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A Numerical Approach to Solve the q-Fractional Boundary Value Problems

Ying Sheng † and Tie Zhang *,†

Department of Mathematics, Northeastern University, Shenyang 110819, China; shengying@mail.neu.edu.cn
* Correspondence: zhangt@mail.neu.edu.cn
† These authors contributed equally to this work.

Abstract: In this present paper, we study the difference method for solving a boundary value problem of the Caputo type q-fractional differential equation. This method is based on the numerical quadrature of the q-fractional derivative and the q-Taylor expansion of related function. We first derive the truncation error boundness of $O(\triangle x_n^2)$-order and prove the existence and uniqueness of the numerical solution. Then, we prove the stability of the numerical solution and give the error estimation. Numerical experiments finally verify the validity of the theoretical analysis.

Keywords: q-fractional differential equation; boundary value problem; difference method; truncation error; stability; error estimation

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1. Introduction

The history of fractional calculus can be dated back to 1695 and it can be applied to the investigation of arbitrary order integrals and derivatives. It has gained quite a lot of interest due to its widespread application in science and engineering fields such as physics, biology, chemistry and economics [1]. For example, Baleanu et al. [2] modeled some processes on real chemical reactions with partial differential equations of the fractional order. They studied a novel modeling of the fractional multiterm boundary value problems on each edge of the graph representation of the glucose molecule and derived some existence results. In addition, a fractional-order derivative can retain the effect of system memory. Therefore, it can describe the processes involving memory and hereditary properties such as electromagnetic waves and heat transfer. For example, Mohammadi et al. [3] used a box model to describe hearing loss in children caused by the mumps virus with the Caputo–Fabrizio fractional derivative. It can also model the transmittance of anthrax between animals [4]. For more works on the application of fractional calculus, we refer readers to [5–7] and the references therein.

A lot has been achieved in the study of fractional calculus, but mostly of a continuous case. It is obvious that the discrete analogues of fractional differential equations are also very useful in applications. Some results concerning the differential equations carry over easily to corresponding results for difference equations while other results seem to be different from their continuous counterparts [8,9]. Therefore, it is necessary to develop fractional differential equations on a discrete time scale [10]. The theory of time scales was first introduced by Stefan Hilger in his PhD thesis in order to unify continuous and discrete analysis [11]. The time scale calculus has a tremendous potential in applications. For example, it can be used to model populations of insects which are continuous while in season, die out in winter while their eggs are dormant or are incubating and then hatch in a new season and can give rise to a nonoverlapping population [10]. A typical time scale is q-geometric set $T_{q,b} = \{0\} \cup \{bq^n, n = 0, 1, \cdots\}$ on which some physical processes occur and the corresponding equations are called q-fractional differential equations.

In the past few years, the q-fractional differential equations based on the q-calculus have been widely studied by engineers and mathematicians. The concept of the q-calculus (also known as quantum calculus) was first proposed by Jackson [12] in 1908. This kind of equation mainly describes some physical processes which occur on $T_{q,b}$ such as quantum dynamics, discrete dynamical systems and discrete stochastic processes [1,10,13–17]. The scale index q of set $T_{q,b}$ is used to describe the discrete path on which the corresponding physical process occurs. With the rapid development of the q-calculus theory, the q-difference operator theory, q-Laplace transform, q-Taylor expansion, q-Bernstein polynomial, q-Sturm–Liouville theory and other related results have been proposed successively. For more details of the q-calculus and the q-fractional calculus, we refer readers to [14,15,18–23]. Compared with the classical fractional calculus, the research of the q-fractional calculus is still immature. On the boundary problems of the q-fractional differential equations, Ferreira [24] proposed a sufficient condition for the existence of nontrivial solutions by using the fixed point theorem of cone compression and properties of Green function. Shahed et al. [25] studied the existence of positive solutions. Liang et al. [26] investigated the existence and uniqueness of solutions for a class of q-fractional differential equations with three point boundary value problems. In [27], by using the Guo–Krasnoselskii fixed point theorem, the authors gave a sufficient condition for the existence of a positive solution for a class of boundary value problems of nonlinear q-fractional difference equations.

On the discrete approximation methods for the initial value problems of q-fractional differential equations, Abdeljawad et al. [28,29] presented a successive iteration method to find the approximation solution. They derived the truncation error bounds, but did not give the stability analysis. Then, Salahshour and Ahmadian et al. [30] investigated the convergence condition of the successive approximation method proposed in [29]. Furthermore, Zhang and Tong [31] proposed a new difference formula by using the piecewise linear interpolation to discretize the Caputo type q-fractional derivative. They proved the unconditional stability of this difference formula and gave the estimate of convergence order. Wu et al. [32] constructed a discrete approximation scheme with the variational iterative method. However, until now, no numerical methods have been presented to solve the boundary value problem of q-fractional differential equations.

In this paper, we present a difference method to solve the boundary value problem of Caputo type q-fractional differential equations: $-{}^cD_q^{\alpha}u(x) + a(x)u(x) = f(x)$. We discretize the q-fractional derivative ${}^cD_q^{\alpha}u(x)$ by using the numerical quadrature and in order to enhance the stability, we further discretize the term $a(x)u(x)$ by means of the q-Taylor expansion. Since the q-fractional differential equations are usually defined on time scale set $T_{q,b}$, our difference scheme must also be established on set $T_{q,b}$, that is, the mesh points are in set $T_{q,b}$. This makes the stability analysis and error estimate much more difficult than that of the usual difference schemes which are established on the selected artificially meshes. We first derive the truncation error bound and prove the existence and uniqueness of the difference solution. Then, we prove the stability and obtain an error estimation of $O(\triangle x_n^2)$ for the difference scheme. Finally, we use numerical examples to illustrate the effectiveness of the difference method.

This paper is organized as follows. We first introduce some notations and relevant operations about q-calculus and q-fractional calculus in Section 2. In Section 3, we establish the difference method for solving a boundary value problem of the Caputo type q-fractional differential equation and derive the boundness of the truncation error. Section 4 is devoted to the stability analysis and error estimation of the difference method. In Section 5, we provide some numerical examples to illustrate the theoretical analysis.

2. Preliminaries

We first introduce some definitions and operations about q-calculus and q-fractional calculus.

Let $\mathbb{N} = \{1, 2, \ldots\}$ be the set of positive integers and $0 < q < 1$. The q-shifted operation is defined as

$$(x-s)_q^{(0)} = 1,\ (x-s)_q^{(m)} = \prod_{k=0}^{m-1}(x - q^k s),\ m \in \mathbb{N}. \tag{1}$$

If $\alpha \in R$ and $\alpha \notin \mathbb{N}$, then

$$(x-s)_q^{(\alpha)} = x^\alpha \prod_{k=0}^{\infty} \frac{x - q^k s}{x - q^{\alpha+k} s}, \, 0 \le s \le x. \tag{2}$$

Denote $\mathbb{C}$ as the set of complex numbers. The q-Gamma function $\Gamma_q(x)$ is defined as

$$\Gamma_q(x) = (1-q)_q^{(x-1)}(1-q)^{1-x}, \ x \in \mathbb{C} \setminus \{-n, n \in \{0\} \cup \mathbb{N}\}. \tag{3}$$

The following notations are defined by

$$[x]_q = \frac{1-q^x}{1-q}, \quad [m]_q! = [m]_q[m-1]_q \cdots [1]_q.$$

Then, we can see that

$$\Gamma_q(1) = 1, \quad \Gamma_q(m+1) = [m]_q!, \quad \Gamma_q(x+1) = [x]_q\Gamma_q(x).$$

For a given $q \in R$, a set $A_q \in R$ is called q-geometric if $qx \in A_q$ whenever $x \in A_q$. That is, $\forall x \in A_q$, A_q includes geometric sequences $\{xq^m\}_{m=0}^{\infty}$ of all. A special q-geometric set is $A_q = \{q^m : m \in \mathbb{Z}\} \cup \{0\}$, where $0 < q < 1$ and $\mathbb{Z}$ is the set of integers.

Definition 1 ([12]). *Let $f(x)$ be a real valued function on set A_q and $0 < q < 1$. Define the q-derivative of $f(x)$ as*

$$D_q f(x) = \frac{d_q f(x)}{d_q x} = \frac{f(x) - f(qx)}{(1-q)x}, \, x \in A_q \setminus \{0\}, \tag{4}$$

$$D_q f(0) = \frac{d_q f(x)}{d_q x}|_{x=0} = \lim_{n\to\infty} \frac{f(xq^n) - f(0)}{xq^n}, \, x \neq 0.$$

On the basis of Definition 1, the high order q-derivative $D_q^n f(x)$ is defined as $D_q^n f(x) = D_q(D_q^{n-1} f(x)), n \ge 2$.

For two real valued functions $f(x)$ and $g(x)$, by a straightforward computation, we have

$$\begin{aligned}
&D_q(af(x) \pm bg(x)) = aD_q f(x) \pm bD_q g(x), \, a, b \in R, \\
&D_q(f(x)g(x)) = g(x)D_q f(x) + f(qx)D_q g(x), \\
&D_q\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)D_q f(x) - f(x)D_q g(x)}{g(x)g(qx)}, \, g(x) \neq 0, \, g(qx) \neq 0.
\end{aligned}$$

Definition 2 ([33]). *Let $f(x)$ be a real valued function defined on set A_q. The q-integral of $f(x)$ is defined by*

$$\int_0^x f(s) d_q s = (1-q) \sum_{n=0}^{\infty} xq^n f(xq^n), \, x \in A_q, \tag{5}$$

$$\int_a^b f(s) d_q s = \int_0^b f(s) d_q s - \int_0^a f(s) d_q s, \, a, b \in A_q. \tag{6}$$

From Definition 2, it is easy to see that

$$|\int_0^b f(s) d_q s| \le \int_0^b |f(s)| d_q s, \, b > 0, \tag{7}$$

$$\int_a^b f(s) d_q s = \int_a^c f(s) d_q s + \int_c^b f(s) d_q s, \, a < c < b. \tag{8}$$

The lemma below gives the operation of q-integration by parts.

Lemma 1 ([31]). *Suppose $f(x)$ and $g(x)$ are two real valued functions defined on set A_q, $0 < q < 1,\ 0 \le a < b,\ a, b \in A_q$, we have*

$$\int_a^b g(qx) D_q f(x) d_q x = (fg)(b) - (fg)(a) - \int_a^b f(x) D_q g(x) d_q x. \tag{9}$$

Introduce the q-Beta function

$$B_q(x, y) = \int_0^1 s^{x-1}(1 - qs)_q^{(y-1)} d_q s, \tag{10}$$

where $x, y \in \mathbb{C}$, $Re(x) > 0$ and $Re(y) > 0$. The q-Gamma and q-Beta functions have the following relation: [14]

$$B_q(x, y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x + y)}.$$

In the following, the concept of q-fractional calculus will be introduced.

Definition 3 ([34]). *Suppose $x \in A_q, a \ge 0$ and $\alpha \ne -1, -2, \ldots$ The α-order Riemann–Liouville q-fractional integral is defined formally by $I^0_{q,a} f(x) = f(x)$ and*

$$I^\alpha_{q,a} f(x) = \frac{1}{\Gamma_q(\alpha)} \int_a^x (x - qs)_q^{(\alpha-1)} f(s) d_q s. \tag{11}$$

Definition 4 ([35]). *Suppose $a \in A_q, a \ge 0$ and $n = \lceil \alpha \rceil$. The α-order Caputo q-fractional derivative of function $f(x) : (a, \infty) \to R$ is defined as*

$$^cD^\alpha_{q,a} f(x) = \begin{cases} I^{-\alpha}_{q,a} f(x), & \alpha \le 0, \\ I^{n-\alpha}_{q,a} D^n_q f(x), & \alpha > 0, \end{cases} \tag{12}$$

where $\lceil \alpha \rceil$ represents the smallest integer which is equal to or greater than α.

For briefness, we use $I^\alpha_q f(x)$ instead of $I^\alpha_{q,0} f(x)$ and $^cD^\alpha_q f(x)$ instead of $^cD^\alpha_{q,0} f(x)$, respectively.

3. The Difference Method and Truncation Error Estimation

In this section, we investigate a difference method to solve a boundary value problem of Caputo type q-fractional differential equations and give the truncation error boundness.

Consider the following problem:

$$\begin{cases} -{}^cD^\alpha_q u(x) + a(x)u(x) = f(x), & 0 < x \le b,\ x \in T_{q,b},\ 0 < q < 1, \\ D_q u(0) = \gamma_1, u(b) = \gamma_2, & 1 < \alpha < 2, \end{cases} \tag{13}$$

where $a(x) \ge 0$. The difference method will be established on a discrete points set $\{x_k\} \subset T_{q,b}$, where $T_{q,b} = \{bq^n : n = 0, 1, \ldots\} \cup \{0\}$ is a q-geometric set.

We first discretize the Caputo q-fractional derivative

$$^cD^\alpha_q u(x) = \frac{1}{\Gamma_q(2 - \alpha)} \int_0^x (x - qs)_q^{(1-\alpha)} D^2_q u(s) d_q s. \tag{14}$$

Let $0 = x_0 < x_1 < \ldots < x_N = b$ be a partition of $[0, b]$ with the point $x_k = bq^{N-k} \in T_{q,b}$. Denote the mesh size $\triangle x_k = x_k - x_{k-1}, 1 \le k \le N$, $N \ge 1$ is a positive integer. At point x_n, using $D_q^2 u(x_k)$ to replace $D_q^2 u(x)$ on interval $[x_{k-1}, x_k]$, we have from (14) that

$$\begin{aligned} {}^c D_q^\alpha u(x_n) &= \frac{1}{\Gamma_q(2-\alpha)} \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} (x_n - qs)_q^{(1-\alpha)} D_q^2 u(s) d_q s \\ &= \frac{1}{\Gamma_q(2-\alpha)} \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} (x_n - qs)_q^{(1-\alpha)} D_q^2 u(x_k) d_q s + R_1^n, \end{aligned} \tag{15}$$

where

$$R_1^n = \frac{1}{\Gamma_q(2-\alpha)} \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} (x_n - qs)_q^{(1-\alpha)} (D_q^2 u(s) - D_q^2 u(x_k)) d_q s. \tag{16}$$

Denoting $v(x) = D_q u(x)$, we have

$$D_q^2 u(x) - D_q^2 u(x_k) = D_q v(x) - D_q v(x_k) = D_q v(x) - \frac{v(x_k) - v(x_{k-1})}{\Delta x_k}. \tag{17}$$

Let $L_{1,k} v(s)$ be the piecewise linear interpolation of $v(s)$

$$L_{1,k} v(s) = \frac{x_k - s}{\triangle x_k} v(x_{k-1}) + \frac{s - x_{k-1}}{\triangle x_k} v(x_k),\ s \in [x_{k-1}, x_k],\ k = 1, 2, \ldots, N. \tag{18}$$

The corresponding interpolation error is

$$R_k(s) = v(s) - L_{1,k} v(s),\ R_k(x_{k-1}) = R_k(x_k) = 0,\ s \in [x_{k-1}, x_k]. \tag{19}$$

Noting that $D_q L_{1,k} v(s) = (v(x_k) - v(x_{k-1}))/\triangle x_k$, we have from (16), (17) and (19) that

$$R_1^n = \frac{1}{\Gamma_q(2-\alpha)} \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} (x_n - qs)_q^{(1-\alpha)} D_q R_k(s) d_q s. \tag{20}$$

Now, using the identity

$$\sum_{k=1}^{n} d_k (x_k - x_{k-1}) = d_n x_n + \sum_{k=1}^{n-1} (d_k - d_{k+1}) x_k - d_1 x_0,$$

we obtain from (15) that (denote $\Gamma_q^\alpha = \Gamma_q(2-\alpha)$)

$$\begin{aligned} {}^c D_q^\alpha u(x_n) &= \frac{1}{\Gamma_q^\alpha} \sum_{k=1}^{n} b_k^{(n)} (v(x_k) - v(x_{k-1})) + R_1^n \\ &= \frac{1}{\Gamma_q^\alpha} [b_n^{(n)} v(x_n) - \sum_{k=1}^{n-1} (b_{k+1}^{(n)} - b_k^{(n)}) v(x_k) - b_1^{(n)} v(x_0)] + R_1^n \\ &= \frac{1}{\Gamma_q^\alpha} [b_n^{(n)} v(x_n) - \sum_{k=1}^{n-1} \frac{b_{k+1}^{(n)} - b_k^{(n)}}{\triangle x_k} (u(x_k) - u(x_{k-1})) - b_1^{(n)} \gamma_1] + R_1^n \\ &= \frac{1}{\Gamma_q^\alpha} [b_n^{(n)} \frac{u(x_n) - u(x_{n-1})}{\triangle x_n} - \sum_{k=1}^{n-1} c_k (u(x_k) - u(x_{k-1})) - b_1^{(n)} \gamma_1] + R_1^n, \end{aligned} \tag{21}$$

where the coefficient

$$b_k^{(n)} = \frac{1}{\triangle x_k} \int_{x_{k-1}}^{x_k} (x_n - qs)_q^{(1-\alpha)} d_q s, \quad c_k = (b_{k+1}^{(n)} - b_k^{(n)})/\triangle x_k. \tag{22}$$

Next, to enhance the stability, we further discrete the term $a(x_n)u(x_n)$ in Equation (13). From (4), we have

$$\begin{aligned} u(x_n) &= u(x_{n-1}) + \triangle x_n D_q u(x_n) \\ &= u(x_{n-1}) + \triangle x_n D_q u(x_{n-1}) + \triangle x_n^2 D_q^2 u(x_n) \\ &= u(x_{n-1}) + \triangle x_n \frac{u(x_{n-1}) - u(x_{n-2})}{\triangle x_{n-1}} + \triangle x_n^2 D_q^2 u(x_n). \end{aligned}$$

Then, (notice that $\triangle x_n / \triangle x_{n-1} = \frac{1}{q}$)

$$a(x_n)u(x_n) = \begin{cases} a(x_1)u(x_0) + a(x_1)\triangle x_1 D_q u(x_0) + R_2^n, & n = 1, \\ a(x_n)u(x_{n-1}) + a(x_n)[u(x_{n-1}) - u(x_{n-2})]/q + R_2^n, & n \geq 2, \end{cases} \tag{23}$$

where the error $R_2^n = a(x_n)\triangle x_n^2 D_q^2 u(x_n)$. Thus, with (21) and (23) we obtain the difference discrete scheme of Problem (13)

$$-\triangle_q^\alpha u(x_n) = f(x_n) - R^n,\ R^n = R_1^n + R_2^n,\ n = 1, 2, \cdots, N, \tag{24}$$

with the boundary value conditions: $D_q u(0) = \gamma_1$, $u(x_N) = \gamma_2$, where the difference operator

$$-\triangle_q^\alpha u(x_1) = \frac{1}{\Gamma_q^\alpha}\left(\frac{b_1^{(n)}}{\triangle x_1}u(x_0) - \frac{b_1^{(n)}}{\triangle x_1}u(x_1) + b_1^{(n)}\gamma_1\right) + a(x_1)u(x_0) + \triangle x_1 a(x_1)\gamma_1, n = 1, \tag{25}$$

$$\begin{aligned} -\triangle_q^\alpha u(x_n) = \frac{1}{\Gamma_q^\alpha}\{&-c_1 u(x_0) - \sum_{k=1}^{n-2}(c_{k+1} - c_k)u(x_k) + \frac{1}{q}\Gamma_q^\alpha a(x_n)u(x_{n-2}) + \\ &+ [\frac{b_n^{(n)}}{\triangle x_n} + c_{n-1} - (1 + \frac{1}{q})\Gamma_q^\alpha a(x_n)]u(x_{n-1}) - \frac{b_n^{(n)}}{\triangle x_n}u(x_n) + b_1^{(n)}\gamma_1\}, 2 \leq n \leq N-1, \end{aligned} \tag{26}$$

$$\begin{aligned} -\triangle_q^\alpha u(x_N) = \frac{1}{\Gamma_q^\alpha}\{&-c_1 u(x_0) - \sum_{k=1}^{N-2}(c_{k+1} - c_k)u(x_k) + \frac{1}{q}\Gamma_q^\alpha a(x_N)u(x_{N-2}) + \\ &+ [\frac{b_N^{(N)}}{\triangle x_N} + c_{N-1} - (1 + \frac{1}{q})\Gamma_q^\alpha a(x_N)]u(x_{N-1}) - \frac{b_N^{(N)}}{\triangle x_N}\gamma_2 + b_1^{(N)}\gamma_1\}, n = N. \end{aligned} \tag{27}$$

Now, we define the difference approximation of Problem (13) by

$$-\triangle_q^\alpha u_n = f_n,\ n = 1, 2, \cdots, N, \tag{28}$$

where $f_n = f(x_n)$. The truncation error of Formula (28) is $R^n = R_1^n + R_2^n$.

In the following, we estimate the truncation error R_n.

Lemma 2 ([31])**.** *Suppose that $v(x)$ is twice q-differentiable on $[x_{k-1}, x_k]$. Then, the error function $R_k(x)$ of linear interpolation can be expressed as follows*

$$R_k(x) = \frac{1}{1+q}D_q^2 v(\xi_k)(x - x_k)(x - x_{k-1}), x \in [x_{k-1}, x_k],\ \xi_k \in (x_{k-1}, x_k),\ 1 \leq k \leq N. \tag{29}$$

Lemma 3. *Suppose $0 < q < 1$, $1 < \alpha < 2$ and D_q is the q-derivative operator of variable s. We have*

$$D_q(x - s)_q^{(1-\alpha)} = -[1 - \alpha]_q (x - qs)_q^{(-\alpha)}, \tag{30}$$

$$|(x - qs)_q^{(-\alpha)}| \leq x^{-\alpha}\frac{1}{1 - q^{\alpha-1}}\frac{1}{1 - q^{2-\alpha}}. \tag{31}$$

Proof. With (4) and (2), we obtain

$$D_q(x-s)_q^{(1-\alpha)} = \frac{(x-s)_q^{(1-\alpha)} - (x-qs)_q^{(1-\alpha)}}{(1-q)s} = \frac{x^{1-\alpha}}{(q-1)s} \lim_{m\to\infty} S_m, \tag{32}$$

where

$$S_m = \prod_{i=0}^{m} \frac{x-q^{i+1}s}{x-q^{i+2-\alpha}s} - \prod_{i=0}^{m} \frac{x-q^{i}s}{x-q^{i+1-\alpha}s}.$$

Further,

$$\begin{aligned} S_m &= \prod_{i=1}^{m} \frac{x-q^{i}s}{x-q^{i+1-\alpha}s} [\frac{x-q^{m+1}s}{x-q^{m+2-\alpha}s} - \frac{x-s}{x-q^{1-\alpha}s}] \\ &= \prod_{i=1}^{m} \frac{x-q^{i}s}{x-q^{i+1-\alpha}s} [\frac{sx(1-q^{1-\alpha})(1-q^{m+1})}{(x-q^{m+2-\alpha}s)(x-q^{1-\alpha}s)}] \\ &= \prod_{i=0}^{m} \frac{x-q^{i+1}s}{x-q^{i+1-\alpha}s} [\frac{sx(1-q^{1-\alpha})(1-q^{m+1})}{(x-q^{m+2-\alpha}s)(x-q^{m+1}s)}] \\ &= \prod_{i=0}^{\infty} \frac{x-q^{i+1}s}{x-q^{i+1-\alpha}s} [\frac{s(1-q^{1-\alpha})}{x}], \ m\to\infty. \end{aligned}$$

Substituting this into (32), it yields

$$D_q(x-s)_q^{(1-\alpha)} = \frac{x^{-\alpha}(1-q^{1-\alpha})}{q-1} \prod_{i=0}^{\infty} \frac{x-q^{i+1}s}{x-q^{i+1-\alpha}s} = -[1-\alpha]_q (x-qs)_q^{(-\alpha)}.$$

Next, we estimate (31). Since

$$(x-qs)_q^{(-\alpha)} = x^{(-\alpha)} \lim_{m\to\infty} S_m^{'}, \ S_m^{'} = \prod_{i=0}^{m} \frac{x-q^{i+1}s}{x-q^{i+1-\alpha}s}, \tag{33}$$

and

$$\max_{0\le s\le x} |\frac{x-qs}{x-q^{1-\alpha}s}| = \max\{1, |\frac{1-q}{1-q^{1-\alpha}}|\} \le \frac{1-q}{1-q^{\alpha-1}},$$

$$\max_{0\le s\le x} \frac{x-q^{i+1}s}{x-q^{i+1-\alpha}s} = \frac{1-q^{i+1}}{1-q^{i+1-\alpha}}, \ i \ge 1.$$

Then,

$$\begin{aligned} |S_m^{'}| &\le \frac{1-q}{1-q^{\alpha-1}} \prod_{i=1}^{m} \frac{1-q^{i+1}}{1-q^{i+1-\alpha}} \\ &= (1-q^{\alpha-1})^{-1}(1-q^{2-\alpha})^{-1} \frac{1-q}{1-q^{3-\alpha}} \frac{1-q^2}{1-q^{4-\alpha}} \cdots \frac{1-q^{m-1}}{1-q^{m+1-\alpha}} (1-q^m)(1-q^{m+1}) \\ &\le (1-q^{\alpha-1})^{-1}(1-q^{2-\alpha})^{-1}. \end{aligned}$$

Substituting the above inequality into (33), we complete the proof. □

Below, we give the truncation error estimation.

Theorem 1. *Suppose $u(x)$ and $D_q^3 u(x)$ are continuous functions on $[0, b]$. Then, the following estimate of the truncation error function of the difference Equation (28) holds:*

$$|R^n| \leq [\frac{1}{4\Gamma_q(2-\alpha)} \frac{1}{q^{\alpha-1}-q} \frac{1}{1-q^2} x_n^{1-\alpha} + a(x_n)] \triangle x_n^2 \max_{0 \leq x \leq x_n} |D_q^3 u(x)|. \tag{34}$$

Proof. Denote $\tilde{R}(s) = R_k(s), s \in [x_{k-1}, x_k], 1 \leq k \leq N$. We have from (20), (9) (19) and Lemma 3 that

$$\begin{aligned}
R_1^n &= \frac{1}{\Gamma_q^\alpha} \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} (x_n - qs)_q^{(1-\alpha)} D_q R_k(s) d_q s \\
&= \sum_{k=1}^{n} \frac{(x_n - qs)_q^{(1-\alpha)}}{\Gamma_q^\alpha} R_k(s)|_{x_{k-1}}^{x_k} - \frac{1}{\Gamma_q^\alpha} \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} D_q (x_n - s)_q^{(1-\alpha)} R_k(s) d_q s \\
&= \frac{[1-\alpha]_q}{\Gamma_q^\alpha} \int_0^{x_n} (x_n - qs)_q^{(-\alpha)} \tilde{R}(s) d_q s.
\end{aligned}$$

With (7), (8), Lemma 2 and Inequality (31), we have

$$\begin{aligned}
|R_1^n| &\leq \frac{|[1-\alpha]_q|}{\Gamma_q^\alpha} \int_0^{x_n} |(x_n - qs)_q^{(-\alpha)} \tilde{R}(s)| d_q s \\
&= \frac{|[1-\alpha]_q|}{\Gamma_q^\alpha} \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} |(x_n - qs)_q^{(-\alpha)} R_k(s)| d_q s \\
&\leq \frac{|[1-\alpha]_q|}{\Gamma_q^\alpha} \frac{1}{1+q} \frac{1}{4} \max_{1 \leq k \leq n} |\triangle x_k|^2 \max_{0 \leq x \leq x_n} |D_q^2 v(x)| \int_0^{x_n} |(x_n - qs)_q^{(-\alpha)}| d_q s \\
&\leq \frac{|[1-\alpha]_q|}{\Gamma_q^\alpha} \frac{1}{1+q} \frac{1}{4} \max_{1 \leq k \leq n} |\triangle x_k|^2 \max_{0 \leq x \leq x_n} |D_q^2 v(x)| x_n^{1-\alpha} \frac{1}{1-q^{\alpha-1}} \frac{1}{1-q^{2-\alpha}} \\
&= \frac{1}{4\Gamma_q^\alpha} \frac{1}{1-q^2} \frac{1}{q^{\alpha-1}-q} x_n^{1-\alpha} \triangle x_n^2 \max_{0 \leq x \leq x_n} |D_q^3 u(x)|.
\end{aligned}$$

From (23) and $R^n = R_1^n + R_2^n$, the proof is completed. □

4. The Stability and the Error Analysis

In this section, we study the stability of the difference formula in (28) and give the error estimation of $u(x_n) - u_n$.

Lemma 4. *Suppose $0 < q < 1, 1 < \alpha < 2$ and $0 \leq s \leq x_n$, then the following property holds:*

$$x_n^{1-\alpha} < (x_n - q^{i+1}s)_q^{(1-\alpha)} \leq (x_n - qs)_q^{(1-\alpha)}, \ i \geq 0. \tag{35}$$

Proof. For the left-hand inequality, we have

$$(x_n - q^{i+1}s)_q^{(1-\alpha)} = x_n^{1-\alpha} \prod_{j=0}^{\infty} \frac{x_n - q^{i+j+1}s}{x_n - q^{i+j+2-\alpha}s} > x_n^{1-\alpha}, \ 0 \leq s \leq x_n, \ i \geq 0.$$

For the right-hand inequality, when $i \geq 1$ (it is obvious for $i = 0$) we have

$$\begin{aligned}
&(x_n - q^{i+1}s)_q^{(1-\alpha)} - (x_n - qs)_q^{(1-\alpha)} \\
&= x_n^{1-\alpha} \prod_{j=0}^{\infty} \frac{x_n - q^{i+j+1}s}{x_n - q^{i+j+2-\alpha}s} - x_n^{1-\alpha} \prod_{j=0}^{\infty} \frac{x_n - q^{j+1}s}{x_n - q^{j+2-\alpha}s} \\
&= x_n^{1-\alpha} \prod_{j=i}^{\infty} \frac{x_n - q^{j+1}s}{x_n - q^{j+2-\alpha}s} (1 - \prod_{j=0}^{i-1} \frac{x_n - q^{j+1}s}{x_n - q^{j+2-\alpha}s}) < 0,
\end{aligned}$$

which completes the proof. $\square$

Lemma 5. *The coefficient series $b_k^{(n)}$ defined by (22) have the following properties*

$$x_n^{(1-\alpha)} < b_1^{(n)} < (x_n - qx_1)_q^{(1-\alpha)}, \tag{36}$$

$$b_k^{(n)} = (x_n - qx_k)_q^{(1-\alpha)}, \quad k = 2, \cdots, n, \quad 2 \le n \le N. \tag{37}$$

Proof. From Lemma 4, we obtain

$$\begin{aligned} b_1^{(n)} &= \frac{1}{\triangle x_1} \int_{x_0}^{x_1} (x_n - qs)_q^{(1-\alpha)} d_q s \\ &= \frac{x_1}{\triangle x_1} (1-q) \sum_{i=0}^{\infty} q^i (x_n - q^{i+1} x_1)_q^{(1-\alpha)}, \end{aligned}$$

$$x_n^{1-\alpha} = x_n^{1-\alpha}(1-q)\sum_{i=0}^{\infty} q^i < b_1^{(n)} < (x_n - qx_1)_q^{(1-\alpha)}(1-q)\sum_{i=0}^{\infty} q^i = (x_n - qx_1)_q^{(1-\alpha)}.$$

This gives (36). Since $x_{k-1} = qx_k, \triangle x_k = x_k - x_{k-1} = x_k(1-q), k \ge 2$, by (22) we have

$$\begin{aligned} b_k^{(n)} &= \frac{1}{\triangle x_k} \int_{x_{k-1}}^{x_k} (x_n - qs)_q^{(1-\alpha)} d_q s \\ &= \frac{1}{\triangle x_k} \int_0^{x_k} (x_n - qs)_q^{(1-\alpha)} d_q s - \frac{1}{\triangle x_k} \int_0^{x_{k-1}} (x_n - qs)_q^{(1-\alpha)} d_q s \\ &= \frac{1}{\triangle x_k}(1-q) \sum_{i=0}^{\infty} x_k q^i (x_n - q^{i+1} x_k)_q^{(1-\alpha)} - \frac{1}{\triangle x_k}(1-q) \sum_{i=0}^{\infty} x_{k-1} q^i (x_n - q^{i+1} x_{k-1})_q^{(1-\alpha)} \\ &= \sum_{i=0}^{\infty} q^i (x_n - q^{i+1} x_k)_q^{(1-\alpha)} - \sum_{i=0}^{\infty} q^{i+1} (x_n - q^{i+2} x_k)_q^{(1-\alpha)} = (x_n - qx_k)_q^{(1-\alpha)}. \end{aligned}$$

This gives (37). $\square$

Lemma 6. *The coefficient series $c_k = (b_{k+1}^{(n)} - b_k^{(n)})/\triangle x_k$ satisfy the following inequality:*

$$0 < c_1 < c_2 < \cdots < c_{n-1}, \quad 2 \le n \le N. \tag{38}$$

Proof. From (36) and (2), we have

$$\begin{aligned} c_1 &= \frac{b_2^{(n)} - b_1^{(n)}}{\triangle x_1} > \frac{1}{\triangle x_1}[(x_n - qx_2)_q^{(1-\alpha)} - (x_n - qx_1)_q^{(1-\alpha)}] \\ &= \frac{1}{\triangle x_1}[(x_n - qx_2)_q^{(1-\alpha)} - (x_n - q^2 x_2)_q^{(1-\alpha)}] \\ &= \frac{1}{\triangle x_1}[x_n^{1-\alpha} \prod_{i=0}^{\infty} \frac{x_n - q^{i+1} x_2}{x_n - q^{i+2-\alpha} x_2} - x_n^{1-\alpha} \prod_{i=0}^{\infty} \frac{x_n - q^{i+2} x_2}{x_n - q^{i+3-\alpha} x_2}] \\ &= \frac{x_n^{1-\alpha}}{\triangle x_1} \prod_{i=1}^{\infty} \frac{x_n - q^{i+1} x_2}{x_n - q^{i+2-\alpha} x_2} (\frac{x_n - qx_2}{x_n - q^{2-\alpha} x_2} - 1) \\ &= \frac{x_n^{1-\alpha}}{\triangle x_1} \frac{qx_2(q^{1-\alpha} - 1)}{x_n - q^{2-\alpha} x_2} \prod_{i=1}^{\infty} \frac{x_n - q^{i+1} x_2}{x_n - q^{i+2-\alpha} x_2} > 0. \end{aligned}$$

Next, since

$$
\begin{aligned}
b_{k+1}^{(n)} - b_k^{(n)} &= (x_n - qx_{k+1})_q^{(1-\alpha)} - (x_n - qx_k)_q^{(1-\alpha)} \\
&= x_n^{1-\alpha} \prod_{i=0}^{\infty} \frac{x_n - q^{i+1}x_{k+1}}{x_n - q^{i+2-\alpha}x_{k+1}} - x_n^{1-\alpha} \prod_{i=0}^{\infty} \frac{x_n - q^{i+2}x_{k+1}}{x_n - q^{i+3-\alpha}x_{k+1}} \\
&= x_n^{1-\alpha} \prod_{i=1}^{\infty} \frac{x_n - q^{i+1}x_{k+1}}{x_n - q^{i+2-\alpha}x_{k+1}} [\frac{x_n - qx_{k+1}}{x_n - q^{2-\alpha}x_{k+1}} - 1],
\end{aligned}
$$

so

$$
\begin{aligned}
c_k - c_{k-1} &= \frac{(b_{k+1}^{(n)} - b_k^{(n)})}{\triangle x_k} - \frac{(b_k^{(n)} - b_{k-1}^{(n)})}{\triangle x_{k-1}} \\
&= \frac{x_n^{1-\alpha}}{\triangle x_k} \prod_{i=1}^{\infty} \frac{x_n - q^{i+1}x_{k+1}}{x_n - q^{i+2-\alpha}x_{k+1}} [\frac{x_n - qx_{k+1}}{x_n - q^{2-\alpha}x_{k+1}} - 1] - \\
&\quad - \frac{x_n^{1-\alpha}}{q\triangle x_k} \prod_{i=1}^{\infty} \frac{x_n - q^{i+2}x_{k+1}}{x_n - q^{i+3-\alpha}x_{k+1}} [\frac{x_n - q^2x_{k+1}}{x_n - q^{3-\alpha}x_{k+1}} - 1] \\
&= \frac{x_n^{1-\alpha}}{\triangle x_k} \prod_{i=2}^{\infty} \frac{x_n - q^{i+1}x_{k+1}}{x_n - q^{i+2-\alpha}x_{k+1}} \{\frac{x_n - q^2x_{k+1}}{x_n - q^{3-\alpha}x_{k+1}} [\frac{x_n - qx_{k+1}}{x_n - q^{2-\alpha}x_{k+1}} - 1] - \\
&\quad - \frac{1}{q}[\frac{x_n - q^2x_{k+1}}{x_n - q^{3-\alpha}x_{k+1}} - 1]\} \\
&\geq \frac{x_n^{1-\alpha}}{\triangle x_k} \{[\frac{x_n - qx_{k+1}}{x_n - q^{2-\alpha}x_{k+1}} - 1] - \frac{1}{q}[\frac{x_n - q^2x_{k+1}}{x_n - q^{3-\alpha}x_{k+1}} - 1]\} \\
&= \frac{x_n^{1-\alpha}}{\triangle x_k} \{[\frac{1 - qs}{1 - q^{2-\alpha}s} - 1] - \frac{1}{q}[\frac{1 - q^2s}{1 - q^{3-\alpha}s} - 1]\},
\end{aligned}
$$

where $s = \frac{x_{k+1}}{x_n}, 0 < s \leq 1$. Let $f(s) = \frac{1-qs}{1-q^{2-\alpha}s} - 1 - \frac{1}{q}\frac{1-q^2s}{1-q^{3-\alpha}s} + \frac{1}{q}$. Then,

$$
f'(s) = (q^{2-\alpha} - q)[\frac{1}{(1 - q^{2-\alpha}s)^2} - \frac{1}{(1 - q^{3-\alpha}s)^2}] > 0.
$$

Since $f(0) = 0$, then $f(s) > 0$, that is, $c_k - c_{k-1} > 0$, $k = 2, 3, \cdots, N-1$. $\square$

According to (25)–(27), we write the difference equations of System (28) as follows:

$$
\begin{cases}
[\frac{b_1^{(n)}}{\triangle x_1} + \Gamma_q^\alpha a(x_1)]u_0 - \frac{b_1^{(n)}}{\triangle x_1}u_1 = \Gamma_q^\alpha f_1 - \Gamma_q^\alpha \triangle x_1 a(x_1)\gamma_1 - b_1^{(n)}\gamma_1, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
-c_1u_0 - (c_2 - c_1)u_1 - \cdots - [c_i - c_{i-1} + \frac{1}{q}\Gamma_q^\alpha a(x_{i+1})]u_{i-1} \\
+[\frac{b_{i+1}^{(n)}}{\triangle x_{i+1}} + c_i + (1 + \frac{1}{q})\Gamma_q^\alpha a(x_{i+1})]u_i - \frac{b_{i+1}^{(n)}}{\triangle x_{i+1}}u_{i+1} = \Gamma_q^\alpha f_{i+1} - b_1^{(n)}\gamma_1, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
-c_1u_0 - (c_2 - c_1)u_1 - (c_3 - c_2)u_2 - \cdots - [c_{N-1} - c_{N-2} + \frac{1}{q}\Gamma_q^\alpha a(x_N)]u_{N-2} \\
+[\frac{b_N^{(N)}}{\triangle x_N} + c_{N-1} + (1 + \frac{1}{q})\Gamma_q^\alpha a(x_N)]u_{N-1} = \Gamma_q^\alpha f_N - b_1^{(N)}\gamma_1 + \frac{b_N^{(N)}}{\triangle x_N}\gamma_2,
\end{cases}
\tag{39}
$$

where $f_i = f(x_i), 1 \leq i \leq N$.

Theorem 2. *The solution of difference equation in (28) exists uniquely.*

Proof. Let A be the coefficient matrix of equations of System (28) with elements a_{ij} $(i, j = 0, 1, \cdots, N-1)$ given in (39). Since $b_k^{(n)} > 0$, $c_{k+1} > c_k > 0$, $a(x) \geq 0$, we have

$$\sum_{j=0}^{N-1} |a_{0j}| = \frac{b_1^{(n)}}{\triangle x_1} \leq a_{00} = \frac{b_1^{(n)}}{\triangle x_1} + \Gamma_q^\alpha a(x_1),$$

$$\begin{aligned}\sum_{j=0, j\neq i}^{N-1} |a_{ij}| &= c_i + \frac{1}{q}\Gamma_q^\alpha a(x_{i+1}) + \frac{b_{i+1}^{(n)}}{\triangle x_{i+1}} \\ &\leq a_{ii} = c_i + (1+\frac{1}{q})\Gamma_q^\alpha a(x_{i+1}) + \frac{b_{i+1}^{(n)}}{\triangle x_{i+1}}, \quad i = 1, 2, \cdots, N-2,\end{aligned}$$

$$\begin{aligned}\sum_{j=0}^{N-2} |a_{N-1,j}| &= c_{N-1} + \frac{1}{q}\Gamma_q^\alpha a(x_N) < a_{N-1,N-1} \\ &= c_{N-1} + (1+\frac{1}{q})\Gamma_q^\alpha a(x_N) + \frac{b_N^{(N)}}{\triangle x_N}.\end{aligned}$$

Therefore, A is diagonally dominant and irreducible (noting that $a_{ij} \neq 0$, $j = i-1, i, i+1$) which implies that A is a invertible matrix [36]. The proof is completed. □

In the following, we give the stability analysis of the difference formula.

Theorem 3. *Let $a(x) \geq a_0 > 0$. Then, the following stability estimation for the solution of the difference equation in (28) holds:*

$$|u_n| \leq \frac{1}{a_0} \max_{1\leq k\leq N} |f(x_k)| + (\frac{1}{\Gamma_q(2-\alpha)a_0} b_1^{(n)} + x_1)|\gamma_1| + |\gamma_2|, \, n \geq 1. \tag{40}$$

Proof. Suppose $|u_i| = \max_{0\leq j\leq N-1} |u_j|$. From (39), we can see that when $i = 0$,

$$[\frac{b_1^{(n)}}{\triangle x_1} + \Gamma_q^\alpha a(x_1)]|u_0| \leq \frac{b_1^{(n)}}{\triangle x_1}|u_1| + \Gamma_q^\alpha |f_1| + \Gamma_q^\alpha \triangle x_1 a(x_1)|\gamma_1| + b_1^{(n)}|\gamma_1|,$$

so,

$$\begin{aligned}|u_0| &\leq \frac{1}{a(x_1)}[|f_1| + \triangle x_1 a(x_1)]|\gamma_1| + \frac{1}{\Gamma_q^\alpha a(x_1)} b_1^{(n)}|\gamma_1| \\ &\leq \frac{1}{a_0}|f_1| + (\frac{1}{\Gamma_q^\alpha a_0} b_1^{(n)} + \triangle x_1)|\gamma_1|.\end{aligned}$$

When $i = N-1$, from (39) we have

$$[\frac{b_N^{(N)}}{\triangle x_N} + \Gamma_q^\alpha a(x_N)]|u_{N-1}| \leq \Gamma_q^\alpha |f_N| + \frac{b_N^{(N)}}{\triangle x_N}|\gamma_2| + b_1^{(N)}|\gamma_1|,$$

$$\begin{aligned}|u_{N-1}| &\leq \frac{1}{\frac{b_N^{(N)}}{\triangle x_N} + \Gamma_q^\alpha a(x_N)}[\Gamma_q^\alpha |f_N| + \frac{b_N^{(N)}}{\triangle x_N}|\gamma_2| + b_1^{(N)}|\gamma_1|] \\ &\leq \frac{1}{a_0}|f_N| + \frac{1}{\Gamma_q^\alpha a_0} b_1^{(N)}|\gamma_1| + |\gamma_2|.\end{aligned}$$

When $1 \leq i \leq N-2$, we have

$$[c_i + \frac{b_{i+1}^{(n)}}{\triangle x_{i+1}} + (1+\frac{1}{q})\Gamma_q^{\alpha} a(x_{i+1})]|u_i|$$
$$\leq [c_1 + (c_2 - c_1) + \cdots + (c_i - c_{i-1}) + \frac{1}{q}\Gamma_q^{\alpha} a(x_{i+1}) + \frac{b_{i+1}^{(n)}}{\triangle x_{i+1}}] \max_{1\leq j\leq N-2} |u_j| + \Gamma_q^{\alpha}|f_i| + b_1^{(n)}|\gamma_1|.$$

Therefore,

$$|u_i| \leq \frac{1}{a(x_{i+1})}|f_i| + \frac{1}{\Gamma_q^{\alpha} a(x_{i+1})} b_1^{(n)}|\gamma_1| \leq \frac{1}{a_0}|f_i| + \frac{1}{\Gamma_q^{\alpha} a_0} b_1^{(n)}|\gamma_1|.$$

Through the three cases of discussion above and noting $\triangle x_1 = x_1 - x_0 = x_1$, the proof is completed. □

Finally, the error estimation is given in the following theorem.

Theorem 4. *Let $u(x)$ and u_n be the solutions of Equations (13) and (28), respectively. Suppose that $u(x)$ and $D_q^3 u(x)$ are both continuous functions on $[0, b]$. Then, the following error estimation holds:*

$$|u(x_n) - u_n| \leq \frac{1}{a_0}[\frac{1}{4\Gamma_q(2-\alpha)}\frac{1}{q^2-1}\frac{1}{q-q^{\alpha-1}} x_n^{1-\alpha} + a(x_n)]\triangle x_n^2 \max_{0\leq x\leq x_n} |D_q^3 u(x)|. \quad (41)$$

Proof. Let error function $e_n = u_n - u(x_n)$. From (24) and (28), we see that e_n satisfies the difference equation: $-\triangle_q^{\alpha} e_n = R^n$ with $\gamma_1 = \gamma_2 = 0$. Thus, we completed the proof by using Theorems 1 and 3. □

5. Numerical Experiment

This section provides two numerical examples to illustrate the effectiveness of the proposed difference formula. The experiments are carried out by using Matlab R2109a.

Example 1. *In this experiment, we solve the following q-fractional differential equation using the difference method (28)*

$$\begin{cases} -{}^cD_q^{11/10} u(x) + (x+2)u(x) = \frac{(1-q^2)x^{2-\alpha}}{1-q^{2-\alpha}\Gamma_q(2-\alpha)} + (x^2-1)(x+2), \\ D_q u(0) = 0,\ u(1) = 0,\ 0 < x \leq 1,\ x \in T_{q,b}. \end{cases} \quad (42)$$

The exact solution is $u(x) = x^2 - 1$. The experiment results are shown in Table 1.

Table 1. Experiment results of problem (42), $q = 3/5, N = 10$.

$x_n = q^{N-n}$	$u(x_n)$	u_n	$\lvert u(x_n) - u_n\rvert$
0.0000	−1.0000	−1.0201	0.0201
$(3/5)^9$	−0.9999	−1.0204	0.0205
$(3/5)^8$	−0.9997	−1.0206	0.0209
$(3/5)^7$	−0.9992	−1.0207	0.0214
$(3/5)^6$	−0.9978	−1.0201	0.0222
$(3/5)^5$	−0.9940	−1.0172	0.0232
$(3/5)^4$	−0.9832	−1.0071	0.0239
$(3/5)^3$	−0.9533	−0.9757	0.0224
$(3/5)^2$	−0.8704	−0.8844	0.0140
$(3/5)^1$	−0.6400	−0.6317	0.0083

Example 2. *In this experiment, we solve the following q-fractional differential equation using the difference method (28)*

$$\begin{cases} -{}^{c}D_q^{13/10}u(x) + 4\cos x u(x) = \frac{(1-q^2)x^{2-\alpha}}{1-q^{2-\alpha}\Gamma_q(2-\alpha)} + 4\cos x(x^2 + x - 1), \\ D_q u(0) = 1,\ u(1) = 1,\ 0 < x \leq 1,\ x \in T_{q,b}. \end{cases} \tag{43}$$

The exact solution is $u(x) = x^2 + x - 1$. The experiment results are shown in Table 2.

Table 2. Experiment results of problem (43), $q = 1/2, N = 10$.

$x_n = q^{N-n}$	$u(x_n)$	u_n	$\lvert u(x_n) - u_n \rvert$
0.0000	−1.0000	−1.0097	0.0097
$(1/2)^9$	−0.9980	−1.0078	0.0097
$(1/2)^8$	−0.9961	−1.0058	0.0097
$(1/2)^7$	−0.9921	−1.0019	0.0098
$(1/2)^6$	−0.9841	−0.9940	0.0098
$(1/2)^5$	−0.9678	−0.9777	0.0099
$(1/2)^4$	−0.9336	−0.9433	0.0097
$(1/2)^3$	−0.8594	−0.8677	0.0083
$(1/2)^2$	−0.6875	−0.6896	0.0021
$(1/2)^1$	−0.2500	−0.2379	0.0121

6. Conclusions

We consider how to solve a Caputo type q-fractional boundary value problem where the order of fractional derivative is $1 < \alpha < 2$. Based on the numerical quadrature and q-Taylor expansion, we discretize the q-fractional equation and derive the truncation error boundness. The unique existence and the stability of the numerical solution are also proved. Finally, we obtain the error estimation and the validity of the theoretical analysis is verified by numerical experiments.

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