\alpha+1]_q-1}{(1+q)([\alpha+1]_q)^2}\right)+\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)|^r}{\chi^{\alpha\varpi_2}}\left(\frac{1}{(1+q)([\alpha+1]_q)^2}\right)\right)^{\frac{1}{r}}\\+A_{10}^{\frac{1}{p}}\left(\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_1)|^r}{\chi^{\alpha\varpi_1}}\left(\frac{q}{1+q}-\frac{(1+q)[\alpha+1]_q-1}{(1+q)([\alpha+1]_q)^2}\right)+\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)|^r}{\chi^{\alpha\varpi_2}}\left(\frac{1}{1+q}-\frac{1}{(1+q)([\alpha+1]_q)^2}\right)\right)^{\frac{1}{r}}\end{array}\right],\quad(29)$$

where

$$A_9=\int\limits_0^{\frac{1}{[\alpha+1]_q}}\left|1-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}\right|{}_0d_q\nu,$$

and

$$A_{10} = \int_{\frac{1}{[\alpha+1]_q}}^{1} |-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}|\,{}_0d_q\nu.$$

Proof. Using Lemma (21), Hölder's inequality and the generalized exponential convexity of $|{}_{\varpi_1}D_q\mathscr{F}|^r$, we have

$$\begin{aligned}
&\left|\mathscr{F}\left(\frac{([\alpha+1]_q-1)\varpi_1+(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))}{[\alpha+1]_q}\right)-\frac{\Gamma_q(\alpha+1)}{\mathcal{R}^{\alpha}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}({}_{\varpi_1}J_q^{\alpha}\mathscr{F})(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))\right|\\
&\leqslant \mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\left[\begin{array}{l}\int_0^{\frac{1}{[\alpha+1]_q}}|1-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}||{}_{\varpi_1}D_q\mathscr{F}(\varpi_1+\nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))|\,{}_0d_q\nu\\
+\int_{\frac{1}{[\alpha+1]_q}}^{1}|-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}||{}_{\varpi_1}D_q\mathscr{F}(\varpi_1+\nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))|\,{}_0d_q\nu\end{array}\right]\\
&\leqslant \mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\left[\begin{array}{l}\left(\int_0^{\frac{1}{[\alpha+1]_q}}|1-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}|^p\,{}_0d_q\nu\right)^{\frac{1}{p}}\left(\int_{\frac{1}{[\alpha+1]_q}}^{1}|{}_{\varpi_1}D_q\mathscr{F}(\varpi_1+\nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))|^r\,{}_0d_q\nu\right)^{\frac{1}{r}}\\
+\left(\int_{\frac{1}{[\alpha+1]_q}}^{1}|-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}|^p\,{}_0d_q\nu\right)^{\frac{1}{p}}\left(\int_{\frac{1}{[\alpha+1]_q}}^{1}|{}_{\varpi_1}D_q\mathscr{F}(\varpi_1+\nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))|^r\,{}_0d_q\nu\right)^{\frac{1}{r}}\end{array}\right]\\
&\leqslant \mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\left[\begin{array}{c}\left(\int_0^{\frac{1}{[\alpha+1]_q}}|1-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}|^p\,{}_0d_q\nu\right)^{\frac{1}{p}}\\
\times\left[\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)|^r}{\chi^{\alpha\varpi_2}}\int_0^{\frac{1}{[\alpha+1]_q}}(1-\nu)\,{}_0d_q\nu+\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)|^r}{\chi^{\alpha\varpi_2}}\int_0^{\frac{1}{[\alpha+1]_q}}\nu\,{}_0d_q\nu\right]^{\frac{1}{r}}\\
+\left(\int_{\frac{1}{[\alpha+1]_q}}^{1}|-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}|^p\,{}_0d_q\nu\right)^{\frac{1}{p}}\\
\times\left[\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_1)|^r}{\chi^{\alpha\varpi_1}}\int_{\frac{1}{[\alpha+1]_q}}^{1}(1-\nu)\,{}_0d_q\nu+\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)|^r}{\chi^{\alpha\varpi_2}}\int_{\frac{1}{[\alpha+1]_q}}^{1}\nu\,{}_0d_q\nu\right]^{\frac{1}{r}}\end{array}\right]
\end{aligned}$$

$$= \mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\left[\begin{array}{c}\left(\int_0^{\frac{1}{[\alpha+1]_q}}|1-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}|^p\,{}_0d_q\nu\right)^{\frac{1}{p}}\\
\times\left(\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_1)|^r}{\chi^{\alpha\varpi_1}}\left(\frac{(1+q)[\alpha+1]_q-1}{(1+q)([\alpha+1]_q)^2}\right)+\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)|^r}{\chi^{\alpha\varpi_2}}\left(\frac{1}{(1+q)([\alpha+1]_q)^2}\right)\right)^{\frac{1}{r}}\\
+\left(\int_{\frac{1}{[\alpha+1]_q}}^{1}|-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}|^p\,{}_0d_q\nu\right)^{\frac{1}{p}}\\
\times\left(\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_1)|^r}{\chi^{\alpha\varpi_1}}\left(\frac{q}{1+q}-\frac{(1+q)[\alpha+1]_q-1}{(1+q)([\alpha+1]_q)^2}\right)+\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)|^r}{\chi^{\alpha\varpi_2}}\left(\frac{1}{1+q}-\frac{1}{(1+q)([\alpha+1]_q)^2}\right)\right)^{\frac{1}{r}}\end{array}\right].$$

This completes the proof. □

Corollary 7. *Under the assumptions of Theorem 8, if we set $\alpha = 1$, then*

$$\left|\mathscr{F}\left(\frac{(1+q)\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}{1+q}\right)-\frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}\int\limits_{\varpi_1}^{\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}\mathscr{F}(\nu)\,{}_{\varpi_1}d_q\nu\right|$$
$$\leqslant q\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\left[A_9^{*\frac{1}{p}}\left(\frac{|\,{}_{\varpi_1}D_q\mathscr{F}(\varpi_1)|^r}{\chi^{\alpha\varpi_1}}\left(\frac{q^2+2q}{(1+q)^3}\right)+\frac{|\,{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)|^r}{\chi^{\alpha\varpi_2}}\left(\frac{1}{(1+q)^3}\right)\right)^{\frac{1}{r}}\right.$$
$$\left.+A_{10}^{*\frac{1}{p}}\left(\frac{|\,{}_{\varpi_1}D_q\mathscr{F}(\varpi_1)|^r}{\chi^{\alpha\varpi_1}}\left(\frac{q^3+q^2-q}{(1+q)^3}\right)+\frac{|\,{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)|^r}{\chi^{\alpha\varpi_2}}\left(\frac{q^2+2q}{(1+q)^3}\right)\right)^{\frac{1}{r}}\right], \tag{30}$$

where

$$A_9^* = \frac{(1-q)}{(1+q)^{p+1}(1-q^{p+1})}$$

and

$$A_{10}^* = \int_{\frac{1}{1+q}}^{1}\left(\frac{1}{q}-\nu\right)^p {}_0d_q\nu.$$

Theorem 9. *Let $\mathscr{F}:\Omega\to\mathbb{R}$ be a continuous function and $\alpha>0$ and ${}_{\varpi_1}D_q\mathscr{F}$ be q-integrable on Ω°. If $|\,{}_{\varpi_1}D_q\mathscr{F}|^r, r\geqslant 1$ is generalized exponential convex on Ω, then*

$$\left|\mathscr{F}\left(\frac{([\alpha+1]_q-1)\varpi_1+(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))}{[\alpha+1]_q}\right)-\frac{\Gamma_q(\alpha+1)}{\mathcal{R}^{\alpha}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}\left(\,{}_{\varpi_1}J_q^{\alpha}\mathscr{F}\right)(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))\right|$$
$$=\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\left[\begin{array}{c} A_9^{1-\frac{1}{r}}\left[A_5\frac{|\,{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)|^r}{\chi^{\alpha\varpi_2}}+A_6\frac{|\,{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)|^r}{\chi^{\alpha\varpi_2}}\right]^{\frac{1}{r}} \\ +A_{10}^{1-\frac{1}{r}}\left[A_7\frac{|\,{}_{\varpi_1}D_q\mathscr{F}(\varpi_1)|^r}{\chi^{\alpha\varpi_1}}+A_8\frac{|\,{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)|^r}{\chi^{\alpha\varpi_2}}\right]^{\frac{1}{r}}\end{array}\right],$$

where

$$A_9 = \int\limits_0^{\frac{1}{[\alpha+1]_q}}\left|1-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}\right|{}_0d_q\nu$$

and

$$A_{10} = \int\limits_{\frac{1}{[\alpha+1]_q}}^{1}|1-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}|{}_0d_q\nu.$$

Proof. Using Lemma (21), power mean integral inequality and the generalized exponential convexity of $|\,{}_{\varpi_1}D_q\mathscr{F}|^r$, we have

$$
\begin{aligned}
&\left|\mathscr{F}\left(\frac{([\alpha+1]_q-1)\varpi_1+(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))}{[\alpha+1]_q}\right)-\frac{\Gamma_q(\alpha+1)}{\mathcal{R}^{\alpha}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}\left({}_{\varpi_1}J_q^{\alpha}\mathscr{F}\right)(\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))\right| \\
&\leqslant \mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\left[\begin{array}{l}
\int_0^{\frac{1}{[\alpha+1]_q}}|1-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}||\,{}_{\varpi_1}D_q\mathscr{F}(\varpi_1+\nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))|\,{}_0d_q\nu \\
+\int_{\frac{1}{[\alpha+1]_q}}^{1}|-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}||\,{}_{\varpi_1}D_q\mathscr{F}(\varpi_1+\nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))|\,{}_0d_q\nu
\end{array}\right] \\
&\leqslant \mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\left[\begin{array}{c}
\left(\int_0^{\frac{1}{[\alpha+1]_q}}|1-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}|\,{}_0d_q\nu\right)^{1-\frac{1}{r}} \\
\times\left(\int_{\frac{1}{[\alpha+1]_q}}^{1}|1-{}_0(1-\Phi_q(\nu)_q^{(\alpha)})||\,{}_{\varpi_1}D_q\mathscr{F}(\varpi_1+\nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))|^r\,{}_0d_q\nu\right)^{\frac{1}{r}} \\
+\left(\int_{\frac{1}{[\alpha+1]_q}}^{1}|-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}|\,{}_0d_q\nu\right)^{1-\frac{1}{r}} \\
\times\left(\int_{\frac{1}{[\alpha+1]_q}}^{1}|-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}||\,{}_{\varpi_1}D_q\mathscr{F}(\varpi_1+\nu\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1))|^r\,{}_0d_q\nu\right)^{\frac{1}{r}}
\end{array}\right] \\
&\leqslant \mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\left[\begin{array}{c}
\left(\int_0^{\frac{1}{[\alpha+1]_q}}|1-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}|\,{}_0d_q\nu\right)^{1-\frac{1}{r}} \\
\times\left(\int_0^{\frac{1}{[\alpha+1]_q}}|1-{}_0(1-\Phi_q(\nu)_q^{(\alpha)})|\left[(1-\nu)\frac{|\,{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)|^r}{\chi^{\alpha\varpi_2}}+\nu\frac{|\,{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)|^r}{\chi^{\alpha\varpi_2}}\right]{}_0d_q\nu\right)^{\frac{1}{r}} \\
+\left(\int_{\frac{1}{[\alpha+1]_q}}^{1}|-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}|\,{}_0d_q\nu\right)^{1-\frac{1}{r}} \\
\times\left(\int_{\frac{1}{[\alpha+1]_q}}^{1}|-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}|\left[(1-\nu)\frac{|\,{}_{\varpi_1}D_q\mathscr{F}(\varpi_1)|^r}{\chi^{\alpha\varpi_1}}+\nu\frac{|\,{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)|^r}{\chi^{\alpha\varpi_2}}\right]{}_0d_q\nu\right)^{\frac{1}{r}}
\end{array}\right] \\
&= \mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\left[\begin{array}{c}
\left(\int_0^{\frac{1}{[\alpha+1]_q}}|1-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}|\,{}_0d_q\nu\right)^{1-\frac{1}{r}} \\
\times\left[\frac{|\,{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)|^r}{\chi^{\alpha\varpi_2}}\int_0^{\frac{1}{[\alpha+1]_q}}|1-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}|(1-\nu)\,{}_0d_q\nu+\frac{|\,{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)|^r}{\chi^{\alpha\varpi_2}}\int_0^{\frac{1}{[\alpha+1]_q}}|1-{}_0(1-\Phi_q(\nu)_q^{(\alpha)})|\nu\,{}_0d_q\nu\right]^{\frac{1}{r}} \\
+\left(\int_{\frac{1}{[\alpha+1]_q}}^{1}|-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}|\,{}_0d_q\nu\right)^{1-\frac{1}{r}} \\
\times\left[\frac{|\,{}_{\varpi_1}D_q\mathscr{F}(\varpi_1)|^r}{\chi^{\alpha\varpi_1}}\int_{\frac{1}{[\alpha+1]_q}}^{1}|-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}|(1-\nu)\,{}_0d_q\nu+\frac{|\,{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)|^r}{\chi^{\alpha\varpi_2}}\int_{\frac{1}{[\alpha+1]_q}}^{1}|-{}_0(1-\Phi_q(\nu))_q^{(\alpha)}|\nu\,{}_0d_q\nu\right]^{\frac{1}{r}}
\end{array}\right].
\end{aligned}
$$

This completes the proof. □

Corollary 8. *Under the assumptions of Theorem 9, if we set $\alpha=1$, then*

$$\left|\mathscr{F}\left(\frac{(1+q)\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}{1+q}\right)-\frac{1}{\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}\int_{\varpi_1}^{\varpi_1+\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)}\mathscr{F}(\nu)\,{}_{\varpi_1}d_q\nu\right|$$
$$\leqslant q\mathcal{R}_{\rho,\lambda,\sigma}(\varpi_2-\varpi_1)\left(\frac{1}{(1+q)^3}\right)\left[\left(\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_1)|^r}{\chi^{\alpha\varpi_1}}\left(\frac{q^2+q}{(1+q+q^2)}\right)+\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)|^r}{\chi^{\alpha\varpi_2}}\left(\frac{1}{(1+q+q^2)}\right)\right)^{\frac{1}{r}}\right.$$
$$\left.+\left(\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_1)|^r}{\chi^{\alpha\varpi_1}}\left(\frac{q^2+q-1}{1+q+q^2}\right)+\frac{|{}_{\varpi_1}D_q\mathscr{F}(\varpi_2)|^r}{\chi^{\alpha\varpi_2}}\left(\frac{2}{1+q+q^2}\right)\right)^{\frac{1}{r}}\right].$$

4. Conclusions

We have introduced the class of generalized exponential convex functions involving Raina's function. We have derived two new identities involving q-Riemann–Liouville fractional integrals. Using these identities, as auxiliary results, we have derived several new q-fractional estimates of trapezoidal-like inequalities, essentially using the class of generalized exponential convex functions. We hope that the ideas within this paper will inspire interested readers. The results of this paper can be extended by using other classes of convexity, for instance by using the exponential preinvexity property of the functions. One can also extend these results using the concepts of post-quantum calculus, which is an interesting problem for future research. It is worth mentioning here that many inequalities e.g., Lipschitz, Hölders, Minkowski, etc., are used to solve the control problems and stability analysis for dynamical systems; for details, see [14–19]. So it can also be an interesting problem for future research to use the inequalities obtained in this paper to solve physical problems.

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