



Article

Fractional Integral Inequalities of Hermite–Hadamard Type for $(h, g; m)$ -Convex Functions with Extended Mittag-Leffler Function

Maja Andrić

Faculty of Civil Engineering, Architecture and Geodesy, University of Split, Matice hrvatske 15,
21000 Split, Croatia; maja.andric@gradst.hr

Abstract: Several fractional integral inequalities of the Hermite–Hadamard type are presented for the class of $(h, g; m)$ -convex functions. Applied fractional integral operators contain extended generalized Mittag-Leffler functions as their kernel, thus enabling new fractional integral inequalities that extend and generalize the known results. As an application, the upper bounds of fractional integral operators for $(h, g; m)$ -convex functions are given.

Keywords: fractional calculus; Mittag-Leffler function; convex function; Hermite–Hadamard inequality

1. Introduction

In recent years, in the field of applied sciences, fractional calculus has been used with different boundary conditions to develop mathematical models relating to real-world problems. This significant interest in the theory of fractional calculus has been stimulated by many of its applications, especially in the various fields of physics and engineering.

Inequalities involving integrals of functions and their derivatives are of great importance in mathematical analysis and its applications. Inequalities containing fractional derivatives have applications in regard to fractional differential equations, especially in establishing the uniqueness of the solutions of initial value problems and their upper bounds. This kind of application motivated the researchers towards the theory of integral inequalities, with the aim of extending and generalizing classical inequalities using different fractional integral operators.

The motivation for this research on Hermite–Hadamard-type integral inequalities was provided by recent studies on these inequalities for different types of integral operators (see [1–8]) and different classes of convexity (see [9–17]). The famous Hermite–Hadamard inequality provides an estimate of the (integral) mean value of a continuous convex function.

Theorem 1 (The Hermite–Hadamard inequality). *Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous convex function. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}.$$

Its fractional version, involving Riemann–Liouville fractional integrals, is given in [18].

Theorem 2 ([18]). *Let $f : [a, b] \rightarrow \mathbb{R}$ be a convex function with $f \in L_1[a, b]$. Then for $\sigma > 0$*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\sigma+1)}{2(b-a)^\sigma} [J_{a+}^\sigma f(b) + J_{b-}^\sigma f(a)] \leq \frac{f(a) + f(b)}{2}.$$

Recall that the left-sided and the right-sided Riemann–Liouville fractional integrals of order $\sigma > 0$ are defined as in [19] for $f \in L_1[a, b]$ with



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$$J_{a+}^{\sigma} f(x) = \frac{1}{\Gamma(\sigma)} \int_a^x (x-t)^{\sigma-1} f(t) dt, \quad x \in (a, b], \quad (1)$$

$$J_{b-}^{\sigma} f(x) = \frac{1}{\Gamma(\sigma)} \int_x^b (t-x)^{\sigma-1} f(t) dt, \quad x \in [a, b). \quad (2)$$

Our aim is to prove Hermite–Hadamard’s inequality in more general settings, and for this we need an extended generalized Mittag-Leffler function with its fractional integral operators and a class of $(h, g; m)$ -convex functions.

The paper is structured as follows. In Section 2, we give preliminary results and definitions that will be used in this paper. In Section 3, several Hermite–Hadamard-type inequalities for $(h, g; m)$ -convex functions using fractional integral operators are presented. Furthermore, several properties and identities of these operators are given. As an application, in Section 4 we derive the upper bounds of fractional integral operators involving $(h, g; m)$ -convex functions. In the last section, Section 5, we present the conclusions of this research.

2. Preliminaries

2.1. An Extended Generalized form of the Mittag-Leffler Function

The Mittag-Leffler function

$$E_{\rho}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\rho n + 1)} \quad (z \in \mathbb{C}, \Re(\rho) > 0)$$

with its generalizations appears as a solution of fractional differential or integral equations. The first generalization for two parameters was carried out by Wiman [8]:

$$E_{\rho, \sigma}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\rho n + \sigma)}, \quad (z, \rho, \sigma \in \mathbb{C}, \Re(\rho) > 0), \quad (3)$$

after which Prabhakar defined the Mittag-Leffler function of three parameters [3]:

$$E_{\rho, \sigma}^{\delta}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_n}{\Gamma(\rho n + \sigma)} \frac{z^n}{n!}, \quad (z, \rho, \sigma, \delta \in \mathbb{C}, \Re(\rho) > 0). \quad (4)$$

Recently we presented in [1] (see also [2]) an extended generalized form of the Mittag-Leffler function $E_{\rho, \sigma, \tau}^{\delta, c, v, r}(z; p)$:

Definition 1 ([1]). Let $\rho, \sigma, \tau, \delta, c \in \mathbb{C}$, $\Re(\rho), \Re(\sigma), \Re(\tau) > 0$, $\Re(c) > \Re(\delta) > 0$ with $p \geq 0$, $r > 0$ and $0 < q \leq r + \Re(\rho)$. Then the extended generalized Mittag-Leffler function $E_{\rho, \sigma, \tau}^{\delta, c, q, r}(z; p)$ is defined by

$$E_{\rho, \sigma, \tau}^{\delta, c, q, r}(z; p) = \sum_{n=0}^{\infty} \frac{B_p(\delta + nq, c - \delta)}{B(\delta, c - \delta)} \frac{(c)_{nq}}{\Gamma(\rho n + \sigma)} \frac{z^n}{(\tau)_{nr}}. \quad (5)$$

Note, we use the generalized Pochhammer symbol $(c)_{nq} = \frac{\Gamma(c+nq)}{\Gamma(c)}$ and an extended beta function $B_p(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} e^{-\frac{p}{t(1-t)}} dt$, where $\Re(x), \Re(y), \Re(p) > 0$.

Remark 1. Several generalizations of the Mittag-Leffler function can be obtained for different parameter choices. For instance, the function (5) is reduced to

- (i) the Salim-Faraj function $E_{\rho, \sigma, \tau}^{\delta, \tau, q}(z)$ for $p = 0$ [5],
- (ii) the Rahman function $E_{\rho, \sigma}^{\delta, q, c}(z; p)$ for $\tau = r = 1$ [4],
- (iii) the Shukla–Prajapati function $E_{\rho, \sigma}^{\delta, q}(z)$ for $p = 0$ and $\tau = r = 1$ [6],
- (iv) the Prabhakar function $E_{\rho, \sigma}^{\delta}(z)$ for $p = 0$ and $\tau = r = q = 1$ [3],

- (v) the Wiman function $E_{\rho,\sigma}(z)$ for $p = 0$ and $\tau = r = q = \delta = 1$ [8],
- (vi) the Mittag-Leffler function $E_{\rho}(z)$ for $p = 0$, $\tau = r = q = \delta = 1$ and $\sigma = 1$.

Next we have corresponding fractional integral operators, the left-sided $\varepsilon_{a^+}^{\omega,\delta,c,q,r} f$ and the right-sided $\varepsilon_{b^-}^{\omega,\delta,c,q,r} f$, where the kernel is a function $E_{\rho,\sigma,\tau}^{\delta,c,q,r}(z; p)$:

Definition 2 ([1]). Let $\omega, \rho, \sigma, \tau, \delta, c \in \mathbb{C}$, $\Re(\rho), \Re(\sigma), \Re(\tau) > 0$, $\Re(c) > \Re(\delta) > 0$ with $p \geq 0$, $r > 0$ and $0 < q \leq r + \Re(\rho)$. Let $f \in L_1[a, b]$ and $x \in [a, b]$. Then the left-sided and the right-sided generalized fractional integral operators $\varepsilon_{a^+}^{\omega,\delta,c,q,r} f$ and $\varepsilon_{b^-}^{\omega,\delta,c,q,r} f$ are defined by

$$\left(\varepsilon_{a^+}^{\omega,\delta,c,q,r} f\right)(x; p) = \int_a^x (x-t)^{\sigma-1} E_{\rho,\sigma,\tau}^{\delta,c,q,r}(\omega(x-t)^\rho; p) f(t) dt, \quad (6)$$

$$\left(\varepsilon_{b^-}^{\omega,\delta,c,q,r} f\right)(x; p) = \int_x^b (t-x)^{\sigma-1} E_{\rho,\sigma,\tau}^{\delta,c,q,r}(\omega(t-x)^\rho; p) f(t) dt. \quad (7)$$

Remark 2. If we apply different parameter choices, then (6) is a generalization of

- (i) the Salim-Faraj fractional integral operator $\varepsilon_{a^+}^{\omega,\delta,q,r} f(x)$ for $p = 0$ [5],
- (ii) the Rahman fractional integral operator $\varepsilon_{a^+}^{\omega,\delta,q,c} f(x; p)$ for $\tau = r = 1$ [4],
- (iii) the Srivastava-Tomovski fractional integral operator $\varepsilon_{a^+}^{\omega,\delta,q} f(x)$ for $p = 0$ and $\tau = r = 1$ [7],
- (iv) the Prabhakar fractional integral operator $\varepsilon(\rho, \sigma; \delta; \omega) f(x)$ for $p = 0$ and $\tau = r = q = 1$ [3],
- (v) the left-sided Riemann-Liouville fractional integral $J_{a^+}^\sigma f(x)$ for $p = \omega = 0$, that is, (1).

We listed reductions for the left-sided fractional integral operator, whereas the analogs are valid for the right-sided.

More details on this generalized form of the Mittag-Leffler function and its fractional integral operators can be found in [1,2]. Here are some results we will use in this study:

Theorem 3 ([1]). If $\alpha, \omega, \rho, \sigma, \tau, \delta, c \in \mathbb{C}$, $\Re(\rho), \Re(\sigma), \Re(\tau) > 0$, $\Re(c) > \Re(\delta) > 0$ with $p \geq 0$, $r > 0$ and $0 < q \leq r + \Re(\rho)$, then for power functions $(t-a)^{\alpha-1}$ and $(b-t)^{\alpha-1}$ follow

$$\left(\varepsilon_{a^+}^{\omega,\delta,c,q,r} (t-a)^{\alpha-1}\right)(x; p) = \Gamma(\alpha) (x-a)^{\alpha+\sigma-1} E_{\rho,\sigma+\alpha,\tau}^{\delta,c,q,r}(\omega(x-a)^\rho; p), \quad (8)$$

$$\left(\varepsilon_{b^-}^{\omega,\delta,c,q,r} (b-t)^{\alpha-1}\right)(x; p) = \Gamma(\alpha) (b-x)^{\alpha+\sigma-1} E_{\rho,\sigma+\alpha,\tau}^{\delta,c,q,r}(\omega(b-x)^\rho; p). \quad (9)$$

If we set $a = 0$ and $x = 1$ in (8), or $b = 1$ and $x = 0$ in (9), then we obtain the following corollary.

Corollary 1 ([1]). If $\alpha, \omega, \rho, \sigma, \tau, \delta, c \in \mathbb{C}$, $\Re(\rho), \Re(\sigma), \Re(\tau) > 0$, $\Re(c) > \Re(\delta) > 0$ with $p \geq 0$, $r > 0$ and $0 < q \leq r + \Re(\rho)$, then

$$\frac{1}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1} (1-t)^{\sigma-1} E_{\rho,\sigma,\tau}^{\delta,c,q,r}(\omega(1-t)^\rho; p) dt = E_{\rho,\sigma+\alpha,\tau}^{\delta,c,q,r}(\omega; p).$$

Setting $\alpha = 1$ in Theorem 3, we obtain following identities for the constant function:

Corollary 2 ([2]). *Let the assumptions of Theorem 3 hold with $\alpha = 1$. Then*

$$\left(\varepsilon_{a^+, \rho, \sigma, \tau}^{\omega, \delta, c, q, r} 1\right)(x; p) = (x - a)^\sigma E_{\rho, \sigma+1, \tau}^{\delta, c, q, r}(\omega(x - a)^\rho; p), \quad (10)$$

$$\left(\varepsilon_{b^-, \rho, \sigma, \tau}^{\omega, \delta, c, q, r} 1\right)(x; p) = (b - x)^\sigma E_{\rho, \sigma+1, \tau}^{\delta, c, q, r}(\omega(b - x)^\rho; p). \quad (11)$$

In this paper, we will use simplified notation to avoid a complicated manuscript form:

$$E(z; p) := E_{\rho, \sigma, \tau}^{\delta, c, q, r}(z; p)$$

and

$$(\varepsilon_{a^+}^\omega f)(x; p) := \left(\varepsilon_{a^+, \rho, \sigma, \tau}^{\omega, \delta, c, q, r} f\right)(x; p),$$

$$(\varepsilon_{b^-}^\omega f)(x; p) := \left(\varepsilon_{b^-, \rho, \sigma, \tau}^{\omega, \delta, c, q, r} f\right)(x; p).$$

Of course, the conditions on all parameters $\rho, \sigma, \tau, \omega, \delta, c, q, r$ are essential and will be added to all theorems.

2.2. A Class of $(h, g; m)$ -Convex Functions

Another direction for the generalization of the Hermite–Hadamard inequality is the use of different classes of convexity. For this we need a class of $(h, g; m)$ -convex functions, the properties of which were recently presented in [14]:

Definition 3 ([14]). *Let h be a nonnegative function on $J \subseteq \mathbb{R}$, $(0, 1) \subseteq J$, $h \not\equiv 0$ and let g be a positive function on $I \subseteq \mathbb{R}$. Furthermore, let $m \in (0, 1]$. A function $f : I \rightarrow \mathbb{R}$ is said to be an $(h, g; m)$ -convex function if it is nonnegative and if*

$$f(tx + m(1 - t)y) \leq h(t)f(x)g(x) + m h(1 - t)f(y)g(y) \quad (12)$$

holds for all $x, y \in I$ and all $t \in (0, 1)$.

If (12) holds in the reversed sense, then f is said to be an $(h, g; m)$ -concave function.

This class unifies a certain range of convexity, enabling generalizations of known results. For different choices of functions h, g and parameter m , a class of $(h, g; m)$ -convex functions is reduced to a class of P -functions [15], h -convex functions [17], m -convex functions [16], $(h - m)$ -convex functions [11], (s, m) -Godunova–Levin functions of the second kind [10], exponentially s -convex functions in the second sense [9], etc. For example, if we set $h(t) = t^s$, $s \in (0, 1]$, $g(x) = e^{-\beta x}$, $\beta \in \mathbb{R}$, then we obtain a class defined in [13]:

A function $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ is called exponentially (s, m) -convex in the second sense if the following inequality holds

$$f(tx + m(1 - t)y) \leq \frac{t^s}{e^{\beta x}} f(x) + \frac{(1 - t)^s}{e^{\beta y}} m f(y) \quad (13)$$

for all $x, y \in I$ and all $t \in [0, 1]$, where $\beta \in \mathbb{R}$, $s, m \in (0, 1]$.

Next we need the Hermite–Hadamard inequality for $(h, g; m)$ -convex functions:

Theorem 4 ([14]). Let f be a nonnegative $(h, g; m)$ -convex function on $[0, \infty)$ where h is a non-negative function on $J \subseteq \mathbb{R}$, $(0, 1) \subseteq J$, $h \not\equiv 0$, g is a positive function on $[0, \infty)$ and $m \in (0, 1]$. If $f, g, h \in L_1[a, b]$, where $0 \leq a < b < \infty$, then the following inequalities hold

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{h\left(\frac{1}{2}\right)}{b-a} \int_a^b \left[f(x)g(x) + mf\left(\frac{x}{m}\right)g\left(\frac{x}{m}\right) \right] dx \\ &\leq \frac{h\left(\frac{1}{2}\right)f(a)g(a)}{b-a} \int_a^b h\left(\frac{b-x}{b-a}\right)g(x) dx \\ &\quad + \frac{mh\left(\frac{1}{2}\right)f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)}{b-a} \int_a^b h\left(\frac{x-a}{b-a}\right)g(x) dx \\ &\quad + \frac{mh\left(\frac{1}{2}\right)f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)}{b-a} \int_a^b h\left(\frac{b-x}{b-a}\right)g\left(\frac{x}{m}\right) dx \\ &\quad + \frac{m^2h\left(\frac{1}{2}\right)f\left(\frac{b}{m^2}\right)g\left(\frac{b}{m^2}\right)}{b-a} \int_a^b h\left(\frac{x-a}{b-a}\right)g\left(\frac{x}{m}\right) dx. \end{aligned} \quad (14)$$

3. Fractional Integral Inequalities of the Hermite–Hadamard Type for $(h, g; m)$ -Convex Functions

The Hermite–Hadamard inequality for $(h, g; m)$ -convex functions is obtained in [14], where some special results are pointed out and several known inequalities are improved upon. In [12], the article that followed, a few more inequalities of the Hermite–Hadamard type are presented. Here we will obtain their fractional generalizations, using (5)–(7), that is, the extended generalized Mittag-Leffler function E with fractional integral operators $\mathfrak{E}_{a+}^{\omega} f$ and $\mathfrak{E}_{b-}^{\omega} f$ in the real domain.

In this section, it is necessary to introduce the following conditions on the parameters and the interval $[a, b]$:

Assumption 1. Let $\omega \in \mathbb{R}$, $\rho, \sigma, \tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r + \rho$. Furthermore, let $0 \leq a < b < \infty$.

We start with the left side, i.e., the first Hermite–Hadamard fractional integral inequality for $(h, g; m)$ -convex functions involving the extended generalized Mittag-Leffler function.

Theorem 5. Let Assumption 1 hold. Let f be a nonnegative $(h, g; m)$ -convex function on $[0, \infty)$, where h is a nonnegative function on $J \subseteq \mathbb{R}$, $(0, 1) \subseteq J$, $h \not\equiv 0$, g is a positive function on $[0, \infty)$ and $m \in (0, 1]$. If $f, g \in L_1[a, \frac{b}{m}]$, then the following inequality holds

$$f\left(\frac{a+b}{2}\right)(\mathfrak{E}_{a+}^{\overline{\omega}} 1)(b; p) \leq h\left(\frac{1}{2}\right) \left[(\mathfrak{E}_{a+}^{\overline{\omega}} f g)(b; p) + m^{\sigma+1} (\mathfrak{E}_{\frac{b}{m}-}^{\overline{\omega}} f g)\left(\frac{a}{m}; p\right) \right], \quad (15)$$

where

$$\overline{\omega} = \frac{\omega}{(b-a)^{\rho}}, \quad \overline{\omega} = \frac{m^{\rho}\omega}{(b-a)^{\rho}}. \quad (16)$$

Proof. Let f be an $(h, g; m)$ -convex function on $[0, \infty)$, $m \in (0, 1]$. Then for $t = \frac{1}{2}$ we have

$$f\left(\frac{x+my}{2}\right) \leq h\left(\frac{1}{2}\right) f(x)g(x) + mh\left(\frac{1}{2}\right) f(y)g(y).$$

Choosing $y \equiv \frac{y}{m}$ we obtain

$$f\left(\frac{x+y}{2}\right) \leq h\left(\frac{1}{2}\right) \left[f(x)g(x) + mf\left(\frac{y}{m}\right)g\left(\frac{y}{m}\right) \right].$$

Let $x = ta + (1-t)b$ and $y = (1-t)a + tb$. Then

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq h\left(\frac{1}{2}\right) \left[f(ta + (1-t)b)g(ta + (1-t)b) \right. \\ &\quad \left. + mf\left((1-t)\frac{a}{m} + t\frac{b}{m}\right)g\left((1-t)\frac{a}{m} + t\frac{b}{m}\right) \right]. \end{aligned}$$

In the following step we will need to multiply both sides of the above inequality by $t^{\sigma-1}\mathbf{E}(\omega t^\rho; p)$ and integrate on $[0, 1]$ with respect to the variable t , which gives us

$$\begin{aligned} &f\left(\frac{a+b}{2}\right) \int_0^1 t^{\sigma-1} \mathbf{E}(\omega t^\rho; p) dt \\ &\leq h\left(\frac{1}{2}\right) \int_0^1 f(ta + (1-t)b)g(ta + (1-t)b) t^{\sigma-1} \mathbf{E}(\omega t^\rho; p) dt \\ &\quad + m h\left(\frac{1}{2}\right) \int_0^1 f\left((1-t)\frac{a}{m} + t\frac{b}{m}\right)g\left((1-t)\frac{a}{m} + t\frac{b}{m}\right) t^{\sigma-1} \mathbf{E}(\omega t^\rho; p) dt. \end{aligned}$$

With substitutions $u = ta + (1-t)b$ and $v = (1-t)\frac{a}{m} + t\frac{b}{m}$ we obtain

$$\begin{aligned} &\frac{1}{(b-a)^\sigma} f\left(\frac{a+b}{2}\right) \int_a^b (b-u)^{\sigma-1} \mathbf{E}(\bar{\omega}(b-u)^\rho; p) du \\ &\leq \frac{h\left(\frac{1}{2}\right)}{(b-a)^\sigma} \int_a^b f(u)g(u)(b-u)^{\sigma-1} \mathbf{E}(\bar{\omega}(b-u)^\rho; p) du \\ &\quad + \frac{m^{\sigma+1} h\left(\frac{1}{2}\right)}{(b-a)^\sigma} \int_{\frac{a}{m}}^{\frac{b}{m}} f(v)g(v)(v-\frac{a}{m})^{\sigma-1} \mathbf{E}(\bar{\omega}(v-\frac{a}{m})^\rho; p) dv. \end{aligned}$$

Since $m \in (0, 1]$, then $a \leq a/m$, $b \leq b/m$ and $[a, b] \subset [a, \frac{b}{m}]$. Therefore, the condition $f, g \in L_1[a, \frac{b}{m}]$ is stated in this theorem. The above inequality can be written as

$$\begin{aligned} &\frac{1}{(b-a)^\sigma} f\left(\frac{a+b}{2}\right) (\mathfrak{E}_{a+}^{\bar{\omega}} 1)(b; p) \\ &\leq \frac{h\left(\frac{1}{2}\right)}{(b-a)^\sigma} \left[(\mathfrak{E}_{a+}^{\bar{\omega}} fg)(b; p) + m^{\sigma+1} (\mathfrak{E}_{\frac{b}{m}-}^{\bar{\omega}} fg)\left(\frac{a}{m}; p\right) \right]. \end{aligned}$$

Note that with Corollary 2 we can obtain the constant $(\mathfrak{E}_{a+}^{\bar{\omega}} 1)(b; p)$. This completes the proof. $\square$

Next we have the second Hermite–Hadamard fractional integral inequality.

Theorem 6. Let the assumptions of Theorem 5 hold with $f, g, h \in L_1[a, \frac{b}{m}]$. Then

$$\begin{aligned}
& (\mathfrak{E}_{a^+}^{\overline{\omega}} f g)(b; p) + m^{\sigma+1} (\mathfrak{E}_{\frac{b}{m}}^{\overline{\omega}} f g)\left(\frac{a}{m}; p\right) \\
& \leq f(a) g(a) \int_a^b h\left(\frac{b-x}{b-a}\right) g(x) (b-x)^{\sigma-1} \mathbf{E}(\overline{\omega}(b-x)^\rho; p) dx \\
& \quad + m f\left(\frac{b}{m}\right) g\left(\frac{b}{m}\right) \int_a^b h\left(\frac{x-a}{b-a}\right) g(x) (b-x)^{\sigma-1} \mathbf{E}(\overline{\omega}(b-x)^\rho; p) dx \\
& \quad + m^{\sigma+1} f\left(\frac{a}{m}\right) g\left(\frac{a}{m}\right) \int_{\frac{a}{m}}^{\frac{b}{m}} h\left(\frac{b-mx}{b-a}\right) g(x) \left(x - \frac{a}{m}\right)^{\sigma-1} \mathbf{E}(\overline{\omega}(x - \frac{a}{m})^\rho; p) dx \\
& \quad + m^{\sigma+2} f\left(\frac{b}{m^2}\right) g\left(\frac{b}{m^2}\right) \int_{\frac{a}{m}}^{\frac{b}{m}} h\left(\frac{mx-a}{b-a}\right) g(x) \left(x - \frac{a}{m}\right)^{\sigma-1} \mathbf{E}(\overline{\omega}(x - \frac{a}{m})^\rho; p) dx, \tag{17}
\end{aligned}$$

where $\overline{\omega}$ and $\overline{\omega}$ are defined by (16).

Proof. Due to the $(h, g; m)$ -convexity of f we have

$$f(ta + (1-t)b) \leq h(t)f(a)g(a) + mh(1-t)f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right).$$

Multiplying both sides of above inequality by $g(ta + (1-t)b)t^{\sigma-1}\mathbf{E}(\omega t^\rho; p)$ and integrating on $[0, 1]$ with respect to the variable t , we obtain

$$\begin{aligned}
& \int_0^1 f(ta + (1-t)b) g(ta + (1-t)b) t^{\sigma-1} \mathbf{E}(\omega t^\rho; p) dt \\
& \leq f(a) g(a) \int_0^1 h(t) g(ta + (1-t)b) t^{\sigma-1} \mathbf{E}(\omega t^\rho; p) dt \\
& \quad + m f\left(\frac{b}{m}\right) g\left(\frac{b}{m}\right) \int_0^1 h(1-t) g(ta + (1-t)b) t^{\sigma-1} \mathbf{E}(\omega t^\rho; p) dt.
\end{aligned}$$

With the substitution $u = ta + (1-t)b$ we obtain

$$\begin{aligned}
& \frac{1}{(b-a)^\sigma} \int_a^b f(u) g(u) (b-u)^{\sigma-1} \mathbf{E}(\overline{\omega}(b-u)^\rho; p) du \\
& \leq \frac{f(a) g(a)}{(b-a)^\sigma} \int_a^b h\left(\frac{b-u}{b-a}\right) g(u) (b-u)^{\sigma-1} \mathbf{E}(\overline{\omega}(b-u)^\rho; p) du \\
& \quad + \frac{m f\left(\frac{b}{m}\right) g\left(\frac{b}{m}\right)}{(b-a)^\sigma} \int_a^b h\left(\frac{u-a}{b-a}\right) g(u) (b-u)^{\sigma-1} \mathbf{E}(\overline{\omega}(b-u)^\rho; p) du,
\end{aligned}$$

that is

$$\begin{aligned}
& (\mathfrak{E}_{a^+}^{\overline{\omega}} f g)(b; p) \\
& \leq f(a) g(a) \int_a^b h\left(\frac{b-u}{b-a}\right) g(u) (b-u)^{\sigma-1} \mathbf{E}(\overline{\omega}(b-u)^\rho; p) du \\
& \quad + m f\left(\frac{b}{m}\right) g\left(\frac{b}{m}\right) \int_a^b h\left(\frac{u-a}{b-a}\right) g(u) (b-u)^{\sigma-1} \mathbf{E}(\overline{\omega}(b-u)^\rho; p) du. \tag{18}
\end{aligned}$$

Again, due to the $(h, g; m)$ -convexity of f we have

$$f\left((1-t)\frac{a}{m} + t\frac{b}{m}\right) \leq h(1-t)f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right) + mh(t)f\left(\frac{b}{m^2}\right)g\left(\frac{b}{m^2}\right).$$

Multiplying both sides of above inequality by $g\left((1-t)\frac{a}{m} + t\frac{b}{m}\right)t^{\sigma-1}\mathbf{E}(\omega t^\rho; p)$ and integrating on $[0, 1]$ with respect to the variable t , we obtain

$$\begin{aligned} & \int_0^1 f\left((1-t)\frac{a}{m} + t\frac{b}{m}\right)g\left((1-t)\frac{a}{m} + t\frac{b}{m}\right)t^{\sigma-1}\mathbf{E}(\omega t^\rho; p)dt \\ & \leq f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)\int_0^1 h(1-t)g\left((1-t)\frac{a}{m} + t\frac{b}{m}\right)t^{\sigma-1}\mathbf{E}(\omega t^\rho; p)dt \\ & \quad + m f\left(\frac{b}{m^2}\right)g\left(\frac{b}{m^2}\right)\int_0^1 h(t)g\left((1-t)\frac{a}{m} + t\frac{b}{m}\right)t^{\sigma-1}\mathbf{E}(\omega t^\rho; p)dt. \end{aligned}$$

With the substitution $v = (1-t)\frac{a}{m} + t\frac{b}{m}$ we obtain

$$\begin{aligned} & \frac{m^\sigma}{(b-a)^\sigma} \int_{\frac{a}{m}}^{\frac{b}{m}} f(v)g(v)\left(v - \frac{a}{m}\right)^{\sigma-1}\mathbf{E}(\bar{\omega}(v - \frac{a}{m})^\rho; p)dv \\ & \leq \frac{m^\sigma f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)}{(b-a)^\sigma} \int_{\frac{a}{m}}^{\frac{b}{m}} h\left(\frac{b-mv}{b-a}\right)g(v)\left(v - \frac{a}{m}\right)^{\sigma-1}\mathbf{E}(\bar{\omega}(v - \frac{a}{m})^\rho; p)dv \\ & \quad + \frac{m^{\sigma+1} f\left(\frac{b}{m^2}\right)g\left(\frac{b}{m^2}\right)}{(b-a)^\sigma} \int_{\frac{a}{m}}^{\frac{b}{m}} h\left(\frac{vm-a}{b-a}\right)g(v)\left(v - \frac{a}{m}\right)^{\sigma-1}\mathbf{E}(\bar{\omega}(v - \frac{a}{m})^\rho; p)dv, \end{aligned}$$

that is

$$\begin{aligned} & (\mathfrak{E}_{\frac{b}{m}}^{\bar{\omega}} f g)\left(\frac{a}{m}; p\right) \\ & \leq f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)\int_{\frac{a}{m}}^{\frac{b}{m}} h\left(\frac{b-mv}{b-a}\right)g(v)\left(v - \frac{a}{m}\right)^{\sigma-1}\mathbf{E}(\bar{\omega}(v - \frac{a}{m})^\rho; p)dv \\ & \quad + m f\left(\frac{b}{m^2}\right)g\left(\frac{b}{m^2}\right)\int_{\frac{a}{m}}^{\frac{b}{m}} h\left(\frac{vm-a}{b-a}\right)g(v)\left(v - \frac{a}{m}\right)^{\sigma-1}\mathbf{E}(\bar{\omega}(v - \frac{a}{m})^\rho; p)dv. \quad (19) \end{aligned}$$

Inequality (17) now follows from (18) and (19). $\square$

In the following we derive fractional integral inequalities of Hermite–Hadamard type for different types of convexity, and state several corollaries, using special functions for h and/or g , and the parameter m . The first consequence of Theorems 5 and 6 obtained via the setting $g \equiv 1$ (i.e., $g(x) = 1$) is the Hermite–Hadamard fractional integral inequality for $(h-m)$ -convex functions given in ([20], Theorem 2.1):

Corollary 3. Let Assumption 1 hold. Let f be a nonnegative $(h-m)$ -convex function on $[0, \infty)$ where h is a nonnegative function on $J \subseteq \mathbb{R}$, $(0, 1) \subseteq J$, $h \not\equiv 0$ and $m \in (0, 1]$. If $f \in L_1[a, \frac{b}{m}]$ and $h \in L_1[0, 1]$, then following inequalities hold

$$\begin{aligned} f\left(\frac{a+b}{2}\right)(\mathfrak{E}_{a^+}^{\bar{\omega}} 1)(b; p) & \leq h\left(\frac{1}{2}\right)\left[(\mathfrak{E}_{a^+}^{\bar{\omega}} f)(b; p) + m^{\sigma+1}(\mathfrak{E}_{\frac{b}{m}}^{\bar{\omega}} f)\left(\frac{a}{m}; p\right)\right] \\ & \leq h\left(\frac{1}{2}\right)(b-a)^\sigma \left\{ \left[f(a) + m^2 f\left(\frac{b}{m^2}\right)\right](\mathfrak{E}_{1^-}^{\omega} h)(0; p) \right. \\ & \quad \left. + \left[m f\left(\frac{a}{m}\right) + m f\left(\frac{b}{m}\right)\right](\mathfrak{E}_{0^+}^{\omega} h)(1; p) \right\}, \quad (20) \end{aligned}$$

where $\bar{\omega}$ and $\bar{\bar{\omega}}$ are defined by (16).

Proof. First we use substitutions $t = \frac{b-x}{b-a}$ and $z = \frac{mx-a}{b-a}$ in Theorem 6, after which we apply identities

$$\int_0^1 h(t) t^{\sigma-1} \mathbf{E}(\omega t^\rho; p) dt = (\mathfrak{E}_1^\omega h)(0; p) \quad (21)$$

and

$$\begin{aligned} & \int_0^1 h(1-t) t^{\sigma-1} \mathbf{E}(\omega t^\rho; p) dt \\ &= \int_0^1 h(z) (1-z)^{\sigma-1} \mathbf{E}(\omega (1-z)^\rho; p) dz = (\mathfrak{E}_0^\omega h)(1; p). \end{aligned} \quad (22)$$

The result now follows from the above and Theorem 5. $\square$

By setting the function $g \equiv 1$ and the parameter $m = 1$, the previous result is reduced to the Hermite–Hadamard fractional integral inequality for h -convex functions:

Corollary 4. Let Assumption 1 hold. Let f be a nonnegative h -convex function on $[0, \infty)$ where h is a nonnegative function on $J \subseteq \mathbb{R}$, $(0, 1) \subseteq J$, $h \not\equiv 0$. If $f \in L_1[a, \frac{b}{m}]$ and $h \in L_1[0, 1]$, then the following inequalities hold

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) (\mathfrak{E}_{a^+}^{\bar{\omega}} 1)(b; p) \\ & \leq h\left(\frac{1}{2}\right) \left[(\mathfrak{E}_{a^+}^{\bar{\omega}} f)(b; p) + (\mathfrak{E}_{b^-}^{\bar{\omega}} f)(a; p) \right] \\ & \leq h\left(\frac{1}{2}\right) (b-a)^\sigma [f(a) + f(b)] [(\mathfrak{E}_1^\omega h)(0; p) + (\mathfrak{E}_0^\omega h)(1; p)], \end{aligned} \quad (23)$$

where $\bar{\omega}$ is defined by (16).

In the following, we set the function $h \equiv \text{id}$, the identity function. With $g \equiv 1$ we obtain the Hermite–Hadamard fractional integral inequality for m -convex functions from ([21], Theorem 3.1):

Corollary 5. Let Assumption 1 hold. Let f be a nonnegative m -convex function on $[0, \infty)$ with $m \in (0, 1]$. If $f \in L_1[a, \frac{b}{m}]$, then the following inequalities hold

$$\begin{aligned} f\left(\frac{a+b}{2}\right) (\mathfrak{E}_{a^+}^{\bar{\omega}} 1)(b; p) & \leq \frac{1}{2} \left[(\mathfrak{E}_{a^+}^{\bar{\omega}} f)(b; p) + m^{\sigma+1} (\mathfrak{E}_{\frac{b}{m}^-}^{\bar{\omega}} f)\left(\frac{a}{m}; p\right) \right] \\ & \leq \frac{(b-a)^\sigma}{2} \left\{ \left[f(a) + m^2 f\left(\frac{b}{m^2}\right) \right] (\mathfrak{E}_1^\omega \text{id})(0; p) \right. \\ & \quad \left. + \left[mf\left(\frac{a}{m}\right) + mf\left(\frac{b}{m}\right) \right] (\mathfrak{E}_0^\omega \text{id})(1; p) \right\}, \end{aligned} \quad (24)$$

where $\bar{\omega}$ and $\bar{\bar{\omega}}$ are defined by (16).

The Hermite–Hadamard fractional integral inequality for convex functions is given in ([21], Theorem 2.1). Here it is a merely a consequence for $h \equiv \text{id}$, $g \equiv 1$ and $m = 1$:

Corollary 6. Let Assumption 1 hold. Let f be a nonnegative convex function on $[0, \infty)$. If $f \in L_1[a, b]$, then the following inequalities hold

$$\begin{aligned} f\left(\frac{a+b}{2}\right) (\mathfrak{E}_{a^+}^{\bar{\omega}} 1)(b; p) & \leq \frac{1}{2} \left[(\mathfrak{E}_{a^+}^{\bar{\omega}} f)(b; p) + (\mathfrak{E}_{b^-}^{\bar{\omega}} f)(a; p) \right] \\ & \leq \frac{f(a) + f(b)}{2} (\mathfrak{E}_{a^+}^{\bar{\omega}} 1)(b; p), \end{aligned} \quad (25)$$

where $\bar{\omega}$ is defined by (16).

Proof. Here we use

$$\begin{aligned}
 & (\mathfrak{E}_{0+}^{\omega} \text{id})(1; p) + (\mathfrak{E}_{1-}^{\omega} \text{id})(0; p) \\
 &= \int_0^1 t^{\sigma} \mathbf{E}(\omega t^{\rho}; p) dt + \int_0^1 t(1-t)^{\sigma-1} \mathbf{E}(\omega(1-t)^{\rho}; p) dt \\
 &= \int_0^1 (1-t)^{\sigma} \mathbf{E}(\omega(1-t)^{\rho}; p) dt + \int_0^1 t(1-t)^{\sigma-1} \mathbf{E}(\omega(1-t)^{\rho}; p) dt \\
 &= \int_0^1 (1-t)^{\sigma-1} \mathbf{E}(\omega(1-t)^{\rho}; p) dt \\
 &= (\mathfrak{E}_{0+}^{\omega} 1)(1; p) = \frac{1}{(b-a)^{\sigma}} (\mathfrak{E}_{a+}^{\omega} 1)(b; p).
 \end{aligned}$$

□

We have presented several Hermite–Hadamard-type inequalities for the $(h, g; m)$ -convex function using fractional integral operators, where the kernel is an extended generalized Mittag-Leffler function. If we apply different parameter choices, as in Remark 2, then we obtain corresponding inequalities for different fractional operators.

Several Properties of Fractional Integral Operators $\mathfrak{E}_{a+}^{\omega} f$ and $\mathfrak{E}_{b-}^{\omega} f$

At the end of this section we give several results for fractional integral operators.

Proposition 1. Let $\omega, \rho, \sigma, \tau, \delta, c \in \mathbb{C}$, $\Re(\rho), \Re(\sigma), \Re(\tau) > 0$, $\Re(c) > \Re(\delta) > 0$ with $p \geq 0$, $r > 0$ and $0 < q \leq r + \Re(\rho)$.

(i) If the function $f \in L_1[a, b]$ is symmetric about $\frac{a+b}{2}$, then

$$(\mathfrak{E}_{a+}^{\omega} f)(b; p) = (\mathfrak{E}_{b-}^{\omega} f)(a; p). \quad (26)$$

In particular,

$$(\mathfrak{E}_{a+}^{\omega} 1)(b; p) = (\mathfrak{E}_{b-}^{\omega} 1)(a; p). \quad (27)$$

(ii) Furthermore,

$$(\mathfrak{E}_{a+}^{\omega} (t-a)^{\alpha-1})(b; p) = (\mathfrak{E}_{b-}^{\omega} (b-t)^{\alpha-1})(a; p), \quad (28)$$

$$(\mathfrak{E}_{a+}^{\omega} (b-t)^{\alpha-1})(b; p) = (\mathfrak{E}_{b-}^{\omega} (t-a)^{\alpha-1})(a; p). \quad (29)$$

In particular,

$$(\mathfrak{E}_{0+}^{\omega} t^{\alpha-1})(1; p) = (\mathfrak{E}_{1-}^{\omega} (1-t)^{\alpha-1})(0; p), \quad (30)$$

$$(\mathfrak{E}_{0+}^{\omega} (1-t)^{\alpha-1})(1; p) = (\mathfrak{E}_{1-}^{\omega} t^{\alpha-1})(0; p). \quad (31)$$

Proof. (i) If the function f is symmetric about $\frac{a+b}{2}$, i.e., $f(t) = f(a+b-t)$ for all $t \in [a, b]$, then, substituting $z = a+b-t$, Equation (26) easily follows:

$$\begin{aligned}
 (\mathfrak{E}_{a+}^{\omega} f)(b; p) &= \int_a^b (b-t)^{\sigma-1} \mathbf{E}(\omega(b-t)^{\rho}; p) f(t) dt \\
 &= \int_a^b (z-a)^{\sigma-1} \mathbf{E}(\omega(z-a)^{\rho}; p) f(a+b-z) dz \\
 &= \int_a^b (z-a)^{\sigma-1} \mathbf{E}(\omega(z-a)^{\rho}; p) f(z) dz = (\mathfrak{E}_{b-}^{\omega} f)(a; p).
 \end{aligned}$$

Note that (27) also follows directly from Corollary 2 if we set $x = b$ in (10) and $x = a$ in (11).

- (ii) Equations (28) and (29) follow with the substitution $z = a + b - t$. Furthermore, (28) follows directly from Theorem 3 if we set $x = b$ in (8) and $x = a$ in (9). The final two equations are obtained for $a = 0$ and $b = 1$.

□

Remark 3. To obtain the Hermite–Hadamard inequality for convex functions involving Riemann–Liouville fractional integrals, given in Theorem 2, first we need to set $p = \omega = 0$ in (5)

$$\mathbf{E}(z; 0) = \sum_{n=0}^{\infty} \frac{(\delta)_{nq}}{\Gamma(\rho n + \sigma)} \frac{z^n}{(\tau)_{nr}}.$$

Since $\mathbf{E}(0; 0) = E_{\rho, \sigma, \tau}^{\delta, c, q, r}(0; 0) = \frac{1}{\Gamma(\sigma)}$, setting $p = \omega = 0$ in (6) we obtain Riemann–Liouville fractional integrals

$$(\mathfrak{E}_{a+}^0 f)(x; 0) = \frac{1}{\Gamma(\sigma)} \int_a^x (x-t)^{\sigma-1} f(t) dt = J_{a+}^{\sigma} f(x),$$

$$(\mathfrak{E}_{b-}^0 f)(x; 0) = \frac{1}{\Gamma(\sigma)} \int_x^b (t-x)^{\sigma-1} f(t) dt = J_{b-}^{\sigma} f(x).$$

Note that a direct consequence of Theorem 3 is

$$(\mathfrak{E}_{0+}^{\omega} \text{id})(1; p) = E_{\rho, \sigma+2, \tau}^{\delta, c, q, r}(\omega; p). \quad (32)$$

For the reader's convenience, we will directly prove this:

$$\begin{aligned} (\mathfrak{E}_{0+}^{\omega} \text{id})(1; p) &= \int_0^1 t(1-t)^{\sigma-1} \mathbf{E}(\omega(1-t)^{\rho}; p) dt \\ &= \int_0^1 t(1-t)^{\sigma-1} \sum_{n=0}^{\infty} \frac{B_p(\delta + nq, c - \delta)}{B(\delta, c - \delta)} \frac{(c)_{nq}}{\Gamma(\rho n + \sigma)} \frac{\omega^n (1-t)^{n\rho}}{(\tau)_{nr}} dt \\ &= \sum_{n=0}^{\infty} \frac{B_p(\delta + nq, c - \delta)}{B(\delta, c - \delta)} \frac{(c)_{nq}}{\Gamma(\rho n + \sigma)} \frac{\omega^n}{(\tau)_{nr}} \int_0^1 t(1-t)^{n\rho + \sigma - 1} dt \\ &= \sum_{n=0}^{\infty} \frac{B_p(\delta + nq, c - \delta)}{B(\delta, c - \delta)} \frac{(c)_{nq}}{\Gamma(\rho n + \sigma)} \frac{\omega^n}{(\tau)_{nr}} B(2, n\rho + \sigma) \\ &= \sum_{n=0}^{\infty} \frac{B_p(\delta + nq, c - \delta)}{B(\delta, c - \delta)} \frac{(c)_{nq}}{\Gamma(\rho n + \sigma)} \frac{\omega^n}{(\tau)_{nr}} \frac{\Gamma(2)\Gamma(n\rho + \sigma)}{\Gamma(2 + n\rho + \sigma)} \\ &= \sum_{n=0}^{\infty} \frac{B_p(\delta + nq, c - \delta)}{B(\delta, c - \delta)} \frac{(c)_{nq}}{\Gamma(\rho n + (\sigma + 2))} \frac{\omega^n}{(\tau)_{nr}} \\ &= E_{\rho, \sigma+2, \tau}^{\delta, c, q, r}(\omega; p). \end{aligned}$$

Hence,

$$(\mathfrak{E}_{0+}^0 \text{id})(1; 0) = \frac{1}{\Gamma(\sigma + 2)}$$

and

$$(\mathfrak{E}_{1-}^0 \text{id})(0; 0) = \int_0^1 t^{\sigma} \mathbf{E}(0; p) dt = \frac{1}{(\sigma + 1)\Gamma(\sigma)}, \quad (33)$$

from which follows

$$(\mathfrak{E}_{0+}^0 \text{id})(1; 0) + (\mathfrak{E}_{1-}^0 \text{id})(0; 0) = \frac{1}{\Gamma(\sigma + 1)}.$$

Finally, if we set $h(x) = x$, $g \equiv 1$, $m = 1$ and $p = \omega = 0$, then Theorems 5 and 6 are reduced to Theorem 2.

4. Applications: Bounds of Fractional Integral Operators for $(h, g; m)$ -Convex Functions

As an application, in this section we obtain the upper bounds of fractional integral operators for $(h, g; m)$ -convex functions.

Assumption 2. Let $\omega \in \mathbb{R}$, $\rho, \sigma, \tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r + \rho$. Let f be a nonnegative $(h, g; m)$ -convex function on $[0, \infty)$ where h is a nonnegative function on $J \subseteq \mathbb{R}$, $(0, 1) \subseteq J$, $h \not\equiv 0$, g is a positive function on $[0, \infty)$, and $m \in (0, 1]$. Furthermore, let $0 \leq a < b < \infty$.

Theorem 7. Let Assumption 2 hold. If $f, g \in L_1[a, b]$ and $h \in L_1[0, 1]$, then for $x \in [a, b]$ the following inequality holds

$$\frac{1}{(x-a)^\sigma} (\mathfrak{E}_{a^+}^{\omega_a} f)(x; p) \leq f(a) g(a) (\mathfrak{E}_{1^-}^{\omega} h)(0; p) + m f\left(\frac{x}{m}\right) g\left(\frac{x}{m}\right) (\mathfrak{E}_{0^+}^{\omega} h)(1; p). \quad (34)$$

where

$$\omega_a = \frac{\omega}{(x-a)^\rho}. \quad (35)$$

Proof. Let f be an $(h, g; m)$ -convex function on $[0, \infty)$, $x \in [a, b]$, $m \in (0, 1]$ and $t \in (0, 1)$. Then, similarly to Theorem 6, we use

$$f(ta + (1-t)x) \leq h(t)f(a)g(a) + mh(1-t)f\left(\frac{x}{m}\right)g\left(\frac{x}{m}\right).$$

Multiplying both sides of the above inequality by $t^{\sigma-1} \mathbf{E}(\omega t^\rho; p)$ and integrating on $[0, 1]$ with respect to the variable t , we obtain

$$\begin{aligned} & \int_0^1 f(ta + (1-t)x) t^{\sigma-1} \mathbf{E}(\omega t^\rho; p) dt \\ & \leq f(a)g(a) \int_0^1 h(t) t^{\sigma-1} \mathbf{E}(\omega t^\rho; p) dt + m f\left(\frac{x}{m}\right) g\left(\frac{x}{m}\right) \int_0^1 h(1-t) t^{\sigma-1} \mathbf{E}(\omega t^\rho; p) dt. \end{aligned}$$

With the substitution $u = ta + (1-t)x$ and identities (21), (22), we obtain the inequality (34). $\square$

Theorem 8. Let Assumption 2 hold. If $f, g \in L_1[a, b]$ and $h \in L_1[0, 1]$, then for $x \in [a, b]$ the following inequality holds

$$\frac{1}{(b-x)^\sigma} (\mathfrak{E}_{b^-}^{\omega_b} f)(x; p) \leq f(b)g(b) (\mathfrak{E}_{1^-}^{\omega} h)(0; p) + m f\left(\frac{x}{m}\right) g\left(\frac{x}{m}\right) (\mathfrak{E}_{0^+}^{\omega} h)(1; p), \quad (36)$$

where

$$\omega_b = \frac{\omega}{(b-x)^\rho}. \quad (37)$$

Proof. Using

$$f(tb + (1-t)x) \leq h(t)f(b)g(b) + mh(1-t)f\left(\frac{x}{m}\right)g\left(\frac{x}{m}\right),$$

the proof follows analogously to that of Theorem 7. $\square$

From the two previous theorems we can directly obtain the following result.

Corollary 7. Let Assumption 2 hold. If $f, g \in L_1[a, b]$ and $h \in L_1[0, 1]$, then for $x \in [a, b]$ the following inequality holds

$$\begin{aligned} & \frac{1}{(x-a)^\sigma} (\mathfrak{E}_{a^+}^{\omega_a} f)(x; p) + \frac{1}{(b-x)^\sigma} (\mathfrak{E}_{b^-}^{\omega_b} f)(x; p) \\ & \leq [f(a)g(a) + f(b)g(b)](\mathfrak{E}_1^\omega h)(0; p) + 2m f\left(\frac{x}{m}\right)g\left(\frac{x}{m}\right)(\mathfrak{E}_{0^+}^\omega h)(1; p). \end{aligned} \quad (38)$$

where ω_a and ω_b are defined by (35) and (37).

If we set $x = b$ in Theorem 7 and $x = a$ in Theorem 8, then we obtain the next fractional integral inequality of the Hermite–Hadamard type.

Theorem 9. Let Assumption 2 hold. If $f, g, h \in L_1[a, b]$, then the following inequalities hold

$$\begin{aligned} & \frac{1}{(b-a)^\sigma} [(\mathfrak{E}_{a^+}^{\bar{\omega}} f)(b; p) + (\mathfrak{E}_{b^-}^{\bar{\omega}} f)(a; p)] \\ & \leq [f(a)g(a) + f(b)g(b)](\mathfrak{E}_1^\omega h)(0; p) \\ & \quad + m \left[f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right) \right] (\mathfrak{E}_{0^+}^\omega h)(1; p), \end{aligned} \quad (39)$$

where $\bar{\omega}$ is defined by (16).

In the following we will extend our interval to $[ma, b]$. Since $m \in (0, 1]$, then $ma \leq a$, $mb \leq b$, and $[a, b] \subset [ma, b]$.

Theorem 10. Let Assumption 2 hold. If $f, g \in L_1[ma, b]$ and $h \in L_1[0, 1]$, then the following inequality holds

$$\begin{aligned} & \frac{1}{(mb-a)^\sigma} [(\mathfrak{E}_{a^+}^{\omega_1} f)(mb; p) + (\mathfrak{E}_{mb^-}^{\omega_1} f)(a; p)] \\ & + \frac{1}{(b-ma)^\sigma} [(\mathfrak{E}_{b^-}^{\omega_2} f)(ma; p) + (\mathfrak{E}_{ma^+}^{\omega_2} f)(b; p)] \\ & \leq (m+1)[f(a)g(a) + f(b)g(b)] [(\mathfrak{E}_1^\omega h)(0; p) + (\mathfrak{E}_{0^+}^\omega h)(1; p)], \end{aligned} \quad (40)$$

where

$$\omega_1 = \frac{\omega}{(mb-a)^\rho}, \quad \omega_2 = \frac{\omega}{(b-ma)^\rho}. \quad (41)$$

Proof. Let f be an $(h, g; m)$ -convex function on $[0, \infty)$, $m \in (0, 1]$ and $t \in (0, 1)$. Then

$$f(ta + m(1-t)b) \leq h(t)f(a)g(a) + mh(1-t)f(b)g(b),$$

$$f((1-t)a + mtb) \leq h(1-t)f(a)g(a) + mh(t)f(b)g(b)$$

and

$$f(tb + m(1-t)a) \leq h(t)f(b)g(b) + mh(1-t)f(a)g(a),$$

$$f((1-t)b + mta) \leq h(1-t)f(b)g(b) + mh(t)f(a)g(a).$$

First we add the above inequalities, i.e.,

$$\begin{aligned} & f(ta + m(1-t)b) + f((1-t)a + mtb) \\ & + f(tb + m(1-t)a) + f((1-t)b + mta) \\ & \leq (m+1)[f(a)g(a) + f(b)g(b)]h(t) \\ & \quad + (m+1)[f(a)g(a) + f(b)g(b)]h(1-t). \end{aligned}$$

Then we use multiplication by $t^{\sigma-1}\mathbf{E}(\omega t^\rho; p)$ and integration on $[0, 1]$ with respect to the variable t to obtain

$$\begin{aligned} & \int_0^1 f(ta + m(1-t)b)t^{\sigma-1}\mathbf{E}(\omega t^\rho; p)dt \\ & + \int_0^1 f((1-t)a + mtb)t^{\sigma-1}\mathbf{E}(\omega t^\rho; p)dt \\ & + \int_0^1 f(tb + m(1-t)a)t^{\sigma-1}\mathbf{E}(\omega t^\rho; p)dt \\ & + \int_0^1 f((1-t)b + mta)t^{\sigma-1}\mathbf{E}(\omega t^\rho; p)dt \\ & \leq (m+1)[f(a)g(a) + f(b)g(b)] \int_0^1 h(t)t^{\sigma-1}\mathbf{E}(\omega t^\rho; p)dt \\ & \quad + (m+1)[f(a)g(a) + f(b)g(b)] \int_0^1 h(1-t)t^{\sigma-1}\mathbf{E}(\omega t^\rho; p)dt. \end{aligned}$$

For the left side of the inequality we need several substitutions. For instance, if we set $u = ta + m(1-t)b$, then we get

$$\begin{aligned} & \int_0^1 f(ta + m(1-t)b)t^{\sigma-1}\mathbf{E}(\omega t^\rho; p)dt \\ & = \frac{1}{(mb-a)^\sigma} \int_a^{mb} f(u)(mb-u)^{\sigma-1}\mathbf{E}(\frac{\omega}{(mb-a)^\rho}(mb-u)^\rho; p)du. \end{aligned}$$

Hence,

$$\begin{aligned} & \frac{1}{(mb-a)^\sigma} \int_a^{mb} f(u)(mb-u)^{\sigma-1}\mathbf{E}(\frac{\omega}{(mb-a)^\rho}(mb-u)^\rho; p)du \\ & + \frac{1}{(mb-a)^\sigma} \int_a^{mb} f(u)(u-a)^{\sigma-1}\mathbf{E}(\frac{\omega}{(mb-a)^\rho}(u-a)^\rho; p)du \\ & + \frac{1}{(b-ma)^\sigma} \int_{ma}^b f(u)(u-ma)^{\sigma-1}\mathbf{E}(\frac{\omega}{(b-ma)^\rho}(u-ma)^\rho; p)du \\ & + \frac{1}{(b-ma)^\sigma} \int_{ma}^b f(u)(b-u)^{\sigma-1}\mathbf{E}(\frac{\omega}{(b-ma)^\rho}(b-u)^\rho; p)du \\ & \leq (m+1)[f(a)g(a) + f(b)g(b)](\mathbf{e}_1^\omega h)(0; p) \\ & \quad + (m+1)[f(a)g(a) + f(b)g(b)](\mathbf{e}_{0+}^\omega h)(1; p), \end{aligned}$$

that is

$$\begin{aligned} & \frac{1}{(mb-a)^\sigma} \left(\mathbf{e}_{a^+}^{\frac{\omega}{(mb-a)^\rho}} f \right) (mb; p) + \frac{1}{(mb-a)^\sigma} \left(\mathbf{e}_{mb^-}^{\frac{\omega}{(mb-a)^\rho}} f \right) (a; p) \\ & + \frac{1}{(b-ma)^\sigma} \left(\mathbf{e}_{b^-}^{\frac{\omega}{(b-ma)^\rho}} f \right) (ma; p) + \frac{1}{(b-ma)^\sigma} \left(\mathbf{e}_{ma^+}^{\frac{\omega}{(b-ma)^\rho}} f \right) (b; p) \\ & \leq (m+1)[f(a)g(a) + f(b)g(b)] [(\mathbf{e}_1^\omega h)(0; p) + (\mathbf{e}_{0+}^\omega h)(1; p)]. \end{aligned}$$

This provides the require inequality. $\square$

Remark 4. With an extended generalized Mittag-Leffler function from Definition 1 and a class of $(h, g; m)$ -convex functions as in Definition 3, for different parameters p, τ, r, q, ω and for different choices of functions h, g and parameter m , we obtain corresponding upper bounds of different fractional operators for different classes of convexity.

5. Conclusions

This research was on Hermite–Hadamard-type inequalities existing in a more general setting. We used a fractional integral operator containing an extended generalized Mittag–Leffler function in the kernel, and obtained Hermite–Hadamard fractional integral inequalities for a class of $(h, g; m)$ -convex functions. Furthermore, we presented the upper bounds of the fractional integral operators for $(h, g; m)$ -convex functions. The obtained results generalize and extend the corresponding inequalities for different classes of convex functions.

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