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The Sharp Bounds of Hankel Determinants for the Families of Three-Leaf-Type Analytic Functions

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Abstract: The theory of univalent functions has shown strong significance in the field of mathematics. It is such a vast and fully applied topic that its applications exist in nearly every field of applied sciences such as nonlinear integrable system theory, fluid dynamics, modern mathematical physics, the theory of partial differential equations, engineering, and electronics. In our present investigation, two subfamilies of starlike and bounded turning functions associated with a three-leaf-shaped domain were considered. These classes are denoted by $\mathcal{BT}_{3l}$ and $\mathcal{S}^*_{3l}$, respectively. For the class $\mathcal{BT}_{3l}$, we study various coefficient type problems such as the first four initial coefficients, the Fekete–Szegö and Zalcman type inequalities and the third-order Hankel determinant. Furthermore, the existing third-order Hankel determinant bounds for the second class will be improved here. All the results that we are going to prove are sharp.

Keywords: analytic (or holomorphic) functions; univalent functions; subordination principle; Schwarz function; coefficient bounds; Hankel determinant

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1. Introduction, Definitions and Preliminaries

For a better understanding of the work studied in this article, we have to provide certain elementary geometric function theory literature. In this regard, we first express the classes of normalized analytic and univalent functions by the letters $\mathcal{A}$ and $\mathcal{S}$, respectively. These classes are defined in the following set-builder form by

$$\mathcal{A} := \left\{ f : f \in \mathcal{H}(\mathbb{E}) \quad \text{and} \quad f(z) = \sum_{j=1}^{\infty} a_j z^j \quad (a_1 = 1) \right\} \tag{1}$$

and

$$\mathcal{S} := \{ f : f \in \mathcal{A} \quad \text{and} \quad f \text{ is univalent in } \mathbb{E} \},$$

where $\mathcal{H}(\mathbb{E})$ stands for the set of analytic functions in the region $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$. The set $\mathcal{S}$ was developed by Köebe [1] in 1907, and it has become a key component of advanced study in this subject. Later in 1916, Bieberbach [2] conjectured the coefficient estimate for the class $\mathcal{S}$ and proved it for the second coefficient. The proof of this conjecture attracted researchers, whose work developed this field immensely. In 1985, de-Branges [3] proved this famous conjecture. From 1916 to 1985, many of the world's most distinguished scholars sought to prove or disprove this claim. As a result, they investigated a number of subfamilies of the class $\mathcal{S}$ of univalent functions that are associated with various image domains. The most fundamental and significant subclasses of the set $\mathcal{S}$ are the families of starlike and convex functions, represented by $\mathcal{S}^*$ and $\mathcal{C}$, respectively.

It is worth noting that Aleman and Constantin [4] recently gave a beautiful interaction between univalent function theory and fluid dynamics. In fact, they demonstrated a simple method for how to use a univalent harmonic map to obtain explicit solutions of incompressible two-dimensional Euler equations.

For the given functions $g_1, g_2 \in \mathcal{A}$, we say $g_1 \prec g_2$, if an analytic function v exists in $\mathbb{E}$ with the restrictions $v(0) = 0$ and $|v(z)| < 1$ such that $g_1(z) = g_2(v(z))$. If g_2 in $\mathbb{E}$ is univalent, then we have the following relationship given by

$$g_1(z) \prec g_2(z) \quad (z \in \mathbb{E}) \quad \Longleftrightarrow \quad g_1(\mathbb{E}) \subset g_2(\mathbb{E}) \text{ with } g_1(0) = g_2(0).$$

In 1992, Ma and Minda [5] presented a unified version of the class $\mathcal{S}^*(\psi)$ using subordination terminology. They introduce the $\mathcal{S}^*(\psi)$ defined by

$$\mathcal{S}^*(\psi) := \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \psi(z) \;\; (z \in \mathbb{E}) \right\}, \tag{2}$$

where ψ is a univalent function with $\psi'(0) > 0$ and $\Re\psi > 0$. In addition, the region $\psi(\mathbb{E})$ is star-shaped about the point $\psi(0) = 1$ and is symmetric along the real line axis. They focused on distortion, growth, and covering theorems, among other interesting properties of functions in this class. Later in 2007, Rosihan et al. [6] determined the sharp bounds of problems involving coefficients for a generalized class of Ma-Minda type starlike functions. The class $\mathcal{S}^*(\psi)$ unifies various sub-families of starlike functions, which are attained by an appropriate choice of ψ. For instance:

(i). By choosing the function

$$\psi(z) = \frac{1 + \mathcal{M}z}{1 + \mathcal{N}z} \qquad (\mathcal{M} \in \mathbb{C}, \; -1 \leqq \mathcal{N} \leqq 0, \; \mathcal{M} \neq \mathcal{N}),$$

we achieve the class

$$\mathcal{S}^*[\mathcal{M}, \mathcal{N}] \equiv \mathcal{S}^*\left(\frac{1 + \mathcal{M}z}{1 + \mathcal{N}z}\right)$$

which was studied in [7]. The above described class, with the limitation $-1 \leq \mathcal{N} < \mathcal{M} \leq 1$, represents the class of Janowski starlike functions investigated in [8]. The special case by taking $\mathcal{M} = 1 - 2\xi_1$ and $\mathcal{N} = -1$ with $0 \leq \xi_1 < 1$ leads to the class $\mathcal{S}^*(\xi_1) \equiv \mathcal{S}^*[1 - 2\xi_1, -1]$ of starlike function of order ξ_1.

(ii). The below listed class

$$\mathcal{SS}^*(\xi_2) \equiv \mathcal{S}^*(\psi(z)), \text{with } \psi(z) = \left(\frac{1 + z}{1 - z}\right)^{\xi_2},$$

for $0 < \xi_2 \leqq 1$ was introduced as the collection of strongly starlike functions of order ξ_2 investigated in [9].

(iii). In [10], Sharma et al. discussed the class $\mathcal{S}^*_{car}$ defined by

$$\mathcal{S}^*_{car} := \mathcal{S}^*(\psi(z)) \qquad \left(\psi(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2\right).$$

Geometrically, it is a subclass of functions $f \in \mathcal{A}$ with

$$Q(z) = \frac{zf'(z)}{f(z)}$$

contained in the cardioid domain given by

$$(9x^2 + 9y^2 - 18x + 5)^2 - 16(9x^2 + 9y^2 - 6x + 1) = 0.$$

(iv). By setting $\psi(z) = 1 + \sin z$, we attain the class $\mathcal{S}^*(\psi(z)) \equiv \mathcal{S}^*_{\sin}$ of starlike functions connected with the eight-shaped domain which was introduced by Cho et al. [11]. Moreover, the below mentioned classes

$$\mathcal{S}^*_{\cos} \equiv \mathcal{S}^*(\cos z) \ \ \& \ \ \mathcal{S}^*_{\cosh} \equiv \mathcal{S}^*(\cosh z),$$

were analyzed, respectively, by Raza and Bano [12] and Alotaibi et al. [13].

(v). By picking $\psi(z) = 1 + \tanh z$, we obtain the class $\mathcal{S}^*_{\tanh}$

$$\mathcal{S}^*_{\tanh} := \mathcal{S}^*(1 + \tanh z),$$

which was established by Ullah et al. [14]. Moreover, they examined the radii results for the class $\mathcal{S}^*_{\tanh}$. Further, in [15] the authors computed third-order Hankel determinant sharp bounds for this class.

Finding bounds for the function coefficients in a given collection has been one of the most fundamental problems in geometric function theory since it impacts geometric features. The constraint on the second coefficient, for example, provides the growth and distortion features. The general form of the Hankel determinant $\Delta_{q,n}(f)$ (with $n, q \in \mathbb{N} = \{1, 2, \ldots\}$) for the function $f \in \mathcal{S}$ was explored by Pommerenke [16,17] in the form of

$$\Delta_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

In fact, the determinants listed below are referred to as first-, second-, and third-order Hankel determinants, respectively.

$$\Delta_{2,1}(f) = a_3 - a_2^2, \tag{3}$$
$$\Delta_{2,2}(f) = a_2 a_4 - a_3^2, \tag{4}$$
$$\Delta_{3,1}(f) = 2a_2 a_3 a_4 - a_3^3 - a_4^2 + a_3 a_5 - a_2^2 a_5. \tag{5}$$

Only a few articles on the Hankel determinant for the class $\mathcal{S}$ can be found in the literature. The earliest recorded sharp inequality for $f \in \mathcal{S}$ is stated by

$$|\Delta_{2,n}(f)| \leqq |\nu|\sqrt{n}, \ \nu \text{ is a constant.}$$

This outcome is due to Hayman [18]. Likewise, for the same set $\mathcal{S}$, the following bounds were derived in [19]:

$$|\Delta_{2,2}(f)| \leq \nu \qquad \left(1 \leq \nu \leqq \frac{11}{3}\right),$$

and

$$|\Delta_{3,1}(f)| \leq \nu \qquad \left(\frac{4}{9} \leqq \nu \leq \frac{32 + \sqrt{285}}{15}\right).$$

The problem of determining the sharp inequalities of Hankel determinants for a certain set of functions attracted the minds of many experts. Janteng et al. [20,21] computed the sharp bound of $|\Delta_{2,2}(f)|$ for the $\mathcal{S}$ sub-families $\mathcal{C}$, $\mathcal{S}^*$ and $\mathcal{BT}$, where $\mathcal{C}$ and $\mathcal{BT}$ are the sets of convex and bounded turning functions. These results are based on estimations provided by

$$|\Delta_{2,2}(f)| \leq \begin{cases} \frac{1}{8}, & \text{for} \quad f \in \mathcal{C}, \\ 1, & \text{for} \quad f \in \mathcal{S}^*, \\ \frac{4}{9}, & \text{for} \quad f \in \mathcal{BT}. \end{cases}$$

For the following two families $\mathcal{S}^*(\xi_1)$ $(0 \leq \xi_1 < 1)$, and $\mathcal{SS}^*(\xi_2)$ $(0 < \xi_2 \leq 1)$, the experts [22,23] obtained that $|\Delta_{2,2}(f)|$ is bounded by $(1-\xi_1)^2$ and ξ_2^2, respectively. This problem was also investigated for different families of bi-univalent functions in [24–29].

The formulae in (3)–(5) make it quite evident that calculating the bound for $|\Delta_{3,1}(f)|$ is significantly more difficult than calculating the bound for $|\Delta_{2,2}(f)|$. Babalola [30] was the first mathematician who studied third-order Hankel determinant for the $\mathcal{C}$, $\mathcal{S}^*$ and $\mathcal{BT}$ families. Though after the Babalola's article, several papers appeared on obtaining the bounds of the determinant $|\Delta_{3,1}(f)|$ for various subclasses of analytic functions. However, Zaprawa's article [31] gained the attention of the readers in which he enhanced Babalola's conclusions by employing a new approach to demonstrate that

$$|\Delta_{3,1}(f)| \leq \begin{cases} \frac{49}{540}, & \text{for} \quad f \in \mathcal{C}, \\ 1, & \text{for} \quad f \in \mathcal{S}^*, \\ \frac{41}{60}, & \text{for} \quad f \in \mathcal{BT}. \end{cases}$$

In addition, he points out that such bounds are not sharp. Later, in 2018, Kwon et al. [32] enhanced the Zaprawa inequality for $f \in \mathcal{S}^*$ by achieving $|\Delta_{3,1}(f)| \leq \frac{8}{9}$, and Zaprawa et al. [33] polished this bound even further in 2021 by proving that $|\Delta_{3,1}(f)| \leq \frac{5}{9}$ for $f \in \mathcal{S}^*$. Moreover, for q-starlike type functions classes, such problems were determined in [34]. Furthermore, the non-sharp bounds of this determinant for the sets $\mathcal{S}^*_{\sin}$ and $\mathcal{S}^*_{car}$ were also computed in the articles [35,36], respectively. They achieved

$$|\Delta_{3,1}(f)| \leq \begin{cases} 0.51856, & \text{for} \quad f \in \mathcal{S}^*_{\sin}, \\ 1.1989, & \text{for} \quad f \in \mathcal{S}^*_{car}. \end{cases}$$

The sharp bounds of the determinant have been sought by many experts, but none have succeeded. Eventually, in 2018, Kowalczyk et al. [37] and Lecko et al. [38] achieved the following sharp bounds of $|\Delta_{3,1}(f)|$ for the sets $\mathcal{C}$ and $\mathcal{S}^*\left(\frac{1}{2}\right)$:

$$|\Delta_{3,1}(f)| \leq \begin{cases} \frac{4}{135}, & \text{for} \quad f \in \mathcal{C}, \\ \frac{1}{9}, & \text{for} \quad f \in \mathcal{S}^*\left(\frac{1}{2}\right). \end{cases}$$

Barukab et al. [39], in the year 2021, computed the sharp bounds of $|\Delta_{3,1}(f)|$ for functions of bounded turning set related with the petal-shaped domain. Later at the end of 2021, Ullah et al. [15] and Wang et al. [40] contributed the following sharp bounds of this determinant:

$$|\Delta_{3,1}(f)| \leq \begin{cases} \frac{1}{16}, & \text{for} \quad f \in \mathcal{BT}_L, \\ \frac{1}{9}, & \text{for} \quad f \in \mathcal{S}^*_{\tanh}, \end{cases}$$

where the family $\mathcal{BT}_L$ is given by

$$\mathcal{BT}_L = \left\{ f \in \mathcal{A} : \quad f'(z) \prec \sqrt{1+z} \qquad (z \in \mathbb{E}) \right\}.$$

The interested readers can look at the work of Srivastava et al. [41] for further contributions in this area. They successfully obtained the bounds of the fourth-order Hankel determinant for various analytic and univalent functions.

In [42], Gandhi introduced a subclass of starlike functions defined by

$$\mathcal{S}^*_{3l} := \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec 1 + \frac{4}{5}z + \frac{1}{5}z^4 \qquad (z \in \mathbb{E}) \right\}.$$

For functions belonging to this class, it means that $\frac{zf'(z)}{f(z)}$ lie in a three-leaf-shaped region in the right-half plane. From the definition of the family $\mathcal{S}^*_{3l}$, the authors [42] deduced that:

$$f \in \mathcal{S}^*_{3l} \Leftrightarrow f(z) = z\exp\left(\int_0^z \frac{u(t)-1}{t}dt\right), \tag{6}$$

for some $u(z) \prec u_0(z)$. By substituting $u(z) = u_0(z)$ in (6), we acquire the function

$$f_0(z) = z\exp\left(\int_0^z \left(\frac{4}{5} + \frac{1}{5}t^3\right)dt\right) = z + \frac{4}{5}z^2 + \cdots. \tag{7}$$

Similar to the definition of $\mathcal{S}^*_{3l}$, we now define a new subfamily of bounded turning functions by the following set builder notation:

$$\mathcal{BT}_{3l} := \left\{f : f \in \mathcal{A} \quad \text{and} \quad f'(z) \prec 1 + \frac{4}{5}z + \frac{1}{5}z^4 \qquad (z \in \mathbb{E})\right\}. \tag{8}$$

For $g(z) = z$, it can be noted that $\mathcal{BT}_{3l}$ is a subclass of functions $f \in \mathcal{S}$ satisfying the condition

$$\frac{zf'(z)}{g(z)} \prec 1 + \frac{4}{5}z + \frac{1}{5}z^4 \qquad (z \in \mathbb{E}).$$

In [43], Shi et al. gave some coefficient estimates on functions belonging to the class $\mathcal{S}^*_{3l}$. However, the bound $|\Delta_{3,1}(f)| \leq \frac{242}{1125}$ that they obtained for the third Hankel determinant is not sharp. In the current paper, we aim to prove some sharp bounds on the coefficient problems associated with $\mathcal{S}^*_{3l}$ and $\mathcal{BT}_{3l}$ using a new method. The main results are organized as follows. The first part ais coefficient problems connected with the newly defined subclass $\mathcal{BT}_{3l}$ of bounded turning functions. In the second part, we give some sharp bounds of third Hankel determinant for the functions in the class $\mathcal{S}^*_{3l}$ which improve the known results.

2. A Set of Lemmas

Before stating the results that are applied in the main contributions, we define the class $\mathcal{P}$ in terms of a set-builder notation:

$$\mathcal{P} = \left\{q \in \mathcal{A} : \quad q(z) \prec \frac{1+z}{1-z} \qquad (z \in \mathbb{E})\right\},$$

where the function q has the below series form:

$$q(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (z \in \mathbb{E}). \tag{9}$$

The subsequent Lemma is essential for the proofs of our main findings. It includes the well-known c_2 formula [44], the c_3 formula credited to Libera and Złotkiewicz [45], and the c_4 formula proven in [46].

Lemma 1. *Let $q \in \mathcal{P}$ be in the form of (9). Then, for $x, \sigma, \rho \in \overline{\mathbb{E}}$ we have*

$$2c_2 = c_1^2 + x\left(4 - c_1^2\right), \tag{10}$$

$$4c_3 = c_1^3 + 2\left(4 - c_1^2\right)c_1 x - c_1\left(4 - c_1^2\right)x^2 + 2\left(4 - c_1^2\right)\left(1 - |x|^2\right)\sigma, \tag{11}$$

$$\begin{aligned} 8c_4 = {} & c_1^4 + (4 - c_1^2)x\left[c_1^2\left(x^2 - 3x + 3\right) + 4x\right] - 4(4 - c_1^2)(1 - |x|^2) \\ & \left[c(x-1)\sigma + \overline{x}\sigma^2 - (1 - |\sigma|^2)\rho\right]. \end{aligned} \tag{12}$$

Lemma 2. *If $q \in \mathcal{P}$ has the form of (9), then*

$$|c_n| \leq 2 \qquad (n \geq 1). \tag{13}$$

and

$$|c_{n+k} - \mu c_n c_k| \leq 2\max(1, |2\mu - 1|) = \begin{cases} 2 & for \quad 0 \leqq \mu \leqq 1; \\ 2|2\mu - 1| & otherwise. \end{cases}. \tag{14}$$

Moreover, if $B \in [0,1]$ with $B(2B-1) \leq D \leq B$, we have

$$\left| c_3 - 2Bc_1c_2 + Dc_1^3 \right| \leq 2. \tag{15}$$

The inequalities (13)–(15) in the last Lemma are taken from [44,47] and [35,36], respectively.

Lemma 3 ([48])**.** *Let α, β, γ and a satisfy the inequalities $0 < \alpha < 1, 0 < a < 1$ and*

$$8a(1-a)\left((\alpha\beta - 2\gamma)^2 + (\alpha(a+\alpha) - \beta)^2\right) + \alpha(1-\alpha)(\beta - 2a\alpha)^2 \leqq 4a\alpha^2(1-\alpha)^2(1-a). \tag{16}$$

If $q \in \mathcal{P}$ is of the form (9), then

$$\left| \gamma c_1^4 + a c_2^2 + 2\alpha c_1 c_3 - \frac{3}{2}\beta c_1^2 c_2 - c_4 \right| \leq 2.$$

3. Coefficient Inequalities and Second Hankel Determinant for the Function Class BT_{3l}

Theorem 1. *If $f \in \mathcal{BT}_{3l}$ has the series expansion (1), then*

$$|a_2| \leq \frac{2}{5}, \tag{17}$$

$$|a_3| \leq \frac{4}{15}, \tag{18}$$

$$|a_4| \leq \frac{1}{5}, \tag{19}$$

$$|a_5| \leq \frac{4}{25}. \tag{20}$$

These bounds are sharp with the extremal functions given by

$$\begin{aligned}
f_1(z) &= \int_0^z \left(1 + \frac{4}{5}(t) + \frac{1}{5}\left(t^4\right)\right)dt = z + \frac{2}{5}z^2 + \frac{1}{25}z^5, \\
f_2(z) &= \int_0^z \left(1 + \frac{4}{5}\left(t^2\right) + \frac{1}{5}\left(t^8\right)\right)dt = z + \frac{4}{15}z^3 + \frac{1}{45}z^9, \\
f_3(z) &= \int_0^z \left(1 + \frac{4}{5}\left(t^3\right) + \frac{1}{5}\left(t^{12}\right)\right)dt = z + \frac{1}{5}z^4 + \frac{1}{65}z^{13}, \\
f_4(z) &= \int_0^z \left(1 + \frac{4}{5}\left(t^4\right) + \frac{1}{5}\left(t^{16}\right)\right)dt = z + \frac{4}{25}z^5 + \frac{1}{85}z^{17}.
\end{aligned}$$

Proof. Let $f \in \mathcal{BT}_{3l}$. Then from the definition, there exists a Schwarz function ω such that

$$f'(z) = 1 + \frac{4}{5}\omega(z) + \frac{1}{5}(\omega(z))^4, \quad (z \in \mathbb{E}).$$

Suppose that $p \in \mathcal{P}$ be described in terms of the Schwarz function ω as

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots. \tag{21}$$

Equivalently, we have

$$\omega(z) = \frac{p(z)-1}{p(z)+1} = \frac{c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots}. \tag{22}$$

From (1), we obtain

$$f'(z) = 1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + 5a_5 z^4 + \cdots. \tag{23}$$

Using the series expansion of (22), we obtain

$$\begin{aligned} 1 + \frac{4}{5}\omega(z) + \frac{1}{5}\omega(z)^4 &= 1 + \left(\frac{2}{5}c_1\right)z + \left(\frac{2}{5}c_2 - \frac{1}{5}c_1^2\right)z^2 + \left(\frac{1}{10}c_1^3 - \frac{2}{5}c_1c_2 + \frac{2}{5}c_3\right)z^3 \\ &\quad + \left(-\frac{3}{80}c_1^4 + \frac{3}{10}c_1^2c_2 - \frac{1}{5}c_2^2 - \frac{2}{5}c_1c_3 + \frac{2}{5}c_4\right)z^4 + \cdots. \end{aligned} \tag{24}$$

Comparing (23) and (24), we find that

$$a_2 = \frac{1}{5}c_1, \tag{25}$$

$$a_3 = \frac{1}{3}\left(\frac{2}{5}c_2 - \frac{1}{5}c_1^2\right), \tag{26}$$

$$a_4 = \frac{1}{4}\left(\frac{1}{10}c_1^3 - \frac{2}{5}c_1c_2 + \frac{2}{5}c_3\right), \tag{27}$$

$$a_5 = \frac{1}{5}\left(-\frac{3}{80}c_1^4 + \frac{3}{10}c_1^2c_2 - \frac{1}{5}c_2^2 - \frac{2}{5}c_1c_3 + \frac{2}{5}c_4\right). \tag{28}$$

The inequalities on a_2, a_3 and a_4 follow directly by using Lemma 2. For a_5, we can rewrite (28) as

$$\begin{aligned} a_5 &= -\frac{2}{25}\left(\frac{3}{32}c_1^4 + \left(\frac{1}{2}\right)c_2^2 + 2\left(\frac{1}{2}\right)c_1c_3 - \frac{3}{2}\left(\frac{1}{2}\right)c_1^2c_2 - c_4\right). \\ &= -\frac{2}{25}\left(\gamma c_1^4 + ac_2^2 + 2\alpha c_1c_3 - \frac{3}{2}\beta c_1^2c_2 - c_4\right), \end{aligned} \tag{29}$$

where

$$\gamma = \frac{3}{32}, \quad a = \frac{1}{2}, \quad \alpha = \frac{1}{2}, \quad \beta = \frac{1}{2}.$$

It can be easily verified that $0 < \alpha < 1$, $0 < a < 1$ and

$$8a(1-a)\left((\alpha\beta - 2\gamma)^2 + (\alpha(a+\alpha) - \beta)^2\right) + \alpha(1-\alpha)(\beta - 2a\alpha)^2 \le 4a\alpha^2(1-\alpha)^2(1-a).$$

Therefore, from Lemma 3 we have

$$|a_5| \le \frac{4}{25}.$$

The proof of Theorem 1 is thus completed. □

Theorem 2. *Let $\gamma \in \mathbb{C}$ and $f \in \mathcal{BT}_{3l}$ be the form of (1). Then the sharp bound of the Fekete–Szegö inequality is*

$$\left|a_3 - \gamma a_2^2\right| \le \max\left\{\frac{4}{15}, \frac{12}{75}|\gamma|\right\}.$$

Proof. By employing (25) and (26), we have

$$\left|a_3 - \gamma a_2^2\right| = \left|\frac{2}{15}c_2 - \frac{1}{15}c_1^2 - \gamma\frac{1}{25}c_1^2\right|.$$

An application of Lemma 2 leads to the desired result. The inequality is sharp with the extremal function given by

$$f_2(z) = \int_0^z \left(1 + \frac{4}{5}\left(t^2\right) + \frac{1}{5}\left(t^8\right)\right)dt = z + \frac{4}{15}z^3 + \frac{1}{45}z^9.$$

□

Theorem 3. *If $f \in \mathcal{BT}_{3l}$ has the form (1), then*

$$|a_2a_3 - a_4| \le \frac{1}{5}.$$

This result is sharp.

Proof. Using (25)–(27), we have

$$|a_2a_3 - a_4| = \frac{1}{10}\left|c_3 - 2\left(\frac{19}{30}\right)c_1c_2 + \frac{23}{60}c_1^3\right|.$$

Let $B = \frac{19}{30}$ and $D = \frac{23}{60}$. It can seen that $0 \le B \leqq 1$, $B \ge D$ and

$$B(2B-1) = \frac{38}{225} \le D = \frac{23}{60}.$$

Applying Lemma 2, we obtain the inequality in Theorem 3. This result is sharp with the extremal function given by

$$f_3(z) = \int_0^z \left(1 + \frac{4}{5}\left(t^3\right) + \frac{1}{5}\left(t^{12}\right)\right)dt = z + \frac{1}{5}z^4 + \frac{1}{65}z^{13}.$$

□

The second-order Hankel determinant $\Delta_{2,2}(f)$ for $f \in \mathcal{BT}_{3l}$ will be estimated next.

Theorem 4. *If $f \in \mathcal{BT}_{3l}$, then*

$$|\Delta_{2,2}(f)| = \left|a_2a_4 - a_3^2\right| \le \frac{16}{225}.$$

The result is sharp.

Proof. By the virtue of (25)–(27), we have

$$\Delta_{2,2}(f) = \frac{1}{1800}c_1^4 - \frac{4}{225}c_2^2 + \frac{1}{50}c_1c_3 - \frac{1}{450}c_1^2c_2.$$

Using (10) and (11) to express c_2 and c_3 in terms of $c_1 = c$ and x, σ in $\mathbb{E}$, we obtain

$$|\Delta_{2,2}(f)| = \left|-\frac{1}{200}c^2\left(4-c^2\right)x^2 - \frac{1}{225}\left(4-c^2\right)^2x^2 + \frac{1}{100}c\left(4-c^2\right)\left(1-|x|^2\right)\sigma\right|.$$

Applying the triangle inequality and using $|x| = b$, $|\sigma| \le 1$, we have

$$|\Delta_{2,2}(f)| \le \frac{1}{200}c^2\left(4-c^2\right)b^2 + \frac{1}{225}\left(4-c^2\right)^2b^2 + \frac{1}{100}c\left(4-c^2\right)\left(1-b^2\right) := \Theta(c,b).$$

It is an easy task to illustrate that $\frac{\partial\Theta}{\partial b} \geq 0$ for $b \in [0,1]$. This means that $\Theta(c,b) \leq \Theta(c,1)$. Thus

$$|\Delta_{2,2}(f)| \leq \frac{1}{200}c^2\left(4-c^2\right) + \frac{1}{225}\left(4-c^2\right)^2 := \varpi(c).$$

By observing that $\varpi'(c) < 0$ for $c \in [0,2]$, we see that $\varpi(c) \leq \varpi(0)$. Thus, we have

$$|\Delta_{2,2}(f)| \leq \frac{16}{225}.$$

Equality is attained by the function given by

$$f_2(z) = \int_0^z \left(1 + \frac{4}{5}\left(t^2\right) + \frac{1}{5}\left(t^8\right)\right)dt = z + \frac{4}{15}z^3 + \frac{1}{45}z^9.$$

□

4. Results on the Third Hankel Determinant of Functions $f \in \mathcal{BT}_{3l}$

Now we study the determinant $\Delta_{3,1}(f)$ for $f \in \mathcal{BT}_{3l}$.

Theorem 5. *If $f \in \mathcal{BT}_{3l}$ has the series expansion (1), then*

$$|\Delta_{3,1}(f)| \leq \frac{1}{25}.$$

The bound is sharp.

Proof. Let $c_1 = c$ and put the values of a_i's from (25)–(28) into (5), we obtain that

$$\begin{aligned}\Delta_{3,1}(f) \;=\; & \frac{1}{1080000}\Big(-211c^6 - 192c^4c_2 + 936c^3c_3 + 528c^2c_2^2 - 9216c^2c_4 \\ & +15840cc_2c_3 - 8320c_2^3 + 11520c_2c_4 - 10800c_3^2\Big).\end{aligned} \tag{30}$$

For some $\rho, x, \sigma \in \overline{\mathbb{E}}$, by substituting $t = 4 - c^2$ in (10)–(12), we have

$$\begin{aligned}
192c^4c_2 &= 96\left(c^6 + c^4tx\right), \\
936c^3c_3 &= -234c^4tx^2 + 468c^3t\left(1-|x|^2\right)\sigma + 468c^4tx + 234c^6, \\
528c^2c_2^2 &= 132c^6 + 264c^4tx + 132c^2t^2x^2, \\
9216c^2c_4 &= 1152c^4tx^3 - 4608c^3tx\left(1-|x|^2\right)\sigma - 4608c^2t\overline{x}\left(1-|x|^2\right)\sigma^2 \\
&\quad -3456c^4tx^2 + 4608c^2t\left(1-|x|^2\right)\left(1-|\sigma|^2\right)\rho + 4608c^3t \\
&\quad \left(1-|x|^2\right)\sigma + 3456c^4tx + 1152c^6 + 4608c^2tx^2, \\
15840cc_2c_3 &= -1980c^2t^2x^3 - 1980c^4tx^2 + 3960ct^2x\left(1-|x|^2\right)\sigma + 3960c^2t^2x^2 \\
&\quad + 3960c^3t\left(1-|x|^2\right)\sigma + 5940c^4tx + 1980c^6, \\
8320c_2^3 &= 1040c^6 + 3120c^4tx + 3120c^2t^2x^2 + 1040t^3x^3,
\end{aligned}$$

$$\begin{aligned}
11520c_2c_4 &= 2880c^2tx^2 + 2880t^2x^3 + 720c^6 + 2880c^4tx + 2880c^3t\left(1-|x|^2\right)\sigma \\
&\quad + 2880c^2t\left(1-|x|^2\right)\left(1-|\sigma|^2\right)\rho + 2160c^2t^2x^2 + 2880ct^2x \\
&\quad \left(1-|x|^2\right)\sigma + 2880t^2x\left(1-|x|^2\right)\left(1-|\sigma|^2\right)\rho - 2160c^4tx^2 \\
&\quad - 2880c^2t\overline{x}\left(1-|x|^2\right)\sigma^2 - 2880c^3tx\left(1-|x|^2\right)\sigma - 2160c^2t^2x^3 \\
&\quad - 2880t^2x\overline{x}\left(1-|x|^2\right)\sigma^2 - 2880ct^2x^2\left(1-|x|^2\right)\sigma \\
&\quad + 720c^4tx^3 + 720c^2t^2x^4, \\
10800c_3^2 &= 675c^2t^2x^4 - 2700ct^2x^2\left(1-|x|^2\right)\sigma - 2700c^2t^2x^3 - 1350c^4tx^2 \\
&\quad + 2700t^2\left(1-|x|^2\right)^2\sigma^2 + 5400ct^2x\left(1-|x|^2\right)\sigma + 2700c^2t^2x^2 \\
&\quad + 2700c^3t\left(1-|x|^2\right)\sigma + 2700c^4tx + 675c^6.
\end{aligned}$$

Putting the above expressions in (30) yields to

$$\begin{aligned}
\Delta_{3,1}(f) &= \frac{1}{1080000}\Big\{-108c^6 + 1728c^3tx\left(1-|x|^2\right)\sigma + 1728c^2t\overline{x}\left(1-|x|^2\right)\sigma^2 - 1728c^2t \\
&\quad \left(1-|x|^2\right)\left(1-|\sigma|^2\right)\rho - 180ct^2x^2\left(1-|x|^2\right)\sigma - 2880t^2x\overline{x}\left(1-|x|^2\right)\sigma^2 \\
&\quad +2880t^2x\left(1-|x|^2\right)\left(1-|\sigma|^2\right)\rho + 1440ct^2x\left(1-|x|^2\right)\sigma + 2880t^2x^3 \\
&\quad -1040t^3x^3 + 180c^4tx + 45c^2t^2x^4 - 1440c^2t^2x^3 + 432c^2t^2x^2 \\
&\quad -2700t^2\left(1-|x|^2\right)^2\sigma^2 - 1728c^2tx^2 - 432c^4tx^3 + 432c^4tx^2\Big\}.
\end{aligned}$$

By virtue of $t = 4 - c^2$, we see that

$$\Delta_{3,1}(f) = \frac{1}{1080000}\Big(v_1(c,x) + v_2(c,x)\sigma + v_3(c,x)\sigma^2 + \Psi(c,x,\sigma)\rho\Big),$$

where

$$\begin{aligned}
v_1(c,x) &= -108c^6 + \left(4-c^2\right)\Big[\left(4-c^2\right)\left(-400c^2x^3 - 1280x^3 + 45c^2x^4 + 432c^2x^2\right) \\
&\quad -1728c^2x^2 + 432c^4x^2 - 432c^4x^3 + 180c^4x\Big], \\
v_2(c,x) &= \left(4-c^2\right)\left(1-|x|^2\right)\Big[\left(4-c^2\right)\left(-180cx^2 + 1440cx\right) + 1728c^3x\Big], \\
v_3(c,x) &= \left(4-c^2\right)\left(1-|x|^2\right)\Big[\left(4-c^2\right)\left(-180|x|^2 - 2700\right) + 1728c^2\overline{x}\Big], \\
\Psi(c,x,\sigma) &= \left(4-c^2\right)\left(1-|x|^2\right)\left(1-|\sigma|^2\right)\Big[2880x\left(4-c^2\right) - 1728c^2\Big].
\end{aligned}$$

By setting $|x| = x$, $|\sigma| = y$ and utilizing the fact $|\rho| \le 1$, we obtain

$$\begin{aligned}
|\Delta_{3,1}(f)| &\le \frac{1}{1080000}\Big(|v_1(c,x)| + |v_2(c,x)|y + |v_3(c,x)|y^2 + |\Psi(c,x,\sigma)|\Big). \\
&\le \frac{1}{1080000}(G(c,x,y)),
\end{aligned} \tag{31}$$

where

$$G(c,x,y) = g_1(c,x) + g_2(c,x)y + g_3(c,x)y^2 + g_4(c,x)\left(1-y^2\right),$$

with

$$\begin{aligned} g_1(c,x) &= 108c^6 + \left(4-c^2\right)[\left(4-c^2\right)\left(400c^2x^3 + 1280x^3 + 45c^2x^4 + 432c^2x^2\right) \\ &\quad + 1728c^2x^2 + 432c^4x^2 + 432c^4x^3 + 180c^4x], \\ g_2(c,x) &= \left(4-c^2\right)\left(1-x^2\right)\left[\left(4-c^2\right)\left(180cx^2 + 1440cx\right) + 1728c^3x\right], \\ g_3(c,x) &= \left(4-c^2\right)\left(1-x^2\right)\left[\left(4-c^2\right)\left(180x^2 + 2700\right) + 1728c^2x\right], \\ g_4(c,x) &= \left(4-c^2\right)\left(1-x^2\right)\left[2880x\left(4-c^2\right) + 1728c^2\right]. \end{aligned}$$

Now, we have to maximize $G(c,x,y)$ in the closed cuboid $\mathrm{Y} : [0,2] \times [0,1] \times [0,1]$. For this, we have to discuss the maximum values of $G(c,x,y)$ in the interior of Y, in the interior of its 6 faces and on its 12 edges.

1. Interior points of cuboid Y:

Let $(c,x,y) \in (0,2) \times (0,1) \times (0,1)$, and differentiating partially $G(c,x,y)$ with respect to y, we have

$$\begin{aligned} \frac{\partial G}{\partial y} &= \left(4-c^2\right)(1-x^2)\left[360y(x-1)\left(\left(4-c^2\right)(x-15) + \frac{48}{5}c^2\right)\right. \\ &\quad \left. + 180c\left(x\left(4-c^2\right)(8+x) + \frac{48}{5}c^2x\right)\right]. \end{aligned}$$

Plugging $\frac{\partial G}{\partial y} = 0$, we obtain

$$y = \frac{180c\left(x(4-c^2)(8+x) + \frac{48}{5}c^2x\right)}{360(x-1)\left((4-c^2)(15-x) - \frac{48}{5}c^2\right)} = y_0.$$

If y_0 is a critical point within Y, then $y_0 \in (0,1)$, which is only achievable if

$$1728c^3x + 180cx\left(4-c^2\right)(8+x) + 360(1-x)\left(4-c^2\right)(15-x) < 3456c^2(1-x). \quad (32)$$

and

$$c^2 > \frac{20(15-x)}{123-5x}. \quad (33)$$

Now, we must find solutions that meet both inequality (32) and (33) for the existence of critical points.

Let $g(x) = \frac{20(15-x)}{123-5x}$. As $g'(x) < 0$ in $(0,1)$, it is noted that $g(x)$ is decreasing over $(0,1)$. Hence $c^2 > \frac{140}{59}$ and an easy calculation indicates that (32) is impossible for all $x \in [\frac{1}{2}, 1)$,Thus there are no critical points of G in $(0,2) \times [\frac{1}{2}, 1) \times (0,1)$.

Suppose that there is a critical point $(\tilde{c}, \tilde{x}, \tilde{y})$ of G existing in the interior of cuboid Y. Clearly it must satisfy that $\tilde{x} < \frac{1}{2}$. From the above discussion, it can also be known that $\tilde{c}^2 > \frac{580}{241}$ and $\tilde{y} \in (0,1)$. In the following, we will prove that $G(\tilde{c}, \tilde{x}, \tilde{y}) < 43200$ in this situation. For $(c,x,y) \in \left(\sqrt{\frac{580}{241}}, 2\right) \times (0, \frac{1}{2}) \times (0,1)$, by invoking $x < \frac{1}{2}$ and $1 - x^2 < 1$, it is not hard to observe that

$$\begin{aligned}
g_1(c,x) &\leq 108c^6 + \left(4-c^2\right)[\left(4-c^2\right)\left(400c^2(1/2)^3 + 1280(1/2)^3 + 45c^2(1/2)^4 + 432c^2(1/2)^2\right) \\
&\quad + 1728c^2(1/2)^2 + 432c^4(1/2)^2 + 432c^4(1/2)^3 + 180c^4(1/2)], \\
&= 108c^6 + \frac{1}{16}\left(4-c^2\right)\left(1459c^4 + 14644c^2 + 10240\right) := \varphi_1(c), \\
g_2(c,x) &\leq \left(4-c^2\right)\left[\left(4-c^2\right)\left(180c(1/2)^2 + 1440c(1/2)\right) + 1728c^3(1/2)\right], \\
&= (4-c^2)\left(99c^3 + 3060c\right) := \varphi_2(c), \\
g_3(c,x) &\leq \left(4-c^2\right)\left[\left(4-c^2\right)\left(180(1/2)^2 + 2700\right) + 1728c^2(1/2)\right], \\
&= (4-c^2)\left(-1881c^2 + 10980\right) := \varphi_3(c), \\
g_4(c,x) &\leq \left(4-c^2\right)\left[2880(1/2)\left(4-c^2\right) + 1728c^2\right]. \\
&= (4-c^2)\left(288c^2 + 5760\right) := \varphi_4(c).
\end{aligned}$$

Therefore, we have

$$G(c,x,y) \leq \varphi_1(c) + \varphi_4(c) + \varphi_2(c)y + [\varphi_3(c) - \varphi_4(c)]y^2 := \Xi_1(c,y).$$

Obviously, it can be seen that

$$\frac{\partial \Xi_1}{\partial y} = \varphi_2(c) + 2[\varphi_3(c) - \varphi_4(c)]y$$

and

$$\frac{\partial^2 \Xi_1}{\partial y^2} = 2[\varphi_3(c) - \varphi_4(c)] = 2(4-c^2)(-2169c^2 + 5220).$$

Since $\varphi_3(c) - \varphi_4(c) \leq 0$ for $c \in \left(\sqrt{\frac{580}{241}}, 2\right)$, we obtain that $\frac{\partial^2 \Xi_1}{\partial y^2} \leq 0$ for $y \in (0,1)$, and thus it follows that

$$\frac{\partial \Xi_1}{\partial y} \geq \frac{\partial \Xi_1}{\partial y}|_{y=1} = (4-c^2)(99c^3 - 4338c^2 + 3060c + 10440) \geq 0, \quad c \in \left(\sqrt{\frac{580}{241}}, 2\right).$$

This implies that

$$\Xi_1(c,y) \leq \Xi_1(c,1) = \varphi_1(c) + \varphi_2(c) + \varphi_3(c) := \iota_1(c).$$

It is easy to calculate that $\iota_1(c)$ attains its maximum value 25,311.25 at $c \approx 1.551335$. Thus, we have

$$G(c,x,y) < 43200, \quad (c,x,y) \in \left(\sqrt{\frac{580}{241}}, 2\right) \times (0, \frac{1}{2}) \times (0,1).$$

Hence $G(\tilde{c}, \tilde{x}, \tilde{y}) < 43{,}200$. This implies that G is less than 43,200 at all the critical points in the interior of Y. Therefore, G has no optimal solution in the interior of Y.

2. Interior of all the six faces of cuboid Y:

(i) On the face $c = 0$, $G(c,x,y)$ becomes to

$$T_1(x,y) = G(0,x,y) = 640\left[32x^3 + \frac{9}{2}(1-x^2)(16x + y^2(x-1)(x-15))\right], \quad x,y \in (0,1).$$

Differentiating partially with respect to y, we obtain

$$\frac{\partial T_1}{\partial y} = 5760y(1-x^2)(x-1)(x-15), \quad x, y \in (0,1).$$

It is easy to see that $T_1(x,y)$ has no critical point in the interval $(0,1) \times (0,1)$.
(ii) On the face $c = 2$, $G(c,x,y)$ yields

$$G(2,x,y) \equiv 6912 < 43200.$$

(iii) On the face $x = 0$, $G(c,x,y)$ becomes

$$T_2(c,y) = G(c,0,y) = 108c^6 + (4-c^2)((10800-2700c^2)y^2 + 1728c^2(1-y^2)).$$

Differentiating $T_2(c,y)$ partially with respect to y, we know that

$$\frac{\partial T_2}{\partial y} = (4-c^2)(21600y - 8856c^2y).$$

Also derivative of $T_2(c,y)$ partially with respect to c is

$$\begin{aligned}\frac{\partial T_2}{\partial c} &= 648c^5 + (4-c^2)(-5400cy^2 + 3456c(1-y^2)) - 3456c^3(1-y^2) \\ &\quad + \left(5400c^3 - 21600c\right)y^2.\end{aligned}$$

By using Newton's methods for the system of nonlinear equations in Maple, we have found no solution to the system of equations in the interval $(0,2) \times (0,1)$. That is, $T_2(c,y)$ has no optimal solution in $(0,2) \times (0,1)$.
(iv) On the face $x = 1$, $G(c,x,y)$ takes the form

$$T_3(c,y) = G(c,1,y) = 108c^6 + (4-c^2)((4-c^2)(1280+877c^2) + 1728c^2 + 1044c^4).$$

Then

$$\frac{\partial T_3}{\partial c} = -354c^5 - 13152c^3 + 21408c.$$

Putting $\frac{\partial T_3}{\partial c} = 0$ and solving we obtain $c \approx 1.2498244295$. Thus we have

$$G(c,1,y) \leq \max T_3(c,y) = 28952.5898 < 43200, \quad (c,y) \in (0,2) \times (0,1).$$

(v) On the face $y = 0$, $G(c,x,y)$ yields

$$\begin{aligned}T_4(c,x) &= G(c,x,0) = 45c^6x^4 - 32c^6x^3 - 360c^4x^4 - 180c^6x - 3072c^4x^3 \\ &\quad + 108c^6 - 1728c^4x^2 + 720c^2x^4 + 3600c^4x + 19200c^2x^3 \\ &\quad - 1728c^4 + 6912c^2x^2 - 23040c^2x - 25600x^3 + 6912c^2 \\ &\quad + 46080x.\end{aligned}$$

Now differentiating partially with respect to c, then with respect to x and simplifying we have

$$\begin{aligned}\frac{\partial T_4}{\partial c} &= 270c^5x^4 - 192c^5x^3 - 1440c^3x^4 - 1080c^5x - 12288c^3x^3 + 648c^5 \\ &\quad - 6912c^3x^2 + 1440cx^4 + 14400c^3x + 38400cx^3 - 6912c^3 \\ &\quad + 13824cx^2 - 46080cx + 13824c. \qquad (34)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial T_4}{\partial x} &= 180c^6x^3 - 96c^6x^2 - 1440c^4x^3 - 180c^6 - 9216c^4x^2 - 3456c^4x \\ &\quad + 2880c^2x^3 + 3600c^4 + 57600c^2x^2 + 13824c^2x \\ &\quad - 23040c^2 - 76800x^2 + 46080. \end{aligned} \tag{35}$$

Applying Newton's methods to the system of nonlinear Equations (34) and (35) in Maple Software, we noticed that the given system of equations has no solution in $(0,2)\times(0,1)$.

(vi) On the face $y=1$, $G(c,x,y)$ reduces to $G(c,x,1)$ given by

$$\begin{aligned}T_5(c,x) &= G(c,x,1) = 45c^6x^4 - 32c^6x^3 - 180c^5x^4 + 288c^5x^3 - 540c^4x^4 \\ &\quad -180c^6x + 180c^5x^2 + 1536c^4x^3 + 1440c^3x^4 + 108c^6 - 288c^5x \\ &\quad - 5976c^4x^2 + 4608c^3x^3 + 2160c^2x^4 - 1008c^4x - 1440c^3x^2 \\ &\quad - 10752c^2x^3 - 2880cx^4 + 2700c^4 - 4608c^3x + 33984c^2x^2 \\ &\quad - 23040cx^3 - 2880x^4 + 6912c^2x + 2880cx^2 + 20480x^3 \\ &\quad - 21600c^2 + 23040cx - 40320x^2 + 43200.\end{aligned}$$

Partial derivative of $T_5(c,x)$ with respect to c and then with respect to x, we have

$$\begin{aligned}\frac{\partial T_5}{\partial c} &= 270c^5x^4 - 192c^5x^3 - 900c^4x^4 + 1440c^4x^3 - 2160c^3x^4 - 1080c^5x \\ &\quad +900c^4x^2 + 6144c^3x^3 + 4320c^2x^4 + 648c^5 - 1440c^4x - 23904c^3x^2 \\ &\quad + 13824c^2x^3 + 4320cx^4 - 4032c^3x - 4320c^2x^2 - 21504cx^3 \\ &\quad - 2880x^4 + 10800c^3 - 13824c^2x + 67968cx^2 - 23040x^3 \\ &\quad + 13824cx + 2880x^2 - 43200c + 23040x.\end{aligned} \tag{36}$$

and

$$\begin{aligned}\frac{\partial T_5}{\partial x} &= 180c^6x^3 - 96c^6x^2 - 720c^5x^3 + 864c^5x^2 - 2160c^4x^3 - 180c^6 \\ &\quad + 360c^5x + 4608c^4x^2 + 5760c^3x^3 - 288c^5 - 11952c^4x \\ &\quad + 13824c^3x^2 + 8640c^2x^3 - 1008c^4 - 2880c^3x - 32256c^2x^2 \\ &\quad - 11520cx^3 - 4608c^3 + 67968c^2x - 69120cx^2 - 11520x^3 \\ &\quad + 6912c^2 + 5760cx + 61440x^2 + 23040c - 80640x.\end{aligned} \tag{37}$$

In Maple Software, we used Newton's techniques to solve the system of nonlinear Equations (36) and (37) and observed that the above system of equations has no solution in $(0,2)\times(0,1)$. Thus $T_5(c,x)$ has no optimal solution in $(0,2)\times(0,1)$.

3. On the Edges of Cuboid Y:

(i) On the edge $x=0$ and $y=0$, then $G(c,x,y)$ becomes

$$G(c,0,0) = 108c^6 - 1728c^4 + 6912c^2 = U_1(c).$$

Clearly,

$$U_1'(c) = 648c^5 - 6912c^3 + 13824c.$$

Putting $U_1'(c)=0$ gives the critical point $c_0 \approx 1.632993161$ at which $G(c,0,0)=U_1(c)$ obtains its maximum. Hence

$$G(c,0,0) \le \max U_1(c) = U_1(c_0) = 8192 < 43200, \quad c \in [0,2].$$

(ii) On the edge $x = 0$ and $y = 1$, then $G(c, x, y)$ takes the form

$$G(c, 0, 1) = 108c^6 + 2700c^4 - 21600c^2 + 43200 = U_2(c).$$

It follows that

$$U_2'(c) = 648c^5 + 10800c^3 - 43200c.$$

As $U_2'(c) < 0$ in $[0, 2]$, we see that $U_2(c)$ is decreasing over $[0, 2]$. Thus $U_2(c)$ has its maxima at $c = 0$. Therefore $\max U_2(c) = U_2(0) = 43200$. Thus

$$G(c, 0, 1) \leq \max U_2(c) = U_2(0) = 43200.$$

(iii) On the edge $c = 0$ and $x = 0$, then $G(c, x, y)$ yields

$$G(0, 0, y) = 43200y^2 = U_3(y) \leq 43200, \quad y \in [0, 1].$$

(iv) On the edges $G(c, 1, 0)$ and $G(c, 1, 1)$, it is noted that $G(c, 1, y)$ is free of y, therefore

$$G(c, 1, 0) = G(c, 1, 1) = -59c^6 - 3288c^4 + 10704c^2 + 20480 = U_4(c).$$

Then

$$U_4'(c) = -354c^5 - 13152c^3 + 21408c.$$

Putting $U_4'(c) = 0$, we obtain the critical point $c_0 \approx 1.249824429$ at which $G(c, 1, 0) = G(c, 1, 1) = U_4(c)$ attains its maximum. Therefore $\max U_4(c) = U_4(c_0) = 28{,}952.5898$. Thus

$$G(c, 1, 0) = G(c, 1, 1) \leq \max U_4(c) = U_4(c_0) = 28952.5898 < 43200, \quad c \in [0, 2].$$

(v) On the edge $c = 0$ and $x = 1$, then $G(c, x, y)$ reduces to

$$G(0, 1, y) = 20480 < 43200, \quad y \in [0, 1].$$

(vi) On the edge $c = 2$, then $G(c, x, y)$ becomes

$$G(2, x, y) \equiv 6912.$$

$G(2, x, y)$ is independent of x and y; therefore

$$G(2, 0, y) = G(2, 1, y) = G(2, x, 0) = G(2, x, 1) = 6912 < 43200, \quad x, y \in [0, 1].$$

(vii) On the edge $c = 0$ and $y = 1$, then $G(c, x, y)$ yields

$$G(0, x, 1) = -2880x^4 + 20480x^3 - 40320x^2 + 43200 = U_5(x).$$

Then

$$U_5'(x) = -11520x^3 + 61440x^2 - 80640x.$$

Since $U_5'(x) < 0$ in $[0, 1]$, it follows that $U_5(x)$ is decreasing over $[0, 1]$. Thus $U_5(x)$ has its maxima at $x = 0$. Therefore $\max U_5(x) = U_5(0) = 43{,}200$. Hence

$$G(0, x, 1) \leq \max U_5(x) = U_5(0) = 43200, \quad x \in [0, 1].$$

(viii) On the edge $c = 0$ and $y = 0$ then $G(c, x, y)$ takes the form

$$G(0, x, 0) = -25600x^3 + 46080x = U_6(x).$$

Then

$$U_6'(x) = -76800x^2 + 46080.$$

The equation $U_6'(x) = 0$ gives the critical point $x_0 \approx 0.774596669$ at which $U_6(x)$ obtains its maximum. Therefore $\max U_6(x) = U_6(x_0) = 23{,}795.60968$. Hence

$$G(0, x, 0) \leq \max U_6(x) = U_6(x_0) = 23795.60968, \quad x \in [0, 1].$$

From above cases we conclude that

$$G(c, x, y) \leq 43200 \quad \text{on } [0, 2] \times [0, 1] \times [0, 1].$$

Using (31), it is clear that

$$|\Delta_{3,1}(f)| \leq \frac{1}{1080000}(G(c, x, y)) \leq \frac{1}{25}.$$

The bound is sharp with the extremal function given by

$$f_3(z) = \int_0^z \left(1 + \frac{4}{5}\left(t^3\right) + \frac{1}{5}\left(t^{12}\right)\right)dt = z + \frac{1}{5}z^4 + \frac{1}{65}z^{13}.$$

□

5. Zalcman Functional

In 1960, Lawrence Zalcman proposed the following conjecture based on a coefficient for the functions belonging to the class $\mathcal{S}$.

$$\left|a_n^2 - a_{2n-1}\right| \leq (n-1)^2.$$

Equality will be obtained when taking the Köebe function or its rotations. The particular case of the familiar Fekete–Szegö inequality will be achieved when we put $n = 2$. For more contributions on this particular topic, see the work [49,50].

Theorem 6. *Let f belong to $\mathcal{BT}_{3l}$ and be of the form (1). Then*

$$\left|a_5 - a_3^2\right| \leq \frac{4}{25}.$$

The inequality is sharp.

Proof. From (26) and (28), we obtain

$$\left|a_5 - a_3^2\right| = \left|-\frac{43}{3600}c_1^4 - \frac{13}{225}c_2^2 - \frac{2}{25}c_1c_3 + \frac{7}{90}c_1^2c_2 + \frac{2}{25}c_4\right|.$$

It follows that

$$\left|a_5 - a_3^2\right| = \frac{2}{25}\left|\frac{43}{288}c_1^4 + \frac{13}{18}c_2^2 + 2\left(\frac{1}{2}\right)c_1c_3 - \frac{3}{2}\left(\frac{35}{54}\right)c_1^2c_2 - c_4\right|. \tag{38}$$

Let $\gamma = \frac{43}{288}$, $a = \frac{13}{18}$,$\alpha = \frac{1}{2}$, $\beta = \frac{35}{54}$. It can be found that $0 < \alpha < 1, 0 < a < 1$ and

$$8a(1-a)\left((\alpha\beta - 2\gamma)^2 + (\alpha(a+\alpha) - \beta)^2\right) + \alpha(1-\alpha)(\beta - 2a\alpha)^2 \leqq 4a\alpha^2(1-\alpha)^2(1-a).$$

From Lemma 3, we have

$$\left|a_5 - a_3^2\right| \leq \frac{4}{25}.$$

The inequality is sharp and is achieved by

$$f_4(z)=\int_0^z\left(1+\frac{4}{5}\left(t^4\right)+\frac{1}{5}\left(t^{16}\right)\right)dt=z+\frac{4}{25}z^5+\frac{1}{85}z^{17}.$$

□

Theorem 7. *If f belongs to $\mathcal{BT}_{3l}$, and has the form (1). Then*

$$\left|a_3a_5-a_4^2\right|\le\frac{1}{25}.$$

This result is sharp.

Proof. Setting (26)–(28) with $c_1=c$, we obtain

$$\begin{aligned}a_3a_5-a_4^2 &= \frac{1}{8000}\left(-c^6+\frac{8}{3}c^3c_3+\frac{16}{3}c^2c_2^2-\frac{128}{3}c^2c_4+\frac{224}{3}cc_2c_3\right.\\ &\quad\left.-\frac{128}{3}c_2^3+\frac{256}{3}c_2c_4-80c_3^2\right).\end{aligned}\tag{39}$$

Using $t=4-c^2$ in (10)–(12), some basic calculations show that

$$\begin{aligned}
\frac{8}{3}c^3c_3 &= \frac{2}{3}c^6+\frac{4}{3}c^4tx-\frac{2}{3}c^4tx^2+\frac{4}{3}c^3t\left(1-|x|^2\right)\sigma,\\
\frac{16}{3}c^2c_2^2 &= \frac{4}{3}c^6+\frac{8}{3}c^4tx+\frac{4}{3}c^2t^2x^2,\\
\frac{128}{3}c^2c_4 &= \frac{16}{3}c^6+\frac{16}{3}c^4tx^3-16c^4tx^2+16c^4tx+\frac{64}{3}tc^2x^2-\frac{64}{3}c^3tx\\
&\quad\left(1-|x|^2\right)\sigma-\frac{64}{3}c^2t\overline{x}\left(1-|x|^2\right)\sigma^2+\frac{64}{3}c^2t\left(1-|x|^2\right)\\
&\quad\left(1-|\sigma|^2\right)\rho+\frac{64}{3}c^3t\left(1-|x|^2\right)\sigma,\\
\frac{224}{3}cc_2c_3 &= \frac{28}{6}c^6+28c^4tx-\frac{28}{3}c^4tx^2+\frac{56}{3}c^3t\left(1-|x|^2\right)\sigma+\frac{56}{3}c^2x^2t^2\\
&\quad-\frac{28}{3}c^2x^3t^2+\frac{56}{3}cxt^2\left(1-|x|^2\right)\sigma,\\
\frac{128}{3}c_2^3 &= \frac{16}{3}c^6+16c^4xt+16c^2x^2t^2+\frac{16}{3}x^3t^3,\\
\frac{256}{3}c_2c_4 &= \frac{16}{3}c^6+\frac{16}{3}c^4x^3t-16c^4x^2t+\frac{64}{3}c^4xt+\frac{64}{3}c^2x^2t-\frac{64}{3}c^3xt\\
&\quad\left(1-|x|^2\right)\sigma-\frac{64}{3}c^2t\overline{x}\left(1-|x|^2\right)\sigma^2+\frac{64}{3}c^2t\left(1-|x|^2\right)\left(1-|\sigma|^2\right)\rho\\
&\quad+\frac{64}{3}c^3t\left(1-|x|^2\right)\sigma+\frac{16}{3}c^2x^4t^2-16c^2x^3t^2+16c^2x^2t^2\\
&\quad+\frac{64}{3}x^3t^2-\frac{64}{3}cx^2t^2\left(1-|x|^2\right)\sigma-\frac{64}{3}xt^2\overline{x}\left(1-|x|^2\right)\sigma^2\\
&\quad+\frac{64}{3}xt^2\left(1-|x|^2\right)\left(1-|\sigma|^2\right)\rho+\frac{64}{3}cxt^2\left(1-|x|^2\right)\sigma,\\
80c_3^2 &= 5c^2x^4t^2-20cx^2t^2\left(1-|x|^2\right)\sigma-20c^2x^3t^2-10c^4x^2t\\
&\quad+20t^2\left(1-|x|^2\right)^2\sigma^2+40cxt^2\left(1-|x|^2\right)\sigma+20c^2x^2t^2\\
&\quad+20c^3t\left(1-|x|^2\right)\sigma+20c^4xt+5c^6.
\end{aligned}$$

Putting the above expressions in (39), we obtain

$$\begin{aligned} a_3a_5 - a_4^2 &= \frac{1}{8000}\left\{\frac{64}{3}x^3t^2 - \frac{16}{3}x^3t^3 + \frac{4}{3}c^4tx + \frac{1}{3}x^4t^2c^2 - \frac{16}{3}x^3t^2c^2 - 20t^2\right. \\ &\quad \left(1-|x|^2\right)^2\sigma^2 - \frac{4}{3}x^2t^2\left(1-|x|^2\right)c\sigma - \frac{64}{3}xt^2\left(1-|x|^2\right)\overline{x}\sigma^2 \\ &\quad \left. + \frac{64}{3}xt^2\left(1-|x|^2\right)\left(1-|\sigma|^2\right)\rho\right\}. \end{aligned}$$

It can be noted that

$$a_3a_5 - a_4^2 = \frac{1}{8000}\left(u_1(c,x) + u_2(c,x)\sigma + u_3(c,x)\sigma^2 + \phi(c,x,\sigma)\rho\right),$$

where

$$\begin{aligned} u_1(c,x) &= \left(4-c^2\right)\left[\left(4-c^2\right)\left(\frac{1}{3}c^2x^4\right) + \frac{4}{3}c^4x\right], \\ u_2(c,x) &= \left(4-c^2\right)\left(1-|x|^2\right)\left[\left(4-c^2\right)\left(-\frac{4}{3}cx^2\right)\right], \\ u_3(c,x) &= \left(4-c^2\right)\left(1-|x|^2\right)\left[\left(4-c^2\right)\left(-\frac{4}{3}|x|^2 - 20\right)\right], \\ \phi(c,x,\sigma) &= \left(4-c^2\right)\left(1-|x|^2\right)\left(1-|\sigma|^2\right)\left[\frac{64}{3}x\left(4-c^2\right)\right]. \end{aligned}$$

By taking $|x| = x, |\sigma| = y$ and utilizing the fact $|\rho| \leq 1$, we obtain

$$\begin{aligned} \left|a_3a_5 - a_4^2\right| &\leq \frac{1}{8000}\left(|u_1(c,x)| + |u_2(c,x)|y + |u_3(c,x)|y^2 + |\phi(c,x,\sigma)|\right). \\ &\leq \frac{1}{8000}(F(c,x,y)), \end{aligned} \tag{40}$$

where

$$F(c,x,y) = f_1(c,x) + f_2(c,x)y + f_3(c,x)y^2 + f_4(c,x)\left(1-y^2\right),$$

with

$$\begin{aligned} f_1(c,x) &= \left(4-c^2\right)\left[\left(4-c^2\right)\left(\frac{1}{3}c^2x^4\right) + \frac{4}{3}c^4x\right], \\ f_2(c,x) &= \left(4-c^2\right)\left(1-x^2\right)\left[\left(4-c^2\right)\left(\frac{4}{3}cx^2\right)\right], \\ f_3(c,x) &= \left(4-c^2\right)\left(1-x^2\right)\left[\left(4-c^2\right)\left(\frac{4}{3}x^2 + 20\right)\right], \\ f_4(c,x) &= \left(4-c^2\right)\left(1-x^2\right)\left[\frac{64}{3}x\left(4-c^2\right)\right]. \end{aligned}$$

Obviously, it can be seen that

$$\frac{\partial F}{\partial y} = f_2(c,x) + 2[f_3(c,x) - f_4(c,x)]y$$

and

$$\frac{\partial^2 F}{\partial y^2} = 2[f_3(c,x) - f_4(c,x)] = \frac{8}{3}(4-c^2)^2(1-x)^2(15-x).$$

Since $f_3(c,x) - f_4(c,x) \geq 0$ for $(c,x) \in [0,2] \times [0,1]$, we obtain that $\frac{\partial^2 F}{\partial y^2} \geq 0$ for $y \in (0,1)$, and thus it follows that

$$\frac{\partial F}{\partial y} \geq \frac{\partial F}{\partial y}|_{y=0} = f_2(c,x) \geq 0, \quad (c,x) \in [0,2] \times [0,1].$$

Therefore, we have

$$F(c,x,y) \leq F(c,x,1) = f_1(c,x) + f_2(c,x) + f_3(c,x) := \tau(c,x).$$

It is not hard to calculate that

$$\begin{aligned} \tau(c,x) &= F(c,x,1) = \frac{1}{3}c^6x^4 - 4c^4x^4 + 16c^2x^4 - \frac{4}{3}c^6x + \frac{16}{3}c^4x \\ &\quad - \frac{4}{3}c^5x^4 + \frac{4}{3}c^5x^2 + \frac{32}{3}c^3x^4 - \frac{32}{3}c^3x^2 - \frac{64}{3}cx^4 \\ &\quad + \frac{64}{3}cx^2 - \frac{56}{3}c^4x^2 + 20c^4 + \frac{448}{3}c^2x^2 - 160c^2 \\ &\quad - \frac{896}{3}x^2 + 320 - \frac{64}{3}x^4. \end{aligned}$$

Partial derivative of $\tau(c,x)$ with respect to c and then with respect to x, we have

$$\begin{aligned} \frac{\partial \tau}{\partial c} &= 2c^5x^4 - 16c^3x^4 + 32cx^4 - 8c^5x + \frac{64}{3}c^3x - \frac{20}{3}c^4x^4 + \frac{20}{3}c^4x^2 \\ &\quad + 32c^2x^4 - 32c^2x^2 - \frac{64}{3}x^4 + \frac{64}{3}x^2 - \frac{224}{3}c^3x^2 + 80c^3 \\ &\quad + \frac{896}{3}cx^2 - 320c \end{aligned} \tag{41}$$

and

$$\begin{aligned} \frac{\partial \tau}{\partial x} &= \frac{4}{3}c^6x^3 - 16c^4x^3 + 64c^2x^3 - \frac{4}{3}c^6 + \frac{16}{3}c^4 - \frac{16}{3}c^5x^3 + \frac{8}{3}c^5x \\ &\quad + \frac{128}{3}c^3x^3 - \frac{64}{3}c^3x - \frac{256}{3}cx^3 + \frac{128}{3}cx - \frac{112}{3}c^4x \\ &\quad + \frac{896}{3}c^2x - \frac{1792}{3}x - \frac{256}{3}x^3. \end{aligned} \tag{42}$$

A numerical calculation, using Maple Software, shows that the system of Equations (41) and (42) has no solution in $(0,2) \times (0,1)$.

For $x = 0$, then $\tau(c,x)$ takes the form

$$\tau(c,0) = 20c^4 - 160c^2 + 320 = B_1(c).$$

Then

$$B_1'(c) = 80c^3 - 320c.$$

Since $B_1'(c) < 0$ in $[0,2]$, it follows that $B_1(c)$ is decreasing over $[0,2]$. Thus $B_1(c)$ has its maxima at $c = 0$. Therefore $\max B_1(c) = B_1(0) = 320$. Thus

$$\tau(c,x) \leq \max B_1(c) = B_1(0) = 320, \quad (c,x) \in [0,2] \times [0,1].$$

For $x = 1$, it is easy to calculate that

$$\tau(c,1) = -c^6 + \frac{8}{3}c^4 + \frac{16}{3}c^2 = B_2(c)$$

and

$$B_2'(c) = -6c^5 + \frac{32}{3}c^3 + \frac{32}{3}c.$$

Putting $B_2'(c) = 0$, we obtain the critical point $c_0 \approx 1.57840302$ at which $B_2(c)$ obtains its maximum. Therefore $\max B_2(c) = B_2(c_0) = 14.37535984$. Thus

$$\tau(c,1) \leq \max B_2(c) = B_2(c_0) = 14.37535984 < 320, \quad c \in [0,2].$$

For $c = 2$, then $\tau(c,x)$ becomes

$$\tau(2,x) \equiv 0 < 320, \quad x \in [0,1].$$

For $c = 0$, then $\tau(c,x)$ reduces to

$$\tau(0,x) = -\frac{64}{3}x^4 - \frac{896}{3}x^2 + 320 = B_3(x).$$

Then

$$B_3'(x) = -\frac{256}{3}x^3 - \frac{1792}{3}x.$$

Since $B_3'(x) < 0$ in $[0,1]$, it is clear that $B_3(x)$ is decreasing over $[0,1]$. Thus $B_3(x)$ has its maxima at $x = 0$. Hence

$$\tau(0,x) \leq \max B_3(x) = B_3(0) = 320, \quad x \in [0,1].$$

Thus from the above cases, we conclude that

$$F(c,x,y) \leq 320 \quad \text{on } [0,2] \times [0,1] \times [0,1].$$

From (31), we know that

$$\left|a_3a_5 - a_4^2\right| \leq \frac{1}{8000}(F(c,x,y)) \leqq \frac{1}{25}.$$

The bound can be achieved with the extremal function given by

$$f_3(z) = \int_0^z \left(1 + \frac{4}{5}\left(t^3\right) + \frac{1}{5}\left(t^{12}\right)\right)dt = z + \frac{1}{5}z^4 + \frac{1}{65}z^{13}.$$

□

6. Sharp Bounds on the Third Hankel Determinant for Functions $f \in \mathcal{S}_{3l}^*$

Next, we will improve the bound $|\Delta_{3,1}(f)| \leq \frac{242}{1125}$ of third Hankel determinant for $f \in \mathcal{S}_{3l}^*$ which was obtained by Shi et al. in [43].

Theorem 8. *If $f \in \mathcal{S}_{3l}^*$ and has the series expansion (1), then*

$$|\Delta_{3,1}(f)| \leq \frac{16}{225}.$$

This result is sharp.

Proof. Let $f \in \mathcal{S}_{3l}^*$. From the definition, there exists a Schwarz function ω such that

$$\frac{zf'(z)}{f(z)} = 1 + \frac{4}{5}\omega(z) + \frac{1}{5}(\omega(z))^4, \quad (z \in \mathbb{E}).$$

Assuming that $p \in \mathcal{P}$. Then it can be written in terms of the Schwarz function $\omega(z)$ as

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots, \tag{43}$$

or equivalently,

$$\omega(z)=\frac{p(z)-1}{p(z)+1}=\frac{c_1z+c_2z^2+c_3z^3+c_4z^4+\cdots}{2+c_1z+c_2z^2+c_3z^3+c_4z^4+\cdots}. \tag{44}$$

From (1), we obtain

$$\begin{aligned}\frac{zf'(z)}{f(z)} &= 1+a_2z+(2a_3-a_2^2)z^2+(3a_4-3a_2a_3+a_2^3)z^3\\ &\quad+(4a_5-2a_3^2-4a_2a_4+4a_2^2a_3-a_2^4)z^4+\cdots.\end{aligned} \tag{45}$$

By simplification and using the series expansion of (44), we obtain

$$\begin{aligned}1+\frac{4}{5}(\omega(z))+\frac{1}{5}(\omega(z))^4 &= 1+\frac{2}{5}c_1z+\left(\frac{2}{5}c_2-\frac{1}{5}c_1^2\right)z^2+\left(\frac{2}{5}c_3+\frac{1}{10}c_1^3-\frac{2}{5}c_1c_2\right)z^3\\ &\quad+\left(\frac{2}{5}c_4-\frac{1}{5}c_2^2-\frac{3}{80}c_1^4+\frac{3}{10}c_1^2c_2-\frac{2}{5}c_1c_3\right)z^4+\cdots.\end{aligned} \tag{46}$$

Comparing like powers of z, z^2, z^3 and z^4 in (45) and (46), we obtain

$$a_2 = \frac{2}{5}c_1, \tag{47}$$

$$a_3 = \frac{1}{5}c_2-\frac{1}{50}c_1^2, \tag{48}$$

$$a_4 = \frac{2}{15}c_3+\frac{1}{250}c_1^3-\frac{4}{75}c_1c_2, \tag{49}$$

$$a_5 = \frac{1}{10}c_4+\frac{81}{40000}c_1^4+\frac{53}{3000}c_1^2c_2-\frac{7}{150}c_1c_3-\frac{3}{100}c_2^2. \tag{50}$$

The third Hankel determinant can be written as

$$\Delta_{3,1}(f)=2a_2a_3a_4-a_3^3-a_4^2+a_3a_5-a_2^2a_5.$$

Let $c_1=c$. It follows that

$$\begin{aligned}\Delta_{3,1}(f) &= \frac{1}{18000000}\Big(-7857c^6-19710c^4c_2+93600c^3c_3-800c^2c_2^2-324000c^2c_4\\ &\quad+472000cc_2c_3-252000c_2^3+360000c_2c_4-320000c_3^2\Big).\end{aligned} \tag{51}$$

Using $t=4-c^2$ in (10)–(12), for some $\rho,x,\sigma\in\overline{\mathbb{E}}$ we obtain

$$\begin{aligned}19710c^4c_2 &= 9855\left(c^6+c^4tx\right),\\ 93600c^3c_3 &= 23400c^6+46800c^4tx-23400c^4tx^2+46800c^3t\left(1-|x|^2\right)\sigma,\\ 800c^2c_2^2 &= 200c^6+400c^4tx+200c^2t^2x^2,\\ 324000c^2c_4 &= 40500c^4tx^3-162000c^3tx\left(1-|x|^2\right)\sigma-162000c^2t\overline{x}\left(1-|x|^2\right)\sigma^2\\ &\quad+162000c^2t\left(1-|x|^2\right)\left(1-|\sigma|^2\right)\rho+162000c^3t\left(1-|x|^2\right)\sigma\\ &\quad+40500c^6+162000c^2tx^2-121500c^4tx^2+121500c^4tx,\\ 472000cc_2c_3 &= -59000c^2t^2x^3-59000c^4tx^2+118000ct^2x\left(1-|x|^2\right)\sigma+59000c^6\\ &\quad+118000c^2t^2x^2+118000c^3t\left(1-|x|^2\right)\sigma+177000c^4tx,\\ 252000c_2^3 &= 31500t^3x^3+94500c^2t^2x^2+94500c^4tx+31500c^6,\end{aligned}$$

$$
\begin{aligned}
360000c_2c_4 &= 90000c^2tx^2 + 90000t^2x^3 + 22500c^6 + 90000c^4tx + 90000c^3t \\
&\quad \left(1-|x|^2\right)\sigma + 90000c^2t\left(1-|x|^2\right)\left(1-|\sigma|^2\right)\rho + 67500c^2t^2x^2 \\
&\quad + 90000t^2x\left(1-|x|^2\right)\left(1-|\sigma|^2\right)\rho - 67500c^4tx^2 - 90000c^3tx \\
&\quad \left(1-|x|^2\right)\sigma - 67500c^2t^2x^3 - 90000t^2x\overline{x}\left(1-|x|^2\right)\sigma^2 + 22500c^2 \\
&\quad t^2x^4 - 90000ct^2x^2\left(1-|x|^2\right)\sigma + 22500c^4tx^3 + 90000ct^2x \\
&\quad \left(1-|x|^2\right)\sigma - 90000c^2t\overline{x}\left(1-|x|^2\right)\sigma^2, \\
320000c_3^2 &= 20000c^2t^2x^4 - 80000ct^2x^2\left(1-|x|^2\right)\sigma - 80000c^2t^2x^3 \\
&\quad +80000t^2\left(1-|x|^2\right)^2\sigma^2 + 160000ct^2x\left(1-|x|^2\right)\sigma + 80000c^2t^2x^2 \\
&\quad +80000c^3t\left(1-|x|^2\right)\sigma + 80000c^4tx + 20000c^6 - 40000c^4tx^2.
\end{aligned}
$$

Setting the above expressions in (51), we obtain

$$
\begin{aligned}
\Delta_{3,1}(f) &= \frac{1}{18000000}\Big\{-5012c^6 + 90000t^2x^3 - 31500t^3x^3 - 72000c^2tx^2 - 18000c^4tx^3 \\
&\quad +7545c^4tx + 2500c^2t^2x^4 - 46500c^2t^2x^3 - 80000t^2\left(1-|x|^2\right)^2\sigma^2 + 12800c^3t \\
&\quad \left(1-|x|^2\right)\sigma + 72000c^3tx\left(1-|x|^2\right)\sigma + 72000c^2t\overline{x}\left(1-|x|^2\right)\sigma^2 - 72000c^2t\left(1-|x|^2\right) \\
&\quad \left(1-|\sigma|^2\right)\rho - 10000ct^2x^2\left(1-|x|^2\right)\sigma - 90000t^2x\overline{x}\left(1-|x|^2\right)\sigma^2 + 48000ct^2x \\
&\quad \left(1-|x|^2\right)\sigma + 90000t^2x\left(1-|x|^2\right)\left(1-|\sigma|^2\right)\rho + 10800c^2x^2t^2 + 11600c^4tx^2\Big\}.
\end{aligned}
$$

Thus, we see

$$
\Delta_{3,1}(f) = \frac{1}{18000000}\left(k_1(c,x) + k_2(c,x)\sigma + k_3(c,x)\sigma^2 + \zeta(c,x,\sigma)\rho\right),
$$

where

$$
\begin{aligned}
k_1(c,x) &= -5012c^6 + \left(4-c^2\right)\Big[\left(4-c^2\right)\Big(10800c^2x^2 + 2500c^2x^4 - 15000c^2x^3 \\
&\quad -36000x^3\Big) + 11600c^4x^2 - 72000c^2x^2 + 7545c^4x - 18000c^4x^3\Big], \\
k_2(c,x) &= \left(4-c^2\right)\left(1-|x|^2\right)\Big[\left(4-c^2\right)\left(-10000cx^2 + 48000cx\right) \\
&\quad +12800c^3 + 72000c^3x\Big], \\
k_3(c,x) &= \left(4-c^2\right)\left(1-|x|^2\right)\left[\left(4-c^2\right)\left(-10000|x|^2 - 80000\right) + 72000c^2\overline{x}\right], \\
\zeta(c,x,\sigma) &= \left(4-c^2\right)\left(1-|x|^2\right)\left(1-|\sigma|^2\right)\left[90000x\left(4-c^2\right) - 72000c^2\right].
\end{aligned}
$$

Taking $|x| = x$, $|\sigma| = y$ and utilizing the fact $|\rho| \le 1$, we obtain

$$
\begin{aligned}
|\Delta_{3,1}(f)| &\le \frac{1}{18000000}\left(|k_1(c,x)| + |k_2(c,x)|y + |k_3(c,x)|y^2 + |\zeta(c,x,\sigma)|\right). \\
&\le \frac{1}{18000000}(Q(c,x,y)),
\end{aligned} \tag{52}
$$

where

$$
Q(c,x,y) = q_1(c,x) + q_2(c,x)y + q_3(c,x)y^2 + q_4(c,x)\left(1-y^2\right)
$$

with

$$\begin{aligned} q_1(c,x) &= 5012c^6 + \left(4-c^2\right)\left[\left(4-c^2\right)\left(10800c^2x^2 + 2500c^2x^4 + 15000c^2x^3\right.\right. \\ &\quad \left.\left. +36000x^3\right) + 11600c^4x^2 + 72000c^2x^2 + 7545c^4x + 18000c^4x^3\right], \\ q_2(c,x) &= \left(4-c^2\right)\left(1-x^2\right)\left[\left(4-c^2\right)\left(10000cx^2 + 48000cx\right) + 12800c^3 + 72000c^3x\right], \\ q_3(c,x) &= \left(4-c^2\right)\left(1-x^2\right)\left[\left(4-c^2\right)\left(10000x^2 + 80000\right) + 72000c^2x\right], \\ q_4(c,x) &= \left(4-c^2\right)\left(1-x^2\right)\left[90000x\left(4-c^2\right) + 72000c^2\right]. \end{aligned}$$

Now, we have to maximize $Q(c,x,y)$ in the closed cuboid $\mathrm{Y} : (0,2) \times (0,1) \times (0,1)$. For this, we have to discuss the maximum values of $Q(c,x,y)$ in the interior of Y, of its 6 faces and on its 12 edges.

1. Interior points of cuboid Y:

Let $(c,x,y) \in (0,2) \times (0,1) \times (0,1)$. Differentiating partially $Q(c,x,y)$ with respect to y, we obtain

$$\begin{aligned} \frac{\partial Q}{\partial y} &= \left(4-c^2\right)(1-x^2)\left[2y\left(2000\left(4-c^2\right)\left(5x^2+40\right) + 36c^2x - 72000c^2 + 90000x\left(4-c^2\right)\right)\right. \\ &\quad \left. +c\left(x\left(4-c^2\right)(48000+10000x) + c^2(72000x+12800)\right)\right]. \end{aligned}$$

Plugging $\frac{\partial Q}{\partial y} = 0$, we find

$$y = \frac{c(5x(4-c^2)(24+5x) + 4c^2(45x+8))}{10(x-1)(5(4-c^2)(8-x) - 36c^2)} = y_0.$$

If y_0 is a critical point inside Y, then $y_0 \in (0,1)$, and this is only achievable if

$$4c^3(45x+8) + 5cx\left(4-c^2\right)(24+5x) + 50(1-x)\left(4-c^2\right)(8-x) < 360c^2(1-x). \quad (53)$$

and

$$c^2 > \frac{20(8-x)}{76-5x}. \quad (54)$$

For the existence of critical points, we must now find solutions that meet both inequalities (53) and (54).

Let $g(x) = \frac{20(8-x)}{76-5x}$. As $g'(x) < 0$ in $(0,1)$, it can be seen that $g(x)$ is decreasing over $(0,1)$. Hence $c^2 > \frac{40}{19}$. It is not hard to verify that the inequality (53) cannot hold true in this situation for $x \in [\frac{1}{2}, 1)$. Thus, there is no such critical point of $Q(c,x,y)$ existing in $(0,2) \times \left(\frac{1}{2}, 1\right) \times (0,1)$.

Suppose that there is a critical point $(\tilde{c}, \tilde{x}, \tilde{y})$ of Q existing in the interior of cuboid Y. Clearly it must satisfy that $\tilde{x} \le \frac{1}{2}$. From the above discussion, it can also be known that $\tilde{c}^2 \ge \frac{300}{147}$ and $\tilde{y} \in (0,1)$. Now we will prove that $Q(\tilde{c}, \tilde{x}, \tilde{y}) < 1{,}280{,}000$. For $(c,x,y) \in \left(\sqrt{\frac{300}{147}}, 2\right) \times (0, \frac{1}{2}) \times (0,1)$, by invoking $x < \frac{1}{2}$ and $1 - x^2 < 1$ it is not hard to observe that

$$\begin{aligned} q_1(c,x) &\leq 5012c^6 + \left(4-c^2\right)\left[\left(4-c^2\right)\left(10800c^2(1/2)^2 + 2500c^2(1/2)^4 + 15000c^2(1/2)^3 + 36000(1/2)^3\right)\right. \\ &\qquad \left. + 11600c^4(1/2)^2 + 72000c^2(1/2)^2 + 7548c^4(1/2) + 18000c^4(1/2)^3\right] \\ &= 5012c^6 + \frac{1}{4}\left(4-c^2\right)\left(16765c^4 + 129700c^2 + 72000\right) := \phi_1(c), \\ q_2(c,x) &\leq \left(4-c^2\right)\left[\left(4-c^2\right)\left(10000c(1/2)^2 + 48000c(1/2)\right) + 12800c^3 + 72000c^3(1/2)\right], \\ &= (4-c^2)\left(22300c^3 + 106000c\right) := \phi_2(c), \\ q_3(c,x) &\leq (4-c^2)\left[\left(4-c^2\right)\left(10000(1/2)^2 + 80000\right) + 72000c^2(1/2)\right], \\ &= (4-c^2)\left(-46500c^2 + 330000\right) := \phi_3(c), \\ q_4(c,x) &\leq (4-c^2)\left[72000c^2 + 90000(1/2)\left(4-c^2\right)\right] = (4-c^2)(180000 + 27000c^2) := \phi_4(c). \end{aligned}$$

Therefore, we have

$$Q(c,x,y) \leq \phi_1(c) + \phi_4(c) + \phi_2(c)y + [\phi_3(c) - \phi_4(c)]y^2 := \Xi_2(c,y).$$

Obviously, it can be seen that

$$\frac{\partial \Xi_2}{\partial y} = \phi_2(c) + 2[\phi_3(c) - \phi_4(c)]y$$

and

$$\frac{\partial^2 \Xi_2}{\partial y^2} = 2[\phi_3(c) - \phi_4(c)] = 2(4-c^2)(-73500c^2 + 150000).$$

Since $\phi_3(c) - \phi_4(c) \leq 0$ for $c \in (\sqrt{\frac{300}{147}}, 2)$, we obtain that $\frac{\partial^2 \Xi_2}{\partial y^2} \leq 0$ for $y \in (0,1)$, and thus it follows that

$$\frac{\partial \Xi_2}{\partial y} \geqq \frac{\partial \Xi_2}{\partial y}|_{y=1} = (4-c^2)(22300c^3 - 147000c^2 + 106000c + 300000) \geq 0, \quad c \in (\sqrt{\frac{300}{147}}, 2).$$

Therefore, we have

$$\Xi_2(c,y) \leq \Xi_2(c,1) = \phi_1(c) + \phi_2(c) + \phi_3(c) := \iota_2(c).$$

It is easy to calculate that $\iota_2(c)$ attains its extremal value 1,126,373 at $c \approx 1.428571$. Thus, we have

$$Q(c,x,y) < 1280000, \quad (c,x,y) \in \left(\sqrt{\frac{300}{147}}, 2\right) \times (0, \frac{1}{2}) \times (0,1).$$

Hence $Q(\tilde{c}, \tilde{x}, \tilde{y}) < 1{,}280{,}000$. This implies that Q is less than 1,280,000 at all the critical points in the interior of Y. Therefore, Q has no optimal solution in the interior of Y.

2. Interior of all the six faces of cuboid Y:

(i) On the face $c = 0$, $Q(c,x,y)$ reduces to

$$\begin{aligned} h_1(x,y) &= Q(0,x,y) = 576000x^3 + (1-x^2)\left[y^2\left(160000x^2 + 1440000x + 1280000\right)\right. \\ &\qquad +1440000x)], \quad x,y \in (0,1). \end{aligned}$$

Now $h_1(x,y)$ differentiating partially with respect to y, we obtain

$$\frac{\partial h_1}{\partial y} = 2y(1-x^2)\left(160000x^2 + 1440000x + 1280000\right), \quad x,y \in (0,1).$$

Thus $h_1(x,y)$ has no critical point in the interval $(0,1) \times (0,1)$.
(ii) On the face $c=2$, $Q(c,x,y)$ yields

$$Q(2,x,y) \equiv 320768 < 1280000, \quad x,y \in (0,1).$$

(iii) On the face $x=0$, $Q(c,x,y)$ becomes

$$\begin{aligned} h_2(c,y) &= Q(c,0,y) = 5012c^6 + (4-c^2)\left(12800c^3y - 72000c^2(1-y^2)\right) \\ &\quad + \left(80000c^4 - 640000c^2 + 1280000\right)y^2. \end{aligned}$$

Differentiating $h_2(c,y)$ partially with respect to y

$$\frac{\partial h_2}{\partial y} = (4-c^2)\left(12800c^3 - 144000c^2y\right) + \left(160000c^4 - 1280000c^2 + 2560000\right)y.$$

Taking $\frac{\partial h_2}{\partial y} = 0$ and solving, we obtain

$$y = \frac{4c^3}{5(19c^2-40)} = y_1.$$

For the provided range of y, y_1 would be a member of $(0,1)$ if $c > c_0$ with $c_0 \approx 1.49903072734$. Also the derivative of $h_2(c,y)$ partially with respect to c is

$$\begin{aligned} \frac{\partial h_2}{\partial c} &= 30072c^5 - 25600c^4y + \left(4-c^2\right)\left(38400c^2y + 144000c\left(1-y^2\right)\right) \\ &\quad - 144000c^3\left(1-y^2\right) - c\left(1280000 - 320000c^2\right)y^2. \end{aligned} \tag{55}$$

By substituting the value of y in (55), plugging $\frac{\partial h_2}{\partial c} = 0$ and simplifying, we obtain

$$\begin{aligned} \frac{\partial h_2}{\partial c} &= 72c\Big(142671c^8 - 2034480c^6 + 9568000c^4 - 18560000c^2 \\ &\quad +12800000) = 0. \end{aligned} \tag{56}$$

A calculation gives the solution of (56) in the interval $(0,1)$ that is $c \approx 1.360226043$. Thus $h_2(c,y)$ has no optimal solution in the interval $(0,2) \times (0,1)$.
(iv) On the face $x=1$, $Q(c,x,y)$ takes the form

$$\begin{aligned} h_3(c,y) &= Q(c,1,y) = 5012c^6 + \left(4-c^2\right)\left(\left(4-c^2\right)\left(28300c^2 + 36000\right) + 37145c^4 \right. \\ &\quad \left. +72000c^2\right). \end{aligned}$$

Then

$$\frac{\partial h_3}{\partial c} = -22998c^5 - 455280c^3 + 905600c.$$

Taking $\frac{\partial h_3}{\partial c} = 0$ and solving, we obtain $c \approx 1.34963183573$. Therefore $\max h_3(c,y) = 999{,}971.4325$. Thus we have

$$Q(c,1,y) \leq \max h_3(c,y) = 999971.4325 < 1280000, \quad (c,y) \in (0,2) \times (0,1).$$

(v) On the face $y = 0$, $Q(c, x, y)$ yields

$$\begin{aligned} h_4(c,x) &= Q(c,x,0) = 2500c^6x^4 - 3000c^6x^3 - 800c^6x^2 - 20000c^4x^4 - 7545c^6x \\ &\quad - 102000c^4x^3 + 5012c^6 - 40000c^4x^2 + 40000c^2x^4 + 120180c^4x \\ &\quad + 672000c^2x^3 - 72000c^4 + 172800c^2x^2 - 720000c^2x - 864000x^3 \\ &\quad + 288000c^2 + 1440000x. \end{aligned}$$

Now, differentiating partially with respect to c, then with respect to x and simplifying, we have

$$\begin{aligned} \frac{\partial h_4}{\partial c} &= 15000c^5x^4 - 18000c^5x^3 - 4800c^5x^2 - 80000c^3x^4 - 45270c^5x \\ &\quad - 408000c^3x^3 + 30072c^5 - 160000c^3x^2 + 80000cx^4 \\ &\quad + 480720c^3x + 1344000cx^3 - 288000c^3 + 345600cx^2 \\ &\quad - 1440000cx + 576000c. \end{aligned} \tag{57}$$

and

$$\begin{aligned} \frac{\partial h_4}{\partial x} &= 10000c^6x^3 - 9000c^6x^2 - 1600c^6x - 80000c^4x^3 - 7545c^6 \\ &\quad - 306000c^4x^2 - 80000c^4x + 160000c^2x^3 + 120180c^4 \\ &\quad + 2016000c^2x^2 + 345600c^2x - 720000c^2 - 2592000x^2 \\ &\quad + 1440000. \end{aligned} \tag{58}$$

Applying Newton's methods to the system of nonlinear Equations (57) and (58) in Maple Software, we found that the given system of equations has no solution in $(0,2) \times (0,1)$.

(vi) On the face $y = 1$, $Q(c, x, y)$ reduces to

$$\begin{aligned} h_5(c,x) &= Q(c,x,1) = 2500c^6x^4 - 3000c^6x^3 - 10000c^5x^4 - 800c^6x^2 \\ &\quad + 24000c^5x^3 - 30000c^4x^4 - 7545c^6x + 22800c^5x^2 \\ &\quad + 60000c^4x^3 + 80000c^3x^4 + 5012c^6 - 24000c^5x \\ &\quad - 182000c^4x^2 + 96000c^3x^3 + 120000c^2x^4 - 12800c^5 \\ &\quad - 41820c^4x - 131200c^3x^2 - 336000c^2x^3 - 160000cx^4 \\ &\quad + 80000c^4 - 96000c^3x + 1020800c^2x^2 - 768000cx^3 \\ &\quad - 160000x^4 + 51200c^3 + 288000c^2x + 160000cx^2 \\ &\quad + 576000x^3 - 640000c^2 + 768000cx - 1120000x^2 \\ &\quad + 1280000. \end{aligned}$$

Partial derivative of $h_5(c, x)$ with respect to c and then with respect to x, we have

$$\begin{aligned} \frac{\partial h_5}{\partial c} &= 15000c^5x^4 - 18000c^5x^3 - 50000c^4x^4 - 4800c^5x^2 + 120000c^4x^3 \\ &\quad - 120000c^3x^4 - 45270c^5x + 114000c^4x^2 + 240000c^3x^3 \\ &\quad + 240000c^2x^4 + 30072c^5 - 120000c^4x - 728000c^3x^2 \\ &\quad + 288000c^2x^3 + 240000cx^4 - 64000c^4 - 167280c^3x \\ &\quad - 393600c^2x^2 - 672000cx^3 - 160000x^4 + 320000c^3 \\ &\quad - 288000c^2x + 2041600cx^2 - 768000x^3 + 153600c^2 \\ &\quad + 576000cx + 160000x^2 - 1280000c + 768000x. \end{aligned} \tag{59}$$

and

$$\begin{aligned}
\frac{\partial h_5}{\partial x} &= 10000c^6x^3 - 9000c^6x^2 - 40000c^5x^3 - 1600c^6x + 72000c^5x^2 \\
&\quad - 120000c^4x^3 - 7545c^6 + 45600c^5x + 180000c^4x^2 \\
&\quad + 320000c^3x^3 - 24000c^5 - 364000c^4x + 288000c^3x^2 \\
&\quad + 480000c^2x^3 - 41820c^4 - 262400c^3x - 1008000c^2x^2 \\
&\quad - 640000cx^3 - 96000c^3 + 2041600c^2x - 2304000cx^2 \\
&\quad - 640000x^3 + 288000c^2 + 320000cx + 1728000x^2 \\
&\quad + 768000c - 2240000x. \qquad (60)
\end{aligned}$$

As mentioned in the above case, we conclude that for the face $y = 0$, the system of Equations (59) and (60) has no solution in $(0,2) \times (0,1)$. Thus $Q(c,x,1)$ has no optimal solution in $(0,2) \times (0,1)$.

3. On the Edges of Cuboid Y:

(i) On the edge $x = 0$ and $y = 0$, then $Q(c,x,y)$ becomes

$$Q(c,0,0) = 5012c^6 - 72000\left(4 - c^2\right)c^2 = m_1(c).$$

It is clear that

$$m_1'(c) = 30072c^5 - 288000c^3 + 57600c.$$

Putting $m_1'(c) = 0$ gives the critical point $c_0 \approx 1.686823152$ at which $Q(c,0,0) = m_1(c)$ obtains its maximum. Therefore $\max m_1(c) = m_1(c_0) = 352{,}004.0398$. Hence

$$Q(c,0,0) \leq \max m_1(c) = m_1(c_0) = 352004.0398 < 1280000, \quad c \in [0,2].$$

(ii) On the edge $x = 0$ and $y = 1$, then $Q(c,x,y)$ takes the form

$$Q(c,0,1) = 5012c^6 - 12800c^5 + 80000c^4 + 51200c^3 - 640000c^2 + 1280000 = m_2(c).$$

Then

$$m_2'(c) = 30072c^5 - 64000c^4 + 320000c^3 + 153600c^2 - 1280000c.$$

As $m_2'(c) < 0$ in $[0,2]$, it is noted that $m_2(c)$ is decreasing over $[0,2]$. Thus $m_2(c)$ has its maxima at $c = 0$. Therefore $\max m_2(c) = m_2(0) = 1{,}280{,}000$. Hence

$$Q(c,0,1) \leq \max m_2(c) = m_2(0) = 1280000 \quad c \in [0,1].$$

(iii) On the edge $c = 0$ and $x = 0$, then $Q(c,x,y)$ yields

$$Q(0,0,y) = 1280000y^2 \leq 1280000, y \in [0,1].$$

(iv) For $Q(c,1,0)$ and $Q(c,1,1)$, as $Q(c,1,y)$ is free of y, it follows that

$$Q(c,1,0) = Q(c,1,1) = -3833c^6 - 113820c^4 + 452800c^2 + 576000 = m_4(c).$$

Then

$$m_4'(c) = -22998c^5 - 455280c^3 + 905600c.$$

Putting $m_4'(c) = 0$, we obtain the critical point $c_0 \approx 1.34963183$ at which $Q(c,1,0) = Q(c,1,1) = m_4(c)$ maximizes, therefore $\max m_4(c) = m_4(c_0) = 999{,}971.435$. Thus

$$Q(c,1,0) = Q(c,1,1) \leq \max m_4(c) = m_4(c_0) = 999971.435 < 12080000, \quad c \in [0,2].$$

(v) On the edge $c = 0$ and $x = 1$, then $Q(c, x, y)$ reduces to

$$Q(0, 1, y) \equiv 576000 < 1280000, \quad y \in [0, 1].$$

(vi) On the edge $c = 2$, then $Q(c, x, y)$ becomes

$$Q(2, x, y) \equiv 320768 < 1280000, \quad (x, y) \in [0, 1] \times [0, 1].$$

(vii) On the edge $c = 0$ and $y = 0$, then $Q(c, x, y)$ yields

$$Q(0, x, 0) = -864000x^3 + 1440000x = m_5(x).$$

Then

$$m_5'(x) = -2592000x^2 + 1440000.$$

The equation $m_5'(x) = 0$ gives the critical point $x_0 \approx 0.74535599$ at which $m_5(x)$ obtains its maximum. Therefore $\max m_5(x) = m_5(x_0) = 715{,}541.7526$. Hence

$$Q(0, x, 0) \leq \max m_5(x) = m_5(x_0) = 715541.7526 < 1280000, \quad x \in [0, 1].$$

(viii) On the edge $c = 0$ and $y = 1$, then $Q(c, x, y)$ takes the form

$$Q(0, x, 1) = -160000x^4 + 576000x^3 - 1120000x^2 + 1280000 = m_6(x).$$

Then

$$m_6'(x) = -640000x^3 + 1728000x^2 - 2240000x.$$

Noting that $m_6'(x) < 0$ in $[0, 1]$, $m_6(x)$ is decreasing over $[0, 1]$. Thus $m_6(x)$ has its maxima at $x = 0$. Therefore $\max m_6(x) = m_6(0) = 1{,}280{,}000$. Hence

$$Q(0, x, 1) \leq \max m_6(x) = m_6(0) = 1280000, \quad x \in [0, 1].$$

From the above cases, we conclude that

$$Q(c, x, y) \leq 1280000 \quad \text{on} \quad [0, 2] \times [0, 1] \times [0.1].$$

Using (52) we see that

$$|\Delta_{3,1}(f)| \leq \frac{1}{18000000}(Q(c, x, y)) \leq \frac{16}{225}.$$

Equality is determined by the extremal function given by

$$z \exp\left(\int_0^z \left(\frac{4}{5}\left(t^2\right) + \frac{1}{5}\left(t^{11}\right)\right) dt\right) = z + \frac{4}{15}z^4 + \cdots.$$

□

Theorem 9. *Let f belong to $\mathcal{S}^*_{3l}$ with the form (1). Then*

$$\left|a_5 - a_3^2\right| \leq \frac{1}{5}.$$

This inequality is the best one.

Proof. From (48) and (50), we obtain

$$\left|a_5 - a_3^2\right| = \left|\frac{13}{8000}c_1^4 - \frac{7}{100}c_2^2 - \frac{7}{150}c_1c_3 + \frac{77}{3000}c_1^2c_2 + \frac{1}{10}c_4\right|.$$

After simplification, we have

$$\left|a_5 - a_3^2\right| = \frac{1}{10}\left|-\frac{13}{800}c_1^4 + \frac{7}{10}c_2^2 + 2\left(\frac{7}{30}\right)c_1c_3 - \frac{3}{2}\left(\frac{77}{450}\right)c_1^2c_2 - c_4\right|. \tag{61}$$

Let $\gamma = -\frac{13}{800}, a = \frac{7}{10}, \alpha = \frac{7}{30}$ and $\beta = \frac{77}{450}$. It can be easily verified that $0 < \alpha < 1$, $0 < a < 1$ and

$$8a(1-a)\left((\alpha\beta - 2\gamma)^2 + (\alpha(a+\alpha) - \beta)^2\right) + \alpha(1-\alpha)(\beta - 2a\alpha)^2 \leqq 4a\alpha^2(1-\alpha)^2(1-a).$$

An application of Lemma 3 leads to

$$\left|a_5 - a_3^2\right| \leq \frac{1}{5}.$$

The equality is obtained by

$$z\exp\left(\int_0^z \left(\frac{4}{5}\left(t^3\right) + \frac{1}{5}\left(t^{15}\right)\right)dt\right) = z + \frac{1}{5}z^5 + \cdots.$$

□

Theorem 10. *If f belongs to $\mathcal{S}_{3l}^*$ and has the expansion* (1), *then*

$$\left|a_3a_5 - a_4^2\right| \leq \frac{16}{225}.$$

This result is sharp.

Proof. Putting (48)–(50) with $c_1 = c$, we obtain

$$\begin{aligned} a_3a_5 - a_4^2 = \frac{1}{18000000}\Big(&-1017c^6 - 2400c^3c_3 + 23200c^2c_2^2 - 36000c^2c_4 + 8610c^4c_2 \\ &+88000cc_2c_3 - 108000c_2^3 + 360000c_2c_4 - 320000c_3^2\Big). \end{aligned} \tag{62}$$

Let $t = 4 - c^2$ in (10), (11) and (12). Now using Lemma 1, we obtain

$$\begin{aligned}
2400c^3c_3 &= 600c^6 + 1200c^4tx - 600c^4tx^2 + 1200c^3t\left(1-|x|^2\right)\sigma, \\
23200c^2c_2^2 &= 5800c^6 + 11600c^4tx + 5800c^2t^2x^2, \\
36000c^2c_4 &= 4500c^6 + 4500c^4tx^3 - 13500c^4tx^2 + 13500c^4tx + 18000tc^2x^2 \\
&\quad - 18000c^3tx\left(1-|x|^2\right)\sigma - 18000c^2t\overline{x}\left(1-|x|^2\right)\sigma^2 \\
&\quad + 18000c^2t\left(1-|x|^2\right)\left(1-|\sigma|^2\right)\rho + 18000c^3t\left(1-|x|^2\right)\sigma, \\
88000cc_2c_3 &= 11000c^6 + 33000c^4tx - 11000c^4tx^2 + 22000c^3t\left(1-|x|^2\right)\sigma \\
&\quad +22000c^2t^2x^2 - 11000c^2t^2x^3 + 22000cxt^2\left(1-|x|^2\right)\sigma, \\
8610c^4c_2 &= 4305c^6 + 4305c^4xt, \\
108000c_2^3 &= 90000c^6 + 40500c^4xt + 40500c^2x^2t^2 + 13500x^3t^3,
\end{aligned}$$

$$\begin{aligned}
360000c_2c_4 &= 22500c^6 + 22500c^4x^3t - 67500c^4x^2t + 90000c^4xt + 90000c^2x^2t \\
&\quad - 90000c^3xt\left(1-|x|^2\right)\sigma - 90000c^2t\overline{x}\left(1-|x|^2\right)\sigma^2 \\
&\quad + 90000c^2t\left(1-|x|^2\right)\left(1-|\sigma|^2\right)\rho + 90000c^3t\left(1-|x|^2\right)\sigma \\
&\quad + 22500c^2x^4t^2 - 67500c^2x^3t^2 + 67500c^2x^2t^2 + 90000x^3t^2 \\
&\quad - 10000cx^2t^2\left(1-|x|^2\right)\sigma - 90000xt^2\overline{x}\left(1-|x|^2\right)\sigma^2 \\
&\quad + 90000xt^2\left(1-|x|^2\right)\left(1-|\sigma|^2\right)\rho + 90000cxt^2\left(1-|x|^2\right)\sigma, \\
320000c_3^2 &= 20000c^6 + 80000c^4xt - 40000c^4x^2t + 80000c^3t\left(1-|x|^2\right)\sigma \\
&\quad +80000c^2x^2t^2 - 80000c^2x^3t^2 + 160000cxt^2\left(1-|x|^2\right)\sigma \\
&\quad +20000c^2x^4t^2 - 80000cx^2t^2\left(1-|x|^2\right)\sigma + 80000t^2\left(1-|x|^2\right)^2\sigma^2.
\end{aligned}$$

Putting the above expressions in (62), we obtain

$$\begin{aligned}
a_3a_5 - a_4^2 &= \frac{1}{18000000}\Big\{-80000t^2\left(1-|x|^2\right)^2\sigma^2 - 24400c^4x^2t + 2500c^2x^4t^2 \\
&\quad +1500c^2x^3t^2 - 25200c^2x^2t^2 + 3705c^4xt + 72000c^2x^2t + 18000c^4x^3t \\
&\quad +90000x^3t^2 - 13500x^3t^3 - 72000c^3xt\left(1-|x|^2\right)\sigma - 72000c^2t\overline{x}\left(1-|x|^2\right)\sigma^2 \\
&\quad +72000c^2t\left(1-|x|^2\right)\left(1-|\sigma|^2\right)\rho - 10000cx^2t^2\left(1-|x|^2\right)\sigma - 90000xt^2\overline{x} \\
&\quad \left(1-|x|^2\right)\sigma^2 + 90000xt^2\left(1-|x|^2\right)\left(1-|\sigma|^2\right)\rho - 48000cxt^2\left(1-|x|^2\right)\sigma \\
&\quad +12800c^3t\left(1-|x|^2\right)\sigma + 3988c^6\Big\}.
\end{aligned}$$

In view of $t = 4 - c^2$, we obtain that

$$a_3a_5 - a_4^2 = \frac{1}{18000000}\left(l_1(c,x) + l_2(c,x)\sigma + l_3(c,x)\sigma^2 + \varsigma(c,x,\sigma)\rho\right),$$

where

$$\begin{aligned}
l_1(c,x) &= 3988c^6 + \left(4-c^2\right)\Big[\left(4-c^2\right)\Big(2500c^2x^4 + 15000c^2x^3 + 36000x^3 \\
&\quad -25200c^2x^2\Big) - 24400c^4x^2 + 3705c^4x + 72000c^2x^2 + 18000c^4x^3\Big], \\
l_2(c,x) &= \left(4-c^2\right)\left(1-|x|^2\right)\Big[\left(4-c^2\right)\left(-48000cx - 10000cx^2\right) \\
&\quad -72000c^3x + 12800c^3\Big], \\
l_3(c,x) &= \left(4-c^2\right)\left(1-|x|^2\right)\Big[\left(4-c^2\right)\left(-80000 - 10000|x|^2\right) - 72000c^2\overline{x}\Big], \\
\varsigma(c,x,\sigma) &= \left(4-c^2\right)\left(1-|x|^2\right)\left(1-|\sigma|^2\right)\Big[90000x\left(4-c^2\right) + 72000c^2\Big].
\end{aligned}$$

Utilizing $|x| = x$, $|\sigma| = y$ and also observing the fact $|\rho| \leq 1$, we obtain

$$\begin{aligned}
\left|a_3a_5 - a_4^2\right| &\leq \frac{1}{18000000}\left(|l_1(c,x)| + |l_2(c,x)|y + |l_3(c,x)|y^2 + |\varsigma(c,x,\sigma)|\right). \\
&\leq \frac{1}{18000000}(S(c,x,y)),
\end{aligned} \tag{63}$$

where

$$S(c,x,y) = s_1(c,x) + s_2(c,x)y + s_3(c,x)y^2 + s_4(c,x)\left(1-y^2\right),$$

with

$$\begin{aligned}
s_1(c,x) &= 3988c^6 + \left(4-c^2\right)\Big[\left(4-c^2\right)\left(2500c^2x^4 + 15000c^2x^3 + 36000x^3\right.\\
&\quad \left.+25200c^2x^2\right) + 24400c^4x^2 + 3705c^4x + 72000c^2x^2 + 18000c^4x^3\Big],\\
s_2(c,x) &= \left(4-c^2\right)\left(1-x^2\right)\Big[\left(4-c^2\right)\left(48000cx + 10000cx^2\right)\\
&\quad +72000c^3x + 12800c^3\Big],\\
s_3(c,x) &= \left(4-c^2\right)\left(1-x^2\right)\Big[\left(4-c^2\right)\left(80000 + 10000x^2\right) + 72000c^2x\Big],\\
s_4(c,x) &= \left(4-c^2\right)\left(1-x^2\right)\Big[90000x\left(4-c^2\right) + 72000c^2\Big].
\end{aligned}$$

Now we have to maximize $S(c,x,y)$ in the closed cuboid Y : $[0,2]\times[0,1]\times[0,1]$. For this, we have to discuss the maximum values of $S(c,x,y)$ in the interior of Y, in the interior of its 6 faces and on its 12 edges.

In the following, we will prove that the maximum value of $S(c,x,y)$ is 1,280,000 in the closed cuboid Y. To prove this, we first discuss the maximum values of $S(c,x,y)$ in the interior of 6 faces and 12 edges of Y.

It is not hard to note that $s_2(c,x) = q_2(c,x)$, $s_3(c,x) = q_3(c,x)$ and $s_4(c,x) = q_4(c,x)$ for all $(c,x)\in[0,2]\times[0,1]$. A simple calculation shows that

$$s_1(c,x) - q_1(c,x) = -1024c^6 + (4-c^2)\Big[14400(4-c^2)c^2x^2 + 12800c^4x^2 - 3840c^4x\Big].$$

It is clear that

$$\varpi(c,0) = -1024c^6 \le 0, \quad c\in[0,2]$$

and thus

$$s_1(c,0) \le q_1(c,0).$$

For $(c,y)\in[0,2]\times[0,1]$, it follows that

$$S(c,0,y) = s_1(c,0) + s_2(c,0)y + s_3(c,0)y^2 + s_4(c,0)(1-y^2) \le Q(c,0,y) \le 1280000.$$

For $x=1$, it is noted that

$$s_2(c,1) = s_3(c,1) = s_4(c,1) \equiv 0, \quad c\in[0,2].$$

Therefore, we have

$$\begin{aligned}
S(c,1,y) = s_1(c,1) = L_1(c) &= 3988c^6 + \left(4-c^2\right)\Big[\left(4-c^2\right)\left(42700c^2 + 36000\right)\\
&\quad +46105c^4 + 72000c^2\Big].
\end{aligned}$$

Then

$$\frac{\partial L_1}{\partial c} = 3498c^5 - 772720c^3 + 1366400c.$$

Putting $\frac{\partial L_1}{\partial c} = 0$ and solving, we obtain $c \approx 1.335172357$. Hence, we obtain that

$$S(c,1,y) \le \max L_1(c) = 1183313.834 < 1280000, \quad (c,y)\in[0,2]\times[0,1].$$

Now we only need to prove that $S(c,x,y)$ does not exceed 1,280,000 in the inside of Y. By observing that

$$\frac{\partial S(c,x,y)}{\partial y} = s_2(c,x) + 2[s_3(c,x) - s_4(c,x)]y = \frac{\partial Q(c,x,y)}{\partial y},$$

we easily find that there are no critical points of S in $(0,2)\times[\frac{1}{2},1)\times(0,1)$ from the proof of Theorem 8.

Suppose that there is a critical point $(\tilde{c},\tilde{x},\tilde{y})$ of S existing in the interior of cuboid Υ. It is clear that $\tilde{x}\leq\frac{1}{2}$. Moreover, it can be seen that $\tilde{c}^2\geq\frac{300}{147}$ and $\tilde{y}\in(0,1)$. For $(c,x,y)\in\left(\sqrt{\frac{300}{147}},2\right)\times(0,\frac{1}{2})\times(0,1)$, by invoking $x<\frac{1}{2}$ and $1-x^2<1$ it is not hard to observe that

$$\begin{aligned}s_1(c,x) \leq\;& 3988c^6+\left(4-c^2\right)\Big[\left(4-c^2\right)\Big(2500c^2(1/2)^4+15000c^2(1/2)^3+36000(1/2)^3\\&+25200c^2(1/2)^2\Big)+24400c^4(1/2)^2+3705c^4(1/2)+72000c^2(1/2)^2+18000c^4(1/2)^3\Big],\\=\;&3988c^6+\frac{1}{4}\left(4-c^2\right)\left(7485c^4+187300c^2+72000\right):=\hat{\phi}_1(c),\end{aligned}$$

and

$$s_2(c,x)\leq\phi_2(c),\quad s_3(c,x)\leq\phi_3(c),\quad s_4(c,x)\leq\phi_4(c).$$

Therefore, we have

$$S(c,x,y)\leq\hat{\phi}_1(c)+\phi_4(c)+\phi_2(c)y+[\phi_3(c)-\phi_4(c)]y^2:=\Xi_3(c,y).$$

It is easily to be seen that

$$\frac{\partial\Xi_3}{\partial y}=\phi_2(c)+2[\phi_3(c)-\phi_4(c)]y=\frac{\partial\Xi_2}{\partial y}\leq 0,\quad y\in(0,1).$$

Thus, we obtain

$$\Xi_3(c,y)\leq\Xi_3(c,1)=\hat{\phi}_1(c)+\phi_2(c)+\phi_3(c):=\iota_3(c),\quad c\in(\sqrt{\frac{300}{147}},2).$$

It is easy to calculate that $\iota_3(c)$ attains its extremal value 1,156,314 at $c\approx 1.428571$. Thus, we have

$$S(c,x,y)<1280000,\quad (c,x,y)\in\left(\sqrt{\frac{300}{147}},2\right)\times(0,\frac{1}{2})\times(0,1).$$

Hence $S(\tilde{c},\tilde{x},\tilde{y})<1{,}280{,}000$. This implies that S is less than 1,280,000 at all the critical points in the interior of Υ. Therefore, S has no optimal solution in the interior of Υ.

From the above discussion, we conclude that

$$S(c,x,y)\leq 1280000\quad\text{on } [0,2]\times[0,1]\times[0,1].$$

In virtue of (63), we can write

$$\left|a_3a_5-a_4^2\right|\leq\frac{1}{18000000}(S(c,x,y))\leq\frac{16}{225}.$$

Equality is achieved by an extremal function

$$z\exp\left(\int_0^z\left(\frac{4}{5}\left(t^2\right)+\frac{1}{5}\left(t^{11}\right)\right)dt\right)=z+\frac{4}{15}z^4+\cdots.$$

□

Example 1. *From (6), one can easily deduce the following functions*

$$f_0(z) = z \exp\left(\frac{16z^3 + z^{12}}{60}\right) = z + \frac{4}{15}z^4 + \frac{8}{225}z^7 + \cdots \tag{64}$$

and

$$f_1(z) = z \exp\left(\frac{16z^4 + z^{16}}{80}\right) = z + \frac{1}{5}z^5 + \frac{1}{50}z^9 + \cdots . \tag{65}$$

Both these functions belong to the class $\mathcal{S}^*_{3l}$. Comparing coefficients of like powers of (64) and (1), we have

$$a_2 = a_3 = a_5 = 0, \ a_4 = \frac{4}{15}.$$

Then, it follows that

$$\left|a_3a_5 - a_4^2\right| = \frac{16}{225}$$

and

$$|\Delta_{3,1}(f)| = \frac{16}{225}.$$

Similarly, using (65), we easily obtain that

$$\left|a_3^2 - a_5\right| = \frac{1}{5}.$$

7. Concluding Remarks and Observations

Due to the great importance of coefficients in the field of function theory, Pommerenke [16,17] proposed the topic of studying the Hankel determinant with entry of coefficients. In the current article, we considered two subfamilies of starlike and bounded turning functions, denoted by $\mathcal{S}^*_{3l}$ and $\mathcal{BT}_{3l}$, respectively. These families of univalent functions were connected by a three-leaf-shaped domain with the quantities $zf'(z)/f(z)$ and $f'(z)$ being subordinated to $1 + \frac{4}{5}z + \frac{1}{5}z^4$. For functions belonging to these classes, we investigated various intriguing problems containing initial coefficients. Among these problems, the sharp bounds of the Hankel determinant are extremely difficult to investigate, and we determined the sharp estimate of this determinant for functions belonging to both classes.

In proving our main results, finding the upper bounds of the Hankel determinant for functions belonging to $\mathcal{S}^*_{3l}$ or $\mathcal{BT}_{3l}$ was transformed into a maximum value problem of a function with three variables in the domain of a cuboid. Based on an analysis of all the possibilities that the maxima might occur, we were able to determine the sharp upper bounds for these families. Numerical analysis was applied since some of the computations are quite complicated. Clearly, this approach may be used to calculate bounds for functions belonging to various subfamilies of univalent functions. However, in most cases, it is not so lucky to obtain such sharp results.

Furthermore, the application of the familiar quantum or fundamental (or q-) calculus, as (for example) in similar recent publications [51–54], might be a promising route for additional research based on our current findings.

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