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Review

A Survey on Recent Results on Lyapunov-Type Inequalities for Fractional Differential Equations

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Abstract: This survey paper is concerned with some of the most recent results on Lyapunov-type inequalities for fractional boundary value problems involving a variety of fractional derivative operators and boundary conditions. Our work deals with Caputo, Riemann–Liouville, ψ-Caputo, ψ-Hilfer, hybrid, Caputo–Fabrizio, Hadamard, Katugampola, Hilfer–Katugampola, p-Laplacian, and proportional fractional derivative operators.

Keywords: Lyapunov-type inequality; fractional derivative and integral operators; boundary conditions; eigenvalue problem; Green's function

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1. Introduction and Preliminaries

Mathematical inequalities play a key role in investigating the qualitative properties of solutions of differential and integral equations. In particular, the Lyapunov inequality serves as an outstanding mathematical tool to establish many important results of a theoretical and applied nature. For a detailed account of Lyapunov inequalities for differential and difference equations and their applications, we refer the reader to the works presented in [1,2].

Differential and integral equations containing fractional-order derivative and integral operators appear in the mathematical models of several real-world processes and phenomena occurring in a variety of fields such as chemistry, physics, biophysics, blood flow problems, control theory, aerodynamics, electrodynamics, signal and image processing, polymer rheology, economics, etc. The overwhelming popularity of fractional differential equations led to a significant interest in the inequalities associated with these equations.

Recently, in the survey [3], the Lyapunov-type inequalities related to fractional boundary value problems were discussed in detail. The survey in [3] was complemented with [4]. In the present survey, we continue our efforts to collect the most recent results on Lyapunov-type inequalities for fractional boundary value problems appearing in the literature after the publication of the surveys [3,4]. Precisely, a comprehensive and up-to-date review of Lyapunov-type inequalities for boundary value problems involving different kinds of fractional derivative operators and boundary conditions will be outlined.

This article is organised as follows. In Section 2, we collect Lyapunov-type inequalities for Caputo-type fractional boundary value problems. Section 3 is concerned with the inequalities for fractional boundary value problems involving the Riemann–Liouville fractional derivative. In Section 4, we discuss the Lyapunov-type inequalities for a nonlinear nonlocal fractional boundary value problem involving the ψ-Caputo fractional derivative. Results on the Lyapunov- and Hartman–Wintner-type for nonlinear fractional hybrid boundary value problems are given in Section 6. Section 7 deals with Lyapunov-type inequalities for Hadamard fractional boundary value problems. In Section 8, Lyapunov-type inequalities for boundary value problems involving Caputo–Fabrizio fractional derivative are presented. Section 9 is concerned with Lyapunov-type inequalities for Katugampola-type fractional boundary value problems, while Section 10 summarises the results on Lyapunov-type inequalities for fractional boundary value problems involving the Hilfer–Katugampola fractional derivative operator. Section 11 deals with Lyapunov-type inequalities for p-Laplacian operators, while Section 12 contains Lyapunov-type inequalities for boundary value problems with fractional proportional derivatives. We present the results without proofs, but provide a complete reference for the details of each result elaborated in this survey for the convenience of the reader.

2. Lyapunov-Type Inequalities for Caputo-Type Fractional Boundary Value Problems

We first provide some basic definitions [5,6] related to the problems addressed in this section.

Definition 1. *(Riemann–Liouville fractional integral) The fractional integral of the Riemann–Liouville-type of order $\alpha \geq 0$ for a function $f : [a,b] \to \mathbb{R}$ is defined by*

$$(I^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds, \ \ \alpha > 0, \ \ t \in [a,b],$$

provided the right-hand side is point–wise defined on $[0,\infty)$. Here, $\Gamma(\alpha)$ is the Euler Gamma function: $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$ and $(I^0 f)(x) = f(x)$.

Definition 2. *(Caputo fractional derivative) The Caputo fractional derivative of order $\alpha \geq 0$ is defined by $({}^C D^0 f)(t) = f(t)$ and*

$$({}^C D^\alpha f)(t) = (I^{m-\alpha} D^m f)(t) \text{ for } \alpha > 0,$$

where m is the smallest integer greater than or equal to α.

In 2020, Ma and Yang [7] discussed the Lyapunov-type inequality for the following fractional boundary value problem:

$$\begin{cases} D^\alpha_{a+} y(t) + q(t)y(t) = 0, \ \ 0 < t < 1, \ \ 1 < \alpha \leq 2, \\ y(0) = \delta y(1), \ \ y'(0) = \gamma y'(1), \ \ \delta, \gamma \in \mathbb{R}, \end{cases} \tag{1}$$

where D^α_{a+} is the Caputo fractional derivative operator and $q \in L(0,1)$ is not identically zero on any compact subinterval of $(0,1)$.

Lemma 1. *A function y is a solution of the boundary value problem (1) if and only if it satisfies the integral equation:*

$$y(t) = \int_0^1 G(t,s)q(s)y(s)ds,$$

where $G(t,s)$ is Green's function given by

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (1-\alpha)\dfrac{\delta\gamma(1-t)+\gamma t}{(1-\gamma)(1-\delta)}(1-s)^{\alpha-2} - \dfrac{\delta}{1-\delta}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ (1-\alpha)\dfrac{\delta\gamma(1-t)+\gamma t}{(1-\gamma)(1-\delta)}(1-s)^{\alpha-2} - \dfrac{\delta}{1-\delta}(1-s)^{\alpha-1}, & 0 \leq t \leq s \leq 1, \end{cases}$$

Lemma 2. *When $\delta \in (0,1)$ and $\gamma \in (0,1)$, Green's function $G(t,s)$ satisfies the following properties:*

(i) $G(t,s) \leq 0, \ \ (t,s) \in [0,1] \times [0,1]$.

(ii) $\max_{0\le t\le 1}|G(t,s)| = -G(1,s) = \dfrac{(1-s)^{\alpha-2}}{\Gamma(\alpha)(1-\delta)(1-\gamma)}\big[\gamma(\alpha-1)+(1-\gamma)(1-s)\big]$, *for* $s\in[0,1]$.

(iii) $\int_0^1 |G(t,s)|ds \le \dfrac{\gamma(\alpha-1)+1}{\Gamma(\alpha+1)(1-\delta)(1-\gamma)}$.

Lemma 3. *When* $\delta\in(1,\infty)$ *and* $\gamma\in(0,1)$, *Green's function* $G(t,s)$ *satisfies the following properties:*

(i) $G(t,s)\ge 0,\ \ (t,s)\in[0,1]\times[0,1]$.

(ii) $\max_{0\le t\le 1}|G(t,s)| = G(0,s) = \dfrac{\delta(1-s)^{\alpha-2}}{\Gamma(\alpha)(\delta-1)(1-\gamma)}\big[\gamma(\alpha-1)+(1-\gamma)(1-s)\big]$, *for* $s\in[0,1]$.

(iii) $\int_0^1 |G(t,s)|ds \le \dfrac{\delta(\gamma\alpha+1-\gamma)}{\Gamma(\alpha+1)(\delta-1)(1-\gamma)}$.

Lemma 4. *When* $\delta\in(0,1)$ *and* $\gamma\in\Big(1,1+\dfrac{(\alpha-1)\delta}{2-\alpha}\Big)$, *Green's function* $G(t,s)$ *satisfies the following properties:*

(i) $G(t,s)\ge 0,\ \ (t,s)\in[0,1]\times[0,1]$.

(ii) $\max_{0\le t\le 1}|G(t,s)| = G(1,s) = \dfrac{(1-s)^{\alpha-2}}{\Gamma(\alpha)(1-\delta)(\gamma-1)}\big[\gamma(\alpha-1)+(1-\gamma)(1-s)\big]$, *for* $s\in[0,1]$.

(iii) $\int_0^1 |G(t,s)|ds \le \dfrac{\gamma(\alpha-1)+1)}{\Gamma(\alpha+1)(1-\delta)(\gamma-1)}$.

Lemma 5. *When* $\delta\in(1,\infty)$ *and* $\gamma\in\Big(1,\dfrac{1}{2-\alpha}\Big]$, *Green's function* $G(t,s)$ *satisfies the following properties:*

(i) $G(t,s)\le 0,\ \ (t,s)\in[0,1]\times[0,1]$.

(ii) *For* $s\in[0,1]$, $\max_{0\le t\le 1}|G(t,s)| \le \dfrac{(1-s)^{\alpha-2}}{\Gamma(\alpha)(\delta-1)(\gamma-1)}$
$$\times\Big\{\delta\gamma(\alpha-1)-\Big[\delta(\gamma-1)-\gamma(2-\alpha)(\delta-1)\Big(\frac{\gamma-1}{\gamma}\Big)^{1/(2-\alpha)}\Big](1-s)\Big\}.$$

(iii) $\int_0^1 |G(t,s)|ds \le \dfrac{1}{\Gamma(\alpha+1)(\delta-1)(\gamma-1)}$
$$\times\Big\{\delta[1+\gamma(\alpha-1)]+(\delta-1)\gamma(2-\alpha)\Big(\frac{\gamma-1}{\gamma}\Big)^{1/(2-\alpha)}\Big\}.$$

In the next theorems, we present the Lyapunov-type inequalities.

Theorem 1. *Suppose that the problem (1) has a nonzero solution* y :

(i) *If* $\delta\in(0,1)$ *and* $\gamma\in(0,1)$, *then*
$$\int_0^1 (1-s)^{\alpha-2}\Big[\gamma(\alpha-1)+(1-\gamma)(1-s)\Big]|q(s)|ds > \Gamma(\alpha)(1-\delta)(1-\gamma).$$

(ii) *If* $\delta\in(1,\infty)$ *and* $\gamma\in(0,1)$, *then*
$$\int_0^1 (1-s)^{\alpha-2}\Big[\gamma(\alpha-1)+(1-\gamma)(1-s)\Big]|q(s)|ds > \frac{\Gamma(\alpha)(\delta-1)(1-\gamma)}{\delta}.$$

(iii) *If* $\delta\in(0,1)$ *and* $\gamma\in\Big(1,1+\dfrac{(\alpha-1)\delta}{2-\alpha}\Big]$, *then*
$$\int_0^1 (1-s)^{\alpha-2}\Big[\gamma(\alpha-1)+(1-\gamma)(1-s)\Big]|q(s)|ds > \Gamma(\alpha)(1-\delta)(\gamma-1).$$

(iv) *If* $\delta \in (1,\infty)$ *and* $\gamma \in \left(1, \dfrac{1}{2-\alpha}\right]$, *then*
$\int_0^1 (1-s)^{\alpha-2}\Big\{\delta\gamma(\alpha-1) - \Big[\delta(\gamma-1) - \gamma(\delta-1)\Big(\dfrac{\gamma-1}{\gamma}\Big)^{1/(2-\alpha)}\Big](1-s)\Big\}|q(s)|ds$
$> \Gamma(\alpha)(\delta-1)(\gamma-1).$

Theorem 2. *(i) When* $\delta \in (0,1)$ *and* $\gamma \in (0,1)$, *if*

$$\int_0^1 (1-s)^{\alpha-2}\Big[\gamma(\alpha-1) + (1-\gamma)(1-s)\Big]|q(s)|ds \le \Gamma(\alpha)(1-\delta)(1-\gamma),$$

then the problem (1) has no nonzero solution.

(ii) *When* $\delta \in (1,\infty)$ *and* $\gamma \in (0,1)$, *if*

$$\int_0^1 (1-s)^{\alpha-2}\Big[\gamma(\alpha-1) + (1-\gamma)(1-s)\Big]|q(s)|ds \le \frac{\Gamma(\alpha)(\delta-1)(1-\gamma)}{\delta},$$

then the problem (1) has no nonzero solution.

(iii) *When* $\delta \in (0,1)$ *and* $\gamma \in \left(1, 1+\frac{(\alpha-1)\delta}{2-\alpha}\right]$, *if*

$$\int_0^1 (1-s)^{\alpha-2}\Big[\gamma(\alpha-1) + (1-\gamma)(1-s)\Big]|q(s)|ds \le \Gamma(\alpha)(1-\delta)(\gamma-1),$$

then the problem (1) has no nonzero solution.

(iv) *When* $\delta \in (1,\infty)$ *and* $\gamma \in \left(1, \frac{1}{2-\alpha}\right]$, *if*
$\int_0^1 (1-s)^{\alpha-2}\Big\{\delta\gamma(\alpha-1) - \Big[\delta(\gamma-1) - \gamma(\delta-1)\Big(\dfrac{\gamma-1}{\gamma}\Big)^{1/(2-\alpha)}\Big](1-s)\Big\}|q(s)|ds$
$\le \Gamma(\alpha)(\delta-1)(\gamma-1),$
then the problem (1) has no nonzero solution.

In 2021, Pourhandi et al. [8] considered the following fractional boundary value problem:

$$\begin{cases} ({}_a^C D^\alpha y)(t) + p(t)y'(t) + q(t)y(t) = 0, \quad a < t < b,\ 2 < \alpha \le 3, \\ y(a) = y'(a) = y(b) = 0, \end{cases} \tag{2}$$

where ${}_a^C D^\alpha y$ is the Caputo fractional derivative, $p \in C^1([a,b],\mathbb{R})$ and $q \in C([a,b],\mathbb{R})$.

Lemma 6. $y \in C^1([a,b],\mathbb{R})$ *is a solution of the fractional boundary value problem (2) if and only if y satisfies the following integral equation:*

$$y(t) = \int_a^b G(s,t)H(s)y(s)ds,$$

where

$$G = [G_1, G_2, G_3], \quad H = \begin{bmatrix} p \\ q \\ p' \end{bmatrix}$$

and

$$G(s,t) = \begin{bmatrix} G_1(s,t) \\ G_2(s,t) \\ G_3(s,t) \end{bmatrix} = \frac{1}{\Gamma(\alpha)} \begin{bmatrix} \{g_{11}(s,t), g_{12}(s,t)\} \\ \{g_{21}(s,t), g_{22}(s,t)\} \\ \{g_{21}(s,t), g_{32}(s,t)\} \end{bmatrix}$$

$$= \frac{1}{\Gamma(\alpha)} \begin{bmatrix} \begin{array}{ll} a \le s \le t \le b, & a \le t \le s \le b \\ \left\{ (\alpha-1)\left(\frac{(t-a)^2}{(b-a)^2}(b-s)^{\alpha-2} - (t-s)^{\alpha-2}\right), \right. & \left. \frac{(\alpha-1)(t-a)^2}{(b-a)^2}(b-s)^{\alpha-2} \right\} \\ \left\{ \frac{(t-a)^2}{(b-a)^2}(b-s)^{\alpha-1} - (t-s)^{\alpha-1}, \right. & \left. \frac{(t-a)^2}{(b-a)^2}(b-s)^{\alpha-1} \right\} \\ \left\{ \frac{-(t-a)^2}{(b-a)^2}(b-s)^{\alpha-1} + (t-s)^{\alpha-1}, \right. & \left. -\frac{(t-a)^2}{(b-a)^2}(b-s)^{\alpha-1} \right\} \end{array} \end{bmatrix}.$$

Moreover, all functions $G_i, i = 1,2,3$ satisfy the following inequalities:

$$|G_1(s,t)| \le \frac{1}{\Gamma(\alpha)}(\alpha-1)(b-a)^{\alpha-2}\max\{g(\alpha),h(\alpha)\}$$

where

$$\max\{g(\alpha),h(\alpha)\} = \begin{cases} g(\alpha), & 2 \le \alpha \le \alpha_0, \\ h(\alpha), & \alpha_0 \le \alpha \le 3, \end{cases} \qquad (g-h)(\alpha_0) = 0, \ \ \alpha_0 \approx 2.427$$

and

$$|G_2(s,t)| = |G_3(s,t)| \le \frac{1}{\Gamma(\alpha)}(b-a)^{\alpha-1}\max\{g(\alpha+1),h(\alpha+1),A(\alpha+1)\},$$

where

$$\begin{aligned} g(\alpha) &= \frac{1}{4}(4-\alpha)^2, \\ h(\alpha) &= \left(\frac{\alpha-2}{2}\right)^{\frac{(\alpha-2)(3-\alpha)}{4-\alpha}} - \left(\frac{\alpha-2}{2}\right)^{\frac{2-(\alpha-2)^2}{4-\alpha}}, \\ A(\alpha) &= 4\alpha^{-\alpha}(\alpha-2)^{\alpha-2}. \end{aligned}$$

In the next theorem, we state the Lyapunov-type inequalities for the boundary value problem (2).

Theorem 3. *If a nontrivial continuously differentiable solution of the boundary value problem (2) exists, then*

$$\int_a^b \Big(|p(s)| + |q(s)| + |p'(s)|\Big)ds \ge \frac{\Gamma(\alpha)(b-a)^{1-\alpha}}{\max\{g(\alpha),h(\alpha),A(\alpha+1)\}}$$

if $\alpha \le b-a+1$ and

$$\int_a^b \Big(|p(s)| + |q(s)| + |p'(s)|\Big)ds \ge \frac{\Gamma(\alpha)(b-a)^{2-\alpha}}{(\alpha-1)\max\{g(\alpha),h(\alpha),A(\alpha+1)\}}$$

if $\alpha \ge b-a+1$, and $g,h,$ and A are given in Lemma 6.

3. Lyapunov-Type Inequalities for Fractional Boundary Value Problems Involving Riemann–Liouville Fractional Derivative

Definition 3. *(Riemann–Liouville fractional derivative) The Riemann–Liouville fractional derivative of order $\alpha \ge 0$ is defined by $(D^0 f)(t) = f(t)$ and*

$$(D^\alpha f)(t) = (D^m\, I^{m-\alpha} f)(t) \text{ for } \alpha > 0,$$

where m is the smallest integer greater than or equal to α.

In 2019, Pathak [9] established Hartman-type and Lyapunov-type inequalities for the following problem:

$$\begin{cases} D^{\alpha}_{a+}y(t) + q(t)y(t) = 0, \ \ a < t < b, \\ y(a) = 0, y'(a) = 0, y'(b) = 0, \end{cases} \tag{3}$$

where D^{α}_{a+} is the Riemann–Liouville fractional derivative operator of order α, $\alpha \in (2,3]$ and $q \in C([a,b],\mathbb{R})$.

Lemma 7. *Let $y \in C([a,b],\mathbb{R})$ be a solution of the boundary value problem (3). Then,*

$$y(t) = \int_a^b G(t,s)q(s)y(s)ds, \tag{4}$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \dfrac{(t-a)^{\alpha-1}(b-s)^{\alpha-2}}{(b-a)^{\alpha-2}} - (t-s)^{\alpha-1}, & a \le s \le t \le b, \\ \dfrac{(t-a)^{\alpha-1}(b-s)^{\alpha-2}}{(b-a)^{\alpha-2}}, & a \le t \le s \le b, \end{cases}$$

which satisfies the following properties:

(i) $G(t,s) \ge 0, \ \text{for} \ \ (t,s) \in [a,b] \times [a,b]$.

(ii) $\max\limits_{t\in[a,b]} G(t,s) = G(b,s) = \dfrac{(b-s)^{\alpha-2}(s-a)}{\Gamma(\alpha)}, \ \ s \in [a,b]$.

The next theorem contains the Hartman–Winter-type inequality.

Theorem 4. *Let $q \in C([a,b],\mathbb{R})$. Assume that the boundary value problem (3) has a solution $y \in C([a,b],\mathbb{R})$ such that $y(t) \neq 0$, for $t \in (a,b)$. Then,*

$$\int_a^b (s-a)(b-s)^{\alpha-2}|q(s)|ds \ge \Gamma(\alpha).$$

We now present a Lyapunov-type inequality.

Theorem 5. *Under the same assumptions as in Theorem 4, we have*

$$\int_a^b |q(s)|ds \ge \frac{\Gamma(\alpha)(\alpha-1)^{\alpha-2}}{(b-a)^{\alpha-1}(\alpha-2)^{\alpha-2}}.$$

In 2020, Bachar and Eltayeb [10] established Hartman-type and Lyapunov-type inequalities for the following problem:

$$\begin{cases} D^{\alpha}_{a+}y(t) - q(t)y(t) = 0, \ \ a < t < b, \\ y(a) = D^{\alpha-3}_{a+}y(a) = D^{\alpha-2}_{a+}y(a) = y''(b) = 0, \end{cases} \tag{5}$$

where D^{α}_{a+} is the Riemann–Liouville fractional of order α, $\alpha \in (3,4]$ and $q \in C([a,b],\mathbb{R})$.

Lemma 8. *Let $y \in C([a,b],\mathbb{R})$ be a solution of the boundary value problem (5). Then,*

$$y(t) = \int_a^b G(t,s)q(s)y(s)ds, \tag{6}$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \left(\frac{b-s}{b-a}\right)^{\alpha-3}(t-a)^{\alpha-1} - (t-s)^{\alpha-1}, & a \le s \le t \le b, \\ \left(\frac{b-s}{b-a}\right)^{\alpha-3}(t-a)^{\alpha-1}, & a \le t \le s \le b, \end{cases}$$

and satisfies the following property:

$$0 \le G(t,s) \le G(b,s), \ \textit{for} \ (t,s) \in [a,b] \times [a,b].$$

The next theorem contains the Hartman–Winter-type inequality.

Theorem 6. *Let $q \in C([a,b],\mathbb{R})$. Assume that the boundary value problem (5) has a solution $y \in C([a,b],\mathbb{R})$ such that $y(t) \neq 0$, for $t \in (a,b)$. Then,*

$$\int_a^b (b-z)^{\alpha-3}(2b-a-z)q^+(z)dz \ge \Gamma(\alpha),$$

where $q^+(z) = \max(q(z),0)$.

We now present a Lyapunov-type inequality.

Theorem 7. *Under the same assumptions as in Theorem 6, we have*

$$\int_a^b q^+(z)dz \ge \frac{\Gamma(\alpha)(\alpha-1)^{\alpha-2}}{2(b-a)^{\alpha-1}\left(\sqrt{(\alpha-1)(\alpha-3)}\right)^{\alpha-3}}.$$

As an application, we give the lower bound for the eigenvalue problem:

$$\begin{cases} D^{\alpha}_{a+}y(t) + \lambda y(t) = 0, \ 0 < t < 1, \ 3 < \alpha \le 4, \\ y(0) = D^{\alpha-3}_{0+}y(0) = D^{\alpha-2}_{0+}y(0) = y''(1) = 0. \end{cases} \tag{7}$$

Corollary 1. *Assume that the eigenvalue problem (7) has a solution $y \in C([a,b],\mathbb{R})$ such that $y(t) \neq 0$, for $t \in (a,b)$. Then,*

$$|\lambda| \ge \frac{(\alpha-2)\Gamma(\alpha+1)}{2}.$$

In 2021, Jonnalagadda and Basua [11] obtained a Lyapunov-type inequality for the following anti-periodic fractional boundary value problem:

$$\begin{cases} (D^{\alpha}_0 y)(t) + q(t)y(t) = 0, \ 0 < t < T, \\ (I^{2-\alpha}_0 y)(0) + (I^{2-\alpha}_0 y)(T) = 0, \ (D^{\alpha-1}_0 y)(0) + (D^{\alpha-1}_0 y)(T) = 0, \end{cases} \tag{8}$$

where $D^{\alpha}_0 y$ is the Riemann–Liouville fractional derivative of order $\alpha \in (1,2]$, $I^{\alpha}_0 y$ is the Riemann–Liouville fractional integral, and $q : [0,T] \to \mathbb{R}$ is a continuous function.

Lemma 9. *The fractional boundary value problem (8) has the unique solution*

$$y(t) = \int_0^T G(t,s)q(s)y(s)ds, \ 0 < t < T,$$

where $G(t,s)$ is the Green function given by

$$G(t,s) = \begin{cases} \frac{t^{\alpha-1}}{2\Gamma(\alpha)} + \frac{t^{\alpha-2}(T-2s)}{4\Gamma(\alpha-1)}, & 0 < t \le s \le T, \\ \frac{t^{\alpha-1}}{2\Gamma(\alpha)} + \frac{t^{\alpha-2}(T-2s)}{4\Gamma(\alpha-1)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 < s \le t \le T. \end{cases}$$

Moreover, the Green function $G(t,s)$ satisfies the inequality:

$$|t^{2-\alpha}G(t,s)| \leq \frac{(3-\alpha)T}{4\Gamma(\alpha)}, \ \forall (t,s) \in [0,T] \times [0,T].$$

We are now able to formulate a Lyapunov-type inequality for the problem (8).

Theorem 8. *If the fractional boundary value problem (8) has a nontrivial solution, then*

$$\int_o^T s^{\alpha-2}|q(s)|ds > \frac{4\Gamma(\alpha)}{(3-\alpha)T}.$$

As an application, we give a lower bound for a fractional eigenvalue problem.

Corollary 2. *Assume that y is a nontrivial solution of the fractional eigenvalue problem:*

$$\begin{cases} (D_0^\alpha y)(t) + \lambda y(t) = 0, 0 < t < T, \\ (I_0^{2-\alpha} y)(0) + (I_0^{2-\alpha} y)(T) = 0, \ (D_0^{\alpha-1} y)(0) + (D_0^{\alpha-1} y)(T) = 0, \end{cases}$$

where $y(t) \neq 0$, for each $t \in (0,T)$. Then,

$$|\lambda| > \frac{4\Gamma(\alpha)(\alpha-1)}{T^\alpha(3-\alpha)}, \ 1 < \alpha \leq 2.$$

In 2021, Zhu et al. [12] obtained a Lyapunov-type inequality for the following m-point fractional boundary value problem:

$$\begin{cases} (D_{0+}^\alpha\big(y''(t)\big) = q(t)f(y(t)), \ 0 < t < 1, \\ y'(0) = y''(0) = y(1) = 0, \ y''(1) - \sum\limits_{i=1}^{m-2} \theta_i y'''(\xi_i) = 0 \end{cases} \tag{9}$$

where $D_0^\alpha y$ is the Riemann–Liouville fractional derivative of order α, $1 < \alpha \leq 2$, $q : [0,1] \to \mathbb{R}$ is a Lebesgue integrable function, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Assume that: $(H_1) \sum\limits_{i=1}^{m-2} \theta_i \xi_i^{\alpha-2} > \dfrac{1}{\alpha-1}$ and $\sum\limits_{i=2}^{m-2} \theta_i \leq \dfrac{(1-\xi_{m-2})^{\alpha-1}}{(\alpha-1)\xi_{m-2}^{\alpha-2}((1-\xi_{m-2})^{\alpha-1}/(\alpha-1)\xi_{m-2}^{\alpha-2})}$.

Lemma 10. *Assume that (H_1) holds. Then, for any $u \in L^1[0,1]$, the fractional boundary value problem:*

$$\begin{cases} (D_{0+}^\alpha\big(y''(t)\big) = u(t), \ 0 < t < 1, \\ y'(0) = y''(0) = y(1) = 0, \ y''(1) - \sum_{i=1}^{m-2} \theta_i y'''(\xi_1) = 0 \end{cases} \tag{10}$$

has the unique solution

$$y(t) = \int_0^1 G(t,s)u(s)ds, \ 0 < t < 1,$$

where $G(t,s)$ is the Green function given by

$$G(t,s)=\frac{1}{\Gamma(\alpha+2)}\begin{cases}(t-s)^{\alpha+1}-(1-s)^{\alpha+1}\\ \quad +1-t^{\alpha+1}-\dfrac{(1-t^{\alpha+1})[1-(1-s)^{\alpha-1}]}{1-(\alpha-1)\sum_{i=1}^{m-2}\theta_i\xi_i^{\alpha-2}}, & s\le t,\ s\le\xi_1,\\ -(1-s)^{\alpha+1}+1-t^{\alpha+1}-\dfrac{(1-t^{\alpha+1})[1-(1-s)^{\alpha-1}]}{1-(\alpha-1)\sum_{i=1}^{m-2}\theta_i\xi_i^{\alpha-2}}, & t\le s\le\xi_i\\ \dfrac{1-t^{\alpha+1}}{1-(\alpha-1)\sum_{i=1}^{m-2}\theta_i\xi_i^{\alpha-2}}\Big[(1-s)^{\alpha-1}\\ \quad -(\alpha-1)\sum_{i=j+1}^{m-2}\theta_i(\xi_i-s)^{\alpha-2}\Big]\\ \quad +(t-s)^{\alpha+1}-(1-s)^{\alpha+1}, & s\le t,\ \xi_j\le s\le\xi_{j+1},\\ & j=1,2,\dots,m-3,\\ \dfrac{1+t^{\alpha+1}}{1-(\alpha-1)\sum_{i=1}^{m-2}\theta_i\xi_i^{\alpha-2}}\Big[(1-s)^{\alpha-1}\\ \quad -(\alpha-1)\sum_{i=j+1}^{m-2}\theta_i(\xi_i-s)^{\alpha-2}\Big]-(1-s)^{\alpha+1}, & t\le s,\ \xi_j\le s\le\xi_{j+1},\\ & j=1,2,\dots,m-3,\\ (t-s)^{\alpha+1}-(1-s)^{\alpha+1}+\dfrac{(1-t^{\alpha+1}(1-s)^{\alpha-1}}{1-(\alpha-1)\sum_{i=1}^{m-2}\theta_i\xi_i^{\alpha-2}}, & \xi_{m-2}\le s\le t,\\ -(1-s)^{\alpha+1}+\dfrac{(1-t^{\alpha+1}(1-s)^{\alpha-1}}{1-(\alpha-1)\sum_{i=1}^{m-2}\theta_i\xi_i^{\alpha-2}}, & \xi_{m-2}\le s,\ t\le s.\end{cases}$$

Moreover, the Green function $G(t,s)$ satisfies the following properties:

(i) $G(t,s)\ge 0,$ *for* $0\le s\le\xi_1.$

(ii) $G(t,s)\le 0,$ *for* $\xi_1<s\le 1.$

(iii) $0\le G(t,s)<\dfrac{1}{\Gamma(\alpha+2)}\Big[1-(1-s)^{\alpha+1}-\dfrac{1-(1-s)^{\alpha-1}}{1-(\alpha-1)\sum_{i=1}^{m-2}\theta_i\xi_i^{\alpha-2}}\Big],$ *for* $0\le s\le\xi_i.$

(iv) $$\frac{1}{\Gamma(\alpha+2)}\Big[\frac{(1-s)^{\alpha-1}-(\alpha-1)\sum_{i=j+1}^{m-2}\theta_i(\xi_i-s)^{\alpha-2}}{1-(\alpha-1)\sum_{i=1}^{m-2}\theta_i\xi_i^{\alpha-2}}-(1-s)^{\alpha+1}\Big]\le G(t,s)\le 0,\ \text{for}\ \xi_j\le s\le\xi_{j+1},\ j=1,2,\dots,m-3.$$

(v) $$\frac{1}{\Gamma(\alpha+2)}\Big[\frac{(1-s)^{\alpha-1}}{1-(\alpha-1)\sum_{i=1}^{m-2}\theta_i\xi_i^{\alpha-2}}-(1-s)^{\alpha+1}\Big]\le G(t,s)\le 0,\ \text{for}\ \xi_{m-2}\le s\le 1.$$

The Lyapunov-type inequality for the m-point fractional boundary value problem (9) is the following:

Theorem 9. *Assume that (H_1) holds. In addition, we suppose that f is a convex function on $\mathbb{R}$. Then, for any nontrivial solution of the problem (9), we have*

$$\int_0^1|q(s)|ds>\frac{\|y\|}{\bar{G}\max_{y\in[y_*,y^*]}|f(y)|},$$

where $y_*=\min_{t\in[0,1]}y(t),\ \ y^*=\max_{t\in[0,1]}y(t)$ *and*

$$\bar{G}=\max\Bigg\{\max_{s\in[0,\xi_1]}\Big(\frac{1}{\Gamma(\alpha+2)}\Big[1-(1-s)^{\alpha+1}-\frac{1-(1-s)^{\alpha-1}}{1-(\alpha-1)\sum_{i=1}^{m-2}\theta_i\xi_i^{\alpha-2}}\Big]\Big),$$
$$\max_{s\in[\xi_j,\xi_{j+1}],1\le j\le m-3}\Big(\frac{1}{\Gamma(\alpha+2)}\Big[\frac{(1-s)^{\alpha-1}-(\alpha-1)\sum_{i=j+1}^{m-2}\theta_i(\xi_i-s)^{\alpha-2}}{1-(\alpha-1)\sum_{i=1}^{m-2}\theta_i\xi_i^{\alpha-2}}-(1-s)^{\alpha+1}\Big]\Big),$$
$$\max_{s\in[\xi_{m-2},1]}\Big(\frac{1}{\Gamma(\alpha+2)}\Big[\frac{(1-s)^{\alpha-1}}{1-(\alpha-1)\sum_{i=1}^{m-2}\theta_i\xi_i^{\alpha-2}}-(1-s)^{\alpha+1}\Big]\Big)\Bigg\}.$$

4. Lyapunov-Type Inequalities for Nonlinear Nonlocal Fractional Boundary Value Problems with ψ-Caputo Fractional Derivative

In [13], Dien studied the sequential nonlocal fractional boundary value problem involving the generalised ψ-Caputo fractional derivative:

$$\begin{cases} ({}^{C}D^{\alpha,\psi}_{a^+}\, {}^{C}D^{\beta,\psi}_{a^+} x)(t) + f(t, x(t)) = 0, & a < t < b, \\ x(a) = 0, \quad x(b) = g(x), \end{cases} \tag{11}$$

where ${}^{C}D^{\alpha,\psi}_{a^+}, {}^{C}D^{\beta,\psi}_{a^+}$ denote the left ψ-Caputo fractional derivatives, $0 < \alpha, \beta \leq 1$ with $\alpha + \beta > 1$ and $\psi \in C^1([a, b], \mathbb{R})$ with $\psi'(t) > 0$, for all $t \in [a, b]$.

We recall that for $\alpha > 0$, $\psi \in C^1([a, b], \mathbb{R})$ with $\psi'(t) > 0$, for all $t \in [a, b]$, and $x \in L^1([a, b], \mathbb{R})$, the left fractional integral of a function x depending on another function ψ is given by

$$I^{\alpha,\psi}_{a^+} x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} x(s) ds.$$

For $x, \psi \in C^n([a, b], \mathbb{R})$ with $\psi'(t) > 0$, for all $t \in [a, b]$, we define the left-side ψ-Caputo fractional derivative of order α of x as

$${}^{C}D^{\alpha,\psi}_{a^+} x(t) = I^{n-\alpha,\psi}_{a^+} \Big(\frac{1}{\psi'(t)} \frac{d}{dt}\Big)^n x(t), \ \alpha \in (n-1, n].$$

Lemma 11. *Let $\psi \in C^1([a, b], \mathbb{R})$ with $\psi'(t) > 0$, for all $t \in [a, b]$. If x is a solution of the problem (11) such that ${}^{C}D^{\beta,\psi}_{a^+} x \in C^1([a, b], \mathbb{R})$, then x is a solution of the following integral equation:*

$$x(t) = \Big(\frac{\psi(t) - \psi(a)}{\psi(b) - \psi(a)}\Big)^\beta g(x) + \int_a^b G(s, t)\psi'(s) f(s, x(s)) ds \tag{12}$$

where $0 < \alpha, \beta \leq 1$ with $\alpha + \beta > 1$ and

$$G(s, t) = \begin{cases} \dfrac{1}{\Gamma(\alpha+\beta)(\psi(b) - \psi(a))^\beta} (\psi(t) - \psi(a))^\beta (\psi(b) - \psi(s))^{\alpha+\beta-1} & \\ -\dfrac{1}{\Gamma(\alpha+\beta)} (\psi(t) - \psi(s))^{\alpha+\beta-1}, & a \leq s \leq t \leq b, \\ \dfrac{1}{\Gamma(\alpha+\beta)(\psi(b) - \psi(a))^\beta} (\psi(t) - \psi(a))^\beta (\psi(b) - \psi(s))^{\alpha+\beta-1}, & a \leq t \leq s \leq b. \end{cases}$$

Moreover, we have

$$\begin{aligned} G_{\max} : &= \max_{a \leq s, t \leq b} |G(s, t)| = \frac{1}{\Gamma(\alpha+\beta)} (\psi(b) - \psi(a))^{\alpha+\beta-1} \\ &\times \max\left\{ \beta^\beta (\alpha+\beta-1)^{\alpha+\beta-1}, \frac{\beta(\alpha+\beta-1)^{\frac{\alpha+\beta-1}{\beta}}}{(\alpha+2\beta-1)^{\frac{\alpha+2\beta-1}{\beta}}} \right\}. \end{aligned} \tag{13}$$

The next theorem contains the Lyapunov-type inequality for the problem (11).

Theorem 10. *Let $0 < \alpha, \beta \leq 1$ with $\alpha + \beta > 1$ and $\psi \in C^1([a, b], \mathbb{R})$ with $\psi'(t) > 0$, for all $t \in [a, b]$. Suppose that:*

(A_1) *There exist a function $q : (a, b) \to \mathbb{R}^+$ and a nondecreasing and concave function $\phi : \mathbb{R} \to \mathbb{R}^+$ such that*

$$|f(t, x)| \leq q(t)\phi(|x|), \ \textit{for all} \ \ t \in (a, b) \ \ \textit{and} \ \ x \in \mathbb{R}.$$

(A_2) *There exists $\kappa \in (0, 1)$ such that*

$$|g(x)| \leq \kappa |x|, \ \textit{for all} \ \ x \in \mathbb{R}.$$

If $\psi'(\cdot)q(\cdot) \in L^1((a,b),\mathbb{R})$ and the problem (11) has a nontrivial solution, then

$$\int_a^b \psi'(s)q(s)ds \geq \frac{1-\kappa}{G_{\max}} \frac{\|x\|}{\phi(\|x\|)}, \tag{14}$$

where $G_{\max}$ is given by (13).

The Lyapunov-type inequalities for the problem (11) immediately follow from Theorem 10, which are expressed in the following corollaries.

Corollary 3. *Let $0 < \alpha, \beta \leq 1$ with $\alpha + \beta > 1$. Suppose that (A_2) holds and that there exists $q : (a,b) \to \mathbb{R}$ such that*

$$f(t,x) = q(t)x, \text{ for all } t \in (a,b).$$

For $\psi(t) = t$, if $q(\cdot) \in L^1((a,b),\mathbb{R})$ and the problem (11) has a nontrivial solution, then

$$\int_a^b |q(s)|ds \geq \frac{1-\kappa}{G_1},$$

where

$$G_1 = \frac{1}{\Gamma(\alpha+\beta)}(b-a)^{\alpha+\beta-1} \max\left\{\beta^\beta(\alpha+\beta-1)^{\alpha+\beta-1}, \frac{\beta(\alpha+\beta-1)^{\frac{\alpha+\beta-1}{\beta}}}{(\alpha+2\beta-1)^{\frac{\alpha+2\beta-1}{\beta}}}\right\}.$$

Corollary 4. *Let $0 < \alpha, \beta \leq 1$ with $\alpha + \beta > 1$. Suppose that (A_2) holds and that there exists $q : (a,b) \to \mathbb{R}$ such that*

$$f(t,x) = q(t)x, \text{ for all } t \in (a,b).$$

For $\psi(t) = \log t$, if $\int_a^b \frac{|q(s)|}{s}ds < +\infty$ and the problem (11) has a nontrivial solution, then

$$\int_a^b \frac{|q(s)|}{s}ds \geq \frac{1-\kappa}{G_2},$$

where

$$G_1 = \frac{1}{\Gamma(\alpha+\beta)}\left(\log\frac{b}{a}\right)^{\alpha+\beta-1} \max\left\{\beta^\beta(\alpha+\beta-1)^{\alpha+\beta-1}, \frac{\beta(\alpha+\beta-1)^{\frac{\alpha+\beta-1}{\beta}}}{(\alpha+2\beta-1)^{\frac{\alpha+2\beta-1}{\beta}}}\right\}.$$

The following corollary, based on the foregoing Lyapunov-type inequality, provides a lower bound for the eigenvalue of a Dirichlet boundary value problem involving sequential fractional derivative operators.

Corollary 5. *Let $0 < \alpha, \beta \leq 1$ with $\alpha + \beta > 1$ and $\psi \in C^1([a,b],\mathbb{R})$ with $\psi'(t) > 0$, for all $t \in [a,b]$. Suppose that (A_2) holds. Suppose further that λ is an eigenvalue of the following problem:*

$$\begin{cases} ({}^C D_{a^+}^{\alpha,\psi}\, {}^C D_{a^+}^{\beta,\psi} x)(t) = \lambda x(t), & a < t < b, \\ x(a) = 0, \quad x(b) = g(x). \end{cases}$$

Then

$$|\lambda| > \frac{1-\kappa}{G_{\max}(\psi(b)-\psi(a))},$$

where $G_{\max}$ is given by (13).

5. Lyapunov- and Hartman–Wintner-Type Inequalities for ψ-Hilfer Fractional Boundary Value Problems

In 2020, Zohra et al. [14] derived Lyapunov and Hartman–Wintner-type inequalities for the fractional boundary value problem:

$$\begin{cases} {}^{H}D_a^{\alpha,\beta,\psi}y(t) + q(t)f(y(t)) = 0, \ \ a < t < b, \, a, b \in \mathbb{R}, \\ y(a) = y(b) = 0, \end{cases} \tag{15}$$

where ${}^{H}D_a^{\alpha,\beta,\psi}$ is the ψ-Hilfer fractional derivative type of order $1 < \alpha < 2$, $0 \leq \beta \leq 1$, $y, \psi \in C^2([a,b],\mathbb{R})$ is such that ψ is strictly increasing, and $f : \mathbb{R} \to \mathbb{R}$, $G : [a,b] \to \mathbb{R}$ are given functions.

Definition 4. *Let $h : [a,b] \to \mathbb{R}$ be a function such that ${}^{RL}I_{a+}^{n-\alpha}h \in AC^n([a,b],\mathbb{R})$. Then, the Hilfer fractional derivative $D^{\alpha,\beta}$ of order $\alpha > 0$ and type $\beta \in [0,1]$ of the function h, existing almost everywhere on $[a,b]$, is defined by*

$$\begin{aligned} {}^{H}D^{\alpha,\beta}h(t) &= {}^{RL}I^{\beta(n-\alpha)}\left(\frac{d}{dt}\right)^n {}^{RL}I^{(n-\alpha)(1-\beta)}h(t) \\ &= {}^{RL}I^{\beta(n-\alpha)}\left(\frac{d}{dt}\right)^n {}^{RL}I^{n-\gamma}h(t) \\ &= {}^{RL}I^{\beta(n-\alpha)}D_{a+}^{\gamma}h(t), \end{aligned}$$

where n is the smallest integer greater than or equal to α and $\gamma = \alpha + \beta(n-\alpha)$.

One can notice that the Hilfer fractional derivative corresponds to the Riemann–Liouville and Caputo fractional derivatives for $\beta = 0$ and $\beta = 1$, respectively.

Definition 5. *Let $-\infty \leq a < b \leq \infty$ and $\alpha > 0$. The left- and right-sided fractional integrals of the function $h \in L^1([a,b],\mathbb{R})$ with respect to an increasing and positive function ψ on (a,b) having a continuous derivative $\psi'(t) > 0$ on (a,b) are defined by*

$$I_{a+}^{\alpha,\psi}h(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)[\psi(t) - \psi(s)]^{\alpha-1}h(s)ds,$$

and

$$I_{b-}^{\alpha,\psi}h(t) = \frac{1}{\Gamma(\alpha)}\int_t^b \psi'(s)[\psi(s) - \psi(t)]^{\alpha-1}h(s)ds.$$

Definition 6. *Let $h : [a,b] \to \mathbb{R}$ be a function such that $\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{a+}^{n-\alpha,\psi}h(t)$ exists almost everywhere on $[a,b]$. Then, the fractional derivative of order $\alpha > 0$ of the function h with respect to the function ψ with $\psi'(t) \neq 0$ is defined by*

$$\begin{aligned} D_{a+}^{\alpha,\psi}h(t) &= \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{a+}^{n-\alpha,\psi}h(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \int_a^t \psi'(s)[\psi(t) - \psi(s)]^{n-\alpha-1}h(s)ds, \end{aligned}$$

and

$$\begin{aligned} D_{b-}^{\alpha,\psi}h(t) &= \left(-\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{b-}^{n-\alpha,\psi}h(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\left(-\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \int_t^b \psi'(s)[\psi(s) - \psi(t)]^{n-\alpha-1}h(s)ds, \end{aligned}$$

where n is the smallest integer greater than or equal to α.

Definition 7. *Let* $E_{\gamma,\psi}([a,b],\mathbb{R}) = \{y : [a,b] \to \mathbb{R}; y \circ \psi^{-1} \in AC^{\gamma}([\psi(a), \psi(b)],\mathbb{R})\}$ *and* $\psi \in C^n([a,b],\mathbb{R})$ *be a strictly increasing function on* $[a,b]$. *The left-sided and right-sided* ψ*-Hilfer fractional derivatives, respectively denoted by* ${}^H D_{a+}^{\alpha,\beta,\psi}(\cdot)$ *and* ${}^H D_{b-}^{\alpha,\beta,\psi}(\cdot)$ *of* $h \in E_{\gamma,\psi}([a,b],\mathbb{R})$ *of order* $\alpha > 0$ *and type* $\beta \in [0,1]$, *are defined by*

$$ {}^H D_{a+}^{\alpha,\beta,\psi} h(t) = I_{a+}^{\beta(n-\alpha),\psi}\Big(\frac{1}{\psi'(t)}\frac{d}{dt}\Big)^n I_{a+}^{(1-\beta)(n-\alpha),\psi} h(t) = I_{a+}^{\gamma-\alpha,\psi} D_{a+}^{\gamma,\psi} h(t), $$

and

$$ {}^H D_{b-}^{\alpha,\beta,\psi} h(t) = I_{b-}^{\beta(n-\alpha),\psi}\Big(-\frac{1}{\psi'(t)}\frac{d}{dt}\Big)^n I_{b-}^{(1-\beta)(n-\alpha),\psi} h(t) = I_{a+}^{\gamma-\alpha,\psi}(-1)^n D_{b-}^{\gamma,\psi} h(t), $$

where n *is the smallest integer greater than or equal to* α *and* $\gamma = \alpha + \beta(n-\alpha)$.

Lemma 12. *Let* n *be the smallest integer greater than or equal to* $\alpha > 0$, $0 \le \beta \le 1$, $\gamma = \alpha + \beta(n-\alpha)$, *and* $h \in C([a,b],\mathbb{R})$. *The function* $y \in E_{\gamma,\psi}([a,b],\mathbb{R}) \cap C([a,b],\mathbb{R})$ *is a solution of the integral equation:*

$$ y(t) = \int_a^b G_{\psi}(t,s)\psi'(s)q(s)h(s)ds, $$

where

$$ G_{\psi}(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \Big[\dfrac{\psi(t)-\psi(a)}{\psi(b)-\psi(a)}\Big]^{\gamma-1}[\psi(b)-\psi(a)]^{\alpha-1} - [\psi(t)-\psi(s)]^{\alpha-1}, & a \le s \le t \le b, \\ \Big[\dfrac{\psi(t)-\psi(a)}{\psi(b)-\psi(a)}\Big]^{\gamma-1}[\psi(b)-\psi(a)]^{\alpha-1}, & a \le t \le s \le b, \end{cases} $$

if and only if y *is a solution of the fractional boundary value problem:*

$$ \begin{cases} {}^H D_a^{\alpha,\beta,\psi} y(t) + q(t)h(t) = 0, & a < t < b, \\ y(a) = y(b) = 0. \end{cases} $$

Lemma 13. *Assume that* $1 < \alpha < 2$, $0 \le \beta \le 1$, $\gamma = \alpha + \beta(2-\alpha)$. *Then, the Green function* $G_{\psi}(t,s)$ *satisfies the following properties:*

(i) $\max_{a \le t \le b} G_{\psi}(t,s) = G_{\psi}(s,s)$, $s \in [a,b]$.

(ii) $G_{\psi}(s,s)$ *has a unique maximum, given by*

$$ \max_{a \le s \le b} G_{\psi}(s,s) = \frac{(\alpha-1)^{\alpha-1}(\gamma-1)^{\gamma-1}[\psi(b)-\psi(a)]^{\alpha-1}}{(\alpha+\gamma-2)^{\alpha+\gamma-2}\Gamma(\alpha)}. $$

The next theorem deals with the Lyapunov-type inequality for the ψ-Hilfer boundary value problem (15).

Theorem 11. *(Lyapunov-type inequality). Suppose that:*

(H_1) *The function* $f : \mathbb{R} \to \mathbb{R}$ *is continuous and sublinear:*

$$ |f(u)| \le \mu|u|, \text{ for each } t \in [a,b], \text{ and all } u \in \mathbb{R} \text{ where } \mu > 0. $$

(H_2) $q \in L^1([a,b],\mathbb{R})$.

If $y \in E_{\gamma,\psi}([a,b],\mathbb{R}) \cap C([a,b],\mathbb{R})$ *is a nontrivial solution of (15), then*

$$ \int_a^b |q(s)|\psi'(s)ds \ge \frac{(\alpha+\gamma-2)^{\alpha+\gamma-2}\Gamma(\alpha)}{\mu(\alpha-1)^{\alpha-1}(\gamma-1)^{\gamma-1}[\psi(b)-\psi(a)]^{\alpha-1}}. $$

Theorem 12. *(Generalised Lyapunov-type inequality). Let* (H_2) *and the following condition hold:*

$(H_3)\, f : \mathbb{R} \to \mathbb{R}$ *is a positive, concave, and nondecreasing function.*
If y is a solution of the problem (15), then

$$\int_a^b |q(s)|\psi'(s)ds > \frac{(\alpha+\gamma-2)^{\alpha+\gamma-2}\Gamma(\alpha)x^*}{\mu(\alpha-1)^{\alpha-1}(\gamma-1)^{\gamma-1}[\psi(b)-\psi(a)]^{\alpha-1}\max_{x(t)\in[x_*,x^*]} f(y(t))},$$

where $x_* = \min_{t\in[a,b]} y(t), \;\; x^* = \max_{t\in[a,b]} y(t)$.

Corollary 6. *Let (H_2) and the following assumptions hold:*
$(H_4)\, f \in C(\mathbb{R}^+, \mathbb{R}^+)$ *is concave and nondecreasing.*
(H_5) *There exist two constants $0 < r_1 < r_2$ such that*

$$f(u) \geq \theta_1 r_1 \;\text{ for }\; u \in [0, r_1], \quad f(u) \leq \theta_2 r_2 \;\text{ for }\; u \in [0, r_2],$$

where

$$\theta_1 = \left(\int_{(2a+b)/3}^{(2b-a)/3} G_\psi(s,s)\psi'(s)q(s)ds\right)^{-1}, \quad \theta_2 = \left(\int_a^b G_\psi(s,s)\psi'(s)q(s)ds\right)^{-1}.$$

If y is a solution of the problem (15), then

$$\int_a^b q(s)\psi'(s)ds \geq \frac{(\alpha+\gamma-2)^{\alpha+\gamma-2}\Gamma(\alpha)r_1}{(\alpha-1)^{\alpha-1}(\gamma-1)^{\gamma-1}[\psi(b)-\psi(a)]^{\alpha-1}f(r_2)}.$$

For illustrating the usefulness of Lyapunov-type inequality given in Theorem 11, we consider the following fractional eigenvalue boundary value problem:

$$\begin{cases} {}^H D_a^{\alpha,\beta,\psi} y(t) + \lambda y(t) = 0, \;\; a < t < b, \\ y(a) = y(b) = 0, \end{cases} \tag{16}$$

where $[a,b] \subset \mathbb{R}$, $\alpha \in (1,2)$, $\beta \in [0,1]$, and $\psi \in C^1([a,b],\mathbb{R})$ with $\psi'(t) > 0$ for all $t \in [a,b]$.
Assume that there exists a nontrivial solution $y_\lambda \in E_{\gamma,\psi}([a,b],\mathbb{R}) \cap C([a,b],\mathbb{R})$ of the problem (16).

Corollary 7. *If λ is an eigenvalue of problem (16), then*

$$|\lambda| > \frac{(\alpha+\gamma-2)^{\alpha+\gamma-2}\Gamma(\alpha)}{(\alpha-1)^{\alpha-1}(\gamma-1)^{\gamma-1}[\psi(b)-\psi(a)]^{\alpha}}.$$

Now, we present a Hartman–Wintner-type inequality.

Theorem 13. *Suppose that (H_2) and (H_4) are satisfied. If the problem (15) has a nontrivial continuous solution y, then*

$$\int_a^b [\psi(s)-\psi(a)]^{\gamma-1}[\psi(b)-\psi(s)]^{\alpha-1}\psi'(s)|q(s)|ds > \frac{\|y\|}{f(\|y\|)}\Gamma(\alpha)[\psi(b)-\psi(a)]^{\gamma-1}.$$

Corollary 8. *Let $f(y) = y$ (linear case) and $q \in L^1([a,b],\mathbb{R}^+)$. Then,*

$$\int_a^b [\psi(s)-\psi(a)]^{\gamma-1}[\psi(b)-\psi(s)]^{\alpha-1}\psi'(s)|q(s)|ds > \Gamma(\alpha)[\psi(b)-\psi(a)]^{\gamma-1}.$$

Corollary 9. *Let (H_2), (H_4), and the following condition hold:*
(H_6) *There exist two constants $0 < r_1 < r_2$ such that*

$$f(u) \geq \theta_1 r_1 \;\text{ for }\; u \in [0, r_1], \quad f(u) \leq \theta_1 r_1 \;\text{ for }\; u \in [0, r_2].$$

If y is a solution of the problem (15), then

$$\int_a^b [\psi(s)-\psi(a)]^{\gamma-1}[\psi(b)-\psi(s)]^{\alpha-1}\psi'(s)|q(s)|ds > \Gamma(\alpha)[\psi(b)-\psi(a)]^{\gamma-1}\frac{r_1}{f(r_2)}.$$

Corollary 10. *If λ is an eigenvalue of problem (16), then*

$$|\lambda| \geq \frac{\Gamma(\alpha)(\psi(b)-\psi(a))^{\gamma-1}}{\int_{\psi(a)}^{\psi(b)}(t-\psi(a))^{\gamma-1}(\psi(b)-s)^{\alpha-1}ds}.$$

6. Lyapunov-Type and Hartman–Wintner-Type Inequalities for Nonlinear Fractional Hybrid Boundary Value Problems

In 2019, Lopez et al. [15] considered the fractional hybrid boundary value problem:

$$\begin{cases} D_{a+}^{\alpha}\Big[\dfrac{y(t)}{f(t,y(t),Hy(t))}\Big] + q(t)Fy(t) = 0, \;\; 0<a<t<b, \\ y(a)=0, \;\; y(b)=0, \end{cases} \tag{17}$$

where D_{a+}^{α} is the Riemann–Liouville fractional derivative operator of order $\alpha \in (1,2]$, $f : [a,b]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}\setminus\{0\}$ is a given function, and the operators F and H map the space of continuous functions $C[a,b]$ into itself, which may be nonlinear, but must exhibit sublinear growth.

Lemma 14. *Let $1<\alpha\leq 2$, $h\in C[a,b]$, $f:[a,b]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}\setminus\{0\}$ be a continuous function and $H : C[a,b]\to C[a,b]$ be an operator (not necessarily linear). Then, the unique solution of the fractional boundary value problem:*

$$\begin{cases} D_{a+}^{\alpha}\Big[\dfrac{y(t)}{f(t,y(t),Hy(t))}\Big] + h(t) = 0, \;\; 0<a<t<b, \;\; 1<\alpha\leq 2, \\ y(a)=0, \;\; y(b)=0, \end{cases} \tag{18}$$

is given by

$$y(t) = f(t,y(t),Hy(t))\int_a^b G(t,s)h(s)ds, \;\; t\in[a,b],$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} \dfrac{(t-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} - (t-s)^{\alpha-1}, & a\leq s\leq t\leq b, \\ \dfrac{(t-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}}, & a\leq t\leq s\leq b, \end{cases}$$

and satisfies the following properties:

(i) G *is continuous on* $[a,b]\times[a,b]$;
(ii) $G(t,s)\geq 0$ *for* $t,s\in[a,b]$;
(iii) $\max\limits_{t,s\in[a,b]} G(t,s) = \dfrac{(b-a)^{\alpha-1}}{2^{2\alpha-2}\Gamma(\alpha)}$.

Theorem 14. *Let $f : [a,b]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}\setminus\{0\}$ be a continuous and bounded function with $M=\sup\{|f(t,x,y)| : t\in[a,b], x,y\in\mathbb{R}\}$, $H,F : C[a,b]\to C[a,b]$ be two operators (not necessarily linear) with $\|Fy\|\leq\lambda\|y\|$ for $y\in C[a,b]$ and some $\lambda>0$, and $q:[a,b]\to\mathbb{R}$ be a continuous function. Then, it follows by the estimate:*

$$\int_a^b |q(s)|ds < \frac{2^{2\alpha-2}\Gamma(\alpha)}{M\lambda(b-a)^{\alpha-1}}$$

that there exists only the trivial solution ($y(t)\equiv 0$) for the the problem (17).

In 2021, Kassymov and Torebek [16] considered the following nonlinear Dirichlet fractional hybrid boundary value problem:

$$\begin{cases} D^{\alpha}_{a+}D^{\alpha}_{b-}\Big(\dfrac{y(t)}{f(t.y(t),Hy(t))}\Big) - q(t)Fy(t) = 0, \;\; a < t < b, \\ y(a) = y(b) = 0, \end{cases} \tag{19}$$

where $1/2 < \alpha \leq 1$, $D^{\alpha}_{a+} = \frac{d}{dt} I^{1-\alpha}_{a+}$ and $D^{\alpha}_{b-}y(t) = I^{1-\alpha}_{b-}y'(t)$ are, respectively, the left Riemann–Liouville and the right Caputo fractional derivatives of order $0 < \alpha \leq 1$ with the left and right Riemann–Liouville fractional integrals respectively given by

$$I^{\alpha}_{a+}y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} y(s) ds$$

and

$$I^{\alpha}_{b-}y(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} y(s) ds, \;\; t \in (a,b).$$

Here f, q, F, and H are given continuous functions such that:

(A) H, F are operators (these operators can be nonlinear) such that $H, F : C[a,b] \to C[a,b]$;
(B) q is a continuous and real-valued function on $[a,b]$;
(C) $f : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ is a continuous function.

Theorem 15. *Assume that $\alpha \in \left(\frac{1}{2}, 1\right]$, Conditions (A)-(C) hold, and y is a solution of (19). Then, the solution of (19) coincides with the solution of the following integral equation:*

$$y(t) = f(t, y(t), Hy(t)) \int_a^b G(t,s) q(s) Fy(s) ds, \tag{20}$$

where

$$G(t,s) = K(t,s) - \frac{K(a,s)K(t,a)}{K(a,a)}$$

and

$$K(t,s) = \frac{1}{\Gamma^2(\alpha)} \int_{\max\{t,s\}}^b (\tau - t)^{\alpha-1} (\tau - s)^{\alpha-1} d\tau.$$

Furthermore, we have

(i) $\sup\limits_{a<s<t} G(t,s) = G(s,s) > 0, \;\; a < s < t < b;$

(ii) $\sup\limits_{a<s<b} G(s,s) < \dfrac{(2^{2-2\alpha} - 1 + 2^{2(1-2\alpha)})}{\Gamma^2(\alpha)(2\alpha-1)} (b-a)^{2\alpha-1}.$

The Lyapunov-type inequality for the nonlinear Dirichlet fractional hybrid boundary value problem (19) is given in the following theorem.

Theorem 16. *Assume that (A)-(C) hold with $\|Fy\| \leq B\|y\|$ for all $y \in C[a,b]$ and $B > 0$. Let $A = \sup_{(t,y,z)} |f(t,y,z)|$, $\alpha \in \left(\frac{1}{2}, 1\right]$ and y be a solution of the boundary value problem (19). Then,*

$$\int_a^b |q(s)| ds > \left(AB \frac{(2^{2-2\alpha} - 1 + 2^{2(1-2\alpha)})}{\Gamma^2(\alpha)(2\alpha-1)} (b-a)^{2\alpha-1} \right)^{-1}.$$

Corollary 11. *Assume that (A)-(C) hold with* $\|Fy\| \le B\|y\|$ *for all* $y \in C[a,b]$ *and* $B > 0$*, and* $A = \sup_{(t,y,z)} |f(t,y,z)|$*,* $\alpha \in \left(\frac{1}{2}, 1\right]$*. If*

$$\int_a^b |q(s)|ds < \left(AB\frac{(2^{2-2\alpha} - 1 + 2^{2(1-2\alpha)})}{\Gamma^2(\alpha)(2\alpha-1)}(b-a)^{2\alpha-1} \right)^{-1},$$

then the boundary value problem (19) has only a trivial solution.

Letting $q^+ = \max\{q, 0\}$, we give the Hartman–Wintner-type inequality for the problem (19).

Theorem 17. *Assume that (A)-(C) hold with* $\|Fy\| \le B\|y\|$ *for all* $y \in C[a,b]$ *and* $B > 0$*. Let* $A = \sup_{(t,y,z)} |f(t,y,z)|$*,* $\alpha \in \left(\frac{1}{2}, 1\right]$ *and* y *be a nontrivial solution of the boundary value problem (19). Then, we have*

$$\begin{aligned} &\int_a^b \left[(b-s)^{2\alpha-1} + (s-a)^{2\alpha-1} - (b-a)^{2\alpha-1} + \frac{(b-s)^{2\alpha-1}(s-a)^{2\alpha-1}}{(b-a)^{2\alpha-1}} \right] q^+(s)ds \\ > \;\; &\Gamma^2(\alpha)(2\alpha-1)(AB)^{-1}. \end{aligned}$$

7. Lyapunov-Type Inequalities for Hadamard Fractional Boundary Value Problems

In 2020, Wang et al. [17] established Lyapunov-type inequalities for Hadamard fractional differential equations, with Sturm–Liouville multi-point and integral boundary conditions given by

$$\begin{cases} {}^H D^\alpha_{a+} y(t) + q(t)y(t) = 0, \;\; 0 < a < t < b, \;\; 1 < \alpha \le 2, \\ y(a) = 0, \;\; \gamma y(b) + \delta y'(b) = \sum\limits_{i=1}^{m-2} \beta_i y(\xi_i), \end{cases} \tag{21}$$

and

$$\begin{cases} {}^H D^\alpha_{a+} y(t) + q(t)y(t) = 0, \;\; 0 < a < t < b, \;\; 1 < \alpha \le 2, \\ y(a) = 0, \;\; \gamma y(b) + \delta y'(b) = \lambda \displaystyle\int_a^b h(s)y(s)ds, \;\; \lambda \ge 0, \end{cases} \tag{22}$$

where ${}^H D^\alpha_{a+}$ is the Hadamard fractional derivative of order α, $\gamma > 0, \delta > 0$, $a < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < b$, $\beta_i \ge 0$ $(i = 1, 2, \ldots, m-2)$, and $h : [a,b] \to [0, \infty)$ with $h \in L^1(a,b)$.

We recall that:

Definition 8. *The fractional integral of the Hadamard-type of order* $\alpha \in \mathbb{R}^+$ *for a continuous function* $f : [a,b] \to \mathbb{R}$ *is defined by*

$${}^H I^\alpha_{a+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log\frac{t}{s} \right)^{\alpha-1} f(s)\frac{ds}{s}, \;\; t \in [a,b].$$

Definition 9. *The Hadamard fractional derivative of order* $\alpha \in \mathbb{R}^+$ *for a continuous function* $f : [a,b] \to \mathbb{R}$ *is defined by*

$${}^H D^\alpha_{a+} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt} \right)^n \int_a^t \left(\log\frac{t}{s} \right)^{n-\alpha-1} f(s)\frac{ds}{s}, \;\; t \in [a,b],$$

where $n - 1 < \alpha < n$*,* $n = [\alpha] + 1$*.*

Let us now set the following notations:

$$\begin{aligned}
\sigma &= \sqrt{(\alpha-1)^2 + \Big(\log\sqrt{\frac{b}{a}}\Big)^2},\\
\rho_1 &= \gamma\Big(\log\frac{b}{a}\Big)^{\alpha-1} + (\alpha-1)\frac{\delta}{b}\Big(\log\frac{b}{a}\Big)^{\alpha-2} > 0,\\
\rho_2 &= \gamma\Big(\log\frac{b}{a}\Big)^{\alpha-1} + (\alpha-1)\frac{\delta}{b}\Big(\log\frac{b}{a}\Big)^{\alpha-2} - \sum_{i=1}^{m-2}\beta_i\Big(\log\frac{\xi_i}{a}\Big)^{\alpha-1} > 0,\\
\rho_3 &= \gamma\Big(\log\frac{b}{a}\Big)^{\alpha-1} + (\alpha-1)\frac{\delta}{b}\Big(\log\frac{b}{a}\Big)^{\alpha-2} - \lambda\int_a^b\Big(\log\frac{t}{a}\Big)^{\alpha-1}h(t)ds > 0.
\end{aligned}$$

Lemma 15. *Assume that $r \in C([a,b],\mathbb{R})$. The Sturm–Liouville–Hadamard fractional boundary value problem:*

$$\begin{cases} {}^{H}D^{\alpha}_{a+}y(t) + r(t) = 0, \;\; 0 < a < t < b, \;\; 1 < \alpha \le 2, \\ y(a) = 0, \;\; \gamma y(b) + \delta y'(b) = \sum_{i=1}^{m-2}\beta_i y(\xi_i), \end{cases} \tag{23}$$

has the unique solution:

$$y(t) = \int_a^b G(t,s)r(s)ds + \frac{1}{\rho_2}\Big(\log\frac{t}{a}\Big)^{\alpha-1}\sum_{i=1}^{m-2}\beta_i\int_a^b G(\xi_i,s)r(s)ds,$$

where $G(t,s)$ is given by

$$G(t,s) = \frac{1}{s\rho_1\Gamma(\alpha)}\begin{cases} \Big(\log\frac{t}{a}\Big)^{\alpha-1}\Big(\log\frac{b}{s}\Big)^{\alpha-2}\Big[\gamma\log\frac{b}{s} + (\alpha-1)\frac{\delta}{b}\Big] - \rho_1\Big(\log\frac{t}{s}\Big)^{\alpha-1}, & a \le s \le t \le b, \\ \Big(\log\frac{t}{a}\Big)^{\alpha-1}\Big(\log\frac{b}{s}\Big)^{\alpha-2}\Big[\gamma\log\frac{b}{s} + (\alpha-1)\frac{\delta}{b}\Big], & a \le t \le s \le b. \end{cases}$$

Lemma 16. *Assume that $g \in C([a,b],\mathbb{R})$. The Sturm–Liouville–Hadamard fractional boundary value:*

$$\begin{cases} {}^{H}D^{\alpha}_{a+}y(t) + g(t) = 0, \;\; 0 < a < t < b, \;\; 1 < \alpha \le 2, \\ y(a) = 0, \;\; \gamma y(b) + \delta y'(b) = \lambda\int_a^b h(s)y(s)ds, \;\; \lambda \ge 0, \end{cases} \tag{24}$$

has the unique solution

$$y(t) = \int_a^b G(t,s)g(s)ds + \frac{\lambda}{\rho_3}\Big(\log\frac{t}{a}\Big)^{\alpha-1}\int_a^b\Big(\int_a^b G(t,s)g(s)ds\Big)h(t)dt,$$

where $h : [a,b] \to [0,\infty)$ with $h \in L^1(a,b)$ and $G(t,s)$ is defined in Lemma 15.

Lemma 17. *The function G defined in Lemma 15 satisfies the following properties:*

(i) $G(t,s) \ge 0$ *on* $[a,b] \times [a,b]$.

(ii) $\max_{t\in[a,b]} G(t,s) = G(s,s) = \frac{1}{s\rho_1\Gamma(\alpha)}\Big(\log\frac{s}{a}\Big)^{\alpha-1}\Big(\log\frac{b}{s}\Big)^{\alpha-2}\Big[\gamma\log\frac{b}{s} + (\alpha-1)\frac{\delta}{b}\Big].$

We now present Lyapunov-type inequalities for the Sturm–Liouville–Hadamard fractional boundary value problem (21).

Theorem 18. *If a nontrivial continuous solution of the Sturm–Liouville–Hadamard fractional boundary value problem (21) exists, then*

$$\begin{aligned}&\int_a^b \frac{1}{s}\Big(\log\frac{s}{a}\Big)^{\alpha-1}\Big(\log\frac{b}{s}\Big)^{\alpha-2}\Big[\gamma\log\frac{b}{s}+(\alpha-1)\frac{\delta}{b}\Big]|q(s)|ds\\ \geq\ & \frac{\rho_1\rho_2}{\rho_2+\sum_{i=1}^{m-2}\beta_i\Big(\log\frac{b}{a}\Big)^{\alpha-1}}\Gamma(\alpha).\end{aligned}$$

Letting $\beta_i = 0,\ i = 1, 2, \ldots, m-2$ in Theorem 18, we have:

Corollary 12. *If there exists a nontrivial continuous solution of the Sturm–Liouville–Hadamard fractional boundary value problem:*

$$\begin{cases} {}^H D^\alpha_{a+} y(t) + q(t)y(t) = 0,\ \ 0 < a < t < b,\ \ 1 < \alpha \leq 2, \\ y(a) = 0,\ \ \gamma y(b) + \delta y'(b) = 0, \end{cases}$$

where $q : [a, b] : \mathbb{R}$ is a continuous function and $\gamma \geq 0, \delta \geq 0, \gamma\delta > 0$, then we have

$$\begin{aligned}&\int_a^b \frac{1}{s}\Big(\log\frac{s}{a}\Big)^{\alpha-1}\Big(\log\frac{b}{s}\Big)^{\alpha-2}\Big[\gamma\log\frac{b}{s}+(\alpha-1)\frac{\delta}{b}\Big]|q(s)|ds\\ \geq\ & \Gamma(\alpha)\Big(\log\frac{b}{a}\Big)^{\alpha-2}\Big[\gamma\log\frac{b}{a}+(\alpha-1)\frac{\delta}{b}\Big].\end{aligned}$$

For $\gamma = 1, \delta = 0$ or $\gamma = 0, \delta = 1$ in Corollary 12, we can obtain the following Lyapunov-type inequalities.

Corollary 13. *If there exists a nontrivial continuous solution of the Sturm–Liouville–Hadamard fractional boundary value problem:*

$$\begin{cases} {}^H D^\alpha_{a+} y(t) + q(t)y(t) = 0,\ \ 0 < a < t < b,\ \ 1 < \alpha \leq 2, \\ y(a) = 0,\ \ y(b) = 0, \end{cases}$$

where $q : [a, b] : \mathbb{R}$ is a continuous function, then

$$\int_a^b \frac{1}{s}\Big(\log\frac{s}{a}\Big)^{\alpha-1}\Big(\log\frac{b}{s}\Big)^{\alpha-1}|q(s)|ds \geq \Gamma(\alpha)\Big(\log\frac{b}{a}\Big)^{\alpha-1}.$$

Corollary 14. *If there exists a nontrivial continuous solution of the Sturm–Liouville–Hadamard fractional boundary value problem:*

$$\begin{cases} {}^H D^\alpha_{a+} y(t) + q(t)y(t) = 0,\ \ 0 < a < t < b,\ \ 1 < \alpha \leq 2, \\ y(a) = 0,\ \ y'(b) = 0, \end{cases}$$

where $q : [a, b] : \mathbb{R}$ is a continuous function, then

$$\int_a^b \frac{1}{s}\Big(\log\frac{s}{a}\Big)^{\alpha-1}\Big(\log\frac{b}{s}\Big)^{\alpha-2}|q(s)|ds \geq \Gamma(\alpha)\Big(\log\frac{b}{a}\Big)^{\alpha-2}.$$

For $\gamma = 1, \delta = 0$ in Theorem 18, we have the following Lyapunov-type inequality.

Corollary 15. *If there exists a nontrivial continuous solution of the Sturm–Liouville–Hadamard fractional boundary value problem*

$$\begin{cases} {}^{H}D^{\alpha}_{a+}y(t) + q(t)y(t) = 0, \ \ 0 < a < t < b, \ \ 1 < \alpha \le 2, \\ y(a) = 0, \ \ y(b) = \displaystyle\sum_{i=1}^{m-2} \beta_i y(\xi_i), \end{cases}$$

where $q : [a,b] : \mathbb{R}$ *is a continuous function,* $a < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < b$, $\beta_i \ge 0$ $(i = 1,2,\ldots,m-2)$ *with* $0 \le \sum_{i=1}^{m-2} \beta_i < 1$, *then*

$$\int_a^b \frac{1}{s}\left(\log\frac{s}{a}\right)^{\alpha-1}\left(\log\frac{b}{s}\right)^{\alpha-1}|q(s)|ds$$
$$\ge \frac{\left(\log\frac{b}{a}\right)^{\alpha-1}\left[\left(\log\frac{b}{a}\right)^{\alpha-1} - \sum_{i=1}^{m-2}\beta_i\left(\log\frac{\xi_i}{a}\right)^{\alpha-1}\right]}{\left(\log\frac{b}{a}\right)^{\alpha-1} - \sum_{i=1}^{m-2}\beta_i\left(\log\frac{\xi_i}{a}\right)^{\alpha-1} + \sum_{i=1}^{m-2}\beta_i\left(\log\frac{b}{a}\right)^{\alpha-1}}\Gamma(\alpha).$$

For $\gamma = 0, \delta = 1$ in Theorem 18, we obtain the following Lyapunov-type inequality.

Corollary 16. *If there exists a nontrivial continuous solution of the Sturm–Liouville–Hadamard fractional boundary value problem:*

$$\begin{cases} {}^{H}D^{\alpha}_{a+}y(t) + q(t)y(t) = 0, \ \ 0 < a < t < b, \ \ 1 < \alpha \le 2, \\ y(a) = 0, \ \ y'(b) = \displaystyle\sum_{i=1}^{m-2} \beta_i y(\xi_i), \end{cases}$$

where $q : [a,b] : \mathbb{R}$ *is a continuous function,* $a < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < b$, $\beta_i \ge 0$ $(i = 1,2,\ldots,m-2)$ *with* $0 \le \sum_{i=1}^{m-2} \beta_i < 1$, *then we have*

$$\int_a^b \frac{1}{s}\left(\log\frac{s}{a}\right)^{\alpha-1}\left(\log\frac{b}{s}\right)^{\alpha-2}|q(s)|ds$$
$$\ge \frac{\left(\log\frac{b}{a}\right)^{\alpha-2}\left[\frac{\alpha-1}{b}\left(\log\frac{b}{a}\right)^{\alpha-2} - \sum_{i=1}^{m-2}\beta_i\left(\log\frac{\xi_i}{a}\right)^{\alpha-1}\right]}{\frac{\alpha-1}{b}\left(\log\frac{b}{a}\right)^{\alpha-2} - \sum_{i=1}^{m-2}\beta_i\left(\log\frac{\xi_i}{a}\right)^{\alpha-1} + \sum_{i=1}^{m-2}\beta_i\left(\log\frac{b}{a}\right)^{\alpha-1}}\Gamma(\alpha).$$

For $\beta = \beta_1 \ge 0$, $\beta_2 = \beta_3 = \ldots = \beta_{m-2} = 0$, $\xi = \xi_1$ in Corollary 16, we obtain the three-point Lyapunov-type inequality.

Corollary 17. *If there exists a nontrivial continuous solution of the Sturm–Liouville–Hadamard fractional boundary value problem:*

$$\begin{cases} {}^{H}D^{\alpha}_{a+}y(t) + q(t)y(t) = 0, \ \ 0 < a < t < b, \ \ 1 < \alpha \le 2, \\ y(a) = 0, \ \ y(b) = \beta y(\xi), \end{cases}$$

where $q : [a,b] : \mathbb{R}$ *is a continuous function,* $a < \xi < b$, $0 \le \beta < 1$, *then*

$$\int_a^b \frac{1}{s}\left(\log\frac{s}{a}\right)^{\alpha-1}\left(\log\frac{b}{s}\right)^{\alpha-1}|q(s)|ds$$
$$\ge \frac{\left(\log\frac{b}{a}\right)^{\alpha-1}\left[\left(\log\frac{b}{a}\right)^{\alpha-1} - \beta\left(\log\frac{\xi}{a}\right)^{\alpha-1}\right]}{\left(\log\frac{b}{a}\right)^{\alpha-1} + \beta\left[\left(\log\frac{b}{a}\right)^{\alpha-1} - \left(\log\frac{\xi}{a}\right)^{\alpha-1}\right]}\Gamma(\alpha).$$

Next, we give Lyapunov-type inequalities for the Sturm–Liouville–Hadamard fractional boundary value problem (22).

Theorem 19. *If a nontrivial continuous solution of the Sturm–Liouville–Hadamard fractional boundary value problem (22) exists, then*

$$\begin{aligned}&\int_a^b \frac{1}{s}\left(\log\frac{s}{a}\right)^{\alpha-1}\left(\log\frac{b}{s}\right)^{\alpha-2}\left[\gamma\log\frac{b}{s}+(\alpha-1)\frac{\delta}{b}\right]|q(s)|ds\\ \geq\ & \frac{\rho_1\rho_3}{\rho_3+\lambda\left(\log\frac{b}{a}\right)^{\alpha-1}\int_a^b h(s)ds}\Gamma(\alpha).\end{aligned}$$

Theorem 20. *If a nontrivial continuous solution of the Sturm–Liouville–Hadamard fractional boundary value problem (22) exists, then*

$$\begin{aligned}&\int_a^b \frac{\gamma\log\frac{b}{s}+(\alpha-1)\frac{\delta}{b}}{\log\frac{b}{s}}|q(s)|ds\\ \geq\ & \frac{\sqrt{ab}\rho_1\rho_3\Gamma(\alpha)}{\rho_3+\lambda\left(\log\frac{b}{a}\right)^{\alpha-1}\int_a^b h(s)ds}\cdot\frac{(2e)^{\alpha-1}(\alpha-1+\sigma)^{\alpha-1}}{(\alpha-1)^{\alpha-1}\left(\log\frac{b}{a}\right)^{2(\alpha-1)}e^{\sigma}}.\end{aligned}$$

In 2021, Wang et al. [18] established Lyapunov-type inequalities for a multipoint Caputo–Hadamard-type fractional boundary value problem:

$$\begin{cases} {}^{C}_{H}D^{\alpha}_{a^+}y(t)+q(t)y(t)=0, \ \ a<t<b,\\ y(a)=0, \ \ y(b)=\sum\limits_{i=1}^{m-2}\beta_i y(\xi_i),\end{cases} \tag{25}$$

where ${}^{C}_{H}D^{\alpha}_{a}$ is the Caputo–Hadamard fractional derivative of order $\alpha\in(1,2)$, $q\in C([a,b],\mathbb{R})$, $\beta_i\geq 0$, $i=1,2,\ldots,m-2$, $a<\xi_1<\xi_2<\ldots<\xi_{m-2}<b$, and $0\leq\sum_{i=1}^{m-2}\beta_i<1$.

Lemma 18. *The boundary value problem (25) has a unique solution y if and only if*

$$y(t)=\int_a^b G(t,s)q(s)y(s)ds+\frac{\log\frac{t}{a}}{\log\frac{b}{a}-\sum_{i=1}^{m-2}\beta_i\log\frac{\xi_i}{a}}\int_a^b\sum_{i=1}^{m-2}\beta_i G(\xi_i,s)q(s)y(s)ds,$$

where $G(t,s)$ is given by

$$G(t,s)=\frac{1}{s\log\frac{b}{a}\Gamma(\alpha)}\begin{cases}\log\frac{t}{a}\left(\log\frac{b}{s}\right)^{\alpha-1}-\log\frac{b}{a}\left(\log\frac{t}{s}\right)^{\alpha-1}, & 0<a\leq s\leq t\leq b,\\ \log\frac{t}{a}\left(\log\frac{b}{s}\right)^{\alpha-1}, & 0<a\leq t\leq s\leq b.\end{cases}$$

Moreover, the function $G(t,s)$ satisfies the following property:

$$|G(t,s)|\leq\frac{1}{a}\frac{(\alpha-1)^{\alpha-1}}{\alpha^{\alpha}\Gamma(\alpha)}\left(\log\frac{b}{a}\right)^{\alpha-1}.$$

In the following theorem, a Lyapunov-type inequality for the fractional boundary value problem (25) is described.

Theorem 21. *If the boundary value problem (25) has a nonzero solution, then*

$$\int_a^b|q(s)|ds\geq a\frac{\alpha^{\alpha}\Gamma(\alpha)}{[(\alpha-1)(\log b-\log a)]^{(\alpha-1)}}\frac{\log\frac{b}{a}-\sum_{i=1}^{m-2}\beta_i\log\frac{\xi_i}{a}}{\log\frac{b}{a}+\sum_{i=1}^{m-2}\beta_i\log\frac{b}{\xi_i}}.$$

In [18], the authors also established the Lyapunov-type inequalities for the Caputo–Hadamard-type fractional differential equation subject to integral boundary conditions:

$$\begin{cases} {}_{H}^{C}D_{a^+}^{\alpha}y(t) + q(t)y(t) = 0, \;\; a < t < b, \\ y(a) = 0, \;\; y(b) = \lambda \displaystyle\int_a^b h(s)y(s)ds, \; \lambda \geq 0, \end{cases} \tag{26}$$

where $h : [a,b] \to [0,\infty)$ with $h \in L^1(a,b)$.

Lemma 19. *The boundary value problem (26) has a unique solution y if and only if*

$$y(t) = \int_a^b G(t,s)q(s)y(s)ds + \frac{\lambda \log\frac{t}{a}}{\log\frac{b}{a} - \lambda\sigma}\int_a^b \Big(\int_a^b G(z,s)h(z)y(z)dz\Big)q(s)y(s)ds,$$

where $\sigma = \int_a^b h(t)\log\frac{t}{a}dt$, $0 \leq \lambda\sigma < \log\frac{b}{a}$, *and* $G(t,s)$ *is defined in Lemma 18.*

Theorem 22. *If the boundary value problem (26) has a nonzero solution, then*

$$\int_a^b |q(s)|ds \geq \frac{a\big[\log\frac{b}{a} - \lambda\sigma\big]}{\log\frac{b}{a} + \lambda\big[\log\frac{b}{a}\int_a^b h(t)dt - \sigma\big]}\frac{\alpha^{\alpha}\Gamma(\alpha)}{[(\alpha-1)(\log b - \log a)]^{(\alpha-1)}}.$$

8. Lyapunov-Type Inequalities for Boundary Value Problems with Fractional Caputo–Fabrizio Derivative

In 2018, Kirane and Torebek [19] obtained a Lyapunov-type inequality for a Dirichlet boundary value problem involving the Caputo–Fabrizio operator given by

$$\begin{cases} {}^{CF}D_a^{\alpha}y(t) + q(t)y(t) = 0, \;\; a < t < b, \\ y(a) = y(b) = 0, \end{cases} \tag{27}$$

where ${}^{CF}D_a^{\alpha}$ is the Caputo–Fabrizio fractional derivative operator of order $\alpha \in (1,2]$ and $q \in C([a,b],\mathbb{R})$.

Definition 10. *Let* $f \in H^n(a,b) = \{y \in L^2(a,b) : y^{(n)} \in L^2(a,b)\}$. *The Caputo–Fabrizio fractional derivative of order* $\alpha \in (n-1,n], n \in \mathbb{N}$ *is defined by*

$$D_a^{\alpha}f(t) = \frac{1}{n-\alpha}\int_a^t \exp\Big(-\frac{\alpha-n+1}{n-\alpha}(t-s)\Big)f^{(n)}(s)ds.$$

Lemma 20. *The function y is a solution of the fractional boundary value problem (27) if and only if y satisfies the following integral equation*

$$y(t) = \int_a^b G(t,s)q(s)y(s)ds,$$

where

$$G(t,s) = \begin{cases} \dfrac{b-t}{b-a}((\alpha-1)(s-a) - 2 + \alpha), & a \leq t \leq s \leq b, \\ \dfrac{s-a}{b-a}((\alpha-1)(s-a) + 2 - \alpha), & a \leq s \leq t \leq b. \end{cases}$$

Moreover, $G(t,s)$ *satisfies the inequality:*

$$|G(t,s)| \leq \frac{((\alpha-1)(b-a) - 2 + \alpha)^2}{4(\alpha-1)(b-a)}.$$

Theorem 23. *Let q be a real and continuous function in $[a,b]$. If the fractional boundary value problem (27) has a nontrivial solution, then*

$$\int_a^b |q(s)|ds > \frac{4(\alpha-1)(b-a)}{((\alpha-1)(b-a)-2+\alpha)^2}.$$

Remark 1. *In [20], Laadjal corrected Theorem 23 as follows:*

Let q be a real and continuous function in $[a,b]$. If the fractional boundary value problem (27) has a nontrivial solution, then

$$\int_a^b |q(s)|ds \geq \begin{cases} \dfrac{1}{2-\alpha}, & b-a < \frac{2-\alpha}{\alpha-1}, \\ \dfrac{4(\alpha-1)(b-a)}{((\alpha-1)(b-a)+2-\alpha)^2}, & b-a \geq \frac{2-\alpha}{\alpha-1}. \end{cases}$$

In 2019, Toprakseven [21] established a Lyapunov-type inequality for a boundary value problem with the Caputo–Fabrizio fractional derivative given by

$$\begin{cases} {}^{CF}D_a^{\sigma} y(t) + q(t)y(t) = 0, & a < t < b, \\ y(a) = y'(a) = 0, & y(b) = 0, \end{cases} \tag{28}$$

where ${}^{CF}D_a^{\alpha}$ is the Caputo–Fabrizio fractional derivative of order $\sigma \in (2,3]$ and $q \in C([a,b],\mathbb{R})$.

Lemma 21. *The boundary value problem (28) has a unique solution y if and only if*

$$y(t) = \int_a^b G(t,s)q(s)y(s)ds,$$

where Green's function $G(t,s)$ is given by

$$G(t,s) = \begin{cases} h_1(t,s), & a \leq t \leq s \leq b, \\ h_2(t,s), & a \leq s \leq t \leq b, \end{cases}$$

where

$$h_1(t,s) = h_2(t,s) - \frac{1}{2(b-a)^2}\left[2(3-\sigma)(t-s)(b-a)^2 + (\sigma-2)(t-s)^2(b-a)^2\right],$$

and

$$h_2(t,s) = \frac{1}{2(b-a)^2}\left[2(3-\sigma)(t-a)^2(b-s) + (\sigma-2)(t-a)^2(b-s)^2\right].$$

Moreover, Green's function $G(t,s)$ satisfies the inequality:

$$G(t,s) \leq \frac{1}{2}\Big[2(3-\sigma)(b-a) + (\sigma-2)(b-a)^2\Big], \quad \sigma \in (2,3], \quad t,s \in [a,b].$$

The following theorem contains a Lyapunov-type inequality for the fractional boundary value problem (28).

Theorem 24. *If the boundary value problem (28) has a nonzero solution, then*

$$\int_a^b |q(s)|ds \geq \frac{2}{2(3-\sigma)(b-a) + (\sigma-2)(b-a)^2}.$$

In 2020, Toprakseven [22] studied Lyapunov-type inequalities for fractional boundary value problems with mixed boundary conditions and involving the fractional Caputo–

Fabrizio fractional derivative. He established a Lyapunov-type inequality for the following boundary value problem:

$$\begin{cases} {}^{CF}D_a^{\beta}y(t) + q(t)y(t) = 0, \ \ a \le t \le b, \\ y\,'(a) = y(b) = 0, \end{cases} \tag{29}$$

where ${}^{CF}D_a^{\beta}$ is the Caputo–Fabrizio differential operator of order $\beta \in (1, 2]$.

Furthermore, the following boundary value problem was investigated:

$$\begin{cases} {}^{CF}D_a^{\sigma}y(t) + q(t)y(t) = 0, \ \ a \le t \le b, \\ y(a) = y''(a) = y(b) = 0, \end{cases} \tag{30}$$

where ${}^{CF}D_a^{\sigma}$ is the Caputo–Fabrizio differential operator of order $\sigma \in (2, 3]$.

Lemma 22. *Let $\alpha \in (0, 1], \beta = \alpha + 1$. Assume that the compatibility condition $q(a)y(a) = 0$ holds. Then, y solves the Caputo–Fabrizio fractional boundary value problem (29) if and only if it solves the integral equation:*

$$y(t) = \int_a^b K(t,s)q(s)y(s)ds,$$

where Green's function $G(t,s)$ is given by

$$G(t,s) = \begin{cases} (1-\alpha) + \alpha(b-t), & a \le s \le t \le b, \\ \alpha(b-s), & a \le t \le s \le b, \end{cases}$$

and satisfies the following inequality

$$|G(t,s)| \le (1-\alpha) + \alpha(b-a), \ \ t,s \in [a,b].$$

We state a Lyapunov-type inequality for boundary value problem (29).

Theorem 25. *Assume that $q \in C([a,b], \mathbb{R})$ and the compatibility condition $q(a)y(a) = 0$ is satisfied. If the Caputo–Fabrizio fractional boundary value problem (29) of order $\beta \in (1, 2]$ has a nonzero solution, then*

$$\int_a^b |q(s)|ds > \frac{1}{(1-\alpha) + \alpha(b-a)}, \ \ \alpha \in (0, 1].$$

Now, we consider the boundary value problem (30).

Lemma 23. *The boundary value problem (30) has a solution y if and only if y has the integral representation:*

$$y(t) = \int_a^b H(t,s)q(s)y(s)ds,$$

where Green's function $H(t,s)$ is given by

$$H(t,s) = \begin{cases} h_1(t,s), & a \le t \le s \le b, \\ h_2(t,s), & a \le s \le t \le b. \end{cases}$$

where

$$\begin{aligned} h_1(t,s) &= \frac{1}{2(b-a)}\big[2(1-\alpha)(s-a)(b-t)+\alpha(s-a)(b-t)^2 \\ &\quad -2(1-\alpha)(s-t)(b-a)-\alpha(s-t)^2(b-a)\big], \\ h_2(t,s) &= \frac{1}{2(b-a)}\big[2(1-\alpha)(s-a)(b-t)+\alpha(s-a)(b-t)^2\big], \end{aligned}$$

and $H(t,s)$ *satisfies the following inequality:*

$$|H(t,s)| \le \frac{1-\alpha}{4}(b-a)+\frac{\alpha}{4}(b-a)^2, \;\; t,s\in[a,b], \;\; \alpha\in(0,1].$$

Theorem 26. *If the Caputo–Fabrizio fractional boundary value problem (30) with* $\sigma = \alpha+2$, $\alpha \in (0,1]$ *and* $q \in C([a,b],\mathbb{R})$ *has a nonzero solution, then*

$$\int_a^b |q(s)|ds > \frac{4}{(1-\alpha)(b-a)+\alpha(b-a)^2}.$$

9. Lyapunov-Type Inequalities for Fractional Boundary Value Problems Involving Katugampola Fractional Derivative

In 2021, Lupinska and Schmeidel [23] considered the Katugampola fractional differential equation under the boundary condition involving the Katugampola fractional derivative:

$$\begin{cases} D^{\alpha,\rho}_{a^+}y(t)+q(t)y(t)=0, \;\; 0\le a<t<b, \; 1<\alpha<2, \; 0<\beta\le 1, \\ y(a)=0, \;\; D^{\beta,\rho}_{a^+}y(b)=0, \end{cases} \tag{31}$$

where $D^{\alpha,\rho}_{a^+}, D^{\beta,\rho}_{a^+}$ is the Katugampola fractional derivative and $q:[a,b]\to\mathbb{R}$ is a continuous function.

We start with some basic concepts related to the problem (31). For $c\in\mathbb{R}$, $p\ge 1$, let $X^p_c(a,b)$ denote the space of all complex-valued Lebesgue-measurable functions x on (a,b) with $\|x\|_{X^p_c}<\infty$, where the norm is defined by $\|x\|_{X^p_c} = \left(\int_a^b |t^c x(t)|^p \frac{dt}{t}\right)^{1/p} < \infty$.

Definition 11. *The left-sided Katugampola fractional integral of order* $\alpha>0$ *and* $\rho>0$ *of* $x\in X^p_c(a,b)$ *for* $0<a<t<b<\infty$ *is defined by*

$$({}^\rho I^\alpha_{a^+}x)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t (t^\rho - s^\rho)^{\alpha-1}s^{\rho-1}x(s)ds, \;\; t\in[a,b].$$

Definition 12. *Let* $\alpha>0$, $n=[\alpha]+1$ *and* $\rho>0$. *The left-sided Katugampola fractional derivative is defined, for* $0\le a<t<b\le\infty$, *by*

$$({}^\rho D^\alpha_{a^+}x)(t) = \delta^n_\rho({}^\rho I^{n-\alpha\alpha}_{a^+}x)(t) = \frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)^n\int_a^t \frac{s^{\rho-1}x(s)}{(t^\rho-s^\rho)^{1-n+\alpha}}ds,$$

where $\delta^n_\rho = \left(t^{1-\rho}\frac{d}{dt}\right)^n$.

We note that the Katugampola fractional derivative generalises the Riemann–Liouville and the Hadamard fractional derivatives, as well as the classical derivative of integer order.

Lemma 24. $y\in C([a,b],\mathbb{R})$ *is a solution of (31) if and only if*

$$y(t) = \int_a^b G(t,s)q(s)y(s)ds,$$

where $G(t,s)$ is Green's function given by

$$G(t,s) = \frac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)} \begin{cases} (t^\rho - a^\rho)^{\alpha-1}\Big(\dfrac{b^\rho - a^\rho}{b^\rho - s^\rho}\Big)^{\beta-\alpha+1}, & a \le t \le s \le b, \\ (t^\rho - a^\rho)^{\alpha-1}\Big(\dfrac{b^\rho - a^\rho}{b^\rho - s^\rho}\Big)^{\beta-\alpha+1} - (t^\rho - s^\rho)^{\alpha-1}, & a \le s \le t \le b, \end{cases}$$

which satisfies the following properties:

(i) $G(t,s) \ge 0, \ \ t,s \in [a,b]$,

(ii) $\max_{t\in[a,b]} G(t,s) = G(s,s) \le \dfrac{4^\beta \max\{a^{\rho-1}, b^{\rho-1}\}}{\Gamma(\alpha)}\Big(\dfrac{b^\rho - a^\rho}{4\rho}\Big), \ \ s \in [a,b].$

Theorem 27. *Let q be a real and continuous function and $\alpha > \beta + 1$. If a nontrivial continuous solution of the fractional boundary value problem (31) exists, then*

$$\int_a^b |q(s)|ds \ge \frac{\Gamma(\alpha)}{4^\beta \max\{a^{\rho-1}, b^{\rho-1}\}}\Big(\frac{4\rho}{b^\rho - a^\rho}\Big)^{\alpha-1}.$$

Since the Katugampola fractional derivative for $\rho = 1$ reduces to the Riemann–Liouville fractional derivative D^α_{a+} and to the Hadamard fractional derivative ${}^H D^\alpha_{a+}$ when $\rho \to 0^+$, we obtain the following results.

Corollary 18. *If a nontrivial continuous solution of the fractional boundary value problem:*

$$\begin{cases} D^\alpha_{a+}y(t) + q(t)y(t) = 0, \ \ 0 \le a < t < b, \ 1 < \alpha < 2, \\ y(a) = 0, \ \ D^\beta_{a+}y(b) = 0, \ \ \beta < \alpha - 1, \end{cases}$$

exists, where q is a real and continuous function, then

$$\int_a^b |q(s)|ds \ge \frac{\Gamma(\alpha)}{4^\beta}\Big(\frac{4}{b-a}\Big)^{\alpha-1}.$$

Corollary 19. *If a nontrivial continuous solution of the fractional boundary value problem:*

$$\begin{cases} {}^H D^\alpha_{a+}y(t) + q(t)y(t) = 0, \ \ 0 \le a < t < b, \ 1 < \alpha < 2, \\ y(a) = 0, \ \ {}^H D^\beta_{a+}y(b) = 0, \ \ \beta < \alpha - 1, \end{cases}$$

exists, where q is a real and continuous function, then

$$\int_a^b |q(s)|ds \ge \frac{\Gamma(\alpha)}{4^\beta \max\{a,b\}}\Big(\frac{\ln\frac{b}{a}}{4}\Big)^{1-\alpha}.$$

In 2021, Jarad et al. [24] studied boundary value problems involving generalised Caputo (Katugampola) fractional derivatives

$$\begin{cases} {}^C D^{\alpha,\rho}_{a+}y(t) + q(t)y(t) = 0, \ \ 0 \le a < t < b, \ 1 < \alpha < 2, \\ y(a) = 0, \ \ y(b) = 0, \end{cases} \tag{32}$$

where ${}^C D^{\alpha,\rho}_{a+}$ is the generalized Caputo (Katugampola) fractional derivative and $q : [a,b] \to \mathbb{R}$ is a continuous function.

Lemma 25. *$y \in C([a,b],\mathbb{R})$ is a solution of (32) if and only if*

$$y(t) = \int_a^b G(t,s)p(s)y(s)ds,$$

where $G(t,s)$ is Green's function given by

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \left(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\right)\left(\frac{b^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}, & a \le t \le s \le b, \\ \left(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\right)\left(\frac{b^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} - \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}, & a \le s \le t \le b, \end{cases}$$

and satisfies the following properties:

(i) $\max\{|G(t,s)| : a \le s,t \le b\} \le G(s,s)$ *for* $s \in [a,b]$.

(ii) $G(t,s)$ *has a unique maximum* $G_{\max}$ *in* $[a,b]$, *given by*

$$G_{\max} = \begin{cases} \left(\frac{L - a^\rho}{b^\rho - a^\rho}\right)\left(\frac{b^\rho - L}{\rho}\right)^{\alpha-1} L^{\frac{\rho-1}{L}}, & N = 0, \\ \frac{((1-\alpha\rho)a^\rho + (2\alpha\rho - 1)b^\rho - M)^{\alpha-1}((1-(\alpha+2)\rho)a^\rho + (2\rho-1)b^\rho + M)}{\Gamma(\alpha)(b^\rho - a^\rho)(2N)^{\frac{N}{\rho}}((\alpha\rho-1)a^\rho + (2\rho-1)b^\rho + M)^{\frac{1-\rho}{\rho}}}, & N \ne 0 \end{cases} \tag{33}$$

for all $s \in [a,b]$, *where*

$$L = \left(\frac{(\rho-1)a^\rho b^\rho}{(2\rho+1)b^\rho - a^\rho}\right)^{\frac{1}{\rho}}, \quad N = (\alpha+1)\rho - 1,$$

and

$$M = \Big(((\alpha\rho-1)a^\rho + (2\rho-1)b^\rho)^2 - 4(1-(\alpha+1)\rho)(1-\rho)a^\rho b^\rho\Big)^{\frac{1}{2}}.$$

Theorem 28. *If a nontrivial continuous solution of the problem (32) exists, then*

$$\int_a^b |p(s)|ds > G_{\max},$$

where $G_{\max}$ *is defined in (33).*

10. Lyapunov-Type Inequalities for Hilfer–Katugampola-Type Fractional Boundary Value Problems

In 2021, Zhang et al. [25] studied multi-point boundary value problems involving the Hilfer–Katugampola fractional derivative given by

$$\begin{cases} {}^\rho D_{a^+}^{\alpha,\beta} y(t) + q(t)y(t) = 0, \ \ 0 < a < t < b, \ 1 < \alpha < 2, \ \rho > 0, \\ y(a) = 0, \ \ y(b) = \sum_{i=1}^{m-2} \gamma_i y(\eta_i), \end{cases} \tag{34}$$

and

$$\begin{cases} {}^\rho D_{a^+}^{\alpha,\beta} y(t) + q(t)y(t) = 0, \ \ 0 < a < t < b, \ 1 < \alpha < 2, \ \rho > 0, \\ y(a) = 0, \ \ t^{1-\rho}\frac{d}{dt}y(t)|_{t=b} = \sum_{i=1}^{m-2} \sigma_i y(\xi_i), \end{cases} \tag{35}$$

where ${}^\rho D_{a^+}^{\alpha,\beta}$ is the Hilfer–Katugampola fractional derivative operator of order α and type $\beta \in [0,1]$, $q \in C([a,b],\mathbb{R})$, $\gamma_i, \sigma_i \ge 0$, $a < \eta_i, \xi_i < b$, $(i = 1,2,\ldots,m-2)$, with $a < \eta_1 < \eta_2 < \ldots < \eta_{m-2} < b$, $a < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < b$ such that the following relations hold:

(A_1) $\sum_{i=1}^{m-2} \gamma_i(\eta_i^\rho - a^\rho)^{1-(2-\alpha)(1-\beta)} < (b^\rho - a^\rho)^{1-(2-\alpha)(1-\beta)}$;

(A_2) $\sum_{i=1}^{m-2} \sigma_i(\xi_i^\rho - a^\rho)^{1-(2-\alpha)(1-\beta)} < [1-(2-\alpha)(1-\beta)]\rho(b^\rho - a^\rho)^{1-(2-\alpha)(1-\beta)}$.

Definition 13. *The left-sided Hilfer–Katugampola fractional derivative of order $\alpha > 0$ and type $\beta \in [0,1]$ of a function x is defined by*

$$({}^{\rho}D_{a^+}^{\alpha,\beta}x)(t) = \left({}^{\rho}I_{a^+}^{\beta(n-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)^n {}^{\rho}I_{a^+}^{(1-\beta)(n-\alpha)}x\right)(t),\ n = [\alpha]+1, \rho > 0.$$

Lemma 26. *Assume that (A_1) holds. The boundary value problem (34) has a unique solution y if and only if*

$$y(t) = \int_a^b G(t,s)q(s)y(s)ds + Q(t)\sum_{i=1}^{m-2}\gamma_i\int_a^b G(\eta_i,s)q(s)y(s)ds,\ t \in [a,b],$$

where

$$Q(t) = \frac{(t^\rho - a^\rho)^{1-(2-\alpha)(1-\beta)}}{(b^\rho - a^\rho)^{1-(2-\alpha)(1-\beta)} - \sum_{i=1}^{m-2}\gamma_i(\eta_i^\rho - a^\rho)^{1-(2-\alpha)(1-\beta)}},\ t \in [a,b],$$

and $G(t,s)$ is Green's function given by

$$G(t,s) = \frac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)(b^\rho - a^\rho)^{1-(2-\alpha)(1-\beta)}}\begin{cases} h_1(t,s), & 0 < a \le s \le t \le b, \\ h_2(t,s), & 0 < a \le t \le s \le b, \end{cases}$$

with

$$\begin{aligned} h_1(t,s) &= (t^\rho - a^\rho)^{1-(2-\alpha)(1-\beta)}(b^\rho - s^\rho)^{\alpha-1} \\ &\quad -(b^\rho - a^\rho)^{1-(2-\alpha)(1-\beta)}(t^\rho - s^\rho)^{\alpha-1}, \\ h_2(t,s) &= (t^\rho - a^\rho)^{1-(2-\alpha)(1-\beta)}(b^\rho - s^\rho)^{\alpha-1}. \end{aligned}$$

Moreover, the function $G(t,s)$ satisfies the following properties:

(i) *$G(t,s)$ is continuous on $[a,b]\times[a,b]$.*

(ii) $|G(t,s)| \le \dfrac{(\alpha-1)^{\alpha-1}[\alpha-1+\beta(2-\alpha)]^{\alpha-1+\beta(2-\alpha)}}{[2(\alpha-1)+\beta(2-\alpha)]^{2(\alpha-1)+\beta(2-\alpha)}\Gamma(\alpha)}\rho^{1-\alpha}s^{\rho-1}(b^\rho - a^\rho)^{\alpha-1}$, *for any* $(t,s) \in [a,b]\times[a,b]$.

The Lyapunov-type inequality for the problems (34) is given in the following theorem.

Theorem 29. *Assume that (A_1) holds. If the boundary value problem (34) has a nonzero solution, then*

$$\int_a^b |q(s)|ds \ge \frac{[2(\alpha-1)+\beta(2-\alpha)]^{2(\alpha-1)+\beta(2-\alpha)}\Gamma(\alpha)\rho^{\alpha-1}}{\Delta_1[1+Q(b)\sum_{i=1}^{m-2}\gamma_i]\max\{a^{\rho-1}, b^{\rho-1}\}},$$

where

$$\Delta_1 = (\alpha-1)^{\alpha-1}[\alpha-1+\beta(2-\alpha)]^{\alpha-1+\beta(2-\alpha)}(b^\rho - a^\rho)^{\alpha-1}.$$

Lemma 27. *Assume that (A_2) holds. The boundary value problem (35) has a unique solution y if and only if*

$$y(t) = \int_a^b K(t,s)q(s)y(s)ds + R(t)\sum_{i=1}^{m-2}\sigma_i\int_a^b K(\xi_i,s)q(s)y(s)ds,\ t \in [a,b],$$

where

$$R(t) = \frac{(t^\rho - a^\rho)^{1-(2-\alpha)(1-\beta)}}{[1-(2-\alpha)(1-\beta)]\rho(b^\rho - a^\rho)^{-(2-\alpha)(1-\beta)} - \sum_{i=1}^{m-2}\sigma_i(\xi_i^\rho - a^\rho)^{1-(2-\alpha)(1-\beta)}},$$

for $t \in [a,b]$, and $K(t,s)$ is Green's function given by

$$K(t,s) = \frac{(b^\rho - s^\rho)^{\alpha-2}\rho^{1-\alpha}s^{\rho-1}}{[1-(2-\alpha)(1-\beta)]\Gamma(\alpha)}\begin{cases} k_1(t,s), & 0 < a \le s \le t \le b, \\ k_2(t,s), & 0 < a \le t \le s \le b, \end{cases}$$

with

$$\begin{aligned} k_1(t,s) &= (\alpha-1)(b^\rho - a^\rho)^{(2-\alpha)(1-\beta)}(t^\rho - a^\rho)^{1-(2-\alpha)(1-\beta)} \\ &\quad -[1-(2-\alpha)(1-\beta)]\frac{(t^\rho - s^\rho)^{\alpha-1}}{(b^\rho - s^\rho)^{\alpha-2}} \\ k_2(t,s) &= (\alpha-1)(b^\rho - a^\rho)^{(2-\alpha)(1-\beta)}(t^\rho - a^\rho)^{1-(2-\alpha)(1-\beta)}. \end{aligned}$$

Moreover, the function $K(t,s)$ satisfies the following properties:

(i) $K(t,s)$ *is continuous on* $[a,b] \times [a,b]$.

(ii) $|K(t,s)| \le \frac{(b^\rho - s^\rho)^{\alpha-2}\rho^{1-\alpha}s^{\rho-1}}{[1-(2-\alpha)(1-\beta)]\Gamma(\alpha)}(b^\rho - a^\rho)\max\{\beta(2-\alpha), \alpha-1\}$, *for any* $(t,s) \in [a,b] \times [a,b]$.

The following theorem contains the Lyapunov-type inequality for the problems (35).

Theorem 30. *Assume that (A_2) holds. If the boundary value problem (35) has a nonzero solution, then*

$$\int_a^b (b^\rho - a^\rho)^{\alpha-2}|q(s)|ds \ge \frac{[1-(2-\alpha)(1-\beta)]\rho^{\alpha-1}\Gamma(\alpha)}{\Delta_2[1 + R(b)\sum_{i=1}^{m-2}\sigma_i]},$$

where

$$\Delta_2 = (b^\rho - a^\rho)\max\{\beta(2-\alpha), \alpha-1\}\max\{a^{\rho-1}, b^{\rho-1}\}.$$

11. Lyapunov-Type Inequalities for Higher-Order Left and Right Fractional *p*-Laplacian Boundary Value Problems

In 2021, Cabada and Khaldi [26] obtained a Lyapunov-type inequality for an iterated boundary value problem

$$\begin{cases} -{}^C D^\alpha_{b^-}(\phi_p(D^\beta_{a^+}y(t))) + \chi(t)\phi_p(y(t)) = 0, \ \ 0 < a < t < b, \\ y^{(i)}(a) = D^{\beta+i}_{a^+}y(b) = 0, \end{cases} \tag{36}$$

where $n-1 < \alpha, \beta \le n$, $n \ge 2$, ${}^C D^\alpha_{b^-}$ and $D^\beta_{a^+}$ refer to the right Caputo and left Riemann–Liouville derivative operators, respectively, $\phi_p(s) = s|s|^{p-2}$, $p > 2$, $\phi_p^{-1} = \phi_q$, $1/p + 1/q = 1$, and $\chi : [a,b] \to \mathbb{R}$ is a continuous function.

Let $g \in L^1[a,b]$ and $\alpha > 0$. Then, we define the left and right fractional integrals respectively as

$$\begin{aligned} I^\alpha_{a^+}g(t) &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}g(s)ds, \\ I^\alpha_{b^-}g(t) &= \frac{1}{\Gamma(\alpha)}\int_t^b (t-s)^{\alpha-1}g(s)ds. \end{aligned}$$

For $g \in AC^n[a,b]$, $n-1 < \alpha \leq n,\ n \geq 1$, the left and right Riemann–Liouville fractional derivatives are respectively given by

$$\begin{aligned} D^{\alpha}_{a^+} g(t) &= \frac{d^n}{dt^n}(I^{n-\alpha}_{a^+} g)(t), \\ D^{\alpha}_{b^-} g(t) &= (-1)^n \frac{d^n}{dt^n}(I^{n-\alpha}_{b^-} g)(t), \end{aligned}$$

while the left and right Caputo fractional derivatives are respectively expressed as

$$\begin{aligned} {}^C D^{\alpha}_{a^+} g(t) &= I^{n-\alpha}_{a^+} g^{(n)}(t), \\ {}^C D^{\alpha}_{b^-} g(t) &= (-1)^n I^{n-\alpha}_{b^-} g^{(n)}(t). \end{aligned}$$

Lemma 28. *A function y is a solution of the boundary value problem (36) if and only if y satisfies the integral equation*

$$y(t) = \frac{1}{(\Gamma(\alpha))^{q-1}\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \phi_q \Big(\int_s^1 (r-s)^{\alpha-1} \chi(r) \phi_p(y(r)dr) \Big) ds.$$

The Lyapunov-type inequality for the fractional boundary value problem (36) is the following:

Theorem 31. *Let y be a nontrivial solution of the problem (36). Then the inequality:*

$$\int_a^b |\chi(s)| ds \geq \frac{\Gamma(\alpha+1)(\Gamma(\beta))^{p-1}}{(b-a)^{(p-1)\beta+\alpha-1}} \Big(\frac{\beta(p-1)-1}{p-2} \Big)^{p-2},$$

holds, where $p > 2$ and $(p-1)\beta \geq 1$.

12. Lyapunov-Type Inequalities via Fractional Proportional Derivatives

In 2019, Abdeljawad et al. [27] considered the following fractional proportional boundary value problem:

$$\begin{cases} {}_aD^{\alpha,\rho} y(t) + q(t)y(t) = 0, \ \ a < t < b,\ 1 < \alpha \leq 2, \rho \in [0,1] \\ y(a) = y(b) = 0, \end{cases} \tag{37}$$

where ${}_aD^{\alpha,\rho}$ is proportional fractional derivative and $q : [a,b] \to \mathbb{R}$ is a continuous function.

Definition 14. *Let $\rho \in (0,1]$ and $\alpha \in \mathbb{C}$ with $Re(\alpha) > 0$. Then, the fractional operator*

$${}_aI^{\alpha,\rho} h(t) = \frac{1}{\rho^{\alpha}\Gamma(\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}(t-s)} (t-s)^{\alpha-1} h(s) ds,\ t > a,$$

is called the left-sided generalized proportional integral of order $\alpha > 0$ of the function h.

Definition 15. *The left generalized proportional fractional derivative of order $\alpha > 0$ and $\rho \in (0,1]$ of the function h is defined by*

$${}_aD^{\alpha,\rho} h(t) = \frac{D^{n,\rho}}{\rho^{n-\alpha}\Gamma(n-\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}(t-s)} (t-s)^{n-\alpha-1} h(s) ds,\ \ \alpha \in \mathbb{C},\ Re(\alpha) > 0,$$

where $\Gamma(\cdot)$ indicates the Gamma function and $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of a real number α.

Definition 16. *The left-sided generalised proportional fractional derivative of the Caputo-type of order $\alpha > 0$ and $\rho \in (0, 1]$ of the function $h \in C^n([a, b], \mathbb{R})$ is defined by*

$$ {}_{a}^{C}D^{\alpha,\rho}h(t) = \frac{1}{\rho^{n-\alpha}\Gamma(n-\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{n-\alpha-1} D^{n,\rho}h(s)ds, \;\; \alpha \in \mathbb{C}, Re(\alpha) > 0, $$

provided the right-hand side exists.

Lemma 29. *y is a solution of the fractional boundary value problem (37) if and only if it satisfies the following integral equation:*

$$ y(t) = \int_a^b G(t,s)q(s)y(s)ds, $$

where G is the Green's function defined by

$$ G(t,s) = \frac{e^{\frac{\rho-1}{\rho}(t-s)}}{\rho^{\alpha}\Gamma(\alpha)} \begin{cases} \dfrac{(t-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} - (t-s)^{\alpha-1}, & a \le s \le t \le b, \\ \dfrac{(t-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}}, & a \le t \le s \le b. \end{cases} $$

Moreover, the function $G(t,s)$, under the condition:

$$ \frac{(\alpha-1)\rho}{1-\rho} \ge b - a, \;\; a \ge 0 \tag{38} $$

has the following properties:

(i) $G(t,s) \ge 0$ *for all* $a \le t, s \le b$;
(ii) $\max_{t\in[a,b]} G(t,s) = G(s,s)$ *for all* $s \in [a,b]$;
(iii) $g(s) = G(s,s)$ *has a unique maximum, given by*

$$ \max_{s\in[a,b]} G(s,s) = \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)\rho^{\alpha}4^{\alpha-1}}. $$

Theorem 32. *Assume that (38) holds. If the fractional proportional boundary value has a solution y in the space $C[a,b] = \{f : [a,b] \to \mathbb{R} : f \text{ is continuous}\}$, then*

$$ \int_a^b |q(s)|ds > \frac{\Gamma(\alpha)\rho^{\alpha}4^{\alpha-1}}{(b-a)^{\alpha-1}}. $$

In [27], the authors also established a Lyapunov inequality for the fractional proportional derivatives in the Caputo sense:

$$ \begin{cases} {}_{a}^{C}D^{\alpha,\rho}y(t) + q(t)y(t) = 0, \;\; a < t < b, \; 1 < \alpha \le 2, \rho \in [0,1], \\ y(a) = y(b) = 0. \end{cases} \tag{39} $$

Lemma 30. *y is a solution of the fractional boundary value problem (39) if and only if it satisfies the following integral equation:*

$$ y(t) = \int_a^b G^C(t,s)q(s)y(s)ds, $$

where G^C is the Green function defined by

$$G^C(t,s) = \frac{e^{\frac{\rho-1}{\rho}(t-s)}}{\rho^\alpha \Gamma(\alpha)} \begin{cases} \dfrac{(t-a)(b-s)^{\alpha-1}}{(b-a)} - (t-s)^{\alpha-1}, & a \le s \le t \le b, \\ \dfrac{(t-a)(b-s)^{\alpha-1}}{(b-a)}, & a \le t \le s \le b. \end{cases}$$

Furthermore, the function G^C, under the condition:

$$\frac{\rho}{1-\rho} \ge b-a, \;\; a \ge 0 \tag{40}$$

satisfies

$$|G^C(t,s)| \le \frac{1}{\rho^\alpha \Gamma(\alpha)} \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}(b-a)^{\alpha-1}, \;\; a \le t,s \le b.$$

The maximum is attained when $t = s = \frac{b+(\alpha-1)a}{\alpha}$.

Theorem 33. *If the fractional proportional boundary value problem (39) has a continuous nontrivial solution, then*

$$\int_a^b |q(s)|ds > \frac{(\rho\alpha)^\alpha \Gamma(\alpha)}{(\alpha-1)^{\alpha-1}(b-a)^{\alpha-1}}.$$

Now, we state a Lyapunov inequality in a larger function space by considering the following weighted proportional boundary value problem in the Riemann sense:

$$\begin{cases} {}_aD^{\alpha,\rho}y(t) + (t-a)^{2-\alpha}q(t)y(t) = 0, \;\; a < t < b, \; 1 < \alpha \le 2, \rho \in (0,1] \\ y(a^+) = y(b) = 0, \end{cases} \tag{41}$$

where $q \in C[a,b]$ and

$$\begin{aligned} y \in \Omega &= C^{\alpha,\rho}_{2-\alpha}[a,b] \\ &= \{f : (a,b] : (t-a)^{2-\alpha} f(t) \in C[a,b], (t-a)^{2-\alpha}{}_aD^{\alpha,\rho} f(t) \in C[a,b]\}. \end{aligned}$$

Lemma 31. *The fractional boundary value problem (41) is equivalent to the following integral equation:*

$$y(t) = \int_a^b H_1(t,s)q(s)(s-a)^{2-\alpha}y(s)ds,$$

where

$$\begin{aligned} H_1(t,s) &= (t-a)^{2-\alpha}G(t,s) \\ &= \frac{e^{\frac{\rho-1}{\rho}(t-s)}}{\rho^\alpha \Gamma(\alpha)} \begin{cases} \dfrac{(t-a)(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} - (t-s)^{\alpha-1}(t-a)^{2-\alpha}, & a \le s \le t \le b, \\ \dfrac{(t-a)(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}}, & a \le t \le s \le b, \end{cases} \end{aligned}$$

which, under the condition (38), has the following properties:

(i) $H_1(t,s) \ge 0$ *for all* $a \le t,s \le b$;
(ii) $\max_{t\in[a,b]} H_1(t,s) = H_1(s,s)$ *for all* $s \in [a,b]$;
(iii) $g(s) = H_1(s,s)$ *has a unique maximum, given by*

$$\max_{s\in[a,b]} H_1(s,s) = \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha \Gamma(\alpha)\rho^\alpha}(b-a).$$

Theorem 34. *Assume that (38) holds and the fractional proportional boundary value problem (41) has a solution y in the space* Ω*. Then*

$$\int_a^b |q(s)|ds > \frac{\alpha^\alpha \Gamma(\alpha)\rho^\alpha}{(\alpha-1)^{\alpha-1}(b-a)}.$$

Since $(t-a)^\beta \in C_{\alpha-1}[a,b]$ for all $\beta > 2-\alpha$, we can consider the following weighted proportional boundary value problem in the Riemann sense:

$$\begin{cases} {}_aD^{\alpha,\rho}y(t) + (t-a)^\beta q(t)y(t) = 0, \;\; a < t < b, \; 1 < \alpha \leq 2, \; \beta > 2-\alpha, \; \rho \in (0,1], \\ y(a^+) = y(b) = 0, \end{cases} \tag{42}$$

where $q \in C[a,b]$ and $y \in \Omega$.

Lemma 32. *For* $\beta \geq 2-\alpha$ *and under the condition:*

$$\frac{\rho}{1-\rho}(\beta+\alpha-1) \geq b-a, \tag{43}$$

the function

$$\begin{aligned} H_\beta(t,s) &= (t-a)^\beta G(t,s) \\ &= \frac{e^{\frac{\rho-1}{\rho}(t-s)}}{\rho^\alpha \Gamma(\alpha)} \begin{cases} \dfrac{(t-a)^{\beta+\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} - (t-s)^{\alpha-1}(t-a)^\beta, & a \leq s \leq t \leq b, \\ \dfrac{(t-a)^{\beta+\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}}, & a \leq t \leq s \leq b, \end{cases} \end{aligned}$$

satisfies the following properties:

(i) $H_\beta(t,s) \geq 0$ *for all* $a \leq t,s \leq b$*;*
(ii) $\max_{t\in[a,b]} H_\beta(t,s) = H_\beta(s,s)$ *for all* $s \in [a,b]$*;*
(iii) $h_\beta(s) = H_\beta(s,s)$ *has a unique maximum, given by*

$$\begin{aligned} \max_{s\in[a,b]} H_\beta(s,s) &= \frac{1}{\Gamma(\alpha)\rho^\alpha(b-a)^{\alpha-1}}\left(\frac{(1-\alpha-\beta)a+(\beta+\alpha-1)b}{2\alpha+\beta-2}\right)^{\beta+\alpha-1} \\ &\quad \times \left(\frac{(\alpha-1)(b-a)}{2\alpha+\beta-2}\right)^{\alpha-1}. \end{aligned}$$

Theorem 35. *Assume that (43) holds and the fractional proportional boundary value problem (42) has a nontrivial solution y in the space* $\Omega_\beta \supseteq \Omega$*. Then,*

$$\int_a^b |q(s)|ds > \frac{1}{h_\beta(s^*)}, \quad s^* = \frac{(\alpha-1)a+(\beta+\alpha-1)b}{\alpha}.$$

13. Conclusions

In this paper, we presented a survey of recent results on Lyapunov-type inequalities for boundary value problems of fractional differential equations. The fractional boundary value problems considered in this survey include different kinds of fractional derivatives and boundary conditions. This survey was prepared keeping in mind the theoretical and practical importance of the inequalities in the context of fractional-order boundary value problems. We believe that the present survey will provide a platform for the researchers working on Lyapunov-type inequalities to learn about the available work on the topic before developing the new results for new fractional boundary value problems.

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