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Neutrosophic Double Controlled Metric Spaces and Related Results with Application

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Abstract: In this paper, the authors introduce the notion of neutrosophic double controlled metric spaces as a generalization of neutrosophic metric spaces. For this purpose, two non-comparable functions, ξ and Γ, are used in triangle inequalities. The authors prove several interesting results for contraction mappings with non-trivial examples. At the end of the paper, the authors prove the existence, and the uniqueness, of the integral equation to support the main result.

Keywords: double controlled metric space; neutrosophic metric space; fixed point results; integral equation

MSC: 47H10; 54H25

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1. Introduction

The concept of metric spaces and the Banach contraction principle are the backbone of the field of fixed-point theory. Axiomatic interpretation of metric space attracts thousands of researchers towards spaciousness. So far, there have been many generalizations on metric spaces. This tells us of the beauty, attraction and expansion of the concept of metric spaces.

The notion of fuzzy sets was proposed by Zadeh [1]. The adjective "fuzzy" seems to be a very popular, and very frequent, one in contemporary studies concerning the logical and set-theoretical foundations of mathematics. The main reason for this quick development is, in our opinion, easy to be understood. The world that surrounds us is full of uncertainty for the following reasons: the information we obtain from the environment, the notions we use, and the data resulting from our observations or measurements are, in general, vague and incorrect. So, every formal description of the real world, or some of its aspects, is, in every case, only an approximation and an idealization of the actual state. Notions like fuzzy sets, fuzzy orderings, fuzzy languages, etc., enable us to handle, and to study, the degree of uncertainty mentioned above in a purely mathematical and formal way.

The concept of fuzzy sets has succeeded in shifting a lot of mathematical structures within its concept. Schweizer and Sklar [2] defined the notion of continuous t-norms. Kramosil and Michalek [3] introduced the notion of fuzzy metric spaces. They applied the concept of fuzziness, via continuous t-norms, to classical notions of metric and metric spaces and compared the notions thus obtained with those resulting from some other, namely probabilistic, statistical generalizations of metric spaces. Garbiec [4] provided the fuzzy interpretation of Banach contraction principle in fuzzy metric spaces. Ur-Reham et al. [5] proved some $\alpha-\phi$-fuzzy cone contraction results with integral type application.

Fuzzy metric spaces only deal with membership functions. An intuitionistic fuzzy metric space was established by Park [6] that is used to deal with both membership and non-membership functions. Konwar [7] presented the concept of an intuitionistic fuzzy b-metric space and proved several fixed-point theorems. Kirişci and Simsek [8] introduced the notion of neutrosophic metric spaces that is used to deal with membership, non-membership and naturalness. Simsek and Kirişci [9] proved some amazing fixed-point results in the context of neutrosophic metric spaces. Sowndrarajan et al. [10] proved some fixed-point results in the setting of neutrosophic metric spaces. Itoh [11] proved an application regarding random differential equations in Banach spaces. Mlaiki [12] coined the concept of controlled metric spaces and proved several fixed-point results for contraction mappings. Sezen [13] presented the notion of controlled fuzzy metric spaces and proved various contraction mapping results. Recently, Saleem et al. [14] introduced the concept of fuzzy double controlled metric spaces. For related articles, see [15–20].

In this paper, the authors used the notion of fuzzy double controlled metric spaces introduced in [14] and neutrosophic metric spaces introduced in [8] to define the notion of neutrosophic double controlled metric spaces. The main objectives of this paper are as follows:

- To introduce the notion of neutrosophic double controlled metric spaces
- To prove several fixed-point theorems for contraction mappings
- To enhance the literature of fuzzy fixed-point theory
- To find the existence of uniqueness of the solution of an integral equation.

2. Preliminaries

In this section, the authors provide some definitions that will be helpful for readers to understand the main section.

Definition 1 ([6]). *A binary operation* $*$: $[0,1] \times [0,1] \to [0,1]$ *is called a continuous triangle norm if:*

1. $\hat{e} * \bar{a} = \bar{a} * \hat{e},\ (\forall)\ \hat{e}, \bar{a} \in [0,1]$;
2. $*$ *is continuous;*
3. $\hat{e} * 1 = \hat{e},\ (\forall)\ \hat{e} \in [0,1]$;
4. $(\hat{e} * \bar{a}) * \tilde{n} = \hat{e} * (\bar{a} * \tilde{n}),\ for\ all\ \hat{e}, \bar{a}, \tilde{n} \in [0,1]$;
5. *If* $\hat{e} \leq \tilde{n}$ *and* $\bar{a} \leq d$, *with* $\hat{e}, \bar{a}, \tilde{n}, d \in [0,1]$, *then* $\hat{e} * \bar{a} \leq \tilde{n} * d$.

Definition 2 ([6]). *A binary operation* $\bigcirc$: $[0,1] \times [0,1] \to [0,1]$ *is called a continuous triangle conorm if:*

1. $\hat{e} \bigcirc \bar{a} = \bar{a} \bigcirc \hat{e},$ for all $\hat{e}, \bar{a} \in [0,1]$;
2. $\bigcirc$ *is continuous;*
3. $\hat{e} \bigcirc 0 = 0,$ for all $\hat{e} \in [0,1]$;
4. $(\hat{e} \bigcirc \bar{a}) \bigcirc \tilde{n} = \hat{e} \bigcirc (\bar{a} \bigcirc \tilde{n}),\ for\ all\ \hat{e}, \bar{a}, \tilde{n} \in [0,1]$;
5. If $\hat{e} \leq \tilde{n}$ *and* $\bar{a} \leq d$, *with* $\hat{e}, \bar{a}, \tilde{n}, d \in [0,1]$, *then* $\hat{e} \bigcirc \bar{a} \leq \tilde{n} \bigcirc d$.

Definition 3 ([11]). *Given* $\xi, \Gamma : \mathfrak{C} \times \mathfrak{C} \to [1, +\infty)$ *are non-comparable functions, if* $\partial : \mathfrak{C} \times \mathfrak{C} \to [0, +\infty)$ *satisfies the following conditions:*

a. $\partial(\kappa, \textsc{n}) = 0$ iff $\kappa = \textsc{n}$;
b. $\partial(\kappa, \textsc{n}) = \partial(\textsc{n}, \kappa)$;
c. $\partial(\kappa, \textsc{n}) \leq \xi(\kappa, \lambda)\partial(\kappa, \lambda) + \Gamma(\lambda, \textsc{n})\partial(\lambda, \textsc{n})$;

for all $\kappa, \textsc{n}, \lambda \in \mathfrak{C}$, *then,* $(\mathfrak{C}, \partial)$ *is said to be a double controlled metric space.*

Definition 4 ([14]). *Suppose* $\mathfrak{C} \neq \varnothing$ *and* $\xi, \Gamma : \mathfrak{C} \times \mathfrak{C} \to [1, +\infty)$ *are given non-comparable functions,* $*$ *is a continuous t-norm and* $\Re$ *is a fuzzy set on* $\mathfrak{C} \times \mathfrak{C} \times (0, +\infty)$ *is said to be a fuzzy double controlled metric on* $\mathfrak{C}$, *for all* $\kappa, \textsc{n}, \lambda \in \mathfrak{C}$ *if:*

i. $\Re(\kappa, \text{N}, 0) = 0$;
ii. $\Re(\kappa, \text{N}, \hat{\text{r}}) = 1$ *for all* $\hat{\text{r}} > 0$, *if and only if* $\kappa = \text{N}$;
iii. $\Re(\kappa, \text{N}, \hat{\text{r}}) = \Re(\text{N}, \kappa, \hat{\text{r}})$;
iv. $\Re(\kappa, \lambda, \hat{\text{r}} + \check{\text{s}}) \geq \Re\Big(\kappa, \text{N}, \frac{\hat{\text{r}}}{\xi(\kappa,\text{N})}\Big) * \Re\Big(\text{N}, \lambda, \frac{\check{\text{s}}}{\Gamma(\text{N},\lambda)}\Big)$;
v. $\Re(\kappa, \text{N}, \cdot) : (0, +\infty) \to [0, 1]$ *is left continuous.*

Then, $(\mathfrak{C}, \Re, \aleph, *)$ *is said to be a fuzzy double controlled metric space.*

Definition 5 ([7]). *Take* $\mathfrak{C} \neq \varnothing$. *Let* $*$ *be a continuous t-norm,* $\bigcirc$ *be a continuous t-conorm,* $b \geq 1$ *and* $\Re, \aleph$ *be fuzzy sets on* $\mathfrak{C} \times \mathfrak{C} \times (0, +\infty)$. *If* $(\mathfrak{C}, \Re, \aleph, *, \bigcirc)$ *fulfils all* $\kappa, \text{N} \in \mathfrak{C}$ *and* $\check{\text{s}}, \hat{\text{r}} > 0$:

I. $\Re(\kappa, \text{N}, \hat{\text{r}}) + \aleph(\kappa, \text{N}, \hat{\text{r}}) \leq 1$;
II. $\Re(\kappa, \text{N}, \hat{\text{r}}) > 0$;
III. $\Re(\kappa, \text{N}, \hat{\text{r}}) = 1 \iff \kappa = \text{N}$;
IV. $\Re(\kappa, \text{N}, \hat{\text{r}}) = \Re(\text{N}, \kappa, \hat{\text{r}})$;
V. $\Re(\kappa, \lambda, b(\hat{\text{r}} + \check{\text{s}})) \geq \Re(\kappa, \text{N}, \hat{\text{r}}) * \Re(\text{N}, \lambda, \check{\text{s}})$;
VI. $\Re(\kappa, \text{N}, \cdot)$ *is a non-decreasing function of* $\mathbb{R}^+$ *and* $\lim_{\hat{\text{r}} \to +\infty} \Re(\kappa, \text{N}, \hat{\text{r}}) = 1$;
VII. $\aleph(\kappa, \text{N}, \hat{\text{r}}) > 0$;
VIII. $\aleph(\kappa, \text{N}, \hat{\text{r}}) = 0 \iff \kappa = \text{N}$;
IX. $\aleph(\kappa, \text{N}, \hat{\text{r}}) = \aleph(\text{N}, \kappa, \hat{\text{r}})$;
X. $\aleph(\kappa, \lambda, b(\hat{\text{r}} + \check{\text{s}})) \leq \aleph(\kappa, \text{N}, \hat{\text{r}}) \bigcirc \aleph(\text{N}, \lambda, \check{\text{s}})$;
XI. $\aleph(\kappa, \text{N}, \cdot)$ *is a non-increasing function of* $\mathbb{R}^+$ *and* $\lim_{\hat{\text{r}} \to +\infty} \aleph(\kappa, \text{N}, \hat{\text{r}}) = 0$,

Then, $(\mathfrak{C}, \Re, \aleph, *, \bigcirc)$ *is an intuitionistic fuzzy b-metric space.*

Definition 6 ([8]). *Let* $\mathfrak{C} \neq \varnothing$, $*$ *is a continuous t-norm,* $\bigcirc$ *be a continuous t-conorm, and* $\Re$, $\aleph, S$ *are neutrosophic sets on* $\mathfrak{C} \times \mathfrak{C} \times (0, +\infty)$ *is said to be a neutrosophic metric on* $\mathfrak{C}$, *if for all* $\kappa, \text{N}, \lambda \in \mathfrak{C}$, *the following conditions are satisfied*:

(1) $\Re(\kappa, \text{N}, \hat{\text{r}}) + \aleph(\kappa, \text{N}, \hat{\text{r}}) + S(\kappa, \text{N}, \hat{\text{r}}) \leq 3$;
(2) $\Re(\kappa, \text{N}, \hat{\text{r}}) > 0$;
(3) $\Re(\kappa, \text{N}, \hat{\text{r}}) = 1$ *for all* $\hat{\text{r}} > 0$, *if and only if* $\kappa = \text{N}$;
(4) $\Re(\kappa, \text{N}, \hat{\text{r}}) = \Re(\text{N}, \kappa, \hat{\text{r}})$;
(5) $\Re(\kappa, \lambda, \hat{\text{r}} + \check{\text{s}}) \geq \Re(\kappa, \text{N}, \hat{\text{r}}) * \Re(\text{N}, \lambda, \check{\text{s}})$;
(6) $\Re(\kappa, \text{N}, \cdot) : (0, +\infty) \to [0, 1]$ *is continuous and* $\lim_{\hat{\text{r}} \to +\infty} \Re(\kappa, \text{N}, \hat{\text{r}}) = 1$;
(7) $\aleph(\kappa, \text{N}, \hat{\text{r}}) < 1$;
(8) $\aleph(\kappa, \text{N}, \hat{\text{r}}) = 0$ *for all* $\hat{\text{r}} > 0$, *if and only if* $\kappa = \text{N}$;
(9) $\aleph(\kappa, \text{N}, \hat{\text{r}}) = \aleph(\text{N}, \kappa, \hat{\text{r}})$;
(10) $\aleph(\kappa, \lambda, \hat{\text{r}} + \check{\text{s}}) \leq \aleph(\kappa, \text{N}, \hat{\text{r}}) \bigcirc \aleph(\text{N}, \lambda, \check{\text{s}})$;
(11) $\aleph(\kappa, \text{N}, \cdot) : (0, +\infty) \to [0, 1]$ *is continuous and* $\lim_{\hat{\text{r}} \to +\infty} \aleph(\kappa, \text{N}, \hat{\text{r}}) = 0$;
(12) $S(\kappa, \text{N}, \hat{\text{r}}) < 1$;
(13) $S(\kappa, \text{N}, \hat{\text{r}}) = 0$ *for all* $\hat{\text{r}} > 0$, *if and only if* $\kappa = \text{N}$;
(14) $S(\kappa, \text{N}, \hat{\text{r}}) = S(\text{N}, \kappa, \hat{\text{r}})$;
(15) $S(\kappa, \lambda, \hat{\text{r}} + \check{\text{s}}) \leq S(\kappa, \text{N}, \hat{\text{r}}) \bigcirc S(\text{N}, \lambda, \check{\text{s}})$;
(16) $S(\kappa, \text{N}, \cdot) : (0, +\infty) \to [0, 1]$ *is continuous and* $\lim_{\hat{\text{r}} \to +\infty} S(\kappa, \text{N}, \hat{\text{r}}) = 0$;
(17) *If* $\hat{\text{r}} \leq 0$, *then* $\Re(\kappa, \text{N}, \hat{\text{r}}) = 0$, $\aleph(\kappa, \text{N}, \hat{\text{r}}) = 1$ *and* $S(\kappa, \text{N}, \hat{\text{r}}) = 1$.

Then, $(\mathfrak{C}, \Re, \aleph, S, *, \bigcirc)$ *is called a neutrosophic metric space.*

3. Main Results

In this part, we present neutrosophic double controlled metric spaces and demonstrate some fixed-point results.

Definition 7. *Let* $\mathfrak{C} \neq \varnothing$ *and* $\xi, \Gamma : \mathfrak{C} \times \mathfrak{C} \to [1, +\infty)$ *be given non-comparable functions,* $*$ *be a continuous t-norm,* $\bigcirc$ *be a continuous t-conorm and* $\Re$, $\aleph, \text{ß}$ *be neutrosophic sets on* $\mathfrak{C} \times \mathfrak{C} \times$

$(0,+\infty)$ is said to be a neutrosophic double controlled metric on $\mathfrak{C}$, if for all $\kappa, \text{N}, \lambda \in \mathfrak{C}$, the following conditions are satisfied:

(i) $\Re(\kappa,\text{N},\hat{r}) + \aleph(\kappa,\text{N},\hat{r}) + ß(\kappa,\text{N},\hat{r}) \leq 3$;
(ii) $\Re(\kappa,\text{N},\hat{r}) > 0$;
(iii) $\Re(\kappa,\text{N},\hat{r}) = 1$ *for all* $\hat{r} > 0$, *if and only if* $\kappa = \text{N}$;
(iv) $\Re(\kappa,\text{N},\hat{r}) = \Re(\text{N},\kappa,\hat{r})$;
(v) $\Re(\kappa,\lambda,\hat{r}+\check{s}) \geq \Re\left(\kappa,\text{N},\frac{\hat{r}}{\xi(\kappa,\text{N})}\right) * \Re\left(\text{N},\lambda,\frac{\check{s}}{\Gamma(\text{N},\lambda)}\right)$;
(vi) $\Re(\kappa,\text{N},\cdot):(0,+\infty)\to[0,1]$ *is continuous and* $\lim_{\hat{r}\to+\infty}\Re(\kappa,\text{N},\hat{r}) = 1$;
(vii) $\aleph(\kappa,\text{N},\hat{r}) < 1$;
(viii) $\aleph(\kappa,\text{N},\hat{r}) = 0$ *for all* $\hat{r} > 0$, *if and only if* $\kappa = \text{N}$;
(ix) $\aleph(\kappa,\text{N},\hat{r}) = \aleph(\text{N},\kappa,\hat{r})$;
(x) $\aleph(\kappa,\lambda,\hat{r}+\check{s}) \leq \aleph\left(\kappa,\text{N},\frac{\hat{r}}{\xi(\kappa,\text{N})}\right) \bigcirc \aleph\left(\text{N},\lambda,\frac{\check{s}}{\Gamma(\text{N},\lambda)}\right)$;
(xi) $\aleph(\kappa,\text{N},\cdot):(0,+\infty)\to[0,1]$ *is continuous and* $\lim_{\hat{r}\to+\infty}\aleph(\kappa,\text{N},\hat{r}) = 0$;
(xii) $ß(\kappa,\text{N},\hat{r}) < 1$;
(xiii) $ß(\kappa,\text{N},\hat{r}) = 0$ *for all* $\hat{r} > 0$, *if and only if* $\kappa = \text{N}$;
(xiv) $ß(\kappa,\text{N},\hat{r}) = ß(\text{N},\kappa,\hat{r})$;
(xv) $ß(\kappa,\lambda,\hat{r}+\check{s}) \leq ß\left(\kappa,\text{N},\frac{\hat{r}}{\xi(\kappa,\text{N})}\right) \bigcirc ß\left(\text{N},\lambda,\frac{\check{s}}{\Gamma(\text{N},\lambda)}\right)$;
(xvi) $ß(\kappa,\text{N},\cdot):(0,+\infty)\to[0,1]$ *is continuous and* $\lim_{\hat{r}\to+\infty}ß(\kappa,\text{N},\hat{r}) = 0$;
(xvii) *If* $\hat{r} \leq 0$, *then* $\Re(\kappa,\text{N},\hat{r}) = 0$, $\aleph(\kappa,\text{N},\hat{r}) = 1$ and $S(\kappa,\text{N},\hat{r}) = 1$.

Then, $(\mathfrak{C},\Re,\aleph,ß,*,\bigcirc)$ is called a neutrosophic double controlled metric space.

Example 1. *Let $\mathfrak{C} = \{1,2,3\}$ and $\xi,\Gamma:\mathfrak{C}\times\mathfrak{C}\to[1,+\infty)$ be two non-comparable functions given by $\xi(\kappa,\text{N}) = \kappa+\text{N}+1$ and $\Gamma(\kappa,\text{N}) = \kappa^2+\text{N}^2+1$. Define $\Re,\aleph,ß:\mathfrak{C}\times\mathfrak{C}\times(0,+\infty)\to[0,1]$ as*

$$\Re(\kappa,\text{N},\hat{r}) = \begin{cases} 1, & if \quad \kappa = \text{N} \\ \frac{\hat{r}}{\hat{r}+\max\{\kappa,\text{N}\}}, & if\ otherwise' \end{cases}$$

$$\aleph(\kappa,\text{N},\hat{r}) = \begin{cases} 0, & if \quad \kappa = \text{N} \\ \frac{\max\{\kappa,\text{N}\}}{\hat{r}+\max\{\kappa,\text{N}\}}, & if\ otherwise' \end{cases}$$

and

$$ß(\kappa,\text{N},\hat{r}) = \begin{cases} 0, & if \quad \kappa = \text{N} \\ \frac{\max\{\kappa,\text{N}\}}{\hat{r}}, & if\ otherwise \end{cases}.$$

Then, $(\mathfrak{C},\Re,\aleph,ß,*,\bigcirc)$ is a neutrosophic double controlled metric space with continuous t-norm $\hat{e}*\bar{a} = \hat{e}\bar{a}$ and continuous t-conorm, $\hat{e}\bigcirc\bar{a} = \max\{\hat{e},\bar{a}\}$.

Proof. Here we prove (v), (x) and (xv) others are obvious.

Let $\kappa = 1$, $\text{N} = 2$ and $\lambda = 3$. Then

$$\Re(1,3,\hat{r}+\check{s}) = \frac{\hat{r}+\check{s}}{\hat{r}+\check{s}+\max\{1,3\}} = \frac{\hat{r}+\check{s}}{\hat{r}+\check{s}+3}.$$

On the other hand,

$$\Re\left(1,2,\frac{\hat{r}}{\xi(1,2)}\right) = \frac{\frac{\hat{r}}{\xi(1,2)}}{\frac{\hat{r}}{\xi(1,2)}+\max\{1,2\}} = \frac{\frac{\hat{r}}{4}}{\frac{\hat{r}}{4}+2} = \frac{\hat{r}}{\hat{r}+8}$$

and

$$\Re\left(2,3,\frac{\check{s}}{\Gamma(2,3)}\right) = \frac{\frac{\check{s}}{\Gamma(2,3)}}{\frac{\check{s}}{\Gamma(2,3)}+\max\{2,3\}} = \frac{\frac{\check{s}}{12}}{\frac{\check{s}}{12}+3} = \frac{\check{s}}{\check{s}+36}.$$

That is,
$$\frac{\hat{r}+\check{s}}{\hat{r}+\check{s}+3} \geq \frac{\hat{r}}{\hat{r}+8}\cdot\frac{\check{s}}{\check{s}+36}.$$
Then it satisfies all $\hat{r}, \check{s} > 0$. Hence,
$$\Re(\kappa,\lambda,\hat{r}+\check{s}) \geq \Re\left(\kappa, \text{N}, \frac{\hat{r}}{\xi(\kappa,\text{N})}\right) * \Re\left(\text{N}, \lambda, \frac{\check{s}}{\Gamma(\text{N},\lambda)}\right).$$
Now,
$$\aleph(1,3,\hat{r}+\check{s}) = \frac{\max\{1,3\}}{\hat{r}+\check{s}+\max\{1,3\}} = \frac{3}{\hat{r}+\check{s}+3}.$$
On the other hand,
$$\aleph\left(1,2,\frac{\hat{r}}{\Gamma(1,2)}\right) = \frac{\max\{1,2\}}{\frac{\hat{r}}{\Gamma(1,2)}+\max\{1,2\}} = \frac{2}{\frac{\hat{r}}{4}+2} = \frac{8}{\hat{r}+8}$$
and
$$\aleph\left(2,3,\frac{\check{s}}{\Gamma(2,3)}\right) = \frac{\max\{2,3\}}{\frac{\check{s}}{\Gamma(2,3)}+\max\{2,3\}} = \frac{3}{\frac{\check{s}}{12}+3} = \frac{36}{\check{s}+36}.$$
That is,
$$\frac{3}{\hat{r}+\check{s}+3} \leq \max\left\{\frac{8}{\hat{r}+8}, \frac{36}{\check{s}+36}\right\}.$$
Then it satisfies all $\hat{r}, \check{s} > 0$. Hence,
$$\aleph(\kappa,\lambda,\hat{r}+\check{s}) \leq \aleph\left(\kappa, \text{N}, \frac{\hat{r}}{\xi(\kappa,\text{N})}\right) \bigcirc \aleph\left(\text{N}, \lambda, \frac{\check{s}}{\Gamma(\text{N},\lambda)}\right).$$
Now,
$$ß(1,3,\hat{r}+\check{s}) = \frac{\max\{1,3\}}{\hat{r}+\check{s}} = \frac{3}{\hat{r}+\check{s}}.$$
On the other hand,
$$ß\left(1,2,\frac{\hat{r}}{\Gamma(1,2)}\right) = \frac{\max\{1,2\}}{\frac{\hat{r}}{\Gamma(1,2)}} = \frac{2}{\frac{\hat{r}}{4}} = \frac{8}{\hat{r}}$$
and
$$ß\left(2,3,\frac{\check{s}}{\Gamma(2,3)}\right) = \frac{\max\{2,3\}}{\frac{\check{s}}{\Gamma(2,3)}} = \frac{3}{\frac{\check{s}}{12}} = \frac{36}{\check{s}}.$$
That is,
$$\frac{3}{\hat{r}+\check{s}} \leq \max\left\{\frac{8}{\hat{r}}, \frac{36}{\check{s}}\right\}.$$
Then it satisfies all $\hat{r}, \check{s} > 0$. Hence,
$$ß(\kappa,\lambda,\hat{r}+\check{s}) \leq ß\left(\kappa, \text{N}, \frac{\hat{r}}{\xi(\kappa,\text{N})}\right) \bigcirc ß\left(\text{N}, \lambda, \frac{\check{s}}{\Gamma(\text{N},\lambda)}\right).$$
Hence, $(\mathfrak{C}, \Re, \aleph, ß, *, \bigcirc)$ is a neutrosophic double controlled metric space. □

Remark 1. *The preceding example also satisfies for continuous t-norm $\hat{e} * \bar{a} = min\{\hat{e}, \bar{a}\}$ and continuous t-conorm $\hat{e} \bigcirc \bar{a} = max\{\hat{e}, \bar{a}\}$.*

Example 2. *Let $\mathfrak{C} = (0, +\infty)$ and $\xi, \Gamma : \mathfrak{C} \times \mathfrak{C} \to [1, +\infty)$ be two non-comparable functions given by $\xi(\kappa, \text{N}) = \kappa + \text{N} + 1$ and $\Gamma(\kappa, \text{N}) = \kappa^2 + \text{N}^2 + 1$.*

Define $\Re, \aleph, \text{ß} : \mathfrak{C} \times \mathfrak{C} \times (0, +\infty) \to [0, 1]$ *as*

$$\Re(\kappa, \text{N}, \hat{\text{r}}) = \frac{\hat{\text{r}}}{\hat{\text{r}} + |\kappa - \text{N}|^2},$$

$$\aleph(\kappa, \text{N}, \hat{\text{r}}) = \frac{|\kappa - \text{N}|^2}{\hat{\text{r}} + |\kappa - \text{N}|^2}, \ \text{ß}(\kappa, \text{N}, \hat{\text{r}}) = \frac{|\kappa - \text{N}|^2}{\hat{\text{r}}}$$

Then, $(\mathfrak{C}, \Re, \aleph, \text{ß}, *, \bigcirc)$ *is a neutrosophic double controlled metric space with continuous t-norm* $\hat{\text{e}} * \bar{\text{a}} = \hat{\text{e}}\bar{\text{a}}$ *and continuous t-conorm* $\hat{\text{e}} \bigcirc \bar{\text{a}} = \max\{\hat{\text{e}}, \bar{\text{a}}\}$.

Remark 2. *The above example also holds for*

$$\xi(\kappa, \text{N}) = \begin{cases} 1 & if \quad \kappa = \text{N}, \\ \frac{1+\max\{\kappa,\text{N}\}}{\min\{\kappa,\text{N}\}} & if \ \kappa \neq \text{N} \end{cases}$$

and

$$\Gamma(\kappa, \text{N}) = \begin{cases} 1 & if \quad \kappa = \text{N}, \\ \frac{1+\max\{\kappa^2,\text{N}^2\}}{\min\{\kappa^2,\text{N}^2\}} & if \ \kappa \neq \text{N} \end{cases}.$$

Remark 3. *The preceding example also satisfies for continuous t-norm* $\hat{\text{e}} * \bar{\text{a}} = min\{\hat{\text{e}}, \bar{\text{a}}\}$ *and continuous t-conorm* $\hat{\text{e}} \bigcirc \bar{\text{a}} = max\{\hat{\text{e}}, \bar{\text{a}}\}$.

Example 3. *Let* $\mathfrak{C} = \{0, 1, 2\}$ *and* $\xi, \Gamma : \mathfrak{C} \times \mathfrak{C} \to [1, +\infty)$ *be given by* $\xi(\kappa, \text{N}) = 1$ *and* $\Gamma(\kappa, \text{N}) = 1$.

The mapping $d : X \times X \to [0, +\infty)$ *defined by* $d(0, 0) = d(1, 1) = d(2, 2) = 0$, $d(0, 1) = d(1, 0) = d(1, 2) = d(2, 1) = 1$ *and* $d(2, 0) = d(0, 2) = m$, *where* $m \geq 2$.

Define $\Re, \aleph, \text{ß} : \mathfrak{C} \times \mathfrak{C} \times (0, +\infty) \to [0, 1]$ *as*

$$\Re(\kappa, \text{N}, \hat{\text{r}}) = \frac{\hat{\text{r}}}{\hat{\text{r}} + d(\kappa, \text{N})},$$

$$\aleph(\kappa, \text{N}, \hat{\text{r}}) = \frac{d(\kappa, \text{N})}{\hat{\text{r}} + d(\kappa, \text{N})}, \ \text{ß}(\kappa, \text{N}, \hat{\text{r}}) = \frac{d(\kappa, \text{N})}{\hat{\text{r}}}.$$

Then we have

$$\Re(\kappa, \lambda, \hat{\text{r}} + \check{\text{s}}) \geq \Re\left(\kappa, \text{N}, \frac{\hat{\text{r}}}{m/2}\right) * \Re\left(\text{N}, \lambda, \frac{\check{\text{s}}}{m/2}\right),$$

$$\aleph(\kappa, \lambda, \hat{\text{r}} + \check{\text{s}}) \leq \aleph\left(\kappa, \text{N}, \frac{\hat{\text{r}}}{m/2}\right) \bigcirc \aleph\left(\text{N}, \lambda, \frac{\check{\text{s}}}{m/2}\right),$$

$$\text{ß}(\kappa, \lambda, \hat{\text{r}} + \check{\text{s}}) \leq \text{ß}\left(\kappa, \text{N}, \frac{\hat{\text{r}}}{m/2}\right) \bigcirc \text{ß}\left(\text{N}, \lambda, \frac{\check{\text{s}}}{m/2}\right).$$

Then $(\mathfrak{C}, \Re, \aleph, \text{ß}, *, \bigcirc)$ *is a neutrosophic double controlled metric space with continuous t-norm* $\hat{\text{e}} * \bar{\text{a}} = \hat{\text{e}}\bar{\text{a}}$ *and continuous t-conorm* $\hat{\text{e}} \bigcirc \bar{\text{a}} = \max\{\hat{\text{e}}, \bar{\text{a}}\}$.

It is easy to see that, when $m = 2$ then $(\mathfrak{C}, \Re, \aleph, \text{ß}, *, \bigcirc)$ *is a neutrosophic metric space and for* $m > 2$, $(\mathfrak{C}, \Re, \aleph, \text{ß}, *, \bigcirc)$ *is a neutrosophic double controlled metric space. This shows that a neutrosophic double controlled metric space is not a neutrosophic metric space but the converse is true.*

Remark 4.

(a) *If we take* $\xi(\kappa, \text{N}) = \Gamma(\text{N}, \lambda) = 1$, *in the above Examples 1 and 2, then the neutrosophic double controlled metric space becomes a neutrosophic metric space.*

(b) *Every fuzzy double controlled metric space is a neutrosophic double controlled metric space of the form $(\mathfrak{C}, \Re, 1-\Re, 1-\Re, *, \bigcirc)$, such that continuous t-norm and continuous t-conorm are associated as $\hat{e} \bigcirc \bar{a} = 1 - ((1-\hat{e}) * (1-\bar{a}))$.*

(c) *All examples of fuzzy double controlled metric spaces in* [7] *are neutrosophic double controlled metric spaces with respect to (b).*

Definition 8. *Let $(\mathfrak{C}, \Re, \aleph, \text{ß}, *, \bigcirc)$ is a neutrosophic double controlled metric space, an open ball is then defined $B(\kappa, r, \hat{r})$ with center κ, radius r, $0 < r < 1$ and $\hat{r} > 0$ as follows:*

$$B(\kappa, r, \hat{r}) = \{\text{N} \in \mathfrak{C} : \Re(\kappa, \text{N}, \hat{r}) > 1 - r,\ \aleph(\kappa, \hat{r}, \text{N}) < r,\ \text{ß}(\kappa, \text{N}, \hat{r}) < r\}.$$

Theorem 1. *Every open ball is an open set in neutrosophic double controlled metric space.*

Proof. Consider $B(\kappa, r, \hat{r})$ be an open ball with center κ and radius r. Assume $r \in B(\kappa, r, \hat{r})$. Therefore, $\Re(\kappa, d, \hat{r}) > 1 - r$, $\aleph(\kappa, d, \hat{r}) < r$, $\text{ß}(\kappa, d, \hat{r}) < r$. There exists $\hat{r}_0 \in (0, \hat{r})$ such that $\Re(\kappa, d, \hat{r}_0) > 1 - r$, $\aleph(\kappa, d, \hat{r}_0) < r$, $\text{ß}(\kappa, d, \hat{r}_0) < r$, due to $\Re(\kappa, d, \hat{r}) > 1 - r$. If we take $r_0 = \Re(\kappa, d, \hat{r}_0)$, then for $r_0 > 1 - r$, $\varepsilon \in (0, 1)$ will exist such that $r_0 > 1 - \varepsilon > 1 - r$. Given r_0 and ε such that $r_0 > 1 - \varepsilon$. Then $r_1, r_2, r_3 \in (0, 1)$ will exist such that $r_0 * r_1 > 1 - \varepsilon$, $(1 - r_0) \bigcirc (1 - r_2) \leq \varepsilon$ and $(1 - r_0) \bigcirc (1 - r_3) \leq \varepsilon$. Choose $r_4 \in \max\{r_1, r_2, r_3\}$. Consider the open ball $B(d,\ 1 - r_4,\ \hat{r} - \hat{r}_0)$. We will show that $B(d,\ 1 - r_4,\ \hat{r} - \hat{r}_0) \subset B(\kappa, r, \hat{r})$. If we take $v \in B(d,\ 1 - r_4,\ \hat{r} - \hat{r}_0)$, then $\Re(d,\ v,\ \hat{r} - \hat{r}_0) > r_4$, $\aleph(d,\ v,\ \hat{r} - \hat{r}_0) < r_4$, $\text{ß}(d,\ v,\ \hat{r} - \hat{r}_0) < r_4$. Then

$$\Re(\kappa, v, \hat{r}) \geq \Re(\kappa, d, \hat{r}_0) * \Re(d, v, \hat{r} - \hat{r}_0) \geq r_0 * r_4 \geq r_0 * r_1 \geq 1 - \varepsilon > 1 - r,$$

$$\aleph(\kappa, v, \hat{r}) \leq \aleph(\kappa, d, \hat{r}_0) \bigcirc \aleph(d, v, \hat{r} - \hat{r}_0) \leq (1 - r_0) \bigcirc (1 - r_4) \leq (1 - r_0) \bigcirc (1 - r_1) \leq \varepsilon < r,$$

$$\text{ß}(\kappa, v, \hat{r}) \leq \text{ß}(\kappa, d, \hat{r}_0) \bigcirc \text{ß}(d, v, \hat{r} - \hat{r}_0) \leq (1 - r_0) \bigcirc (1 - r_4) \leq (1 - r_0) \bigcirc (1 - r_1) \leq \varepsilon < r.$$

It shows that $v \in B(\kappa, r, \hat{r})$ and $B(d,\ 1 - r_4,\ \hat{r} - \hat{r}_0) \subset B(\kappa, r, \hat{r})$.

Now we will examine the fact that a neutrosophic double controlled metric space is not continuous. □

Example 4. *Let $\mathfrak{C} = \{0, 1, 2\}$ and $\xi, \Gamma : \mathfrak{C} \times \mathfrak{C} \to [1, +\infty)$ be given by $\xi(\kappa, \text{N}) = \kappa + \text{N} + 1$ and $\Gamma(\kappa, \text{N}) = \kappa^2 + \text{N}^2 + 1$.*

Define $\Re, \aleph, \text{ß} : \mathfrak{C} \times \mathfrak{C} \times (0, +\infty) \to [0, 1]$ as

$$\Re(\kappa, \text{N}, \hat{r}) = \frac{\hat{r}}{\hat{r} + d(\kappa, \text{N})},$$

$$\aleph(\kappa, \text{N}, \hat{r}) = \frac{d(\kappa, \text{N})}{\hat{r} + d(\kappa, \text{N})},\quad \text{ß}(\kappa, \text{N}, \hat{r}) = \frac{d(\kappa, \text{N})}{\hat{r}}.$$

The mapping $d : \mathfrak{C} \times \mathfrak{C} \to [0, +\infty)$ defined by

$$d(\kappa, \text{N}) = \begin{cases} 0, & if \quad \kappa = \text{N} \\ 2(\kappa + \text{N})^2, & if\ \kappa, \text{N} \in [0, 1] \\ \frac{1}{2}(\kappa + \text{N})^2, & \text{otherwise.} \end{cases}$$

Then, $(\mathfrak{C}, \Re, \aleph, \text{ß}, *, \bigcirc)$ is a neutrosophic double controlled metric space with continuous t-norm $\hat{e} * \bar{a} = \hat{e}\bar{a}$ and continuous t-conorm $\hat{e} \bigcirc \bar{a} = \max\{\hat{e}, \bar{a}\}$. To illustrate the discontinuity, we have

$$\lim_{n \to +\infty} \Re\left(0, 1 - \frac{1}{n}, \hat{r}\right) = \lim_{n \to +\infty} \frac{\hat{r}}{\hat{r} + 2\left(1 - \left(\frac{1}{n}\right)\right)^2} = \frac{\hat{r}}{\hat{r} + 2} = \Re(0, 1, \hat{r}),$$

$$\lim_{n\to+\infty}\aleph\left(0,1-\frac{1}{n},\hat{r}\right)=\lim_{n\to+\infty}\frac{2\left(1-\left(\frac{1}{n}\right)\right)^2}{\hat{r}+2\left(1-\left(\frac{1}{n}\right)\right)^2}=\frac{2}{\hat{r}+2}=\aleph(0,1,\hat{r}),$$

$$\lim_{n\to+\infty}ß\left(0,1-\frac{1}{n},\hat{r}\right)=\lim_{n\to+\infty}\frac{2\left(1-\left(\frac{1}{n}\right)\right)^2}{\hat{r}}=\frac{2}{\hat{r}}=ß(0,1,\hat{r}).$$

However, since

$$\lim_{n\to+\infty}\Re\left(1,1-\frac{1}{n},\hat{r}\right)=\lim_{n\to+\infty}\frac{\hat{r}}{\hat{r}+2\left(2-\left(\frac{1}{n}\right)\right)^2}=\frac{\hat{r}}{\hat{r}+8}\neq 1=\Re(1,1,\hat{r}),$$

$$\lim_{n\to+\infty}\aleph\left(1,1-\frac{1}{n},\hat{r}\right)=\lim_{n\to+\infty}\frac{2\left(2-\left(\frac{1}{n}\right)\right)^2}{\hat{r}+2\left(2-\left(\frac{1}{n}\right)\right)^2}=\frac{8}{\hat{r}+8}\neq 0=\aleph(1,1,\hat{r}),$$

$$\lim_{n\to+\infty}ß\left(1,1-\frac{1}{n},\hat{r}\right)=\lim_{n\to+\infty}\frac{2\left(2-\left(\frac{1}{n}\right)\right)^2}{\hat{r}}=\frac{8}{\hat{r}}\neq 0=ß(1,1,\hat{r}).$$

One can assert that $(\mathfrak{C},\Re,\aleph,ß,*,\bigcirc)$ *is not continuous.*

Note, we are assuming the case in which the neutrosophic double controlled metric space $(\mathfrak{C},\Re,\aleph,ß,*,\bigcirc)$ *is a Hausdorff and continuous. The continuity of the neutrosophic double controlled metric space* $(\mathfrak{C},\Re,\aleph,ß,*,\bigcirc)$ *means the continuity of the involved functions* $\Re$, $\aleph$ *and* $ß$.

Definition 9. *Let* $(\mathfrak{C},\Re,\aleph,ß,*,\bigcirc)$ *is a neutrosophic double controlled metric space and* $\{\kappa_n\}$ *be a sequence in* $\mathfrak{C}$. *Then* $\{\kappa_n\}$ *is said to be:*

(a) *a convergent exists if there exists* $\kappa\in\mathfrak{C}$ *such that*

$$\lim_{n\to+\infty}\Re(\kappa_n,\ \kappa,\ \hat{r})=1,\ \lim_{n\to+\infty}\aleph(\kappa_n,\ \kappa,\ \hat{r})=0,\ \lim_{n\to+\infty}ß(\kappa_n,\ \kappa,\ \hat{r})=0 \text{ for all } \hat{r}>0,$$

(b) *a Cauchy sequence, if and only if for each* $\bar{a}>0,\ \hat{r}>0$, *there exists* $n_0\in\mathbb{N}$ *such that*

$$\Re\left(\kappa_n,\ \kappa_{n+q},\ \hat{r}\right)\geq 1-\bar{a},\ \aleph\left(\kappa_n,\ \kappa_{n+q},\ \hat{r}\right)\leq\bar{a},\aleph\left(\kappa_n,\ \kappa_{n+q},\ \hat{r}\right)\leq\bar{a} \text{ for all } n,m\geq n_0,$$

If every Cauchy sequence convergent in $\mathfrak{C}$, *then* $(\mathfrak{C},\Re,\aleph,ß,*,\bigcirc)$ *is called a complete neutrosophic double controlled metric space.*

Lemma 1. *Let* $\{\kappa_n\}$ *be a Cauchy sequence in neutrosophic double controlled metric space* $(\mathfrak{C},\Re,\aleph,ß,*,\bigcirc)$ *such that* $\kappa_n\neq\kappa_m$ *whenever* $m,n\in\mathbb{N}$ *with* $n\neq m$. *Then the sequence* $\{\kappa_n\}$ *can converge to, at most, one limit point.*

Proof. Contrarily, assume that $\kappa_n\to\kappa$ and $\kappa_n\to\text{N}$, *for* $\kappa\neq\text{N}$. Then, $\lim_{n\to+\infty}\Re(\kappa_n,\ \kappa,\ \hat{r})=1,\ \lim_{n\to+\infty}\aleph(\kappa_n,\ \kappa,\ \hat{r})=0,\ \lim_{n\to+\infty}ß(\kappa_n,\ \kappa,\ \hat{r})=0$, *and* $\lim_{n\to+\infty}\Re(\kappa_n,\ \text{N},\ \hat{r})=1,\ \lim_{n\to+\infty}\aleph(\kappa_n,\ \text{N},\ \hat{r})=0,\ \lim_{n\to+\infty}ß(\kappa_n,\ \text{N},\ \hat{r})=0$, for all $\hat{r}>0$. Suppose

$$\Re(\kappa,\text{N},\hat{r})\geq\Re\left(\kappa,\ \kappa_n,\frac{\hat{r}}{2\xi(\kappa,\kappa_n)}\right)*\Re\left(\kappa_n,\text{N},\frac{\hat{r}}{2\Gamma(\kappa_n,\text{N})}\right)\to 1*1,\text{ as } n\to+\infty,$$

$$\aleph(\kappa,\text{N},\hat{r})\leq\aleph\left(\kappa,\ \kappa_n,\frac{\hat{r}}{2\xi(\kappa,\kappa_n)}\right)\bigcirc\aleph\left(\kappa_n,\text{N},\frac{\hat{r}}{2\Gamma(\kappa_n,\text{N})}\right)\to 0\bigcirc 0,\text{ as } n\to+\infty,$$

$$ß(\kappa,\text{N},\hat{r})\leq ß\left(\kappa,\ \kappa_n,\frac{\hat{r}}{2\xi(\kappa,\kappa_n)}\right)\bigcirc ß\left(\kappa_n,\text{N},\frac{\hat{r}}{2\Gamma(\kappa_n,\text{N})}\right)\to 0\bigcirc 0,\text{ as } n\to+\infty.$$

That is $\Re(\kappa, \textsc{n}, \hat{\mathrm{r}}) \geq 1 * 1 = 1$, $\aleph(\kappa, \textsc{n}, \hat{\mathrm{r}}) \leq 0 \bigcirc 0 = 0$ and $\text{ß}(\kappa, \textsc{n}, \hat{\mathrm{r}}) \leq 0 \bigcirc 0 = 0$. Hence $\kappa = \textsc{n}$, that is, the sequence converges to, at most, one limit point. □

Lemma 2. *Let* $(\mathfrak{C}, \Re, \aleph, \text{ß}, *, \bigcirc)$ *is a neutrosophic double controlled metric space. If for some* $0 < \theta < 1$ *and for any* $\kappa, \textsc{n} \in \mathfrak{C}$, $\hat{\mathrm{r}} > 0$,

$$\Re(\kappa, \textsc{n}, \hat{\mathrm{r}}) \geq \Re\left(\kappa, \textsc{n}, \frac{\hat{\mathrm{r}}}{\theta}\right), \ \aleph(\kappa, \textsc{n}, \hat{\mathrm{r}}) \leq \aleph\left(\kappa, \textsc{n}, \frac{\hat{\mathrm{r}}}{\theta}\right), \ \text{ß}(\kappa, \textsc{n}, \hat{\mathrm{r}}) \leq \text{ß}\left(\kappa, \textsc{n}, \frac{\hat{\mathrm{r}}}{\theta}\right) \tag{1}$$

then $\kappa = \textsc{n}$.

Proof. (1) implies that

$$\Re(\kappa, \textsc{n}, \hat{\mathrm{r}}) \geq \Re\left(\kappa, \textsc{n}, \frac{\hat{\mathrm{r}}}{\theta^n}\right), \ \aleph(\kappa, \textsc{n}, \hat{\mathrm{r}}) \leq \aleph\left(\kappa, \textsc{n}, \frac{\hat{\mathrm{r}}}{\theta^n}\right), \ \text{ß}(\kappa, \textsc{n}, \hat{\mathrm{r}}) \leq \text{ß}\left(\kappa, \textsc{n}, \frac{\hat{\mathrm{r}}}{\theta^n}\right), n \in \mathbb{N}, \ \hat{\mathrm{r}} > 0.$$

Now

$$\Re(\kappa, \textsc{n}, \hat{\mathrm{r}}) \geq \lim_{n \to +\infty} \Re\left(\kappa, \textsc{n}, \frac{\hat{\mathrm{r}}}{\theta^n}\right) = 1, \aleph(\kappa, \textsc{n}, \hat{\mathrm{r}}) \leq \lim_{n \to +\infty} \aleph\left(\kappa, \textsc{n}, \frac{\hat{\mathrm{r}}}{\theta^n}\right) = 0$$

$$\text{ß}(\kappa, \textsc{n}, \hat{\mathrm{r}}) \leq \lim_{n \to +\infty} \text{ß}\left(\kappa, \textsc{n}, \frac{\hat{\mathrm{r}}}{\theta^n}\right) = 0, \ \hat{\mathrm{r}} > 0.$$

Also, by dint of (iii), (viii) and (xiii), that is, $\kappa = \textsc{n}$.

Now, we will prove the neutrosophic double controlled Banach contraction theorem. □

Theorem 2. *Suppose* $(\mathfrak{C}, \Re, \aleph, \text{ß}, *, \bigcirc)$ *is a complete neutrosophic double controlled metric space in the company of* $\xi, \Gamma : \mathfrak{C} \times \mathfrak{C} \to [1, +\infty)$ *with* $0 < \theta < 1$ *and suppose that*

$$\lim_{\hat{\mathrm{r}} \to +\infty} \Re(\kappa, \textsc{n}, \hat{\mathrm{r}}) = 1, \ \lim_{\hat{\mathrm{r}} \to +\infty} \aleph(\kappa, \textsc{n}, \hat{\mathrm{r}}) = 0 \text{ and } \lim_{\hat{\mathrm{r}} \to +\infty} \text{ß}(\kappa, \hat{\mathrm{r}}, \textsc{n}) = 0 \tag{2}$$

for all $\kappa, \textsc{n} \in \mathfrak{C}$ and $\hat{\mathrm{r}} > 0$. *Let* $\text{þ} : \mathfrak{C} \to \mathfrak{C}$ *be a mapping satisfying*

$$\Re(\text{þ}\kappa, \text{þ}\textsc{n}, \theta\hat{\mathrm{r}}) \geq \Re(\kappa, \textsc{n}, \hat{\mathrm{r}}),$$

$$\aleph(\text{þ}\kappa, \text{þ}\textsc{n}, \theta\hat{\mathrm{r}}) \leq \aleph(\kappa, \textsc{n}, \hat{\mathrm{r}}) \text{ and } \text{ß}(\text{þ}\kappa, \text{þ}\textsc{n}, \theta\hat{\mathrm{r}}) \leq \text{ß}(\kappa, \textsc{n}, \hat{\mathrm{r}}) \tag{3}$$

for all $\kappa, \textsc{n} \in \mathfrak{C}$ *and* $\hat{\mathrm{r}} > 0$. *Then* þ *has a unique fixed point.*

Proof. Let κ_0 be a point of $\mathfrak{C}$ and define a sequence κ_n by $\kappa_n = \text{þ}^n \kappa_0 = \text{þ}\kappa_{n-1}$, $n \in \mathbb{N}$. By utilising (2) for all $\hat{\mathrm{r}} > 0$, we obtain

$$\Re(\kappa_n, \kappa_{n+1}, \theta\hat{\mathrm{r}}) = \Re(\text{þ}\kappa_{n-1}, \text{þ}\kappa_n, \theta\hat{\mathrm{r}}) \geq \Re(\kappa_{n-1}, \kappa_n, \hat{\mathrm{r}}) \geq \Re\left(\kappa_{n-2}, \kappa_{n-1}, \frac{\hat{\mathrm{r}}}{\theta}\right)$$

$$\geq \Re\left(\kappa_{n-3}, \kappa_{n-2}, \frac{\hat{\mathrm{r}}}{\theta^2}\right) \geq \cdots \geq \Re\left(\kappa_0, \kappa_1, \frac{\hat{\mathrm{r}}}{\theta^{n-1}}\right),$$

$$\aleph(\kappa_n, \kappa_{n+1}, \theta\hat{\mathrm{r}}) = \aleph(\text{þ}\kappa_{n-1}, \text{þ}\kappa_n, \theta\hat{\mathrm{r}}) \leq \aleph(\kappa_{n-1}, \kappa_n, \hat{\mathrm{r}}) \leq \aleph\left(\kappa_{n-2}, \kappa_{n-1}, \frac{\hat{\mathrm{r}}}{\theta}\right)$$

$$\leq \aleph\left(\kappa_{n-3}, \kappa_{n-2}, \frac{\hat{\mathrm{r}}}{\theta^2}\right) \leq \cdots \leq \aleph\left(\kappa_0, \kappa_1, \frac{\hat{\mathrm{r}}}{\theta^{n-1}}\right)$$

and

$$\text{ß}(\kappa_n, \kappa_{n+1}, \theta\hat{\mathrm{r}}) = \text{ß}(\text{þ}\kappa_{n-1}, \text{þ}\kappa_n, \theta\hat{\mathrm{r}}) \leq \text{ß}(\kappa_{n-1}, \kappa_n, \hat{\mathrm{r}}) \leq \text{ß}\left(\kappa_{n-2}, \kappa_{n-1}, \frac{\hat{\mathrm{r}}}{\theta}\right)$$

$$\leq \text{ß}\left(\kappa_{n-3}, \kappa_{n-2}, \frac{\hat{\mathrm{r}}}{\theta^2}\right) \leq \cdots \leq \text{ß}\left(\kappa_0, \kappa_1, \frac{\hat{\mathrm{r}}}{\theta^{n-1}}\right).$$

We obtain

$$\Re(\kappa_n, \kappa_{n+1}, \theta\hat{\mathrm{r}}) \geq \Re\left(\kappa_0, \kappa_1, \frac{\hat{\mathrm{r}}}{\theta^{n-1}}\right),$$

$$\aleph(\kappa_n, \kappa_{n+1}, \theta\hat{\mathrm{r}}) \leq \aleph\left(\kappa_0, \kappa_1, \frac{\hat{\mathrm{r}}}{\theta^{n-1}}\right) \text{ and } \text{ß}(\kappa_n, \kappa_{n+1}, \theta\hat{\mathrm{r}}) \leq \text{ß}\left(\kappa_0, \kappa_1, \frac{\hat{\mathrm{r}}}{\theta^{n-1}}\right) \tag{4}$$

for any $q \in \mathbb{N}$, using (v), (x) and (xv), we deduce

$$\begin{aligned}
&\Re\big(\kappa_n, \kappa_{n+q}, \hat{\mathrm{r}}\big) \geq \Re\left(\kappa_n, \kappa_{n+1}, \tfrac{\hat{\mathrm{r}}}{2(\xi(\kappa_n,\kappa_{n+1}))}\right) * \Re\left(\kappa_{n+1}, \kappa_{n+q}, \tfrac{\hat{\mathrm{r}}}{2\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\right)}\right) \\
&\geq \Re\left(\kappa_n, \kappa_{n+1}, \tfrac{\hat{\mathrm{r}}}{2(\xi(\kappa_n,\kappa_{n+1}))}\right) * \Re\left(\kappa_{n+1}, \kappa_{n+2}, \tfrac{\hat{\mathrm{r}}}{(2)^2\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\xi(\kappa_{n+1},\kappa_{n+2})\right)}\right) \\
&*\Re\left(\kappa_{n+2}, \kappa_{n+q}, \tfrac{\hat{\mathrm{r}}}{(2)^2\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\right)}\right) \\
&\geq \Re\left(\kappa_n, \kappa_{n+1}, \tfrac{\hat{\mathrm{r}}}{2(\xi(\kappa_n,\kappa_{n+1}))}\right) * \Re\left(\kappa_{n+1}, \kappa_{n+2}, \tfrac{\hat{\mathrm{r}}}{(2)^2\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\xi(\kappa_{n+1},\kappa_{n+2})\right)}\right) \\
&*\Re\left(\kappa_{n+2}, \kappa_{n+3}, \tfrac{\hat{\mathrm{r}}}{(2)^3\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\xi(\kappa_{n+2},\kappa_{n+3})\right)}\right) \\
&*\Re\left(\kappa_{n+3}, \kappa_{n+q}, \tfrac{\hat{\mathrm{r}}}{(2)^3\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+3},\kappa_{n+q}\right)\right)}\right)
\end{aligned}$$

$$\begin{gathered}
\Re\big(\kappa_n, \kappa_{n+q}, \hat{\mathrm{r}}\big) \geq \Re\left(\kappa_n, \kappa_{n+1}, \frac{\hat{\mathrm{r}}}{2(\xi(\kappa_n, \kappa_{n+1}))}\right) \\
* \Re\left(\kappa_{n+1}, \kappa_{n+2}, \frac{\hat{\mathrm{r}}}{(2)^2\left(\Gamma\left(\kappa_{n+1}, \kappa_{n+q}\right)\xi(\kappa_{n+1}, \kappa_{n+2})\right)}\right) \\
* \Re\left(\kappa_{n+2}, \kappa_{n+3}, \frac{\hat{\mathrm{r}}}{(2)^3\left(\Gamma\left(\kappa_{n+1}, \kappa_{n+q}\right)\Gamma\left(\kappa_{n+2}, \kappa_{n+q}\right)\xi(\kappa_{n+2}, \kappa_{n+3})\right)}\right) \\
* \Re\left(\kappa_{n+3}, \kappa_{n+4}, \frac{\hat{\mathrm{r}}}{(2)^4\left(\Gamma\left(\kappa_{n+1}, \kappa_{n+q}\right)\Gamma\left(\kappa_{n+2}, \kappa_{n+q}\right)\Gamma\left(\kappa_{n+3}, \kappa_{n+q}\right)\xi(\kappa_{n+3}, \kappa_{n+4})\right)}\right) * \cdots * \\
\Re\left(\kappa_{n+q-2}, \kappa_{n+q-1}, \frac{\hat{\mathrm{r}}}{(2)^{q-1}\left(\Gamma\left(\kappa_{n+1}, \kappa_{n+q}\right)\Gamma\left(\kappa_{n+2}, \kappa_{n+q}\right)\cdots\Gamma\left(\kappa_{n+q-2}, \kappa_{n+q}\right)\xi\left(\kappa_{n+q-2}, \kappa_{n+q-1}\right)\right)}\right) \\
* \Re\left(\kappa_{n+q-1}, \kappa_{n+q}, \frac{\hat{\mathrm{r}}}{(2)^{q-1}\left(\Gamma\left(\kappa_{n+1}, \kappa_{n+q}\right)\Gamma\left(\kappa_{n+2}, \kappa_{n+q}\right)\cdots\Gamma\left(\kappa_{n+q-1}, \kappa_{n+q}\right)\right)}\right),
\end{gathered}$$

$$\begin{aligned}
&\aleph\big(\kappa_n, \kappa_{n+q}, \hat{\mathrm{r}}\big) \leq \aleph\left(\kappa_n, \kappa_{n+1}, \tfrac{\hat{\mathrm{r}}}{2(\xi(\kappa_n,\kappa_{n+1}))}\right) \bigcirc \aleph\left(\kappa_{n+1}, \kappa_{n+q}, \tfrac{\hat{\mathrm{r}}}{2\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\right)}\right) \\
&\leq \aleph\left(\kappa_n, \kappa_{n+1}, \tfrac{\hat{\mathrm{r}}}{2(\xi(\kappa_n,\kappa_{n+1}))}\right) \bigcirc \aleph\left(\kappa_{n+1}, \kappa_{n+2}, \tfrac{\hat{\mathrm{r}}}{(2)^2\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\xi(\kappa_{n+1},\kappa_{n+2})\right)}\right) \\
&\bigcirc\aleph\left(\kappa_{n+2}, \kappa_{n+q}, \tfrac{\hat{\mathrm{r}}}{(2)^2\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\right)}\right) \\
&\leq \aleph\left(\kappa_n, \kappa_{n+1}, \tfrac{\hat{\mathrm{r}}}{2(\xi(\kappa_n,\kappa_{n+1}))}\right) \bigcirc \aleph\left(\kappa_{n+1}, \kappa_{n+2}, \tfrac{\hat{\mathrm{r}}}{(2)^2\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\xi(\kappa_{n+1},\kappa_{n+2})\right)}\right) \\
&\bigcirc\aleph\left(\kappa_{n+2}, \kappa_{n+3}, \tfrac{\hat{\mathrm{r}}}{(2)^3\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\xi(\kappa_{n+2},\kappa_{n+3})\right)}\right) \\
&\bigcirc\aleph\left(\kappa_{n+3}, \kappa_{n+q}, \tfrac{\hat{\mathrm{r}}}{(2)^3\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+3},\kappa_{n+q}\right)\right)}\right)
\end{aligned}$$

$$\aleph\big(\kappa_n, \kappa_{n+q}, \hat{\mathrm{r}}\big) \leq \aleph\left(\kappa_n, \kappa_{n+1}, \frac{\hat{\mathrm{r}}}{2(\xi(\kappa_n, \kappa_{n+1}))}\right)$$

$$\bigcirc\aleph\left(\kappa_{n+1},\kappa_{n+2},\frac{\hat{\mathrm{r}}}{(2)^2\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\xi(\kappa_{n+1},\kappa_{n+2})\right)}\right)$$
$$\bigcirc\aleph\left(\kappa_{n+2},\kappa_{n+3},\frac{\hat{\mathrm{r}}}{(2)^3\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\xi(\kappa_{n+2},\kappa_{n+3})\right)}\right)$$
$$\bigcirc\aleph\left(\kappa_{n+3},\kappa_{n+4},\frac{\hat{\mathrm{r}}}{(2)^4\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+3},\kappa_{n+q}\right)\xi\left(\kappa_{n+3},\kappa_{n+4}\right)\right)}\right)\bigcirc\cdots\bigcirc$$
$$\aleph\left(\kappa_{n+q-2},\kappa_{n+q-1},\frac{\hat{\mathrm{r}}}{(2)^{q-1}\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\cdots\Gamma\left(\kappa_{n+q-2},\kappa_{n+q}\right)\xi\left(\kappa_{n+q-2},\kappa_{n+q-1}\right)\right)}\right)$$
$$\bigcirc\aleph\left(\kappa_{n+q-1},\kappa_{n+q},\frac{\hat{\mathrm{r}}}{(2)^{q-1}\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+3},\kappa_{n+q}\right)\cdots\Gamma\left(\kappa_{n+q-1},\kappa_{n+q}\right)\right)}\right)$$

and

$$\begin{aligned}
&\text{ß}\left(\kappa_n,\kappa_{n+q},\hat{\mathrm{r}}\right)\le\text{ß}\left(\kappa_n,\kappa_{n+1},\frac{\hat{\mathrm{r}}}{2(\xi(\kappa_n,\kappa_{n+1}))}\right)\bigcirc\text{ß}\left(\kappa_{n+1},\kappa_{n+q},\frac{\hat{\mathrm{r}}}{2\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\right)}\right)\\
&\le\text{ß}\left(\kappa_n,\kappa_{n+1},\frac{\hat{\mathrm{r}}}{2(\xi(\kappa_n,\kappa_{n+1}))}\right)\bigcirc\text{ß}\left(\kappa_{n+1},\kappa_{n+2},\frac{\hat{\mathrm{r}}}{(2)^2\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\xi(\kappa_{n+1},\kappa_{n+2})\right)}\right)\\
&\bigcirc\text{ß}\left(\kappa_{n+2},\kappa_{n+q},\frac{\hat{\mathrm{r}}}{(2)^2\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\right)}\right)\\
&\le\text{ß}\left(\kappa_n,\kappa_{n+1},\frac{\hat{\mathrm{r}}}{2(\xi(\kappa_n,\kappa_{n+1}))}\right)\bigcirc\text{ß}\left(\kappa_{n+1},\kappa_{n+2},\frac{\hat{\mathrm{r}}}{(2)^2\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\xi(\kappa_{n+1},\kappa_{n+2})\right)}\right)\\
&\bigcirc\text{ß}\left(\kappa_{n+2},\kappa_{n+3},\frac{\hat{\mathrm{r}}}{(2)^3\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\xi(\kappa_{n+2},\kappa_{n+3})\right)}\right)\\
&\bigcirc\text{ß}\left(\kappa_{n+3},\kappa_{n+q},\frac{\hat{\mathrm{r}}}{(2)^3\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+3},\kappa_{n+q}\right)\right)}\right)
\end{aligned}$$

$$\text{ß}\left(\kappa_n,\kappa_{n+q},\hat{\mathrm{r}}\right)\le\text{ß}\left(\kappa_n,\kappa_{n+1},\frac{\hat{\mathrm{r}}}{2\left(\xi\left(\kappa_n,\kappa_{n+1}\right)\right)}\right)$$
$$\bigcirc\text{ß}\left(\kappa_{n+1},\kappa_{n+2},\frac{\hat{\mathrm{r}}}{(2)^2\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\xi\left(\kappa_{n+1},\kappa_{n+2}\right)\right)}\right)$$
$$\bigcirc\text{ß}\left(\kappa_{n+2},\kappa_{n+3},\frac{\hat{\mathrm{r}}}{(2)^3\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\xi\left(\kappa_{n+2},\kappa_{n+3}\right)\right)}\right)$$
$$\bigcirc\text{ß}\left(\kappa_{n+3},\kappa_{n+4},\frac{\hat{\mathrm{r}}}{(2)^4\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+3},\kappa_{n+q}\right)\xi\left(\kappa_{n+3},\kappa_{n+4}\right)\right)}\right)\bigcirc\cdots\bigcirc$$
$$\text{ß}\left(\kappa_{n+q-2},\kappa_{n+q-1},\frac{\hat{\mathrm{r}}}{(2)^{q-1}\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\cdots\Gamma\left(\kappa_{n+q-2},\kappa_{n+q}\right)\xi\left(\kappa_{n+q-2},\kappa_{n+q-1}\right)\right)}\right)$$
$$\bigcirc\text{ß}\left(\kappa_{n+q-1},\kappa_{n+q},\frac{\hat{\mathrm{r}}}{(2)^{q-1}\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+3},\kappa_{n+q}\right)\cdots\Gamma\left(\kappa_{n+q-1},\kappa_{n+q}\right)\right)}\right).$$

Using (4) in the above inequalities, we deduce

$$\Re\left(\kappa_n,\kappa_{n+q},\hat{\mathrm{r}}\right)\ge\Re\left(\kappa_0,\kappa_1,\frac{\hat{\mathrm{r}}}{2(\theta)^{n-1}\left(\xi\left(\kappa_n,\kappa_{n+1}\right)\right)}\right)$$
$$*\,\Re\left(\kappa_0,\kappa_1,\frac{\hat{\mathrm{r}}}{(2)^2(\theta)^n\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\xi\left(\kappa_{n+1},\kappa_{n+2}\right)\right)}\right)$$
$$*\,\Re\left(\kappa_0,\kappa_1,\frac{\hat{\mathrm{r}}}{(2)^3(\theta)^{n+1}\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\xi\left(\kappa_{n+2},\kappa_{n+3}\right)\right)}\right)$$

$$* \Re\left(\kappa_0, \kappa_1, \frac{\hat{\mathrm{r}}}{(2)^4(\theta)^{n+2}\left(\Gamma\left(\kappa_{n+1}, \kappa_{n+q}\right)\Gamma\left(\kappa_{n+2}, \kappa_{n+q}\right)\Gamma\left(\kappa_{n+3}, \kappa_{n+q}\right)\xi\left(\kappa_{n+3}, \kappa_{n+4}\right)\right)}\right) * \cdots *$$

$$\Re\left(\kappa_0, \kappa_1, \frac{\hat{\mathrm{r}}}{(2)^{q-1}(\theta)^{n+q-2}\left(\Gamma\left(\kappa_{n+1}, \kappa_{n+q}\right)\Gamma\left(\kappa_{n+2}, \kappa_{n+q}\right)\cdots\Gamma\left(\kappa_{n+q-2}, \kappa_{n+q}\right)\xi\left(\kappa_{n+q-2}, \kappa_{n+q-1}\right)\right)}\right)$$

$$* \Re\left(\kappa_0, \kappa_1, \frac{\hat{\mathrm{r}}}{(2)^{q-1}(\theta)^{n+q-1}\left(\Gamma\left(\kappa_{n+1}, \kappa_{n+q}\right)\Gamma\left(\kappa_{n+2}, \kappa_{n+q}\right)\Gamma\left(\kappa_{n+3}, \kappa_{n+q}\right)\cdots\Gamma\left(\kappa_{n+q-1}, \kappa_{n+q}\right)\right)}\right),$$

$$\aleph\left(\kappa_n, \kappa_{n+q}, \hat{\mathrm{r}}\right) \leq \aleph\left(\kappa_0, \kappa_1, \frac{\hat{\mathrm{r}}}{2(\theta)^{n-1}\left(\xi\left(\kappa_n, \kappa_{n+1}\right)\right)}\right)$$

$$\bigcirc\, \aleph\left(\kappa_0, \kappa_1, \frac{\hat{\mathrm{r}}}{(2)^2(\theta)^{n}\left(\Gamma\left(\kappa_{n+1}, \kappa_{n+q}\right)\xi\left(\kappa_{n+1}, \kappa_{n+2}\right)\right)}\right)$$

$$\bigcirc\, \aleph\left(\kappa_0, \kappa_1, \frac{\hat{\mathrm{r}}}{(2)^3(\theta)^{n+1}\left(\Gamma\left(\kappa_{n+1}, \kappa_{n+q}\right)\Gamma\left(\kappa_{n+2}, \kappa_{n+q}\right)\xi\left(\kappa_{n+2}, \kappa_{n+3}\right)\right)}\right)$$

$$\bigcirc\, \aleph\left(\kappa_0, \kappa_1, \frac{\hat{\mathrm{r}}}{(2)^4(\theta)^{n+2}\left(\Gamma\left(\kappa_{n+1}, \kappa_{n+q}\right)\Gamma\left(\kappa_{n+2}, \kappa_{n+q}\right)\Gamma\left(\kappa_{n+3}, \kappa_{n+q}\right)\xi\left(\kappa_{n+3}, \kappa_{n+4}\right)\right)}\right)$$

$$\bigcirc \cdots \bigcirc$$

$$\aleph\left(\kappa_0, \kappa_1, \frac{\hat{\mathrm{r}}}{(2)^{q-1}(\theta)^{n+q-2}\left(\Gamma\left(\kappa_{n+1}, \kappa_{n+q}\right)\Gamma\left(\kappa_{n+2}, \kappa_{n+q}\right)\cdots\Gamma\left(\kappa_{n+q-2}, \kappa_{n+q}\right)\xi\left(\kappa_{n+q-2}, \kappa_{n+q-1}\right)\right)}\right)$$

$$\bigcirc \aleph\left(\kappa_0, \kappa_1, \frac{\hat{\mathrm{r}}}{(2)^{q-1}(\theta)^{n+q-1}\left(\Gamma\left(\kappa_{n+1}, \kappa_{n+q}\right)\Gamma\left(\kappa_{n+2}, \kappa_{n+q}\right)\Gamma\left(\kappa_{n+3}, \kappa_{n+q}\right)\cdots\Gamma\left(\kappa_{n+q-1}, \kappa_{n+q}\right)\right)}\right)$$

and

$$ß\left(\kappa_n, \kappa_{n+q}, \hat{\mathrm{r}}\right) \leq ß\left(\kappa_0, \kappa_1, \frac{\hat{\mathrm{r}}}{2(\theta)^{n-1}\left(\xi\left(\kappa_n, \kappa_{n+1}\right)\right)}\right)$$

$$\bigcirc\, ß\left(\kappa_0, \kappa_1, \frac{\hat{\mathrm{r}}}{(2)^2(\theta)^{n}\left(\Gamma\left(\kappa_{n+1}, \kappa_{n+q}\right)\xi\left(\kappa_{n+1}, \kappa_{n+2}\right)\right)}\right)$$

$$\bigcirc\, ß\left(\kappa_0, \kappa_1, \frac{\hat{\mathrm{r}}}{(2)^3(\theta)^{n+1}\left(\Gamma\left(\kappa_{n+1}, \kappa_{n+q}\right)\Gamma\left(\kappa_{n+2}, \kappa_{n+q}\right)\xi\left(\kappa_{n+2}, \kappa_{n+3}\right)\right)}\right)$$

$$\bigcirc\, ß\left(\kappa_0, \kappa_1, \frac{\hat{\mathrm{r}}}{(2)^4(\theta)^{n+2}\left(\Gamma\left(\kappa_{n+1}, \kappa_{n+q}\right)\Gamma\left(\kappa_{n+2}, \kappa_{n+q}\right)\Gamma\left(\kappa_{n+3}, \kappa_{n+q}\right)\xi\left(\kappa_{n+3}, \kappa_{n+4}\right)\right)}\right)$$

$$ß\left(\kappa_0, \kappa_1, \frac{\hat{\mathrm{r}}}{(2)^{q-1}(\theta)^{n+q-2}\left(\Gamma\left(\kappa_{n+1}, \kappa_{n+q}\right)\Gamma\left(\kappa_{n+2}, \kappa_{n+q}\right)\cdots\Gamma\left(\kappa_{n+q-2}, \kappa_{n+q}\right)\xi\left(\kappa_{n+q-2}, \kappa_{n+q-1}\right)\right)}\right)$$

$$\bigcirc\, ß\left(\kappa_0, \kappa_1, \frac{\hat{\mathrm{r}}}{(2)^{q-1}(\theta)^{n+q-1}\left(\Gamma\left(\kappa_{n+1}, \kappa_{n+q}\right)\Gamma\left(\kappa_{n+2}, \kappa_{n+q}\right)\Gamma\left(\kappa_{n+3}, \kappa_{n+q}\right)\cdots\Gamma\left(\kappa_{n+q-1}, \kappa_{n+q}\right)\right)}\right)$$

$$\bigcirc \cdots \bigcirc$$

Using (2), for $n \to +\infty,$ we deduce

$$\lim_{n\to+\infty} \Re\left(\kappa_n, \kappa_{n+q}, \hat{\mathrm{r}}\right) = 1 * 1 * \cdots * 1 = 1,$$

$$\lim_{n\to+\infty} \aleph\left(\kappa_n, \kappa_{n+q}, \hat{\mathrm{r}}\right) = 0 \bigcirc 0 \bigcirc \cdots \bigcirc 0 = 0$$

and

$$\lim_{n\to+\infty} ß\left(\kappa_n, \kappa_{n+q}, \hat{\mathrm{r}}\right) = 0 \bigcirc 0 \bigcirc \cdots \bigcirc 0 = 0.$$

i.e., $\{\kappa_n\}$ is a Cauchy sequence. Since $(\mathfrak{C}, \Re, \aleph, ß, *, \bigcirc)$ is a complete neutrosophic double controlled metric space, there exists

$$\lim_{n\to+\infty} \kappa_n = \kappa.$$

Now look into the fact that κ is a fixed point of þ, utilizing (v), (x), (xv) and (2), we get

$$\Re(\kappa, þ\kappa, \hat{\mathrm{r}}) \geq \Re\left(\kappa, \kappa_{n+1}, \frac{\hat{\mathrm{r}}}{2(\xi(\kappa, \kappa_{n+1}))}\right) * \Re\left(\kappa_{n+1}, þ\kappa, \frac{\hat{\mathrm{r}}}{2(\Gamma(\kappa_{n+1}, þ\kappa))}\right)$$

$$\Re(\kappa, þ\kappa, \hat{\mathrm{r}}) \geq \Re\left(\kappa, \kappa_{n+1}, \frac{\hat{\mathrm{r}}}{2(\xi(\kappa, \kappa_{n+1}))}\right) * \Re\left(þ\kappa_n, þ\kappa, \frac{\hat{\mathrm{r}}}{2(\Gamma(\kappa_{n+1}, þ\kappa))}\right)$$

$$\Re(\kappa, þ\kappa, \hat{\mathrm{r}}) \geq \Re\left(\kappa, \kappa_{n+1}, \frac{\hat{\mathrm{r}}}{2(\xi(\kappa, \kappa_{n+1}))}\right) * \Re\left(\kappa_n, \kappa, \frac{\hat{\mathrm{r}}}{2\theta(\Gamma(\kappa_{n+1}, þ\kappa))}\right) \to 1 = 1 \text{ as } n \to +\infty,$$

$$\aleph(\kappa, þ\kappa, \hat{\mathrm{r}}) \leq \aleph\left(\kappa, \kappa_{n+1}, \frac{\hat{\mathrm{r}}}{2(\xi(\kappa, \kappa_{n+1}))}\right) \bigcirc \aleph\left(\kappa_{n+1}, þ\kappa, \frac{\hat{\mathrm{r}}}{2(\Gamma(\kappa_{n+1}, þ\kappa))}\right)$$

$$\aleph(\kappa, þ\kappa, \hat{\mathrm{r}}) \leq \aleph\left(\kappa, \kappa_{n+1}, \frac{\hat{\mathrm{r}}}{2(\xi(\kappa, \kappa_{n+1}))}\right) \bigcirc \aleph\left(þ\kappa_n, þ\kappa, \frac{\hat{\mathrm{r}}}{2(\Gamma(\kappa_{n+1}, þ\kappa))}\right)$$

$$\aleph(\kappa, þ\kappa, \hat{\mathrm{r}}) \leq \aleph\left(\kappa, \kappa_{n+1}, \frac{\hat{\mathrm{r}}}{2(\xi(\kappa, \kappa_{n+1}))}\right) \bigcirc \aleph\left(\kappa_n, \kappa, \frac{\hat{\mathrm{r}}}{2\theta(\Gamma(\kappa_{n+1}, þ\kappa))}\right) \to 0 \bigcirc 0 = 0$$

as $n \to +\infty$, and

$$ß(\kappa, þ\kappa, \hat{\mathrm{r}}) \leq ß\left(\kappa, \kappa_{n+1}, \frac{\hat{\mathrm{r}}}{2(\xi(\kappa, \kappa_{n+1}))}\right) \bigcirc ß\left(\kappa_{n+1}, þ\kappa, \frac{\hat{\mathrm{r}}}{2(\Gamma(\kappa_{n+1}, þ\kappa))}\right)$$

$$ß(\kappa, þ\kappa, \hat{\mathrm{r}}) \leq ß\left(\kappa, \kappa_{n+1}, \frac{\hat{\mathrm{r}}}{2(\xi(\kappa, \kappa_{n+1}))}\right) \bigcirc ß\left(þ\kappa_n, þ\kappa, \frac{\hat{\mathrm{r}}}{2(\Gamma(\kappa_{n+1}, þ\kappa))}\right)$$

$$ß(\kappa, þ\kappa, \hat{\mathrm{r}}) \leq ß\left(\kappa, \kappa_{n+1}, \frac{\hat{\mathrm{r}}}{2(\xi(\kappa, \kappa_{n+1}))}\right) \bigcirc ß\left(\kappa_n, \kappa, \frac{\hat{\mathrm{r}}}{2\theta(\Gamma(\kappa_{n+1}, þ\kappa))}\right) \to 0 \bigcirc 0 = 0$$

as $n \to +\infty$. Hence, $þ\kappa = \kappa$.

Now, we examine the uniqueness. Let $þñ = ñ$ for some $ñ \in \mathfrak{C}$, then

$$1 \geq \Re(ñ, \kappa, \hat{\mathrm{r}}) = \Re(þñ, þ\kappa, \hat{\mathrm{r}}) \geq \Re\left(ñ, \kappa, \frac{\hat{\mathrm{r}}}{\theta}\right) = \Re\left(þñ, þ\kappa, \frac{\hat{\mathrm{r}}}{\theta}\right)$$

$$\geq \Re\left(ñ, \kappa, \frac{\hat{\mathrm{r}}}{\theta^2}\right) \geq \cdots \geq \Re\left(ñ, \kappa, \frac{\hat{\mathrm{r}}}{\theta^n}\right) \to 1 \text{ as } n \to +\infty,$$

$$0 \leq \aleph(ñ, \kappa, \hat{\mathrm{r}}) = \aleph(þñ, þ\kappa, \hat{\mathrm{r}}) \leq \aleph\left(ñ, \kappa, \frac{\hat{\mathrm{r}}}{\theta}\right) = \aleph\left(þñ, þ\kappa, \frac{\hat{\mathrm{r}}}{\theta}\right)$$

$$\leq \aleph\left(ñ, \kappa, \frac{\hat{\mathrm{r}}}{\theta^2}\right) \leq \cdots \leq \aleph\left(ñ, \kappa, \frac{\hat{\mathrm{r}}}{\theta^n}\right) \to 0 \text{ as } n \to +\infty,$$

and

$$0 \leq ß(ñ, \kappa, \hat{\mathrm{r}}) = ß(þñ, þ\kappa, \hat{\mathrm{r}}) \leq ß\left(ñ, \kappa, \frac{\hat{\mathrm{r}}}{\theta}\right) = ß\left(þñ, þ\kappa, \frac{\hat{\mathrm{r}}}{\theta}\right)$$

$$\leq ß\left(ñ, \kappa, \frac{þ}{\theta^2}\right) \leq \cdots \leq ß\left(ñ, \kappa, \frac{þ}{\theta^n}\right) \to 0 \text{ as } n \to +\infty,$$

by using (iii), (viii) and (xiii), $\kappa = ñ$. $\square$

Corollary 1. *Suppose* $(\mathfrak{C}, \Re, \aleph, ß, *, \bigcirc)$ *is a complete neutrosophic double controlled metric space in the company of* $\xi, \Gamma : \mathfrak{C} \times \mathfrak{C} \to [1, +\infty)$ *with* $0 < \theta < 1$ *and suppose that*

$$\lim_{\hat{\mathrm{r}} \to +\infty} \Re(\kappa, \mathrm{N}, \hat{\mathrm{r}}) = 1, \lim_{\hat{\mathrm{r}} \to +\infty} \aleph(\kappa, \mathrm{N}, \hat{\mathrm{r}}) = 0 \text{ and } \lim_{\hat{\mathrm{r}} \to +\infty} ß(\kappa, \mathrm{N}, \hat{\mathrm{r}}) = 0$$

for all $\kappa, \mathrm{N} \in \mathfrak{C}$ *and* $\hat{\mathrm{r}} > 0$. *Let* $\text{þ} : \mathfrak{C} \to \mathfrak{C}$ *be a mapping satisfying*

$$\Re(\text{þ}\kappa, \text{þ}\mathrm{N}, \theta\hat{\mathrm{r}}) \geq \min\{\Re(\kappa, \mathrm{N}, \hat{\mathrm{r}}), \Re(\kappa, \text{þ}\kappa, \hat{\mathrm{r}}), \Re(\mathrm{N}, \text{þ}\mathrm{N}, \hat{\mathrm{r}})\},$$

$$\aleph(\text{þ}\kappa, \text{þ}\mathrm{N}, \theta\hat{\mathrm{r}}) \leq \min\{\aleph(\kappa, \mathrm{N}, \hat{\mathrm{r}}), \aleph(\kappa, \text{þ}\kappa, \hat{\mathrm{r}}), \aleph(\mathrm{N}, \text{þ}\mathrm{N}, \hat{\mathrm{r}})\}$$

$$\text{and } ß(\text{þ}\kappa, \text{þ}\mathrm{N}, \theta\hat{\mathrm{r}}) \leq \min\{ß(\kappa, \mathrm{N}, \hat{\mathrm{r}}), ß(\kappa, \text{þ}\kappa, \hat{\mathrm{r}}), ß(\mathrm{N}, \text{þ}\mathrm{N}, \hat{\mathrm{r}})\}$$

for all $\kappa, \mathrm{N} \in \mathfrak{C}$ *and* $\hat{\mathrm{r}} > 0$. *Then,* þ *has a unique fixed point.*

Proof. Easy to prove by using Theorem 1 and Lemma 2. □

Definition 10. *Let* $(\mathfrak{C}, \Re, \aleph, ß, *, \bigcirc)$ *be a neutrosophic double controlled metric space. A map* $\text{þ} : \mathfrak{C} \to \mathfrak{C}$ *is an ND-controlled contraction if there exists* $0 < \theta < 1$, *such that*

$$\frac{1}{\Re(\text{þ}\kappa, \text{þ}\mathrm{N}, \hat{\mathrm{r}})} - 1 \leq \theta\left[\frac{1}{\Re(\kappa, \mathrm{N}, \hat{\mathrm{r}})} - 1\right] \tag{5}$$

$$\aleph(\text{þ}\kappa, \text{þ}\mathrm{N}, \hat{\mathrm{r}}) \leq \theta\aleph(\kappa, \mathrm{N}, \hat{\mathrm{r}}), \tag{6}$$

and

$$ß(\text{þ}\kappa, \text{þ}\mathrm{N}, \hat{\mathrm{r}}) \leq \theta ß(\kappa, \mathrm{N}, \hat{\mathrm{r}}), \tag{7}$$

for all $\kappa, \mathrm{N} \in \mathfrak{C}$ *and* $\hat{\mathrm{r}} > 0$.

Now, we prove the theorem for ND-controlled contraction.

Theorem 3. *Let* $(\mathfrak{C}, \Re, \aleph, ß, *, \bigcirc)$ *be a complete neutrosophic double controlled metric space with* $\xi, \Gamma : \mathfrak{C} \times \mathfrak{C} \to [1, +\infty)$ *and suppose that*

$$\lim_{\hat{\mathrm{r}} \to +\infty} \Re(\kappa, \mathrm{N}, \hat{\mathrm{r}}) = 1, \lim_{\hat{\mathrm{r}} \to +\infty} \aleph(\kappa, \mathrm{N}, \hat{\mathrm{r}}) = 0 \text{ and } \lim_{\hat{\mathrm{r}} \to +\infty} ß(\kappa, \mathrm{N}, \hat{\mathrm{r}}) = 0 \tag{8}$$

for all $\kappa, \mathrm{N} \in \mathfrak{C}$ *and* $\hat{\mathrm{r}} > 0$. *Let* $\text{þ} : \mathfrak{C} \to \mathfrak{C}$ *be a ND-controlled contraction. Further, suppose that for an arbitrary* $\kappa_0 \in \mathfrak{C}$, *and* $n, q \in \mathbb{N}$, *where* $\kappa_n = \text{þ}^n\kappa_0 = \text{þ}\kappa_{n-1}$. *Then,* þ *has a unique fixed point.*

Proof. Let κ_0 be a point of $\mathfrak{C}$ and define a sequence κ_n by $\kappa_n = \text{þ}^n\kappa_0 = \text{þ}\kappa_{n-1}$, $n \in \mathbb{N}$. By using (5), (6) and (7) for all $\hat{\mathrm{r}} > 0$, $n > q$, we deduce

$$\frac{1}{\Re(\kappa_n, \kappa_{n+1}, \hat{\mathrm{r}})} - 1 = \frac{1}{\Re(\text{þ}\kappa_{n-1}, \kappa_n, \hat{\mathrm{r}})} - 1$$

$$\leq \theta\left[\frac{1}{\Re(\kappa_{n-1}, \kappa_n, \hat{\mathrm{r}})} - 1\right] = \frac{\theta}{\Re(\kappa_{n-1}, \kappa_n, \hat{\mathrm{r}})} - \theta$$

$$\Rightarrow \frac{1}{\Re(\kappa_n, \kappa_{n+1}, \hat{\mathrm{r}})} \leq \frac{\theta}{\Re(\kappa_{n-1}, \kappa_n, \hat{\mathrm{r}})} + (1-\theta)$$

$$\leq \frac{\theta^2}{\Re(\kappa_{n-2}, \kappa_{n-1}, \hat{\mathrm{r}})} + \theta(1-\theta) + (1-\theta)$$

Carrying on in this manner, we deduce

$$\frac{1}{\Re(\kappa_n, \kappa_{n+1}, \hat{\mathrm{r}})} \leq \frac{\theta^n}{\Re(\kappa_0, \kappa_1, \hat{\mathrm{r}})} + \theta^{n-1}(1-\theta) + \theta^{n-2}(1-\theta) + \cdots + \theta(1-\theta) + (1-\theta)$$
$$\leq \frac{\theta^n}{\Re(\kappa_0, \kappa_1, \hat{\mathrm{r}})} + \left(\theta^{n-1} + \theta^{n-2} + \cdots + 1\right)(1-\theta) \leq \frac{\theta^n}{\Re(\kappa_0, \kappa_1, \hat{\mathrm{r}})} + (1-\theta^n)$$

We obtain

$$\frac{1}{\frac{\theta^n}{\Re(\kappa_0,\kappa_1,\hat{\mathrm{r}})}+(1-\theta^n)} \leq \Re(\kappa_n,\kappa_{n+1},\hat{\mathrm{r}}) \tag{9}$$

$$\begin{aligned}&\aleph(\kappa_n,\kappa_{n+1},\hat{\mathrm{r}}) = \aleph(\text{þ}\kappa_{n-1},\kappa_n,\hat{\mathrm{r}}) \leq \theta\aleph(\kappa_{n-1},\kappa_n,\hat{\mathrm{r}}) = \aleph(\text{þ}\kappa_{n-2},\kappa_{n-1},\hat{\mathrm{r}})\\ &\leq \theta^2\aleph(\kappa_{n-2},\kappa_{n-1},\hat{\mathrm{r}}) \leq \cdots \leq \theta^n\aleph(\kappa_0,\kappa_1,\hat{\mathrm{r}})\end{aligned} \tag{10}$$

and

$$\begin{aligned}&\text{ß}(\kappa_n,\kappa_{n+1},\hat{\mathrm{r}}) = \text{ß}(\text{þ}\kappa_{n-1},\kappa_n,\hat{\mathrm{r}}) \leq \theta\text{ß}(\kappa_{n-1},\kappa_n,\hat{\mathrm{r}}) = \text{ß}(\text{þ}\kappa_{n-2},\kappa_{n-1},\hat{\mathrm{r}})\\ &\leq \theta^2\text{ß}(\kappa_{n-2},\kappa_{n-1},\hat{\mathrm{r}}) \leq \cdots \leq \theta^n\text{ß}(\kappa_0,\kappa_1,\hat{\mathrm{r}})\end{aligned} \tag{11}$$

for any $q \in \mathbb{N}$, using (v), (x) and (xv), we deduce

$$\begin{aligned}
&\Re(\kappa_n,\kappa_{n+q},\hat{\mathrm{r}}) \geq \Re\left(\kappa_n,\kappa_{n+1},\frac{\hat{\mathrm{r}}}{2(\xi(\kappa_n,\kappa_{n+1}))}\right) * \Re\left(\kappa_{n+1},\kappa_{n+q},\frac{\hat{\mathrm{r}}}{2\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\right)}\right)\\
&\geq \Re\left(\kappa_n,\kappa_{n+1},\frac{\hat{\mathrm{r}}}{2(\xi(\kappa_n,\kappa_{n+1}))}\right) * \Re\left(\kappa_{n+1},\kappa_{n+2},\frac{\hat{\mathrm{r}}}{(2)^2\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\xi(\kappa_{n+1},\kappa_{n+2})\right)}\right)\\
&*\Re\left(\kappa_{n+2},\kappa_{n+q},\frac{\hat{\mathrm{r}}}{(2)^2\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\right)}\right)\\
&\geq \Re\left(\kappa_n,\kappa_{n+1},\frac{\hat{\mathrm{r}}}{2(\xi(\kappa_n,\kappa_{n+1}))}\right) * \Re\left(\kappa_{n+1},\kappa_{n+2},\frac{\hat{\mathrm{r}}}{(2)^2\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\xi(\kappa_{n+1},\kappa_{n+2})\right)}\right)\\
&*\Re\left(\kappa_{n+2},\kappa_{n+3},\frac{\hat{\mathrm{r}}}{(2)^3\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\xi(\kappa_{n+2},\kappa_{n+3})\right)}\right)\\
&*\Re\left(\kappa_{n+3},\kappa_{n+q},\frac{\hat{\mathrm{r}}}{(2)^3\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+3},\kappa_{n+q}\right)\right)}\right)
\end{aligned}$$

$$\begin{aligned}
\Re(\kappa_n,\kappa_{n+q},\hat{\mathrm{r}}) &\geq \Re\left(\kappa_n,\kappa_{n+1},\frac{\hat{\mathrm{r}}}{2(\xi(\kappa_n,\kappa_{n+1}))}\right)\\
&* \Re\left(\kappa_{n+1},\kappa_{n+2},\frac{\hat{\mathrm{r}}}{(2)^2\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\xi(\kappa_{n+1},\kappa_{n+2})\right)}\right)\\
&* \Re\left(\kappa_{n+2},\kappa_{n+3},\frac{\hat{\mathrm{r}}}{(2)^3\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\xi(\kappa_{n+2},\kappa_{n+3})\right)}\right)\\
&* \Re\left(\kappa_{n+3},\kappa_{n+4},\frac{\hat{\mathrm{r}}}{(2)^4\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+3},\kappa_{n+q}\right)\xi(\kappa_{n+3},\kappa_{n+4})\right)}\right) * \cdots *\\
&\Re\left(\kappa_{n+q-2},\kappa_{n+q-1},\frac{\hat{\mathrm{r}}}{(2)^{q-1}\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\cdots\Gamma\left(\kappa_{n+q-2},\kappa_{n+q}\right)\xi\left(\kappa_{n+q-2},\kappa_{n+q-1}\right)\right)}\right)\\
&* \Re\left(\kappa_{n+q-1},\kappa_{n+q},\frac{\hat{\mathrm{r}}}{(2)^{q-1}\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+3},\kappa_{n+q}\right)\cdots\Gamma\left(\kappa_{n+q-1},\kappa_{n+q}\right)\right)}\right),
\end{aligned}$$

$$\aleph(\kappa_n,\kappa_{n+q},\hat{\mathrm{r}}) \leq \aleph\left(\kappa_n,\kappa_{n+1},\frac{\hat{\mathrm{r}}}{2(\xi(\kappa_n,\kappa_{n+1}))}\right) \bigcirc \aleph\left(\kappa_{n+1},\kappa_{n+q},\frac{\hat{\mathrm{r}}}{2\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\right)}\right)$$

$$\begin{aligned}
\aleph(\kappa_n,\kappa_{n+q},\hat{\mathrm{r}}) &\leq \aleph\left(\kappa_n,\kappa_{n+1},\frac{\hat{\mathrm{r}}}{2(\xi(\kappa_n,\kappa_{n+1}))}\right)\\
&\bigcirc\aleph\left(\kappa_{n+1},\kappa_{n+2},\frac{\hat{\mathrm{r}}}{(2)^2\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\xi(\kappa_{n+1},\kappa_{n+2})\right)}\right)\\
&\bigcirc\aleph\left(\kappa_{n+2},\kappa_{n+q},\frac{\hat{\mathrm{r}}}{(2)^2\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\right)}\right)\\
&\leq \aleph\left(\kappa_n,\kappa_{n+1},\frac{\hat{\mathrm{r}}}{2(\xi(\kappa_n,\kappa_{n+1}))}\right) \bigcirc \aleph\left(\kappa_{n+1},\kappa_{n+2},\frac{\hat{\mathrm{r}}}{(2)^2\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\xi(\kappa_{n+1},\kappa_{n+2})\right)}\right)\\
&\bigcirc\aleph\left(\kappa_{n+2},\kappa_{n+3},\frac{\hat{\mathrm{r}}}{(2)^3\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\xi(\kappa_{n+2},\kappa_{n+3})\right)}\right)\\
&\bigcirc\aleph\left(\kappa_{n+3},\kappa_{n+q},\frac{\hat{\mathrm{r}}}{(2)^3\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+3},\kappa_{n+q}\right)\right)}\right)
\end{aligned}$$

$$\aleph \leq \aleph\left(\kappa_n, \kappa_{n+1}, \frac{\hat{\mathrm{r}}}{2(\xi(\kappa_n,\kappa_{n+1}))}\right) \bigcirc \aleph\left(\kappa_{n+1}, \kappa_{n+2}, \frac{\hat{\mathrm{r}}}{(2)^2\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\xi(\kappa_{n+1},\kappa_{n+2})\right)}\right)$$
$$\bigcirc\aleph\left(\kappa_{n+2}, \kappa_{n+3}, \frac{\hat{\mathrm{r}}}{(2)^3\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\xi(\kappa_{n+2},\kappa_{n+3})\right)}\right)$$
$$\bigcirc\aleph\left(\kappa_{n+3}, \kappa_{n+4}, \frac{\hat{\mathrm{r}}}{(2)^4\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+3},\kappa_{n+q}\right)\xi(\kappa_{n+3},\kappa_{n+4})\right)}\right) \bigcirc \cdots \bigcirc$$
$$\aleph\left(\kappa_{n+q-2}, \kappa_{n+q-1}, \frac{\hat{\mathrm{r}}}{(2)^{q-1}\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\cdots\Gamma\left(\kappa_{n+q-2},\kappa_{n+q}\right)\xi\left(\kappa_{n+q-2},\kappa_{n+q-1}\right)\right)}\right)$$
$$\bigcirc\aleph\left(\kappa_{n+q-1}, \kappa_{n+q}, \frac{\hat{\mathrm{r}}}{(2)^{q-1}\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+3},\kappa_{n+q}\right)\cdots\Gamma\left(\kappa_{n+q-1},\kappa_{n+q}\right)\right)}\right)$$

and

$$ß\left(\kappa_n, \kappa_{n+q}, \hat{\mathrm{r}}\right) \leq ß\left(\kappa_n, \kappa_{n+1}, \frac{\hat{\mathrm{r}}}{2(\xi(\kappa_n,\kappa_{n+1}))}\right) \bigcirc ß\left(\kappa_{n+1}, \kappa_{n+q}, \frac{\hat{\mathrm{r}}}{2\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\right)}\right)$$

$$ß\left(\kappa_n, \kappa_{n+q}, \hat{\mathrm{r}}\right) \leq ß\left(\kappa_n, \kappa_{n+1}, \frac{\hat{\mathrm{r}}}{2(\xi(\kappa_n,\kappa_{n+1}))}\right)$$
$$\bigcirc ß\left(\kappa_{n+1}, \kappa_{n+2}, \frac{\hat{\mathrm{r}}}{(2)^2\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\xi(\kappa_{n+1},\kappa_{n+2})\right)}\right)$$
$$\bigcirc ß\left(\kappa_{n+2}, \kappa_{n+q}, \frac{\hat{\mathrm{r}}}{(2)^2\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\right)}\right)$$
$$\leq ß\left(\kappa_n, \kappa_{n+1}, \frac{\hat{\mathrm{r}}}{2(\xi(\kappa_n,\kappa_{n+1}))}\right) \bigcirc ß\left(\kappa_{n+1}, \kappa_{n+2}, \frac{\hat{\mathrm{r}}}{(2)^2\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\xi(\kappa_{n+1},\kappa_{n+2})\right)}\right)$$
$$\bigcirc ß\left(\kappa_{n+2}, \kappa_{n+3}, \frac{\hat{\mathrm{r}}}{(2)^3\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\xi(\kappa_{n+2},\kappa_{n+3})\right)}\right)$$
$$\bigcirc ß\left(\kappa_{n+3}, \kappa_{n+q}, \frac{\hat{\mathrm{r}}}{(2)^3\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+3},\kappa_{n+q}\right)\right)}\right)$$

$$ß\left(\kappa_n, \kappa_{n+q}, \hat{\mathrm{r}}\right) \leq ß\left(\kappa_n, \kappa_{n+1}, \frac{\hat{\mathrm{r}}}{2(\xi(\kappa_n,\kappa_{n+1}))}\right)$$
$$\bigcirc ß\left(\kappa_{n+1}, \kappa_{n+2}, \frac{\hat{\mathrm{r}}}{(2)^2\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\xi(\kappa_{n+1},\kappa_{n+2})\right)}\right)$$
$$\bigcirc ß\left(\kappa_{n+2}, \kappa_{n+3}, \frac{\hat{\mathrm{r}}}{(2)^3\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\xi(\kappa_{n+2},\kappa_{n+3})\right)}\right)$$
$$\bigcirc ß\left(\kappa_{n+3}, \kappa_{n+4}, \frac{\hat{\mathrm{r}}}{(2)^4\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+3},\kappa_{n+q}\right)\xi(\kappa_{n+3},\kappa_{n+4})\right)}\right) \bigcirc \cdots \bigcirc$$
$$ß\left(\kappa_{n+q-2}, \kappa_{n+q-1}, \frac{\hat{\mathrm{r}}}{(2)^{q-1}\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\cdots\Gamma\left(\kappa_{n+q-2},\kappa_{n+q}\right)\xi\left(\kappa_{n+q-2},\kappa_{n+q-1}\right)\right)}\right)$$
$$\bigcirc ß\left(\kappa_{n+q-1}, \kappa_{n+q}, \frac{\hat{\mathrm{r}}}{(2)^{q-1}\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+3},\kappa_{n+q}\right)\cdots\Gamma\left(\kappa_{n+q-1},\kappa_{n+q}\right)\right)}\right).$$

$$\Re\left(\kappa_n, \kappa_{n+q}, \hat{\mathrm{r}}\right) \geq \frac{1}{\frac{\theta^n}{\Re\left(\kappa_0,\kappa_1,\frac{\hat{\mathrm{r}}}{2\left(\xi\left(\kappa_n,\kappa_{n+1}\right)\right)}\right)}+(1-\theta^n)}$$
$$*\frac{1}{\frac{\theta^{n+1}}{\Re\left(\kappa_0,\kappa_1,\frac{\hat{\mathrm{r}}}{(2)^2\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\xi\left(\kappa_{n+1},\kappa_{n+2}\right)\right)}\right)}+\left(1-\theta^{n+1}\right)}$$
$$*\frac{1}{\frac{\theta^{n+2}}{\Re\left(\kappa_0,\kappa_1,\frac{\hat{\mathrm{r}}}{(2)^3\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\xi\left(\kappa_{n+2},\kappa_{n+3}\right)\right)}\right)}+(1-\theta^{n+2})} * \cdots *$$
$$\frac{1}{\frac{\theta^{n+q-2}}{\Re\left(\kappa_0,\kappa_1,\frac{\hat{\mathrm{r}}}{(2)^{q-1}\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\cdots\Gamma\left(\kappa_{n+q-2},\kappa_{n+q}\right)\xi\left(\kappa_{n+q-2},\kappa_{n+q-1}\right)\right)}\right)}+\left(1-\theta^{n+q-2}\right)}$$
$$*\frac{1}{\frac{\theta^{n+q-1}}{\Re\left(\kappa_0,\kappa_1,\frac{\hat{\mathrm{r}}}{(2)^{q-1}\left(\Gamma\left(\kappa_{n+1},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+2},\kappa_{n+q}\right)\Gamma\left(\kappa_{n+3},\kappa_{n+q}\right)\cdots\Gamma\left(\kappa_{n+q-1},\kappa_{n+q}\right)\right)}\right)}+\left(1-\theta^{n+q-1}\right)},$$

$$\aleph\left(\kappa_n, \kappa_{n+q}, \hat{\mathrm{r}}\right) \leq \theta^n \aleph\left(\kappa_0, \kappa_1, \frac{\hat{\mathrm{r}}}{2\left(\xi\left(\kappa_n, \kappa_{n+1}\right)\right)}\right)$$
$$\bigcirc\, \theta^{n+1}\aleph\left(\kappa_0, \kappa_1, \frac{\hat{\mathrm{r}}}{(2)^2\left(\Gamma\left(\kappa_{n+1}, \kappa_{n+q}\right)\xi\left(\kappa_{n+1}, \kappa_{n+2}\right)\right)}\right)$$
$$\bigcirc\, \theta^{n+2}\aleph\left(\kappa_0, \kappa_1, \frac{\hat{\mathrm{r}}}{(2)^3\left(\Gamma\left(\kappa_{n+1}, \kappa_{n+q}\right)\Gamma\left(\kappa_{n+2}, \kappa_{n+q}\right)\xi\left(\kappa_{n+2}, \kappa_{n+3}\right)\right)}\right) \bigcirc \cdots \bigcirc$$
$$\theta^{n+q-2}\aleph\left(\kappa_0, \kappa_1, \frac{\hat{\mathrm{r}}}{(2)^{q-1}\left(\Gamma\left(\kappa_{n+1}, \kappa_{n+q}\right)\Gamma\left(\kappa_{n+2}, \kappa_{n+q}\right)\cdots\Gamma\left(\kappa_{n+q-2}, \kappa_{n+q}\right)\xi\left(\kappa_{n+q-2}, \kappa_{n+q-1}\right)\right)}\right)$$
$$\bigcirc\, \theta^{n+q-1}\aleph\left(\kappa_0, \kappa_1, \frac{\hat{\mathrm{r}}}{(2)^{q-1}\left(\Gamma\left(\kappa_{n+1}, \kappa_{n+q}\right)\Gamma\left(\kappa_{n+2}, \kappa_{n+q}\right)\Gamma\left(\kappa_{n+3}, \kappa_{n+q}\right)\cdots\Gamma\left(\kappa_{n+q-1}, \kappa_{n+q}\right)\right)}\right)$$

and

$$\text{ß}\left(\kappa_n, \kappa_{n+q}, \hat{\mathrm{r}}\right) \leq \theta^n \text{ß}\left(\kappa_0, \kappa_1, \frac{\hat{\mathrm{r}}}{2\left(\xi\left(\kappa_n, \kappa_{n+1}\right)\right)}\right)$$
$$\bigcirc\, \theta^{n+1}\text{ß}\left(\kappa_0, \kappa_1, \frac{\hat{\mathrm{r}}}{(2)^2\left(\Gamma\left(\kappa_{n+1}, \kappa_{n+q}\right)\xi\left(\kappa_{n+1}, \kappa_{n+2}\right)\right)}\right)$$
$$\bigcirc\, \theta^{n+2}\text{ß}\left(\kappa_0, \kappa_1, \frac{\hat{\mathrm{r}}}{(2)^3\left(\Gamma\left(\kappa_{n+1}, \kappa_{n+q}\right)\Gamma\left(\kappa_{n+2}, \kappa_{n+q}\right)\xi\left(\kappa_{n+2}, \kappa_{n+3}\right)\right)}\right) \bigcirc \cdots \bigcirc$$
$$\theta^{n+q-2}\text{ß}\left(\kappa_0, \kappa_1, \frac{\hat{\mathrm{r}}}{(2)^{q-1}\left(\Gamma\left(\kappa_{n+1}, \kappa_{n+q}\right)\Gamma\left(\kappa_{n+2}, \kappa_{n+q}\right)\cdots\Gamma\left(\kappa_{n+q-2}, \kappa_{n+q}\right)\xi\left(\kappa_{n+q-2}, \kappa_{n+q-1}\right)\right)}\right)$$
$$\bigcirc\, \theta^{n+q-1}\text{ß}\left(\kappa_0, \kappa_1, \frac{\hat{\mathrm{r}}}{(2)^{q-1}\left(\Gamma\left(\kappa_{n+1}, \kappa_{n+q}\right)\Gamma\left(\kappa_{n+2}, \kappa_{n+q}\right)\Gamma\left(\kappa_{n+3}, \kappa_{n+q}\right)\cdots\Gamma\left(\kappa_{n+q-1}, \kappa_{n+q}\right)\right)}\right).$$

Therefore,

$$\lim_{n\to+\infty} \Re\left(\kappa_n, \kappa_{n+q}, \hat{\mathrm{r}}\right) = 1 * 1 * \cdots * = 1,$$
$$\lim_{n\to+\infty} \aleph\left(\kappa_n, \kappa_{n+q}, \hat{\mathrm{r}}\right) = 0 \bigcirc 0 \bigcirc \cdots \bigcirc 0 = 0,$$

and

$$\lim_{n\to+\infty} \text{ß}\left(\kappa_n, \kappa_{n+q}, \hat{\mathrm{r}}\right) = 0 \bigcirc 0 \bigcirc \cdots \bigcirc 0 = 0$$

i.e., $\{\kappa_n\}$ is a CS. Since $(\mathfrak{C}, \Re, \aleph, \text{ß}, *, \bigcirc)$ be a complete neutrosophic double controlled metric space, there exists

$$\lim_{n\to+\infty} \kappa_n = \kappa$$

Now, we examine that κ is a fixed point of þ, utilising (v), (x) and (xv), we get

$$\frac{1}{\Re(\text{þ}\kappa_n, \text{þ}\kappa, \hat{\mathfrak{r}})} - 1 \leq \theta\left[\frac{1}{\Re(\kappa_n, \kappa, \hat{\mathfrak{r}})} - 1\right] = \frac{\theta}{\Re(\kappa_n, \kappa, \hat{\mathfrak{r}})} - \theta$$

$$\Rightarrow \frac{1}{\frac{\theta}{\Re(\kappa_n,\kappa,\hat{\mathfrak{r}})} + (1-\theta)} \leq \Re(\text{þ}\kappa_n, \text{þ}\kappa, \hat{\mathfrak{r}}).$$

Using the above inequality, we obtain

$$\begin{aligned}
&\Re(\kappa, \text{þ}\kappa, \hat{\mathfrak{r}}) \geq \Re\left(\kappa, \kappa_{n+1}, \tfrac{\hat{\mathfrak{r}}}{2\xi(\kappa,\kappa_{n+1})}\right) * \Re\left(\kappa_{n+1}, \text{þ}\kappa, \tfrac{\hat{\mathfrak{r}}}{2\Gamma(\kappa_{n+1},\text{þ}\kappa)}\right) \\
&\geq \Re\left(\kappa, \kappa_{n+1}, \tfrac{\hat{\mathfrak{r}}}{2\xi(\kappa,\kappa_{n+1})}\right) * \Re\left(\text{þ}\kappa_n, \text{þ}\kappa, \tfrac{\hat{\mathfrak{r}}}{2\Gamma(\kappa_{n+1},\text{þ}\kappa)}\right) \\
&\geq \Re\left(\kappa_n, \kappa_{n+1}, \tfrac{\hat{\mathfrak{r}}}{(2\xi(\kappa,\kappa_{n+1}))}\right) * \frac{1}{\frac{\theta}{\Re\left(\kappa_n,\kappa,\frac{\hat{\mathfrak{r}}}{2\Gamma(\kappa_{n+1},\text{þ}\kappa)}\right)} + (1-\theta)} \to 1 * 1 = 1 \text{ as } n \to +\infty \\
&\aleph(\kappa, \text{þ}\kappa, \hat{\mathfrak{r}}) \leq \aleph\left(\kappa, \kappa_{n+1}, \tfrac{\hat{\mathfrak{r}}}{2\xi(\kappa,\kappa_{n+1})}\right) \bigcirc \aleph\left(\kappa_{n+1}, \text{þ}\kappa, \tfrac{\hat{\mathfrak{r}}}{2\Gamma(\kappa_{n+1},\text{þ}\kappa)}\right) \\
&\leq \aleph\left(\kappa, \kappa_{n+1}, \tfrac{\hat{\mathfrak{r}}}{2\xi(\kappa,\kappa_{n+1})}\right) \bigcirc \aleph\left(\hat{\mathfrak{r}}\kappa_n, \hat{\mathfrak{r}}\kappa, \tfrac{\hat{\mathfrak{r}}}{2\Gamma(\kappa_{n+1},\text{þ}\kappa)}\right) \\
&\leq \aleph\left(\kappa_n, \kappa_{n+1}, \tfrac{\hat{\mathfrak{r}}}{2\xi(\kappa,\kappa_{n+1})}\right) \bigcirc \theta\aleph\left(\kappa_n, \kappa, \tfrac{\hat{\mathfrak{r}}}{2\Gamma(\kappa_{n+1},\text{þ}\kappa)}\right) \to 0 \bigcirc 0 = 0 \text{ as } n \to +\infty
\end{aligned}$$

and

$$\begin{aligned}
&\text{ß}(\kappa, \text{þ}\kappa, \hat{\mathfrak{r}}) \leq \text{ß}\left(\kappa, \kappa_{n+1}, \tfrac{\hat{\mathfrak{r}}}{2\xi(\kappa,\kappa_{n+1})}\right) \bigcirc \text{ß}\left(\kappa_{n+1}, \text{þ}\kappa, \tfrac{\hat{\mathfrak{r}}}{2\Gamma(\kappa_{n+1},\text{þ}\kappa)}\right) \\
&\leq \text{ß}\left(\kappa, \kappa_{n+1}, \tfrac{\hat{\mathfrak{r}}}{2\xi(\kappa,\kappa_{n+1})}\right) \bigcirc \text{ß}\left(\text{þ}\kappa_n, \text{þ}\kappa, \tfrac{\hat{\mathfrak{r}}}{2\Gamma(\kappa_{n+1},\text{þ}\kappa)}\right) \\
&\leq \text{ß}\left(\kappa_n, \kappa_{n+1}, \tfrac{\hat{\mathfrak{r}}}{2\xi(\kappa,\kappa_{n+1})}\right) \bigcirc \theta\text{ß}\left(\kappa_n, \kappa, \tfrac{\hat{\mathfrak{r}}}{2\Gamma(\kappa_{n+1},\text{þ}\kappa)}\right) \to 0 \bigcirc 0 = 0 \text{ as } n \to +\infty.
\end{aligned}$$

Hence, $\text{þ}\kappa = \kappa$. Now, we examine the uniqueness. Let $\text{þ}\tilde{n} = \tilde{n}$ for some $\tilde{n} \in \mathfrak{C}$, then

$$\begin{aligned}
&\tfrac{1}{\Re(\kappa,\tilde{n},\hat{\mathfrak{r}})} - 1 = \tfrac{1}{\Re(\text{þ}\kappa,\text{þ}\tilde{n},\hat{\mathfrak{r}})} - 1 \\
&\leq \theta\left[\tfrac{1}{\Re(\kappa,\tilde{n},\hat{\mathfrak{r}})} - 1\right] < \tfrac{1}{\Re(\kappa,\tilde{n},\hat{\mathfrak{r}})} - 1
\end{aligned}$$

is a contradiction,

$$\aleph(\kappa, \tilde{n}, \hat{\mathfrak{r}}) = \aleph(\text{þ}\kappa, \text{þ}\tilde{n}, \hat{\mathfrak{r}}) \leq \theta\aleph(\kappa, \tilde{n}, \hat{\mathfrak{r}}) < \aleph(\kappa, \tilde{n}, \hat{\mathfrak{r}})$$

is a contradiction, and,

$$\text{ß}(\kappa, \tilde{n}, \hat{\mathfrak{r}}) = \text{ß}(\text{þ}\kappa, \text{þ}\tilde{n}, \hat{\mathfrak{r}}) \leq \theta\text{ß}(\kappa, \tilde{n}, \hat{\mathfrak{r}}) < \text{ß}(\kappa, \tilde{n}, \hat{\mathfrak{r}})$$

Is a contradiction. Therefore, we must have $\Re(\kappa, \tilde{n}, \hat{\mathfrak{r}}) = 1$, $\aleph(\kappa, \tilde{n}, \hat{\mathfrak{r}}) = 0$ and $\text{ß}(\kappa, \tilde{n}, \hat{\mathfrak{r}}) = 0$, hence, $\kappa = \tilde{n}$. □

Example 5. *Let* $\mathfrak{C} = [0,1]$ *and* $\xi, \Gamma : \mathfrak{C} \times \mathfrak{C} \to [1, +\infty)$ *be two non-comparable functions given by*

$$\xi(\kappa, \text{N}) = \begin{cases} 1 & if \quad \kappa = \text{N}, \\ \frac{1+\max\{\kappa,\text{N}\}}{1+\min\{\kappa,\text{N}\}} & if\ \kappa \neq \text{N} \end{cases}$$

and

$$\Gamma(\kappa, \text{N}) = \begin{cases} 1 & if \quad \kappa = \text{N}, \\ \frac{1+\max\{\kappa^2,\text{N}^2\}}{1+\min\{\kappa^2,\text{N}^2\}} & if\ \kappa \neq \text{N} \end{cases}.$$

Define $\Re, \aleph, \text{ß} : \mathfrak{C} \times \mathfrak{C} \times (0, +\infty) \to [0, 1]$ *as*

$$\Re(\kappa, \textsc{n}, \hat{\mathrm{r}}) = \frac{\hat{\mathrm{r}}}{\hat{\mathrm{r}} + |\kappa - \textsc{n}|^2}, \ \ \aleph(\kappa, \textsc{n}, \hat{\mathrm{r}}) = \frac{|\kappa - \textsc{n}|^2}{\hat{\mathrm{r}} + |\kappa - \textsc{n}|^2},$$

$$\aleph(\kappa, \textsc{n}, \hat{\mathrm{r}}) = \frac{|\kappa - \textsc{n}|^2}{\hat{\mathrm{r}}}$$

Then, $(\mathfrak{C}, \Re, \aleph, \text{ß}, *, \bigcirc)$ *is a complete neutrosophic double controlled metric space with continuous t-norm* $\hat{\mathrm{e}} * \bar{\mathrm{a}} = \hat{\mathrm{e}}\bar{\mathrm{a}}$ *and continuous t-conorm* $\hat{\mathrm{e}} \bigcirc \bar{\mathrm{a}} = \max\{\hat{\mathrm{e}}, \bar{\mathrm{a}}\}$.

Define $\text{þ} : \mathfrak{C} \to \mathfrak{C}$ by $\text{þ}(\kappa) = \frac{1-2^{-\kappa}}{3}$ *and take* $\theta \in \left[\frac{1}{2}, 1\right)$, *then*

$$\begin{aligned}
&\Re(\text{þ}\kappa, \text{þ}\textsc{n}, \theta\hat{\mathrm{r}}) = \Re\left(\tfrac{1-2^{-\kappa}}{3}, \tfrac{1-2^{-\textsc{n}}}{3}, \theta\hat{\mathrm{r}}\right) \\
&= \frac{\theta\hat{\mathrm{r}}}{\theta\hat{\mathrm{r}} + \left|\frac{1-2^{-\kappa}}{3} - \frac{1-2^{-\textsc{n}}}{3}\right|^2} = \frac{\theta\hat{\mathrm{r}}}{\theta\hat{\mathrm{r}} + \frac{|2^{-\kappa} - 2^{-\textsc{n}}|^2}{9}} \\
&\geq \frac{\theta\hat{\mathrm{r}}}{\theta\hat{\mathrm{r}} + \frac{|\kappa - \textsc{n}|^2}{9}} = \frac{9\theta\hat{\mathrm{r}}}{9\theta\hat{\mathrm{r}} + |\kappa - \textsc{n}|^2} \geq \frac{\hat{\mathrm{r}}}{\hat{\mathrm{r}} + |\kappa - \textsc{n}|^2} = \Re(\kappa, \textsc{n}, \hat{\mathrm{r}}), \\
&\aleph(\text{þ}\kappa, \text{þ}\textsc{n}, \theta\hat{\mathrm{r}}) = \aleph\left(\tfrac{1-2^{-\kappa}}{3}, \tfrac{1-2^{-\textsc{n}}}{3}, \theta\hat{\mathrm{r}}\right) \\
&= \frac{\left|\frac{1-2^{-\kappa}}{3} - \frac{1-2^{-\textsc{n}}}{3}\right|^2}{\theta\hat{\mathrm{r}} + \left|\frac{1-2^{-\kappa}}{3} - \frac{1-2^{-\textsc{n}}}{3}\right|^2} = \frac{\frac{|2^{-\kappa} - 2^{-\textsc{n}}|^2}{9}}{\theta\hat{\mathrm{r}} + \frac{|2^{-\kappa} - 2^{-\textsc{n}}|^2}{9}} \\
&= \frac{|2^{-\kappa} - 2^{-\textsc{n}}|^2}{9\theta\hat{\mathrm{r}} + |2^{-\kappa} - 2^{-\textsc{n}}|^2} \leq \frac{|\kappa - \textsc{n}|^2}{9\theta\hat{\mathrm{r}} + |\kappa - \textsc{n}|^2} \leq \frac{|\kappa - \textsc{n}|^2}{\hat{\mathrm{r}} + |\kappa - \textsc{n}|^2} = \aleph(\kappa, \textsc{n}, \hat{\mathrm{r}})
\end{aligned}$$

and

$$\begin{aligned}
&\text{ß}(\text{þ}\kappa, \text{þ}\textsc{n}, \theta\hat{\mathrm{r}}) = \text{ß}\left(\tfrac{1-2^{-\kappa}}{3}, \tfrac{1-2^{-\textsc{n}}}{3}, \theta\hat{\mathrm{r}}\right) \\
&= \frac{\left|\frac{1-2^{-\kappa}}{3} - \frac{1-2^{-\textsc{n}}}{3}\right|^2}{\theta\hat{\mathrm{r}}} = \frac{\frac{|2^{-\kappa} - 2^{-\textsc{n}}|^2}{9}}{\theta\hat{\mathrm{r}}} \\
&= \frac{|2^{-\kappa} - 2^{-\textsc{n}}|^2}{9\theta\hat{\mathrm{r}}} \leq \frac{|\kappa - \textsc{n}|^2}{9\theta\hat{\mathrm{r}}} \leq \frac{|\kappa - \textsc{n}|^2}{\hat{\mathrm{r}}} = \text{ß}(\kappa, \textsc{n}, \hat{\mathrm{r}}).
\end{aligned}$$

As a result, all of the conditions of Theorem 1 are met, and 0 is the only fixed point for þ.

4. Application to Fredholm Integral Equation

Suppose $\mathfrak{C} = C([\mathrm{c}, \ \mathrm{a}], \ \mathbb{R})$ is the set of real value continuous functions defined on $[\mathrm{c}, \ \mathrm{a}]$. Suppose the integral equation:

$$\kappa(\tau) = \Lambda(\tau) + \delta \int_{\mathrm{c}}^{\mathrm{a}} Л(\tau, \ v)\kappa(\tau)dv \text{ for } \tau, \ v \in [\mathrm{c}, \ \mathrm{a}] \tag{12}$$

where $\delta > 0, \ \Lambda(v)$ is a fuzzy function of $v : v \in [\mathrm{c}, \mathrm{a}]$ and $Л : C([\mathrm{c}, \ \mathrm{a}] \times \mathbb{R}) \to \mathbb{R}^+$. Define $\Re$ and $\aleph$ by means of

$$\Re(\kappa(\tau), \ \textsc{n}(\tau), \ \hat{\mathrm{r}}) = \sup_{\tau \in [\mathrm{c}, \ \mathrm{a}]} \frac{\hat{\mathrm{r}}}{\hat{\mathrm{r}} + |\kappa(\tau) - \textsc{n}(\tau)|^2} \text{ for all } \kappa, \ \textsc{n} \in \mathfrak{C} \text{ and } \hat{\mathrm{r}} > 0,$$

$$\aleph(\kappa(\tau), \ \textsc{n}(\tau), \ \hat{\mathrm{r}}) = 1 - \sup_{\tau \in [\mathrm{c}, \ \mathrm{a}]} \frac{\hat{\mathrm{r}}}{\hat{\mathrm{r}} + |\kappa(\tau) - \textsc{n}(\tau)|^2} \text{ for all } \kappa, \ \textsc{n} \in \mathfrak{C} \text{ and } \hat{\mathrm{r}} > 0,$$

and

$$\text{ß}(\kappa(\tau), \ \textsc{n}(\tau), \ \hat{\mathrm{r}}) = \sup_{\tau \in [\mathrm{c}, \ \mathrm{a}]} \frac{|\kappa(\tau) - \textsc{n}(\tau)|^2}{\hat{\mathrm{r}}} \ \text{ for all } \kappa, \ \textsc{n} \in \mathfrak{C} \text{ and } \hat{\mathrm{r}} > 0,$$

with continuous t-norm and continuous t-conorm define by $\hat{e} * \bar{a} = \hat{e}.\bar{a}$ and $\hat{e} \bigcirc \bar{a} = \max\{\hat{e}, \bar{a}\}$. Define $\xi, \Gamma : \mathfrak{C} \times \mathfrak{C} \to [1, +\infty)$ as

$$\xi(\kappa, \text{N}) = \begin{cases} 1 & if \quad \kappa = \text{N}; \\ \frac{1+\max\{\kappa, \text{N}\}}{\min\{\kappa, \text{N}\}} & if\ \kappa \neq \text{N} \neq 0 \end{cases};$$

$$\Gamma(\kappa, \text{N}) = \begin{cases} 1 & if \quad \kappa = \text{N}, \\ \frac{1+\max\{\kappa^2, \text{N}^2\}}{\min\{\kappa^2, \text{N}^2\}} & if\ \kappa \neq \text{N} \end{cases}.$$

Then $(\mathfrak{C}, \Re, \aleph, ß, *, \bigcirc)$ is a complete neutrosophic double controlled metric space.

Suppose that $|Л(\tau, v)\kappa(\tau) - Л(\tau, v)\text{N}(\tau)| \leq |\kappa(\tau) - \text{N}(\tau)|$ for $\kappa, \text{N} \in \mathfrak{C}$, $\theta \in (0, 1)$ and $\forall\ \tau, v \in [\text{N}, a]$. Also, let $Л(\tau, v)\left(\delta \int_c^a dv\right)^2 \leq \theta < 1$. Then, the integral Equation (12) has a unique solution.

Proof. Define $þ : \mathfrak{C} \to \mathfrak{C}$ by

$$þ\kappa(\tau) = \Lambda(\tau) + \delta \int_c^a Л(\tau, v)c(\tau)dv \text{ for all } \tau, v \in [c, a].$$

Now, for all $\kappa, \text{N} \in \mathfrak{C}$, we deduce

$$\begin{aligned}
&\Re(þ\kappa(\tau), þ\text{N}(\tau), \theta\,\hat{r}) = \sup_{\tau \in [c, a]} \frac{\theta\hat{r}}{\theta\hat{r} + \left|þ\kappa(\tau) - þ\text{N}(\tau)\right|^2} \\
&= \sup_{\tau \in [c, a]} \frac{\theta\hat{r}}{\theta\hat{r} + \left|\Lambda(\tau) + \delta \int_c^a Л(\tau, v)c(\tau)dv - \Lambda(\tau) - \delta \int_c^a Л(\tau, v)c(\tau)dv\right|^2} \\
&= \sup_{\tau \in [c, a]} \frac{\theta\hat{r}}{\theta\hat{r} + \left|\delta \int_c^a Л(\tau, v)c(\tau)dv - \delta \int_c^a Л(\tau, v)c(\tau)dv\right|^2} \\
&= \sup_{\tau \in [c, a]} \frac{\theta\hat{r}}{\theta\hat{r} + |Л(\tau, v)\kappa(\tau) - Л(\tau, v)\text{N}(\tau)|^2 \left(\delta \int_c^a dv\right)^2} \\
&\geq \sup_{\tau \in [c, a]} \frac{\hat{r}}{\hat{r} + |\kappa(\tau) - \text{N}(\tau)|^2} \\
&\geq \Re(\kappa(\tau), \text{N}(\tau), \hat{r}),
\end{aligned}$$

$$\begin{aligned}
&\aleph(þ\kappa(\tau), þ\text{N}(\tau), \theta\,\hat{r}) = 1 - \sup_{\tau \in [c, a]} \frac{\theta\hat{r}}{\theta\hat{r} + \left|þ\kappa(\tau) - þ\text{N}(\tau)\right|^2} \\
&= 1 - \sup_{\tau \in [c, a]} \frac{\theta\hat{r}}{\theta\hat{r} + \left|\Lambda(\tau) + \delta \int_c^a Л(\tau, v)c(\tau)dv - \Lambda(\tau) - \delta \int_c^a Л(\tau, v)c(\tau)dv\right|^2} \\
&= 1 - \sup_{\tau \in [c, a]} \frac{\theta\hat{r}}{\theta\hat{r} + \left|\delta \int_c^a Л(\tau, v)c(\tau)dv - \delta \int_c^a Л(\tau, v)c(\tau)dv\right|^2} \\
&= 1 - \sup_{\tau \in [c, a]} \frac{\theta\hat{r}}{\theta\hat{r} + |Л(\tau, v)\kappa(\tau) - Л(\tau, v)\text{N}(\tau)|^2 \left(\delta \int_c^a dv\right)^2} \\
&\leq 1 - \sup_{\tau \in [c, a]} \frac{\hat{r}}{\hat{r} + |\kappa(\tau) - \text{N}(\tau)|^2} \\
&\leq \aleph(\kappa(\tau), \text{N}(\tau), \hat{r}),
\end{aligned}$$

and

$$
\begin{aligned}
&ß(þ\kappa(\tau),\ C\text{N}(\tau),\theta\,\hat{\mathfrak{r}}) = \sup_{\tau\in[\mathsf{c},\,\mathsf{a}]} \frac{\left|þ\kappa(\tau)-þ\text{N}(\tau)\right|^2}{\theta\hat{\mathfrak{r}}} \\
&= \sup_{\tau\in[\mathsf{c},\,\mathsf{a}]} \frac{\left|\Lambda(\tau)+\delta\int_{\mathsf{c}}^{\mathsf{a}} Л(\tau,\,v)\mathsf{c}(\tau)dv-\Lambda(\tau)-\delta\int_{\mathsf{c}}^{\mathsf{a}} Л(\tau,\,v)\mathsf{c}(\tau)dv\right|^2}{\theta\hat{\mathfrak{r}}} \\
&= \sup_{\tau\in[\mathsf{c},\,\mathsf{a}]} \frac{\left|\delta\int_{\mathsf{c}}^{\mathsf{a}} Л(\tau,\,v)\mathsf{c}(\tau)dv-\delta\int_{\mathsf{c}}^{\mathsf{a}} Л(\tau,\,v)\mathsf{c}(\tau)dv\right|^2}{\theta\hat{\mathfrak{r}}} \\
&= \sup_{\tau\in[\mathsf{c},\,\mathsf{a}]} \frac{|Л(\tau,\,v)\kappa(\tau)-Л(\tau,\,v)\text{N}(\tau)|^2\left(\delta\int_{\mathsf{c}}^{\mathsf{a}} dv\right)^2}{\theta\hat{\mathfrak{r}}} \\
&\leq \sup_{\tau\in[\mathsf{c},\,\mathsf{a}]} \frac{|\kappa(\tau)-\text{N}(\tau)|^2}{\hat{\mathfrak{r}}} \\
&\leq ß(\kappa(\tau),\ \text{N}(\tau),\ \hat{\mathfrak{r}}).
\end{aligned}
$$

As a result, all of the conditions of Theorem 1 are satisfied and operator þ has a unique fixed point. This indicates that an integral Equation (12) has a unique solution. □

Example 6. *Assume the following non-linear integral equation*

$$\kappa(\tau) = |\sin\tau| + \frac{1}{8}\int_0^1 v\kappa(v)dv, \text{ for all } v \in [0,\ 1]$$

Then it has a solution in $\mathfrak{C}$.

Proof. Let $þ : \mathfrak{C} \to \mathfrak{C}$ be defined by

$$þ\kappa(\tau) = |\sin\tau| + \frac{1}{8}\int_0^1 v\kappa(v)dv,$$

and set $Л(\tau,\ v)\kappa(\tau) = \frac{1}{8}v\kappa(v)$ and $Л(\tau,\ v)\text{N}(\tau) = \frac{1}{8}v\text{N}(v)$, where $\kappa,\ \text{N} \in \mathfrak{C}$, and for all $\tau,\ v \in [0,1]$. Then we have

$$
\begin{aligned}
&|Л(\tau,\ v)\kappa(\tau) - Л(\tau,\ v)\text{N}(\tau)| = \left|\tfrac{1}{8}v\kappa(v) - \tfrac{1}{8}v\text{N}(v)\right| \\
&= \tfrac{v}{8}|\kappa(v) - \text{N}(v)| \leq |\kappa(v) - \text{N}(v)|.
\end{aligned}
$$

Furthermore, see that $\left(\frac{1}{8}\int_0^1 vdv\right)^2 = \left(\frac{1}{64}\left(\frac{(1)^2}{2} - \frac{(0)^2}{2}\right)\right)^2 = \frac{1}{256} = \theta \leq 1$, where $\delta = \frac{1}{8}$. Then, it is easy to see that all other conditions of the above application are easy to examine and the above problem has a solution in $\mathfrak{C}$. □

5. Conclusions

This paper introduced the concept of neutrosophic double controlled fuzzy metric spaces, as well as various new types of fixed-point theorems that can be proved in this novel environment. Furthermore, we offered a non-trivial example to show that the proposed solutions are viable. We have complemented our work with an application that shows how the developed approach outperforms the literature-based methods. Our conclusions and conceptions augment a generalized number of previously published results, since our structure is more general than the class of fuzzy and double controlled fuzzy spaces.

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