



Article

The Traveling Wave Solutions in a Mixed-Diffusion Epidemic Model

Ru Hou ¹ and Wen-Bing Xu ^{2,*}¹ School of Mathematical Sciences, Peking University, Beijing 100871, China; rhou@pku.edu.cn² School of Mathematical Sciences, Capital Normal University, Beijing 100048, China

* Correspondence: 6919@cnu.edu.cn

Abstract: In this paper, we study the traveling wave solution of an epidemic model with mixed diffusion. First, we give two definitions of the minimum wave speeds and prove that they are equivalent. Second, the existence, decaying behavior, and uniqueness of traveling wave fronts are obtained. Third, the signs of minimum wave speeds are studied, and further, in two specific cases of the dispersal kernel, we show how to identify the signs of minimum wave speeds.

Keywords: traveling wave solutions; epidemic model; mixed-diffusion; nonlocal dispersal; minimum wave speeds

1. Introduction

This paper is devoted to studying the following epidemic model:

$$\begin{cases} u_t = u_{xx} - \alpha u + h(v), & x \in \mathbb{R}, t \in \mathbb{R}, \\ v_t = K * v - v - \beta v + g(u), & x \in \mathbb{R}, t \in \mathbb{R}, \end{cases} \quad (1)$$

where $u(t, x)$ and $v(t, x)$ in biology stand for the spatial concentration of an infectious agent and the spatial density of the infectious human population, respectively; $\alpha > 0$ and $\beta > 0$ denote the natural death rates of the infectious agent and infectious humans; $h(v)$ means the growth of the infectious agent caused by infectious humans; and $g(u)$ is the infection rate of the human population under the assumption that the total susceptible human population is a constant during the evolution of the epidemic. The model (1) describes a positive feedback interaction between the concentration of infectious agent and the infectious human population; that is, a high concentration of infectious agent leads to a large infection rate in the human population, and as more people are infected, the growth rate of the infectious agent increases. This model is an extension to the classical SEIR (susceptible–exposed–infectious–recovered) model. There is a widely adopted numerical approach to the solution of epidemic phenomena based on the modification of SEIR model and similar ones, and the very recent contributions include [1–3].

The model (1) is a mixed-diffusion variant of the following classical epidemic model:

$$\begin{cases} u_t = d_1 u_{xx} - \alpha u + h(v), & x \in \mathbb{R}, t \in \mathbb{R}, \\ v_t = d_2 v_{xx} - \beta v + g(u), & x \in \mathbb{R}, t \in \mathbb{R}, \end{cases} \quad (2)$$

which was proposed by Capasso and Maddalena [4,5] to model the spread of cholera in the European Mediterranean regions in 1973. In (2), the diffusions of infectious agent and infectious human are described by the classical diffusion operators u_{xx} and v_{xx} . However, in (1), the diffusion of infectious human is represented by the nonlocal dispersal operator

$$K * v(x) - v(x) = \int_{\mathbb{R}} K(x - y)(v(y) - v(x))dy,$$



Citation: Hou, R.; Xu, W.-B. The Traveling Wave Solutions in a Mixed-Diffusion Epidemic Model. *Fractal Fract.* **2022**, *6*, 217. <https://doi.org/10.3390/fractalfract6040217>

Academic Editors: Hacı Mehmet Baskonus and Natália Martins

Received: 15 March 2022

Accepted: 6 April 2022

Published: 11 April 2022

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



Copyright: © 2022 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

where $K(x - y)$ can be viewed as the probability of individuals moving from location y to location x (see [6]). Compared to the classical diffusion operator, the nonlocal dispersal operator describes the movements between not only adjacent but also nonadjacent spatial locations. Here the nonlocal dispersal of v can be thought as the long-distance movements of infectious humans across cities or countries by air traffic and other long-distance transportation. If the diffusion of infectious agent is also nonlocal, then (1) reduces to the following nonlocal dispersal model:

$$\begin{cases} u_t = d_1(K_1 * u - u) - \alpha u + h(v), & x \in \mathbb{R}, t \in \mathbb{R}, \\ v_t = d_2(K_2 * v - v) - \beta v + g(u), & x \in \mathbb{R}, t \in \mathbb{R}. \end{cases} \tag{3}$$

In (3), the nonlocal dispersal operator $K_1 * u - u$ means that long-distance movements of infectious agent happen; for example, the infectious agent can move among countries through the transportation of imported food or the flow of international rivers.

The wave propagation phenomena, which are associated with the studies of traveling wave solutions and spreading speeds of systems (2) and (3), have been widely studied in the literature. For example, Hsu and Yang [7] considered the existence, uniqueness, and decaying behavior of traveling wave fronts of (2), and Wu and Hsu [8] studied the entire solutions of (2). We also refer to [9,10] for the traveling wave solution of (2) in the case $d_2 = 0$, and [11,12] for traveling wave solutions of a more general system that includes (2) as a special case. For the nonlocal dispersal model (3), we assume that K_i satisfies $\int_{\mathbb{R}} K_i(x)e^{\lambda x} dx < +\infty$ for $\lambda \in \mathbb{R}$. Li, Xu, and Zhang [13]; and Meng, Yu, and Hsu [14] studied the traveling wave solutions and entire solutions. We also refer to [15,16] for the traveling wave solutions and spreading speed of (3) in the case $d_2 = 0$. The spreading speed of (3) was studied by Bao et al. [17], Hu et al. [18], and Xu et al. [19].

The study of the following scalar dispersal equation with reaction:

$$u_t = \mathcal{A}u + f(u), \quad x \in \mathbb{R}, t > 0 \tag{4}$$

is also closely related to (1) and (3), where \mathcal{A} is a dispersal operator. There are various forms of \mathcal{A} , such as classical diffusion $\mathcal{A}u = \Delta u$, nonlocal dispersal $\mathcal{A}u = K * u - u$, fractional Laplacian $\mathcal{A}u = -(-\Delta)^\alpha u$ (see [20] for a recent review), and variable-order Riemann–Liouville fractional derivatives defined by [21,22]

$$\begin{aligned} {}_{x_L}D_x^{\alpha(x,t)} u(x,t) &= \frac{1}{\Gamma(2 - \alpha(x,t))} \left[\frac{\partial^2}{\partial \xi^2} \int_{x_L}^{\xi} \frac{u(\eta,t)}{(\xi - \eta)^{\alpha(x,t)-1}} d\eta \right]_{\xi=x}, \\ {}_x D_{x_R}^{\alpha(x,t)} u(x,t) &= \frac{1}{\Gamma(2 - \alpha(x,t))} \left[\frac{\partial^2}{\partial \xi^2} \int_{\xi}^{x_R} \frac{u(\eta,t)}{(\eta - \xi)^{\alpha(x,t)-1}} d\eta \right]_{\xi=x}, \end{aligned}$$

where $\Gamma(\cdot)$ is the gamma function. The variability and transition of fractional orders contribute to the detailed description of highly heterogeneous systems and complex phenomena. Such scenarios have motivated the formulation of variable-order fractional operators and related algorithms—for example, [23]. Consider the monostable case with f satisfying the following Fisher–KPP condition

$$f(0) = f(1) = 0, \quad f(u) > 0 \text{ for } u \in (0,1), \quad f'(0) > 0, \quad f(u) \leq f'(0)u \text{ for } u \in (0,1).$$

The different forms of \mathcal{A} usually cause distinct wave propagation phenomena of (4).

- (i) When $\mathcal{A}u = \Delta u$, (4) is a classical reaction-diffusion equation and there is a unique traveling wave front for any speed $c \geq 2\sqrt{f'(0)}$, but no traveling wave solution for the speed $c \in (0, 2\sqrt{f'(0)})$.
- (ii) When $\mathcal{A}u = -(-\Delta)^\alpha u$, (4) is a fractional diffusion equation with reaction, and there is no traveling wave solution for any speed $c \in \mathbb{R}$. Moreover, it was shown that the front position propagates exponentially; see, e.g., [24–26]. To the best of our knowledge,

there is no result about the propagation dynamics of variable-order fractional diffusion equations, and our work could possibly provide some basis for this topic.

For the case $\mathcal{A}u = K * u - u$, the properties of K determine whether (4) admits a traveling wave front spreading at a finite speed or has exponentially propagating front position. More precisely, when the symmetric kernel K satisfies $\int_{\mathbb{R}} K(x)e^{\lambda x} dx < +\infty$ for $\lambda \in \mathbb{R}$, there exists $c^* > 0$ such that (4) has a unique traveling wave front for any speed $c \geq c^*$, and no traveling wave solution with the speed $c \in (0, c^*)$; see [27–33]. However, when K is “heavy-tailed”, in the sense that $|K'(x)| = o(K(x))$ as $|x| \rightarrow +\infty$, there is no traveling wave solution for any speed $c \in \mathbb{R}$ and the spatial propagation of front position is accelerated; see, e.g., [34–37]. In particular, when $K(x) \sim |x|^{-\beta}$ as $|x| \rightarrow +\infty$, the front position propagates exponentially, which means the nonlocal dispersal case with an algebraic-tailed kernel has similar wave propagation properties to the case $\mathcal{A}u = -(-\Delta)^\alpha u$.

To the best of our knowledge, there is no result about the traveling wave solutions of the mixed-diffusion model (1), although its background in biology is clear; that is, the movements of an infectious agent are local, but the long-distance movements of infectious human happen. Herein, we consider the traveling wave solutions and the minimum wave speeds in monostable system (1). A traveling wave solution of (1) is a solution of the special form $(u(x, t), v(x, t)) = (\phi(x - ct), \psi(x - ct))$, which can be regarded as the dispersal process of epidemic from outbreak to an endemic. Usually, a non-decreasing or non-increasing traveling wave solution is called a traveling wave front. Note that we use the form $(u(x, t), v(x, t)) = (\phi(x - ct), \psi(x - ct))$ to represent not only non-increasing but also non-decreasing traveling wave front. Therefore, no matter whether a traveling wave solution is non-increasing or non-decreasing, when its speed is positive, it propagates from left to right along the x -axis, and when its speed is negative, it propagates from right to left on the x -axis. In this paper, we study the “light-tailed” dispersal kernel, namely, $\int_{\mathbb{R}} K(x)e^{\lambda x} dx < +\infty$, for $\lambda \in \mathbb{R}$. Our results can be summarized from three angles.

First, we give two definitions of the minimum wave speeds. The first definition is related to the principal eigenvalue of a linear operator derived from (1), and this definition is common in the study of traveling wave solutions and spreading speeds in (2) and (3), and other related systems (see, e.g., [17–19,38,39]). The second definition is related to the root number of an eigenvalue equation, and this definition is used to study the traveling wave solutions in [7,13]. Moreover, we prove that these two definitions are equivalent.

Second, we consider the traveling wave solutions of (1). Motivated by the works of [7,13,40,41], we change the traveling wave solution problem into investigating the fixed point of a nonlinear operator, and the existence of traveling wave front is obtained by constructing a pair of upper and lower solutions and applying the Schauder’s fixed point theorem. The decaying behavior and uniqueness of traveling wave fronts are also obtained.

Third, we study the signs of minimum wave speeds. In (1), the kernel function $K(\cdot)$ is assumed to be asymmetric. As stated in [32], asymmetric kernels may induce non-positive minimal wave speed. Thus, it is significant to study the signs of minimum wave speeds, which determine whether it happens that the asymmetric kernel changes the propagation direction of traveling wave solutions. Motivated by the work of [19] for (3), we show that the signs of minimum wave speeds of (1) depend only on the number of elements in some set, which is further applied to two specific forms of $K(\cdot)$ (i.e., normal distribution and uniform distribution). For these two specific forms, the study of signs of minimum wave speeds is quite different from that considered in [19] for (3), because in this work for (1) we consider the influences of the asymmetric dispersal of v under the assumption that u has symmetric local diffusion, but in [19] for (3), the authors study the influences of symmetric nonlocal kernel of v when u has asymmetric nonlocal dispersal. We show that when $K(\cdot)$ is normal distribution or uniform distribution, the signs of minimum wave speeds depend only on μ and $\frac{\mu}{\sqrt{\sigma}}$ where $\mu \in \mathbb{R}$ is the expectation and σ is the variance of K , which is different from the results obtained in [19] for (3). Thus, the study for the cases of normal distribution and uniform distribution in this paper is a new result to understand the influences of asymmetric dispersal on the signs of minimum wave speeds.

The rest of this paper is organized as follows. In Section 2, we give two definitions of minimum wave speeds and prove they are equivalent. Section 3 presents the existence, uniqueness, and decaying behavior of traveling wave fronts of (1). Section 4 deals with the signs of minimum wave speeds, and the results for two specific forms of K are given.

2. Two Definitions of Minimum Wave Speeds

In the section, we give two definitions of minimum wave speeds and prove that they are equivalent. First we state the assumptions. Assume that

(A1) $g(\cdot)$ and $h(\cdot)$ are two functions in $C^1([0, 1]) \cap C^{1+\delta_0}([0, p_0])$ with $\delta_0 > 0$ and $p_0 \in (0, 1)$, and $g(0) = h(0) = 0, h(1)/\alpha = g(1)/\beta = 1, h(g(u)/\beta) - \alpha u > 0$ for $u \in (0, 1)$;

(A2) $0 < g(u) \leq g'(0)u, g'(u) \geq 0$ for $u \in (0, 1)$; $0 < h(v) \leq h'(0)v, h'(v) \geq 0$ for $v \in (0, 1)$.

Then (1) is a monostable system with equilibria $(u, v) \equiv (0, 0)$ and $(u, v) \equiv (1, 1)$, and there exists no equilibrium (u, v) satisfying $0 < u, v < 1$. We can easily check that $h'(0)g'(0)u \geq h'(0)g(u) \geq \beta h(g(u)/\beta) > \alpha\beta u$ for $u \in (0, 1)$, which implies that

$$h'(0)g'(0) > \alpha\beta.$$

We assume that $K(\cdot)$ is a continuous and nonnegative function satisfying

(K) $\int_{\mathbb{R}} K(x)dx = 1, \int_{\mathbb{R}} K(x)e^{\lambda x}dx < +\infty$ for $\lambda \in \mathbb{R}$, and there exist $x_1 > 0$ and $x_2 < 0$ such that $K(x_1) > 0$ and $K(x_2) > 0$.

Note that we do not assume that $K(\cdot)$ is symmetric.

2.1. The First Definition

We denote

$$a(\lambda) = \lambda^2 - \alpha, \quad b(\lambda) = \int_{\mathbb{R}} K(x)e^{\lambda x}dx - 1 - \beta \text{ for } \lambda \in \mathbb{R}.$$

Consider the matrix

$$E(\lambda) = \begin{pmatrix} a(\lambda) & h'(0) \\ g'(0) & b(\lambda) \end{pmatrix}.$$

Let $\chi(\lambda)$ be the large one of the two eigenvalues of $E(\lambda)$, namely,

$$\chi(\lambda) = \frac{1}{2} \left[a(\lambda) + b(\lambda) + \sqrt{(a(\lambda) - b(\lambda))^2 + 4g'(0)h'(0)} \right],$$

and then $(a(\lambda) - \chi(\lambda))(b(\lambda) - \chi(\lambda)) = h'(0)g'(0)$. Denote

$$c(\lambda) = \frac{\chi(\lambda)}{\lambda} \text{ for } \lambda \neq 0. \tag{5}$$

It follows that

$$[a(\lambda) - \lambda c(\lambda)][b(\lambda) - \lambda c(\lambda)] = h'(0)g'(0). \tag{6}$$

Theorem 1. *We have the following statements about $c(\lambda)$:*

(i) $c(\lambda)$ satisfies that

$$\lim_{\lambda \rightarrow 0^+} c(\lambda) = \lim_{\lambda \rightarrow +\infty} c(\lambda) = +\infty, \quad \lim_{\lambda \rightarrow 0^-} c(\lambda) = \lim_{\lambda \rightarrow -\infty} c(\lambda) = -\infty;$$

(ii) *There are two unique constants $\lambda_R^* > 0$ and $\lambda_L^* < 0$ such that*

$$c'(\lambda) \begin{cases} > 0, & \lambda \in (\lambda_R^*, +\infty), \\ = 0, & \lambda = \lambda_R^*, \\ < 0, & \lambda \in (0, \lambda_R^*), \end{cases} \quad \text{and} \quad c'(\lambda) \begin{cases} > 0, & \lambda \in (-\infty, \lambda_L^*), \\ = 0, & \lambda = \lambda_L^*, \\ < 0, & \lambda \in (\lambda_L^*, 0); \end{cases}$$

(iii) If we set

$$c_R^* \triangleq c(\lambda_R^*) = \min_{\lambda > 0} \{c(\lambda)\}, \quad c_L^* \triangleq c(\lambda_L^*) = \max_{\lambda < 0} \{c(\lambda)\},$$

then $c_L^* < c_R^*$ holds.

Proof. Since $a(0) = -\alpha$ and $b(0) = -\beta$, we get from $h'(0)g'(0) > \alpha\beta$ that

$$\chi(0) = \frac{1}{2} \left[-\alpha - \beta + \sqrt{(\alpha - \beta)^2 + 4g'(0)h'(0)} \right] > 0,$$

which implies that $\lim_{\lambda \rightarrow 0^+} c(\lambda) = +\infty$ and $\lim_{\lambda \rightarrow 0^-} c(\lambda) = -\infty$. Note that

$$\lim_{\lambda \rightarrow +\infty} \frac{a(\lambda)}{\lambda} = +\infty, \quad \lim_{\lambda \rightarrow +\infty} \frac{b(\lambda)}{\lambda} = +\infty,$$

and then

$$\lim_{\lambda \rightarrow +\infty} c(\lambda) = \lim_{\lambda \rightarrow +\infty} \frac{\chi(\lambda)}{\lambda} = +\infty.$$

Similarly, it holds that $\lim_{\lambda \rightarrow -\infty} c(\lambda) = -\infty$.

The proofs of (ii) and (iii) are similar to the counterpart in the proof of [19] (Theorem 2.1). \square

2.2. The Second Definition

Consider the function

$$\Delta_c(\lambda) = A_c(\lambda)B_c(\lambda) - h'(0)g'(0), \quad c \in \mathbb{R}, \lambda \in \mathbb{R},$$

where

$$A_c(\lambda) = a(\lambda) - c\lambda = \lambda^2 - c\lambda - \alpha,$$

$$B_c(\lambda) = b(\lambda) - c\lambda = \int_{\mathbb{R}} K(x)e^{\lambda x} dx - 1 - c\lambda - \beta.$$

We can easily check that

$$A_c(0) = -\alpha < 0, \quad B_c(0) = -\beta < 0, \quad A_c(\pm\infty) = B_c(\pm\infty) = +\infty,$$

and

$$A_c''(\lambda) = 2 > 0, \quad B_c''(\lambda) = \int_{\mathbb{R}} K(x)e^{\lambda x} x^2 dx > 0.$$

Then, there exist four unique constants $\eta^-, \zeta^- \in (-\infty, 0)$ and $\eta^+, \zeta^+ \in (0, +\infty)$ such that

$$A_c(\lambda) \begin{cases} < 0, & \lambda \in (\eta^-, \eta^+), \\ = 0, & \lambda = \eta^- \text{ or } \eta^+, \\ > 0, & \lambda \in (-\infty, \eta^-) \cup (\eta^+, +\infty), \end{cases} \quad B_c(\lambda) \begin{cases} < 0, & \lambda \in (\zeta^-, \zeta^+), \\ = 0, & \lambda = \zeta^- \text{ or } \zeta^+, \\ > 0, & \lambda \in (-\infty, \zeta^-) \cup (\zeta^+, +\infty), \end{cases}$$

and

$$A_c'(\lambda) \begin{cases} < 0, & \lambda \in (-\infty, \eta^-), \\ > 0, & \lambda \in (\eta^+, +\infty), \end{cases} \quad B_c'(\lambda) \begin{cases} < 0, & \lambda \in (-\infty, \zeta^-), \\ > 0, & \lambda \in (\zeta^+, +\infty). \end{cases}$$

Denote

$$\lambda_1 = \min\{\eta^-, \zeta^-\}, \quad \lambda_2 = \max\{\eta^-, \zeta^-\}, \quad \lambda_3 = \min\{\eta^+, \zeta^+\}, \quad \lambda_4 = \max\{\eta^+, \zeta^+\},$$

and it follows that $\lambda_1 < \lambda_2 < 0 < \lambda_3 < \lambda_4$.

Theorem 2. For sufficiently large $c \in \mathbb{R}$, $\Delta_c(\lambda) = 0$ has exactly three different positive roots and one negative root. For sufficiently small $c \in \mathbb{R}$, $\Delta_c(\lambda) = 0$ has exactly three different negative roots and one positive root.

Proof. We have that

$$\Delta'_c(\lambda) = A'_c(\lambda)B_c(\lambda) + A_c(\lambda)B'_c(\lambda) \begin{cases} < 0, & \lambda \in (-\infty, \lambda_1), \\ > 0, & \lambda \in (\lambda_4, +\infty). \end{cases}$$

Since

$$\Delta_c(\lambda_1) = \Delta_c(\lambda_4) = -h'(0)g'(0) < 0 \text{ and } \Delta_c(\pm\infty) = +\infty,$$

then $\Delta_c(\lambda) = 0$ has a unique root in $(-\infty, \lambda_1)$ and a unique root in $(\lambda_4, +\infty)$. In $[\lambda_1, \lambda_2]$ or $[\lambda_3, \lambda_4]$, since $A_c(\lambda)B_c(\lambda) \leq 0$, we have that $\Delta_c(\lambda) \leq -h'(0)g'(0) < 0$, which implies that $\Delta_c(\lambda) = 0$ has no root in $[\lambda_1, \lambda_2]$ or $[\lambda_3, \lambda_4]$.

Next consider the roots in (λ_2, λ_3) . Note that

$$\begin{aligned} A_c\left(\frac{1}{\sqrt{c}}\right) &= \frac{1}{c} - \sqrt{c} - \alpha \rightarrow -\infty, & \text{as } c \rightarrow +\infty, \\ B_c\left(\frac{1}{\sqrt{c}}\right) &= \int_{\mathbb{R}} K(x)e^{\frac{x}{\sqrt{c}}} dx - 1 - \sqrt{c} - \beta \rightarrow -\infty, & \text{as } c \rightarrow +\infty, \\ A_c\left(-\frac{1}{\sqrt{-c}}\right) &= -\frac{1}{c} - \sqrt{-c} - \alpha \rightarrow -\infty, & \text{as } c \rightarrow -\infty, \\ B_c\left(-\frac{1}{\sqrt{-c}}\right) &= \int_{\mathbb{R}} K(x)e^{-\frac{x}{\sqrt{-c}}} dx - 1 - \sqrt{-c} - \beta \rightarrow -\infty, & \text{as } c \rightarrow -\infty. \end{aligned}$$

Then

$$\begin{aligned} \Delta_c\left(\frac{1}{\sqrt{c}}\right) &\rightarrow +\infty \quad \text{as } c \rightarrow +\infty, \\ \Delta_c\left(-\frac{1}{\sqrt{-c}}\right) &\rightarrow +\infty \quad \text{as } c \rightarrow -\infty. \end{aligned}$$

We can easily check that

$$\Delta_c(0) = \alpha\beta - h'(0)g'(0) < 0, \quad \Delta_c(\lambda_2) = \Delta_c(\lambda_3) = -h'(0)g'(0) < 0.$$

Therefore, for sufficiently large $c \in \mathbb{R}$, $\Delta_c(\lambda) = 0$ has at least one root in $(0, \frac{1}{\sqrt{c}})$ and at least one root in $(\frac{1}{\sqrt{c}}, \lambda_3)$. Similarly, for sufficiently small $c \in \mathbb{R}$, $\Delta_c(\lambda) = 0$ has at least one root in $(-\frac{1}{\sqrt{-c}}, 0)$ and at least one root in $(\lambda_2, -\frac{1}{\sqrt{-c}})$.

We now prove that $\Delta_c(\lambda) = 0$ has at most two roots in (λ_2, λ_3) . When λ belongs to (λ_2, λ_3) and satisfies

$$\Delta'_c(\lambda) = A'_c(\lambda)B_c(\lambda) + A_c(\lambda)B'_c(\lambda) = 0,$$

we have that $A'_c(\lambda)B'_c(\lambda) \leq 0$ since $A_c(\lambda) < 0$ and $B_c(\lambda) < 0$, which along with $A''_c(\lambda) > 0$ and $B''_c(\lambda) > 0$, implies that

$$\Delta''_c(\lambda) = A''_c(\lambda)B_c(\lambda) + 2A'_c(\lambda)B'_c(\lambda) + A_c(\lambda)B''_c(\lambda) < 0.$$

Then the function $\Delta_c(\cdot)$ has only a unique maximum point and no minimum point in (λ_2, λ_3) . By Rolle's theorem, we can obtain the existence of maximum point of $\Delta_c(\cdot)$ in (λ_2, λ_3) , which is denoted by λ_M . Moreover, we have that $\Delta'_c(\lambda) > 0$ for (λ_2, λ_M) , and $\Delta'_c(\lambda) < 0$ for (λ_M, λ_3) . Therefore, $\Delta_c(\lambda) = 0$ has at most two roots in (λ_2, λ_3) , which (if exist) are in (λ_2, λ_M) and (λ_M, λ_3) , respectively.

Therefore, for sufficiently large $c \in \mathbb{R}$, $\Delta_c(\lambda) = 0$ has exactly three different positive roots and one negative root. For sufficiently small $c \in \mathbb{R}$, $\Delta_c(\lambda) = 0$ has exactly three different negative roots and one positive root. \square

Definition 1. From Theorem 2, we can define

$$C_R^* \triangleq \inf\{c \in \mathbb{R} \mid \Delta_c(\lambda) = 0 \text{ has three different positive roots}\},$$

$$C_L^* \triangleq \sup\{c \in \mathbb{R} \mid \Delta_c(\lambda) = 0 \text{ has three different negative roots}\}.$$

Let

$$M(c) = \text{the number of positive roots of } \Delta_c(\lambda) = 0,$$

$$N(c) = \text{the number of negative roots of } \Delta_c(\lambda) = 0.$$

A simple calculation implies

$$\frac{\partial}{\partial c} \Delta_c(\lambda) = A_c(\lambda) \frac{\partial}{\partial c} B_c(\lambda) + B_c(\lambda) \frac{\partial}{\partial c} A_c(\lambda) = -\lambda[A_c(\lambda) + B_c(\lambda)].$$

Since $A_c(\lambda) < 0$ and $B_c(\lambda) < 0$ for $\lambda \in (\lambda_2, \lambda_3)$, we have that $\frac{\partial}{\partial c} \Delta_c(\lambda) > 0$ for $\lambda \in (0, \lambda_3)$, and $\frac{\partial}{\partial c} \Delta_c(\lambda) < 0$ for $\lambda \in (\lambda_2, 0)$. Since $\Delta_c(\lambda)$ with $\lambda \in (0, \lambda_3)$ is strictly increasing in c , the proof of Theorem 2 implies that the root number of $\Delta_c(\lambda) = 0$ in $(0, \lambda_3)$ is non-decreasing in c . Definition 1 shows that

$$M(c) = \begin{cases} 3, & c > C_R^*, \\ 2, & c = C_R^*, \\ 1, & c < C_R^*. \end{cases} \tag{7}$$

Similarly, we have that

$$N(c) = \begin{cases} 3, & c < C_L^*, \\ 2, & c = C_L^*, \\ 1, & c > C_L^*. \end{cases} \tag{8}$$

2.3. Equivalence of Two Definitions

Theorem 3. $c_R^* = C_R^*$ and $c_L^* = C_L^*$.

Proof. First, we prove $c_R^* \geq C_R^*$. For any $c > c_R^*$, Theorem 1 implies that there are two positive constants $\xi_1(c) \in (0, \lambda_R^*)$ and $\xi_2(c) \in (\lambda_R^*, +\infty)$ such that

$$c = c(\xi_1(c)) = c(\xi_2(c)).$$

From (6), it follows that

$$[a(\xi_1(c)) - c\xi_1(c)][b(\xi_1(c)) - c\xi_1(c)] = h'(0)g'(0),$$

$$[a(\xi_2(c)) - c\xi_1(c)][b(\xi_2(c)) - c\xi_1(c)] = h'(0)g'(0),$$

which mean that

$$\Delta_c(\xi_1(c)) = \Delta_c(\xi_2(c)) = 0,$$

Then $\xi_1(c)$ and $\xi_2(c)$ are two different positive roots of $\Delta_c(\lambda) = 0$, and $M(c) \geq 2$ for any $c > c_R^*$. From (7) it follows that $c_R^* \geq C_R^*$.

Second, we prove $c_R^* \leq C_R^*$. For any $c > C_R^*$, by the proof of Theorem 2 and Definition 1, $\Delta_c(\lambda) = 0$ has two different positive roots in $(0, \lambda_3)$, and we denote them by $\eta_1(c)$ and $\eta_2(c)$ with $\eta_1(c) < \eta_2(c)$. Then for $i = 1, 2$, we have that

$$\Delta_c(\eta_i(c)) = A_c(\eta_i(c))B_c(\eta_i(c)) - h'(0)g'(0) = 0,$$

where $A_c(\eta_i(c)) = a(\eta_i(c)) - c\eta_i(c) < 0$, and $B_c(\eta_i(c)) = b(\eta_i(c)) - c\eta_i(c) < 0$.

It follows that

$$[c\eta_i(c)]^2 - c\eta_i(c)[a(\eta_i(c)) + b(\eta_i(c))] + a(\eta_i(c))b(\eta_i(c)) - h'(0)g'(0) = 0.$$

Note that if $c\eta_i(c) = \frac{1}{2}[a(\eta_i(c)) + b(\eta_i(c)) - \sqrt{[a(\eta_i(c)) - b(\eta_i(c))]^2 + 4h'(0)g'(0)}]$, then

$$A_c(\eta_i(c)) = a(\eta_i(c)) - c\eta_i(c) > 0 \text{ and } B_c(\eta_i(c)) = b(\eta_i(c)) - c\eta_i(c) > 0,$$

which is a contradiction. Thus, we have that

$$c\eta_i(c) = \frac{1}{2} \left[a(\eta_i(c)) + b(\eta_i(c)) + \sqrt{[a(\eta_i(c)) - b(\eta_i(c))]^2 + 4h'(0)g'(0)} \right] = \chi(\eta_i(c)),$$

which implies that

$$c = \frac{\chi(\eta_i(c))}{\eta_i(c)} = c(\eta_i(c)) \geq \max_{\lambda > 0} \{c(\lambda)\} = c_R^*.$$

We obtain $C_R^* \geq c_R^*$.

Therefore, it holds that $c_R^* = C_R^*$, and the proof of $c_L^* = C_L^*$ is similar. \square

Theorem 3 shows that the first definition for c_R^* and c_L^* in Theorem 1 (iii) is equivalent to the second definition for C_R^* and C_L^* in Definition 1. Thus, we use c_R^* and c_L^* for the minimum wave speeds in the rest of the paper.

3. Traveling Wave Solutions

In this section, we consider the traveling wave solutions of (1). First, we introduce the notations for the standard order in \mathbb{R}^2 . For $U = (u_1, u_2)^T$ and $V = (v_1, v_2)^T$, we denote $U \leq V$ if $u_1 \leq v_1$ and $u_2 \leq v_2$; $U < V$ if $U \leq V$ but $U \neq V$; and $U \ll V$ if $u_1 < v_1$ and $u_2 < v_2$.

Substituting $(u(x, t), v(x, t)) = (\phi(x - ct), \psi(x - ct))$ into (1) and letting $\xi = x - ct$, we can get that

$$\begin{cases} \phi''(\xi) + c\phi'(\xi) - \alpha\phi(\xi) + h(\psi)(\xi) = 0, & \xi \in \mathbb{R}, \\ K * \psi(\xi) - \psi(\xi) + c\psi'(\xi) - \beta\psi(\xi) + g(\phi)(\xi) = 0, & \xi \in \mathbb{R}. \end{cases} \tag{9}$$

Let $\Phi(\xi) = (\phi(\xi), \psi(\xi))$, $\xi \in \mathbb{R}$. For a non-increasing traveling wave front, we assume the boundary condition

$$\Phi(-\infty) = (1, 1), \quad \Phi(+\infty) = (0, 0), \tag{10}$$

and for a non-decreasing traveling wave front, we assume the boundary condition

$$\Phi(-\infty) = (0, 0), \quad \Phi(+\infty) = (1, 1). \tag{11}$$

Define $F(\Phi)(\xi) = (f_1(\Phi), f_2(\Phi))(\xi)$ satisfying

$$\begin{aligned} f_1(\Phi)(\xi) &= \frac{1}{\lambda_2 - \lambda_1} \left[\int_{-\infty}^{\xi} e^{\lambda_1(\xi-s)} h(\psi)(s) ds + \int_{\xi}^{+\infty} e^{\lambda_2(\xi-s)} h(\psi)(s) ds \right], \\ f_2(\Phi)(\xi) &= -\frac{e^{\frac{(1+\beta)\xi}{c}}}{c} \int_{-\infty}^{\xi} e^{-\frac{(1+\beta)s}{c}} [K * \psi(s) + g(\phi)(s)] ds, \end{aligned}$$

where λ_1 and λ_2 are the roots of the equation $\lambda^2 + c\lambda - \alpha = 0$, namely,

$$\lambda_1 = \frac{-c - \sqrt{c^2 + 4\alpha}}{2}, \quad \lambda_2 = \frac{-c + \sqrt{c^2 + 4\alpha}}{2}.$$

Note that $F(\Phi)(\cdot)$ satisfies that

$$\begin{cases} (f_1(\Phi))'' + c(f_1(\Phi))' - \alpha(f_1(\Phi)) + h(\psi) = 0, \\ K * f_2(\Phi) - f_2(\Phi) + c(f_2(\Phi))' - \beta f_2(\Phi) + g(\phi) = 0. \end{cases}$$

Therefore, a fixed point of F is a solution of (9).

Definition 2. A continuous function $\Psi = (\phi, \psi) : \mathbb{R} \rightarrow \mathbb{R}^2$ is called an upper solution of (9), if $\phi(\xi)$ is twice continuously differentiable and $\psi(\xi)$ is continuously differentiable on $\mathbb{R} \setminus \mathbb{T}$, where $\mathbb{T} = \{T_i\}_{i=1, \dots, m}$ is a set containing countable points in \mathbb{R} , and they satisfy

$$\begin{cases} \phi''(\xi) + c\phi'(\xi) - \alpha\phi(\xi) + h(\psi)(\xi) \leq 0, & \xi \in \mathbb{R} \setminus \mathbb{T}, \\ K * \psi(\xi) - \psi(\xi) + c\psi'(\xi) - \beta\psi(\xi) + g(\phi)(\xi) \leq 0, & \xi \in \mathbb{R} \setminus \mathbb{T}, \end{cases} \tag{12}$$

A lower solution of (9) is defined similarly by reversing the inequality in (12).

Next we consider the non-increasing traveling wave front satisfying (10). Define

$$C_{[0,1]}(\mathbb{R}, \mathbb{R}^2) = \{\Phi = (\phi, \psi) \in C(\mathbb{R}, \mathbb{R}^2) \mid \phi(\xi) \in [0, 1] \text{ and } \psi(\xi) \in [0, 1] \text{ for } \xi \in \mathbb{R}\}.$$

The following result reduces the existence of the solution of (9) to the existence of a pair of upper and lower solutions satisfying some additional conditions.

Theorem 4. Assume that (A1), (A2), and (H) hold. If (9) has an upper solution $\bar{\Phi} = (\bar{\phi}, \bar{\psi}) \in C_{[0,1]}(\mathbb{R}, \mathbb{R}^2)$ and a lower solution $\underline{\Phi} = (\underline{\phi}, \underline{\psi}) \in C_{[0,1]}(\mathbb{R}, \mathbb{R}^2)$ satisfying

- (a) $\sup_{s \geq \xi} \{\Phi(s)\} \leq \bar{\Phi}(\xi)$ for any $\xi \in \mathbb{R}$,
- (b) (9) has no constant solution on $(0, \inf_{\xi \in \mathbb{R}} \bar{\Phi}(\xi)) \cup [\sup_{\xi \in \mathbb{R}} \underline{\Phi}(\xi), 1)$,
- (c) $\bar{\Phi}'(\xi+) \leq \bar{\Phi}'(\xi-)$ for $\xi \in \mathbb{R}$,
- (d) $\underline{\Phi}'(\xi+) \geq \underline{\Phi}'(\xi-)$ for $\xi \in \mathbb{R}$,

then (9) has a non-increasing solution satisfying (10), which is a traveling wave front of (1).

Proof. The proof is similar to the proofs of [40] (Theorem 2.2) and [41] (Theorem 3.2), where Schauder’s fixed point theorem is applied to obtain the fixed point of F . The properties for f_1 can be studied by the method in [40] (Lemmas 2.3 and 2.4). The properties for f_2 can be obtained from [41] (Lemmas 3.3, 3.5, 3.6, and 3.7). Thus, we omit the details. \square

By Theorem 4, we can obtain the following results of traveling wave fronts.

Theorem 5. Assume (A1), (A2), and (H) hold. Then for $c \geq c_R^*$, (1) has a non-increasing traveling wave front $(u(x, t), v(x, t)) = (\phi(x - ct), \psi(x - ct))$ satisfying (10), and for $c \leq c_L^*$, (1) has a non-decreasing traveling wave front $(u(x, t), v(x, t)) = (\phi(x - ct), \psi(x - ct))$ satisfying (11). Moreover, there exists no traveling wave solution of (1) with $c \in (c_L^*, c_R^*)$.

Proof. We consider the existence of a non-increasing traveling wave front satisfying (10) and the existence of a non-decreasing traveling wave front satisfying (11) to be similar. By Theorem 2 and Definition 1, when $c > c_R^*$, there are two different positive roots of $\Delta_c(\lambda) = 0$ in $(0, \lambda_3)$, and we denote them by $\mu_1(c)$ and $\mu_2(c)$ with $\mu_1(c) < \mu_2(c)$. Define

$$l(c) \triangleq -\frac{A_c(\mu_1(c))}{h'(0)} = -\frac{g'(0)}{B_c(\mu_1(c))} > 0.$$

Consider δ satisfying

$$0 < \delta < \min\{\delta_0, \mu_2(c)/\mu_1(c) - 1\},$$

where δ_0 is the constant in (A1). We define

$$\begin{aligned} \bar{\phi}_c(\xi) &\triangleq \min\{1, e^{-\mu_1(c)\xi}\}, & \bar{\psi}_c(\xi) &\triangleq \min\{1, l(c)e^{-\mu_1(c)\xi}\}, \\ \underline{\phi}_c(\xi) &\triangleq \max\{0, e^{-\mu_1(c)\xi} - qe^{-\mu_1(c)(1+\delta)\xi}\}, \\ \underline{\psi}_c(\xi) &\triangleq \max\{0, l(c)e^{-\mu_1(c)\xi} - qL(c, \delta)e^{-\mu_1(c)(1+\delta)\xi}\}, \end{aligned}$$

where q is a sufficiently large constant, and $L(c, \delta)$ satisfies

$$\frac{g'(0)}{-B_c((1 + \delta)\mu_1(c))} < L(c, \delta) < \frac{-A_c((1 + \delta)\mu_1(c))}{h'(0)}.$$

By the methods in [13] (Lemma 2.7) and [8] (Lemma 2.2), we can get that $\bar{\Phi}_c(\xi) \triangleq (\bar{\phi}_c(\xi), \bar{\psi}_c(\xi))$ is an upper solution and $\underline{\Phi}_c(\xi) \triangleq (\underline{\phi}_c(\xi), \underline{\psi}_c(\xi))$ is a lower solution of (1). Note that $\bar{\Phi}_c(\cdot)$ is non-increasing and $\underline{\Phi}_c(\xi) \leq \bar{\Phi}_c(\xi)$ for any $\xi \in \mathbb{R}$. Then we have

$$\sup_{s \geq \xi} \{\underline{\Phi}_c(s)\} \leq \sup_{s \geq \xi} \{\bar{\Phi}_c(s)\} = \bar{\Phi}_c(\xi) \text{ for } \xi \in \mathbb{R},$$

which implies the condition (a) in Theorem 4 holds. Recall that there is no equilibrium (u, v) of (1) satisfying $0 < u, v < 1$, and then the condition (b) holds. The conditions (c) and (d) can be easily checked. By Theorem 4, (1) has a non-increasing traveling wave front $(\phi(x - ct), \psi(x - ct))$ with $c > c_R^*$ satisfying (10).

Now we consider the case $c = c_R^*$. Let $\{c_n\}$ satisfy $c_n > c_R^*$ and $c_n \rightarrow c_R^*$ as $n \rightarrow +\infty$. Then there is a sequence of non-increasing continuous functions $\{\Phi_n\} = \{(\phi_n, \psi_n)\}$ satisfying

$$\begin{cases} \phi_n'' + c_n \phi_n' - \alpha \phi_n + h(\psi_n) = 0, \\ K * \psi_n - \psi_n + c_n \psi_n' - \beta \psi_n + g(\phi_n) = 0, \\ \lim_{\xi \rightarrow -\infty} (\phi_n, \psi_n) = (1, 1), \quad \lim_{\xi \rightarrow \infty} (\phi_n, \psi_n) = (0, 0). \end{cases} \tag{13}$$

By the methods in the proof of [7] (Theorem 1.1), the function sequences $\{\phi_n\}, \{\phi_n'\}$, and $\{\phi_n''\}$ are uniformly bounded and equicontinuous on \mathbb{R} ; and by the methods in the proof of [13] (Theorem 2.1), the function sequences $\{\psi_n\}$ and $\{\psi_n'\}$ are uniformly bounded and equicontinuous on \mathbb{R} . By the Arzela–Ascoli theorem, we can find a subsequence of $\{n\}$ denoted by $\{n_k\}$ such that

$$c_{n_k} \rightarrow c^*, \phi_{n_k} \rightarrow \phi_{c^*}, \phi_{n_k}' \rightarrow \phi_{c^*}', \phi_{n_k}'' \rightarrow \phi_{c^*}'', \psi_{n_k} \rightarrow \psi_{c^*}, \psi_{n_k}' \rightarrow \psi_{c^*}'.$$

From (13), it follows that

$$\begin{cases} \phi_{c^*}'' + c^* \phi_{c^*}' - \alpha \phi_{c^*} + h(\psi_{c^*}) = 0, \\ K * \psi_{c^*} - \psi_{c^*} + c^* \psi_{c^*}' - \beta \psi_{c^*} + g(\phi_{c^*}) = 0, \\ \lim_{\xi \rightarrow -\infty} (\phi_{c^*}, \psi_{c^*}) = (1, 1), \quad \lim_{\xi \rightarrow \infty} (\phi_{c^*}, \psi_{c^*}) = (0, 0). \end{cases}$$

Then (ϕ_{c^*}, ψ_{c^*}) is a traveling wave front of (1) with $c = c_R^*$ satisfying (10).

Finally, the proof of the nonexistence of traveling wave solution with $c \in (c_L^*, c_R^*)$ is similar to the counterpart in [13] (Theorem 2.1) or [7] (Theorem 1.1). \square

By Theorem 2 and Definition 1, when $c \geq c_R^*$, $\Delta_c(\lambda) = 0$ has three positive roots, and let μ_c be the smallest one. When $c \leq c_L^*$, $\Delta_c(\lambda) = 0$ has three negative roots, and let ν_c be the largest one. Define

$$\bar{\Delta}_c(\lambda) = A_c(\lambda)B_c(\lambda) - g'(1)h'(1), \quad c \in \mathbb{R}, \lambda \in \mathbb{R}.$$

Now suppose that $g'(1)h'(1) < \alpha\beta$. It follows that $\bar{\Delta}_c(0) > 0$. By a similar argument for $\Delta_c(\lambda) = 0$ in the proof of Theorem 2, there are two positive roots and two negative roots of $\bar{\Delta}_c(\lambda) = 0$. Let γ_c^1 be the larger negative root, and let γ_c^2 be the smaller positive root. The following two theorems study the decaying behavior and uniqueness of traveling wave fronts.

Theorem 6. *Let (A1), (A2), and (H) hold, and suppose that $g'(1)h'(1) < \alpha\beta$. For $c \geq c_R^*$ and $c \neq 0$, let $(u(x, t), v(x, t)) = (\phi(x - ct), \psi(x - ct))$ be a non-increasing traveling wave front of (1) satisfying (10). Then there exist $\mathbf{D}_1, \mathbf{D}_2$, and $\mathbf{D}_3 \in \mathbb{R}^+ \times \mathbb{R}^+$ such that*

$$\lim_{\xi \rightarrow +\infty} \frac{(\phi(\xi), \psi(\xi))}{e^{-\mu_c \xi}} = \mathbf{D}_1, \quad \text{when } c > c_R^*, \tag{14}$$

$$\lim_{\xi \rightarrow +\infty} \frac{(\phi(\xi), \psi(\xi))}{\xi e^{-\mu_c \xi}} = \mathbf{D}_2, \quad \text{when } c = c_R^*, \tag{15}$$

$$\lim_{\xi \rightarrow -\infty} \frac{(1 - \phi(\xi), 1 - \psi(\xi))}{e^{-\gamma_c^1 \xi}} = \mathbf{D}_3, \quad \text{when } c \geq c_R^*. \tag{16}$$

Similarly, for $c \leq c_L^*$ and $c \neq 0$, let $(u(x, t), v(x, t)) = (\phi(x - ct), \psi(x - ct))$ be a non-decreasing traveling wave front of (1) satisfying (11). Then there exist $\mathbf{D}_4, \mathbf{D}_5$, and $\mathbf{D}_6 \in \mathbb{R}^+ \times \mathbb{R}^+$ such that

$$\lim_{\xi \rightarrow -\infty} \frac{(\phi(\xi), \psi(\xi))}{e^{-\nu_c \xi}} = \mathbf{D}_4, \quad \text{when } c < c_L^*, \tag{17}$$

$$\lim_{\xi \rightarrow -\infty} \frac{(\phi(\xi), \psi(\xi))}{\xi e^{-\nu_c \xi}} = \mathbf{D}_5, \quad \text{when } c = c_L^*, \tag{18}$$

$$\lim_{\xi \rightarrow +\infty} \frac{(1 - \phi(\xi), 1 - \psi(\xi))}{e^{-\gamma_c^2 \xi}} = \mathbf{D}_6, \quad \text{when } c \geq c_L^*. \tag{19}$$

Theorem 7. *Under the same assumptions of Theorem 6, the traveling wave front of (1) with (10) or (11) is unique up to translation, in the sense that, for any $c \in (c_R^*, +\infty) \cup (-\infty, c_L^*)$, if $(\phi_1(x), \psi_1(x))$ and $(\phi_2(x), \psi_2(x))$ are two solutions of (9) and at least one of them is continuous, then there exists $\xi_0 \in \mathbb{R}$ such that*

$$(\phi_1(\xi + \xi_0), \psi_1(\xi + \xi_0)) = (\phi_2(\xi), \psi_2(\xi)), \quad \xi \in \mathbb{R}.$$

The proof of Theorem 6 is similar to Theorem 2.2 in [13], and we give a scheme here. By a similar argument for (27) in [13], there are two constants $\gamma > 0$ and $M > 0$ such that

$$\phi(\xi) \leq Me^{-\gamma \xi}, \quad \psi(\xi) \leq Me^{-\gamma \xi}, \quad \xi \in \mathbb{R}.$$

Define

$$U(\lambda) = \int_{\mathbb{R}} \phi(\xi) e^{\lambda \xi} d\xi, \quad V(\lambda) = \int_{\mathbb{R}} \psi(\xi) e^{\lambda \xi} d\xi, \quad 0 < \text{Re} \lambda < \gamma.$$

Multiplying (9) by $e^{\lambda \xi}$ and integrating it over \mathbb{R} , we obtain that

$$\begin{pmatrix} A_c(\lambda) & h'(0) \\ g'(0) & B_c(\lambda) \end{pmatrix} \begin{pmatrix} U(\lambda) \\ V(\lambda) \end{pmatrix} = \begin{pmatrix} H_\psi(\lambda) \\ G_\phi(\lambda) \end{pmatrix},$$

where

$$H_\psi(\lambda) \triangleq \int_{\mathbb{R}} [h'(0)\psi(\xi) - h(\psi(\xi))] e^{\lambda \xi} d\xi, \quad G_\phi(\lambda) \triangleq \int_{\mathbb{R}} [g'(0)\phi(\xi) - g(\phi(\xi))] e^{\lambda \xi} d\xi.$$

It follows that

$$\int_0^{+\infty} \phi(\xi)e^{\lambda\xi}d\xi = \frac{B_c(\lambda)H_\psi(\lambda) - h'(0)G_\phi(\lambda) - \Delta_c(\lambda) \int_{-\infty}^0 \phi(\xi)e^{\lambda\xi}d\xi}{\Delta_c(\lambda)},$$

$$\int_0^{+\infty} \psi(\xi)e^{\lambda\xi}d\xi = \frac{A_c(\lambda)G_\phi(\lambda) - g'(0)H_\psi(\lambda) - \Delta_c(\lambda) \int_{-\infty}^0 \psi(\xi)e^{\lambda\xi}d\xi}{\Delta_c(\lambda)}.$$

If ϕ and ψ are monotonous, we can get (14) and (15) by Ikehara’s theorem. If not, define $(\tilde{\phi}(\xi), \tilde{\psi}(\xi)) = (\phi(\xi)e^{-p\xi}, \psi(\xi)e^{-p\xi})$ for $x \in \mathbb{R}$. When p is large enough, $\tilde{\phi}$ and $\tilde{\psi}$ are monotonous since ϕ' and ψ' are bounded. Then we can get (14) and (15) by applying Ikehara’s theorem to $\tilde{\phi}$ and $\tilde{\psi}$. The result (16) can be similarly proved by considering $(1 - \phi(\xi), 1 - \psi(\xi))$ as $\xi \rightarrow -\infty$. The proofs of (17), (18), and (19) are similar to (14), (15), and (16), respectively.

The proof of Theorem 7 is based on the following claim: if $(\phi_1, \psi_1) \leq (\phi_2, \psi_2)$ on \mathbb{R} , then either $(\phi_1, \psi_1) < (\phi_2, \psi_2)$ or $(\phi_1, \psi_1) \equiv (\phi_2, \psi_2)$ on \mathbb{R} , where (ϕ_1, ψ_1) and (ϕ_2, ψ_2) are two solutions of (9) with (10) or (11). If there exists $x_0 \in \mathbb{R}$ satisfying $\psi_1(x_0) = \psi_2(x_0)$, since x_0 is a maximum point of $\psi_1(\cdot) - \psi_2(\cdot)$, we have that

$$\int_{\mathbb{R}} K(x_0 - y)(\psi_1(y) - \psi_2(y))dy + g(\phi_1(x_0)) - g(\phi_2(x_0)) = 0.$$

It follows from $(\phi_1, \psi_1) \leq (\phi_2, \psi_2)$ and $g' > 0$ that $\phi_1(x_0) = \phi_2(x_0)$ and $\psi_1(y) = \psi_2(y)$ for any $y \in \{y \mid x_0 - y \in \text{supp}(K)\}$, which implies that $\psi_1(y) = \psi_2(y)$ for $y \in \mathbb{R}$ (by redefining x_0 and repeating this process if $\text{supp}(K) \neq \mathbb{R}$). By the uniqueness of solution for the equation $\phi'' + c\phi' - \alpha\phi + h(\psi_1) = 0$ with $\phi(x_0) = \phi_1(x_0)$, we get $\phi_1(x) = \phi_2(x)$ for $x \in \mathbb{R}$. Hence, it holds that $(\phi_1, \psi_1) \equiv (\phi_2, \psi_2)$ on \mathbb{R} . If there exists $y_0 \in \mathbb{R}$ satisfying $\phi_2(y_0) = \phi_1(y_0)$, we have that

$$\phi_1''(y_0) - \phi_2''(y_0) + h(\psi_1(y_0)) - h(\psi_2(y_0)) = 0.$$

It follows from $\phi_1''(y_0) - \phi_2''(y_0) \leq 0$ and $h' > 0$ that $\psi_1(y_0) = \psi_2(y_0)$, which implies that $(\phi_1, \psi_1) \equiv (\phi_2, \psi_2)$ on \mathbb{R} by the argument above. The claim is proved, and Theorem 7 is proved by a similar method to Theorem 1.2 in [7].

4. The Signs of Minimum Wave Speeds

In this section, we show how to identify the signs of c_R^* and c_L^* . Recall that

$$a(\lambda) = \lambda^2 - \alpha, \quad b(\lambda) = \int_{\mathbb{R}} K(x)e^{\lambda x}dx - 1 - \beta \text{ for } \lambda \in \mathbb{R}.$$

Define

$$\Lambda \triangleq \{\lambda \in \mathbb{R} \mid a(\lambda)b(\lambda) \geq g'(0)h'(0), a(\lambda) < 0, b(\lambda) < 0\}.$$

Theorem 8. We have that either $\Lambda \subseteq \mathbb{R}^+$ or $\Lambda \subseteq \mathbb{R}^-$, and

- (i) $c_L^* < c_R^* < 0 \Leftrightarrow \text{int}(\Lambda) \cap \mathbb{R}^+ \neq \emptyset$;
- (ii) $c_L^* < c_R^* = 0 \Leftrightarrow \Lambda \cap \mathbb{R}^+$ is a singleton set;
- (iii) $c_L^* < 0 < c_R^* \Leftrightarrow \Lambda = \emptyset$;
- (iv) $0 = c_L^* < c_R^* \Leftrightarrow \Lambda \cap \mathbb{R}^-$ is a singleton set;
- (v) $0 < c_L^* < c_R^* \Leftrightarrow \text{int}(\Lambda) \cap \mathbb{R}^- \neq \emptyset$.

Proof. The proofs of “ \Leftarrow ” in (i)–(v) are similar to the proof of [19] (Theorem 2.2). Now we prove “ \Rightarrow ” for (i)–(v). By Theorem 1, we get $c_L^* < c_R^*$. Then the relationships among $0, c_L^*$, and c_R^* in (i)–(v) are all possible cases. By [19] (Theorem 2.2), Λ is an empty set or a closed interval without 0 in \mathbb{R} . Then, either $\Lambda \subseteq \mathbb{R}^+$ or $\Lambda \subseteq \mathbb{R}^-$. We have that the conditions of Λ in (i)–(v) contain all possible cases, which means that Λ must satisfy one of the conditions in (i)–(v). Therefore, the proofs of “ \Rightarrow ” can be obtained from “ \Leftarrow ” for (i)–(v). \square

Next we give two specific forms of the kernel function $K(\cdot)$. For each case, we show how to apply Theorem 8 to identify the signs of c_R^* and c_L^* .

4.1. Normal Distribution

Assume that $K(x)$ satisfies that

$$K(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(x-\mu)^2}{2\sigma}\right) \text{ with } \mu \in \mathbb{R}, \sigma > 0.$$

Let $r = \mu/\sqrt{2\sigma}$. When $\mu \neq 0$, $K(\cdot)$ can be regarded as a function with parameters μ and r , namely,

$$K(x) = \begin{cases} \frac{r}{\mu\sqrt{\pi}} \exp\left(-\frac{r^2(x-\mu)^2}{\mu^2}\right), & \text{when } \mu \neq 0; \\ \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{x^2}{2\sigma}\right), & \text{when } \mu = 0. \end{cases}$$

Define

$$\mathcal{K} \triangleq \alpha(\beta + 1 - \exp(-r^2)) / (g'(0)h'(0)) > 0. \tag{20}$$

Corollary 1. For any fixed r satisfying $\mathcal{K} > 1$, there is a constant $\mu^* > 0$ such that

- (i) $\mu > \mu^* \Leftrightarrow$ the propagation to left fails, namely, $0 < c_L^* < c_R^*$;
- (ii) $\mu = \mu^* \Leftrightarrow 0 = c_L^* < c_R^*$;
- (iii) $-\mu^* < \mu < \mu^* \Leftrightarrow$ the propagation to both left and right happens, namely, $c_L^* < 0 < c_R^*$;
- (iv) $\mu = -\mu^* \Leftrightarrow c_L^* < c_R^* = 0$;
- (v) $\mu < -\mu^* \Leftrightarrow$ the propagation to right fails, namely, $c_L^* < c_R^* < 0$.

Moreover, we have that $\mu^* \in (0, \bar{\mu})$ where

$$\bar{\mu} = \frac{2r^2}{\sqrt{\alpha\left(1 - \frac{1}{\mathcal{K}}\right)}}.$$

For any r satisfying $\mathcal{K} \leq 1$, it holds that $c_L^* < 0 < c_R^*$ for any $\mu \in \mathbb{R}$.

Proof. We first consider the case $\mathcal{K} > 1$. Note that when $\mu = 0$, it holds that $r = 0$ and $\mathcal{K} = \alpha\beta / (g'(0)h'(0)) < 1$. Now we consider the case $\mu > 0$, and the case $\mu < 0$ can be obtained similarly. Some calculations imply that

$$a(\lambda) = \lambda^2 - \alpha, \quad b(\lambda) = \int_{\mathbb{R}} K(x)e^{\lambda x} dx - 1 - \beta = \exp\left(\mu\lambda + \frac{\mu^2\lambda^2}{4r^2}\right) - 1 - \beta.$$

Define

$$\Lambda_a = \{\lambda \in \mathbb{R} \mid a(\lambda) < 0\}, \quad \Lambda_b = \{\lambda \in \mathbb{R} \mid b(\lambda) < 0\}.$$

It follows that $\Lambda \in \Lambda_a \cap \Lambda_b$. For $\mu > 0$, since

$$a(0)b(0) = \alpha\beta < h'(0)g'(0), \quad \left. \frac{d}{d\lambda}(a(\lambda)b(\lambda)) \right|_{\lambda=0} = -\mu\alpha < 0,$$

the proof of [19] (Theorem 2.2) shows that $\Lambda \subseteq \mathbb{R}^-$ for $\mu > 0$.

When $\mathcal{K} > 1$ and $\mu \geq \bar{\mu}$, we have that

$$\alpha\left(1 - \frac{1}{\mathcal{K}}\right) \geq \frac{4r^4}{\mu^2}. \tag{21}$$

Define

$$\lambda_0 = -\frac{2r^2}{\mu}.$$

From (20) and (21), it follows that

$$\begin{aligned} a(\lambda_0)b(\lambda_0) &= \left(\frac{4r^4}{\mu^2} - \alpha\right) \left(\exp(-r^2) - 1 - \beta\right) \\ &= -\left(\frac{4r^4}{\mu^2} - \alpha\right) \frac{\mathcal{K}}{\alpha} g'(0)h'(0) \\ &\geq g'(0)h'(0). \end{aligned}$$

It is easy to check that

$$\begin{aligned} \frac{\partial}{\partial \lambda} (a(\lambda)b(\lambda)) \Big|_{\lambda=\lambda_0} &= a'(\lambda_0)b(\lambda_0) + a(\lambda_0)b'(\lambda_0) \\ &= 2\lambda_0(\exp(-r^2) - 1 - \beta) + (\lambda_0^2 - \alpha) \exp(-r^2) \left(\mu + \frac{\mu^2 \lambda_0}{2r^2}\right) \\ &= 2\lambda_0(\exp(-r^2) - 1 - \beta) > 0. \end{aligned}$$

Then there is $\epsilon > 0$ such that $(\lambda_0, \lambda_0 + \epsilon) \subseteq \Lambda$, which implies that $0 < c_L^* < c_R^*$ for $\mu \geq \bar{\mu}$.
When $\mathcal{K} > 1$ and $\mu \in (0, \bar{\mu})$, we have that

$$\alpha \left(1 - \frac{1}{\mathcal{K}}\right) < \frac{4r^4}{\mu^2},$$

which implies that

$$a(\lambda_0)b(\lambda_0) = -\left(\frac{4r^4}{\mu^2} - \alpha\right) \frac{\mathcal{K}}{\alpha} g'(0)h'(0) < g'(0)h'(0).$$

When $\lambda \in \Lambda_a \cap \Lambda_b \cap (-\infty, \lambda_0)$, we can get that

$$\begin{aligned} \frac{d}{d\lambda} (a(\lambda)b(\lambda)) &= a'(\lambda)b(\lambda) + a(\lambda)b'(\lambda) \\ &= 2\lambda b(\lambda) + a(\lambda) \exp\left(\mu\lambda + \frac{\mu^2 \lambda^2}{4r^2}\right) \left(\mu + \frac{\mu^2}{2r^2} \lambda\right) > 0. \end{aligned}$$

It follows that

$$a(\lambda)b(\lambda) < a(\lambda_0)b(\lambda_0) < g'(0)h'(0) \quad \text{for any } \lambda \in \Lambda_a \cap \Lambda_b \cap (-\infty, \lambda_0).$$

Then we have that $\Lambda \cap \Lambda_a \cap \Lambda_b \cap (-\infty, \lambda_0) \neq \emptyset$, which implies

$$\Lambda \subseteq \Lambda_a \cap \Lambda_b \cap (\lambda_0, 0) \quad \text{for } \mu \in (0, \bar{\mu}).$$

Consider a function $\Lambda(\cdot) : \mu \mapsto \Lambda$ which is from \mathbb{R}^+ to the set that consists of all closed intervals in \mathbb{R} . We can check that

$$\frac{\partial}{\partial \mu} (a(\lambda)b(\lambda)) = a(\lambda)\lambda \left(1 + \frac{\lambda\mu}{2r^2}\right) \exp\left(\mu\lambda + \frac{\mu^2 \lambda^2}{4r^2}\right) > 0 \quad \text{for } \lambda \in \Lambda_a \cap (\lambda_0, 0).$$

It follows that

$$\Lambda(\mu) \subseteq \Lambda(\mu') \quad \text{for any } 0 < \mu < \mu' < \bar{\mu}, \quad (22)$$

and this inclusion is strict when $\Lambda(\mu') \neq \emptyset$.

Note that we have already obtained $\text{int}(\Lambda) \neq \emptyset$ when $\mu \geq \bar{\mu}$. Now consider the case $\mu \rightarrow 0^+$. For fixed r satisfying $\mathcal{K} > 1$, it holds that $\lambda_0 \rightarrow -\infty$ as $\mu \rightarrow 0^+$. Since $\alpha\beta < g'(0)h'(0)$, there exists $\mu > 0$ sufficiently small such that

$$\lambda_0 < -\sqrt{\alpha}, \quad \text{and } 1 - \exp\left(-\mu\sqrt{\alpha} + \frac{\mu^2\alpha}{4r^2}\right) < \frac{g'(0)h'(0) - \alpha\beta}{\alpha}.$$

It follows that

$$-\alpha b(-\sqrt{\alpha}) = \alpha\left(\beta + 1 - \exp\left(-\mu\sqrt{\alpha} + \frac{\mu^2\alpha}{4r^2}\right)\right) < h'(0)g'(0).$$

Note that $(-\sqrt{\alpha}, 0) = \Lambda_a \cap \mathbb{R}^-$. Since $\lambda_0 < -\sqrt{\alpha}$, we have that $b(\cdot)$ is increasing in $(-\sqrt{\alpha}, 0)$ and $b(\lambda) > b(-\sqrt{\alpha})$ for $\lambda \in (-\sqrt{\alpha}, 0)$. Note that $a(\lambda) \geq -\alpha$ for $\lambda \in (-\sqrt{\alpha}, 0)$, and then

$$a(\lambda)b(\lambda) \leq -\alpha b(-\sqrt{\alpha}) < h'(0)g'(0) \text{ for } \lambda \in \Lambda_a \cap \mathbb{R}^-.$$

We get that Λ is an empty set for sufficiently small $\mu > 0$. Therefore, there exists $\mu^* \in (0, \bar{\mu})$ such that

$$\Lambda \begin{cases} \text{is an empty set,} & \text{when } \mu \in (0, \mu^*); \\ \text{is a singleton set in } \mathbb{R}^-, & \text{when } \mu = \mu^*; \\ \text{has interior points in } \mathbb{R}^-, & \text{when } \mu > \mu^*. \end{cases}$$

Considering $\tilde{K}(x) = K(-x)$ and $\nu = -\mu$, we get by a similar argument as above that

$$\Lambda \begin{cases} \text{is an empty set,} & \text{when } \mu \in (-\mu^*, 0); \\ \text{is a singleton set in } \mathbb{R}^+, & \text{when } \mu = -\mu^*; \\ \text{has interior points in } \mathbb{R}^+, & \text{when } \mu < -\mu^*. \end{cases}$$

By Theorem 8, we have proved “ \Rightarrow ” for (i)–(v) in Corollary 1. Note that the relationships among $0, c_L^*$, and c_R^* in (i)–(v) are all possible cases; and the relationships among μ, μ^* , and $-\mu^*$ in (i)–(v) are also all possible cases. Thus, the proofs of “ \Leftarrow ” can be obtained from “ \Rightarrow ” for (i)–(v).

Next, consider the case $\mathcal{K} \leq 1$. When $\mu = 0$, it is easy to check that

$$a(\lambda)b(\lambda) = (\lambda^2 - \alpha)\left(\exp\left(\frac{\sigma}{2}\lambda^2\right) - 1 - \beta\right) \leq \alpha\beta < g'(0)h'(0) \quad \text{for } \lambda \in \Lambda_a \cap \Lambda_b,$$

which implies $c_L^* < 0 < c_R^*$. When $\mathcal{K} \leq 1$ and $\mu \neq 0$, since

$$a(\lambda) \geq a(0) = -\alpha, \quad b(\lambda) \geq b(\lambda_0) = \exp(-r^2) - 1 - \beta \quad \text{for } \lambda \in \mathbb{R},$$

it holds that

$$a(\lambda)b(\lambda) \leq \alpha(\beta + 1 - \exp(-r^2)) \quad \text{for } \lambda \in \Lambda_a \cap \Lambda_b.$$

In the above inequality, the equality holds only if $\lambda_0 = -2r^2/\mu = -\mu/\sigma = 0$, which implies $\mu = 0$. Then, when $\mu \neq 0$, we have that

$$a(\lambda)b(\lambda) < \alpha(\beta + 1 - \exp(-r^2)) = \mathcal{K}g'(0)h'(0) \leq g'(0)h'(0) \quad \text{for } \lambda \in \Lambda_a \cap \Lambda_b,$$

which implies $c_L^* < 0 < c_R^*$. \square

4.2. Uniform Distribution

Suppose that $K(\cdot)$ is given by

$$K(x) = \begin{cases} \frac{1}{A - B}, & \text{for } x \in [B, A], \\ 0, & \text{for } x \notin [B, A], \end{cases}$$

where the constants $A \in \mathbb{R}^+$ and $B \in \mathbb{R}^-$ stand for the farthest distances of the movements of infectious agents during a unit time period to the right and left of x -axis, respectively. Some calculations imply that

$$a(\lambda) = \lambda^2 - \alpha, \quad b(\lambda) = \begin{cases} \frac{e^{A\lambda} - e^{B\lambda}}{(A - B)\lambda} - 1 - \beta, & \lambda \neq 0, \\ -\beta, & \lambda = 0. \end{cases}$$

Next, we state the following lemma whose proof can be found in [42] (Lemma 5.3).

Lemma 1. Define $\omega(x) = (x - 1)e^x, x \in \mathbb{R}$. There is a unique non-zero continuous function $r \mapsto z_r$ from $(0, +\infty)$ to $(-\infty, 1)$ with $z_r \neq 0$ such that $\omega(z_r) = \omega(-rz_r)$ for any $r > 0$. Moreover, when $r > 1$,

$$\omega(x) - \omega(-rx) < 0 \text{ for } x < z_r \text{ with } x \neq 0, \quad \omega(x) - \omega(-rx) > 0 \text{ for } x > z_r,$$

and when $r \in (0, 1)$,

$$\omega(x) - \omega(-rx) < 0 \text{ for } x < z_r, \quad \omega(x) - \omega(-rx) > 0 \text{ for } x > z_r \text{ with } x \neq 0.$$

We also have that z_r is strictly increasing in $r \in (0, +\infty)$, and $z_r = 0$ when $r = 1$. Then it holds that $(r - 1)z_r > 0$ for $r \neq 1$.

Denote $r = -A/B > 0$, and let z_r be the constant defined in Lemma 1. It follows from Lemma 1 that

$$b'(\lambda) = \frac{1}{(A - B)\lambda^2}(\omega(A\lambda) - \omega(B\lambda)) \begin{cases} \leq 0, & \lambda \in (-\infty, z_r/B), \\ = 0, & \lambda = z_r/B, \\ \geq 0, & \lambda \in (z_r/B, +\infty), \end{cases} \tag{23}$$

and

$$b\left(\frac{z_r}{B}\right) = \min\{b(z), z \in \mathbb{R}\} = \frac{e^{z_r} - e^{-rz_r}}{(1 + r)z_r} - 1 - \beta < 0. \tag{24}$$

When $r \neq 1$, it holds that $z_r \neq 0$ and we denote

$$\mathcal{K} \triangleq \frac{-\alpha}{g'(0)h'(0)} b\left(\frac{z_r}{B}\right) = \frac{-\alpha}{g'(0)h'(0)} \left[\frac{e^{z_r} - e^{-rz_r}}{(1 + r)z_r} - 1 - \beta \right] > 0.$$

When $r = 1$, since $\min\{b(\lambda); \lambda \in \mathbb{R}\} = -\beta$, we can simply denote $\mathcal{K} = \alpha\beta / (g'(0)h'(0)) < 1$.

We denote

$$\mu = \frac{A + B}{2} \in \mathbb{R},$$

which implies that when $r \neq 1, A = \frac{2r\mu}{r-1}$ and $B = -\frac{2\mu}{r-1}$. When $r \neq 1$ (i.e., $\mu \neq 0$), $K(\cdot)$ can be regarded as a function with parameters μ and r , namely,

$$K(x) = \begin{cases} \frac{r - 1}{2\mu(r + 1)}, & \text{for } x \in \left[-\frac{2\mu}{r - 1}, \frac{2r\mu}{r - 1}\right], \\ 0, & \text{for } x \notin \left[-\frac{2\mu}{r - 1}, \frac{2r\mu}{r - 1}\right]. \end{cases}$$

Note that \mathcal{K} depends only on r and is independent of μ .

Corollary 2. All results in Corollary 1 hold for the uniform distribution case after the definition of $\bar{\mu}$ is replaced by

$$\bar{\mu} = \frac{(r-1)z_r}{2\sqrt{\alpha\left(1-\frac{1}{\mathcal{K}}\right)}} > 0 \quad \text{when } \mathcal{K} > 1.$$

Proof. Although the proof uses the idea similar to the proof of Corollary 1, we still give some details because some important calculations are different. When $r = 1$, recall that $\mathcal{K} = \alpha\beta/(g'(0)h'(0)) < 1$ and $z_r = 0$, which imply that

$$b(\lambda) \geq \min_{\lambda \in \mathbb{R}}\{b(\lambda)\} = b(0) = -\beta, \quad \lambda \in \mathbb{R}.$$

Since $a(\lambda) \geq -\alpha$ for $\lambda \in \mathbb{R}$, we have that

$$a(\lambda)b(\lambda) \leq \alpha\beta < g'(0)h'(0), \quad \lambda \in \Lambda_a \cap \Lambda_b,$$

which implies that $\Lambda = \emptyset$ when $r = 1$, namely, $c_L^* < 0 < c_R^*$.

When $\mathcal{K} > 1$ we only consider the case $\mu > 0$ (i.e., $r > 1$), and the case $\mu < 0$ (i.e., $r < 1$) is similar. It holds that

$$b'(0) = \lim_{\lambda \rightarrow 0} \frac{b(\lambda) - b(0)}{\lambda} = \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \left[\frac{e^{A\lambda} - e^{B\lambda}}{(A-B)\lambda} - 1 \right] = \frac{A+B}{2} = \mu.$$

For $\mu > 0$, since

$$a(0)b(0) = \alpha\beta < h'(0)g'(0), \text{ and } \left. \frac{d}{d\lambda}(a(\lambda)b(\lambda)) \right|_{\lambda=0} = a(0)b'(0) = -\mu\alpha < 0,$$

we have that $\Lambda \subseteq \mathbb{R}^-$ for $\mu > 0$.

When $\mathcal{K} > 1$ and $\mu \geq \bar{\mu}$, we have that

$$\mu^2 \geq \left(\frac{(r-1)z_r}{2} \right)^2 \frac{1}{\alpha\left(1-\frac{1}{\mathcal{K}}\right)}.$$

Denote

$$\lambda_0 = \frac{z_r}{B} = -\frac{(r-1)z_r}{2\mu}.$$

Then

$$\begin{aligned} a(\lambda_0)b(\lambda_0) &= \left[\left(\frac{(r-1)z_r}{2\mu} \right)^2 - \alpha \right] b\left(\frac{z_r}{B}\right) \\ &= -\left[\left(\frac{(r-1)z_r}{2\mu} \right)^2 - \alpha \right] \frac{\mathcal{K}}{\alpha} g'(0)h'(0) \geq g'(0)h'(0). \end{aligned}$$

By (23) and (24), we have that $b(\lambda_0) < 0$ and $b'(\lambda_0) = 0$. Thus, it holds that

$$\left. \frac{\partial}{\partial \lambda}(a(\lambda)b(\lambda)) \right|_{\lambda=\lambda_0} = a'(\lambda_0)b(\lambda_0) + a(\lambda_0)b'(\lambda_0) = 2\lambda_0 b(\lambda_0) > 0.$$

Then, $\text{int}(\Lambda) \cap \mathbb{R}^- \neq \emptyset$, which implies that $0 < c_L^* < c_R^*$ for $\mu \geq \bar{\mu}$.

When $\mathcal{K} > 1$ and $\mu \in (0, \bar{\mu})$, we have that

$$\mu^2 < \left(\frac{(r-1)z_r}{2} \right)^2 \frac{1}{\alpha\left(1-\frac{1}{\mathcal{K}}\right)},$$

and then

$$a(\lambda_0)b(\lambda_0) = - \left[\left(\frac{(r-1)z_r}{2\mu} \right)^2 - \alpha \right] \frac{\mathcal{K}}{\alpha} g'(0)h'(0) < g'(0)h'(0).$$

When $\lambda \leq \lambda_0$, we have that $b'(\lambda) \leq 0$ by (23). When $\lambda \in \Lambda_a \cap \Lambda_b \cap (-\infty, \lambda_0)$, it holds that

$$\frac{d}{d\lambda} (a(\lambda)b(\lambda)) = a'(\lambda)b(\lambda) + a(\lambda)b'(\lambda) = 2\lambda b(\lambda) + a(\lambda)b'(\lambda) \geq 0.$$

It follows that

$$a(\lambda)b(\lambda) \leq a(\lambda_0)b(\lambda_0) < g'(0)h'(0) \quad \text{for } \lambda \in \Lambda_a \cap \Lambda_b \cap (-\infty, \lambda_0).$$

Thus, $\Lambda \subseteq \Lambda_a \cap \Lambda_b \cap (\lambda_0, 0)$ for $\mu \in (0, \bar{\mu})$. Let $\Lambda(\cdot) : \mu \mapsto \Lambda$ be the function from \mathbb{R}^+ to the set that consists of all closed intervals in \mathbb{R} . Note that

$$\begin{aligned} b(\lambda) &= \frac{e^{A\lambda} - e^{B\lambda}}{(A - B)\lambda} - 1 - \beta \\ &= \frac{r-1}{2\lambda(r+1)} \frac{1}{\mu} \left[\exp\left(\frac{2r\lambda\mu}{r-1}\right) - \exp\left(-\frac{2\lambda\mu}{r-1}\right) \right] - 1 - \beta, \quad \lambda \neq 0. \end{aligned}$$

Then, we have

$$\begin{aligned} \frac{\partial}{\partial \mu} (b(\lambda)) &= \frac{r-1}{2\lambda(r+1)} \frac{1}{\mu^2} \left[\left(\frac{2r\lambda\mu}{r-1} - 1 \right) \exp\left(\frac{2r\lambda\mu}{r-1}\right) + \left(\frac{2\lambda\mu}{r-1} + 1 \right) \exp\left(-\frac{2\lambda\mu}{r-1}\right) \right] \\ &= \frac{r-1}{2\lambda(r+1)} \frac{1}{\mu^2} \left[(A\lambda - 1)e^{A\lambda} - (B\lambda - 1)e^{B\lambda} \right] \\ &= \frac{r-1}{2\lambda(r+1)} \frac{1}{\mu^2} [\omega(-rB\lambda) - \omega(B\lambda)]. \end{aligned}$$

When $\lambda \in (\lambda_0, 0)$, we have that $B\lambda \in (0, z_r)$, which implies along with Lemma 1 and $r > 1$ that

$$\frac{\partial}{\partial \mu} (b(\lambda)) < 0, \quad \lambda \in (\lambda_0, 0).$$

It follows that

$$\frac{\partial}{\partial \mu} (a(\lambda)b(\lambda)) = a(\lambda) \frac{\partial}{\partial \mu} (b(\lambda)) > 0 \quad \text{for } \lambda \in \Lambda_a \cap (\lambda_0, 0).$$

Then,

$$\Lambda(\mu) \subseteq \Lambda(\mu') \quad \text{for any } 0 < \mu < \mu' < \bar{\mu},$$

and this inclusion is strict when $\Lambda(\mu') \neq \emptyset$.

Some calculations show that

$$\lim_{\mu \rightarrow 0^+} b(-\sqrt{\alpha}) = -\beta.$$

Since $\alpha\beta < g'(0)h'(0)$, there exists μ sufficiently small such that

$$-\alpha b(-\sqrt{\alpha}) < g'(0)h'(0), \quad \text{and} \quad \lambda_0 = \frac{z_r}{B} = -\frac{(r-1)z_r}{2\mu} < -\sqrt{\alpha}.$$

By (24), $b(\lambda)$ is increasing in $\lambda \in (\lambda_0, +\infty)$. We have that

$$a(\lambda)b(\lambda) < -\alpha b(\lambda) \leq -\alpha b(-\sqrt{\alpha}) < g'(0)h'(0) \quad \text{for } \lambda \in \Lambda_a \cap \Lambda_b \cap \mathbb{R}^-.$$

Then, $\Lambda = \emptyset$ for $\mu > 0$ sufficiently small. The rest of the proof is similar to the counterpart in the proof of Corollary 1. \square

Remark 1. Note that the parameter r for the normal distribution in Section 4.1 is defined by $\mu/\sqrt{2\sigma}$, where σ is variance and μ is expectation. For the uniform distribution, the variance $\text{Var}(K)$ of K is $(A - B)^2/12$, and the expectation is $\mu = (A + B)/2$. Consider

$$R \triangleq \frac{\mu}{\sqrt{\text{Var}(K)}} = \frac{\sqrt{3}(r-1)}{r+1} \text{ with } r = -\frac{A}{B} > 0.$$

Since the function $r \mapsto (r-1)/(r+1)$ from \mathbb{R}^+ to $(-1, 1)$ is bijection and increasing, we can use $\frac{2\sqrt{3}}{\sqrt{3-R}} - 1$ in place of r in Corollary 2. From the results in Sections 4.1 and 4.2, we see that $\frac{\mu}{\sqrt{\text{Var}(K)}}$ (or with some coefficient) and μ are important parameters to describe whether an asymmetric kernel changes the signs of minimum wave speeds.

5. Conclusions

We studied traveling wave solutions of an epidemic model with mixed diffusion. We gave two definitions of the minimum wave speeds, and the equivalence of these two definitions was proved. The existence, decaying behavior, and uniqueness of traveling wave fronts were obtained. We also presented how to identify the signs of minimum wave speeds and apply them to two specific forms of the kernel function, namely, normal distribution and uniform distribution. Our study indicates that in these two scenarios, the asymmetric nonlocal kernel may induce non-positive minimal wave speed and standing wave solution whose wave speed is zero. However, for general dispersal kernel $K(\cdot)$ with the expectation μ and the variance σ , it is unknown whether the parameters $\frac{\mu}{\sqrt{\sigma}}$ and μ can determine the signs of minimum wave speeds, and this interesting question will be the topic of future research.

Author Contributions: Conceptualization and formal analysis, W.-B.X.; validation and writing, R.H. All authors have read and agreed to the published version of the manuscript.

Funding: This research of R. Hou was funded by China Postdoctoral Science Foundation, grant number BX20200011.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

References

1. Badshah, N.; Akbar, A. Stability analysis of fractional order SEIR model for malaria disease in Khyber Pakhtunkhwa. *Demonstr. Math.* **2021**, *54*, 326–334. [\[CrossRef\]](#)
2. Cherraf, A.; Li, M.; Moulai-Khatir, A. Interaction tumor-immune model with time-delay and immuno-chemotherapy protocol. In *Rendiconti del Circolo Matematico di Palermo Series 2*; Springer: Berlin/Heidelberg, Germany, 2022. [\[CrossRef\]](#)
3. Hoang, M.T. Dynamical analysis of two fractional-order SIQRA malware propagation models and their discretizations. In *Rendiconti del Circolo Matematico di Palermo Series 2*; Springer: Berlin/Heidelberg, Germany, 2022. [\[CrossRef\]](#)
4. Capasso, V.; Maddalena, L. Convergence to equilibrium states for a reaction-diffusion system modelling the spatial spread of a class of bacterial and viral diseases. *J. Math. Biol.* **1981**, *13*, 173–184. [\[CrossRef\]](#) [\[PubMed\]](#)
5. Capasso, V.; Maddalena, L. A nonlinear diffusion system modelling the spread of oro-faecal diseases. In *Nonlinear Phenomena in Mathematical Sciences*; Academic Press: New York, NY, USA, 1982; pp. 207–217.
6. Fife, P. Some nonclassical trends in parabolic and parabolic-like evolutions. In *Trends in Nonlinear Analysis*; Springer: Berlin/Heidelberg, Germany, 2003; pp. 153–191.
7. Hsu, C.-H.; Yang, T.-S. Existence, uniqueness, monotonicity and asymptotic behaviour of travelling waves for epidemic models. *Nonlinearity* **2013**, *26*, 121–139. [\[CrossRef\]](#)
8. Wu, S.-L.; Hsu, C.-H. Existence of entire solutions for delayed monostable epidemic models. *Trans. Am. Math. Soc.* **2016**, *368*, 6033–6062. [\[CrossRef\]](#)

9. Xu, D.; Zhao, X.-Q. Bistable waves in an epidemic model. *J. Dynam. Differ. Equ.* **2004**, *16*, 679–707. [[CrossRef](#)]
10. Zhao, X.-Q.; Wang, W. Fisher waves in an epidemic model. *Discret. Contin. Dyn. Syst. B* **2004**, *4*, 1117–1128. [[CrossRef](#)]
11. Volpert, A.I.; Volpert, V.A.; Volpert, V.A. Traveling Wave Solutions of Parabolic Systems. In *Translations of Mathematical Monographs*; American Mathematical Society: Providence, RI, USA, 1994.
12. Weinberger, H.F. On spreading speeds and traveling waves for growth and migration models in a periodic habitat. *J. Math. Biol.* **2002**, *45*, 511–548. [[CrossRef](#)]
13. Li, W.-T.; Xu, W.-B.; Zhang, L. Traveling waves and entire solutions for an epidemic model with asymmetric dispersal. *Discret. Contin. Dyn. Syst.* **2017**, *37*, 2483–2512. [[CrossRef](#)]
14. Meng, Y.; Yu, Z.; Hsu, C.-H. Entire solutions for a delayed nonlocal dispersal system with monostable nonlinearities. *Nonlinearity* **2019**, *32*, 1206–1236. [[CrossRef](#)]
15. Wang, J.-B.; Li, W.-T.; Sun, J.-W. Global dynamics and spreading speeds for a partially degenerate system with non-local dispersal in periodic habitats. *Proc. R. Soc. Edinb. Sect. A Math.* **2018**, *148*, 849–880. [[CrossRef](#)]
16. Zhang, L.; Li, W.-T.; Wu, S.-L. Multi-type entire solutions in a nonlocal dispersal epidemic model. *J. Dyn. Differ. Equ.* **2016**, *28*, 189–224. [[CrossRef](#)]
17. Bao, X.; Li, W.-T.; Shen, W.; Wang, Z.-C. Spreading speeds and linear determinacy of time dependent diffusive cooperative/competitive systems. *J. Differ. Equ.* **2018**, *265*, 3048–3091. [[CrossRef](#)]
18. Hu, C.; Kuang, Y.; Li, B.; Liu, H. Spreading speeds and traveling wave solutions in cooperative integral-differential systems. *Discret. Contin. Dyn. Syst. B* **2015**, *20*, 1663–1684. [[CrossRef](#)]
19. Xu, W.-B.; Li, W.-T.; Ruan, S. Spatial propagation in an epidemic model with nonlocal diffusion: The influences of initial data and dispersals. *Sci. China Math.* **2020**, *63*, 2177–2206. [[CrossRef](#)]
20. Lischke, A.; Pang, G.; Gulian, M.; Song, F.; Glusa, C.; Zheng, X.; Mao, Z.; Cai, W.; Meerschaert, M.M.; Ainsworth, M.; et al. What is the fractional Laplacian? A comparative review with new results. *J. Comput. Phys.* **2020**, *404*, 109009. [[CrossRef](#)]
21. Pang, H.-K.; Sun, H.-W. A Fast Algorithm for the Variable-Order Spatial Fractional Advection-Diffusion Equation. *J. Sci. Comput.* **2021**, *87*, 15. [[CrossRef](#)]
22. Zhuang, P.; Liu, F.; Anh, V.; Turner, I. Numerical method for the variable-order fractional advection diffusion equation with a nonlinear source term. *SIAM J. Numer. Anal.* **2009**, *47*, 1760–1781. [[CrossRef](#)]
23. Zheng, X.; Wang, H. An error estimate of a numerical approximation to a hidden-memory variable-order space-time fractional diffusion equation. *SIAM J. Numer. Anal.* **2020**, *58*, 2492–2514. [[CrossRef](#)]
24. Cabré, X.; Roquejoffre, J.M. The influence of fractional diffusion in Fisher-KPP equations. *Commun. Math. Phys.* **2013**, *320*, 679–722. [[CrossRef](#)]
25. Coulon, A.C.; Yangari, M. Exponential propagation for fractional reaction-diffusion cooperative systems with fast decaying initial conditions. *J. Dyn. Differ. Equ.* **2017**, *29*, 799–815. [[CrossRef](#)]
26. Felmer, P.; Yangari, M. Fast propagation for fractional KPP equations with slowly decaying initial conditions. *SIAM J. Math. Anal.* **2013**, *45*, 662–678. [[CrossRef](#)]
27. Yagisita, H. Existence and nonexistence of traveling waves for a nonlocal monostable equation. *Publ. Res. Inst. Math. Sci.* **2009**, *45*, 925–953. [[CrossRef](#)]
28. Bates, P.W.; Fife, P.C.; Ren, X.; Wang, X. Traveling waves in a convection model for phase transitions. *Arch. Ration. Mech. Anal.* **1997**, *138*, 105–136. [[CrossRef](#)]
29. Carr, J.; Chmaj, A. Uniqueness of travelling waves for nonlocal monostable equations. *Proc. Am. Math. Soc.* **2004**, *132*, 2433–2439. [[CrossRef](#)]
30. Chen, X. Existence, uniqueness, and asymptotic stability of traveling waves in nonlocal evolution equations. *Adv. Differ. Equ.* **1997**, *2*, 125–160.
31. Chen, X.; Guo, J.-S. Uniqueness and existence of traveling waves for discrete quasilinear monostable dynamics. *Math. Ann.* **2003**, *326*, 123–146. [[CrossRef](#)]
32. Coville, J.; Dávila, J.; Martínez, S. Nonlocal anisotropic dispersal with monostable nonlinearity. *J. Differ. Equ.* **2008**, *244*, 3080–3118. [[CrossRef](#)]
33. Schumacher, K. Travelling-front solutions for integro-differential equations. I. *J. Reine Angew. Math.* **1980**, *316*, 54–70. [[CrossRef](#)]
34. Alfaro, M.; Coville, J. Propagation phenomena in monostable integro-differential equations: Acceleration or not? *J. Differ. Equ.* **2017**, *263*, 5727–5758. [[CrossRef](#)]
35. Finkelshtein, D.; Tkachov, P. Accelerated nonlocal nonsymmetric dispersion for monostable equations on the real line. *Appl. Anal.* **2019**, *98*, 756–780. [[CrossRef](#)]
36. Garnier, J. Accelerating solutions in integro-differential equations. *SIAM J. Math. Anal.* **2011**, *43*, 1955–1974. [[CrossRef](#)]
37. Garnier, J.; Hamel, F.; Roques, L. Transition fronts and stretching phenomena for a general class of reaction-dispersion equations. *Discret. Contin. Dyn. Syst.* **2017**, *37*, 743–756. [[CrossRef](#)]
38. Weinberger, H.F. Long-time behavior of a class of biological models. *SIAM J. Math. Anal.* **1982**, *13*, 353–396. [[CrossRef](#)]
39. Weinberger, H.F.; Lewis, M.A.; Li, B. Analysis of linear determinacy for spread in cooperative models. *J. Math. Biol.* **2002**, *45*, 183–218. [[CrossRef](#)]
40. Ma, S. Traveling wavefronts for delayed reaction-diffusion systems via a fixed point theorem. *J. Differ. Equ.* **2001**, *171*, 294–314. [[CrossRef](#)]

41. Pan, S.; Li, W.-T.; Lin, G. Travelling wave fronts in nonlocal delayed reaction-diffusion systems and applications. *Z. Angew. Math. Phys.* **2009**, *60*, 377–392. [[CrossRef](#)]
42. Xu, W.-B.; Li, W.-T.; Ruan, S. Spatial propagation in nonlocal dispersal Fisher-KPP equations. *J. Funct. Anal.* **2021**, *280*, 108957. [[CrossRef](#)]