fractal and fractional

Article

Dynamical Behaviors of an Environmental Protection Expenses Model in Protected Areas with Two Delays

Jun He [1,2], Ping Yang [3,*] and Jinde Cao [4,5]

1 School of Energy and Environment, Southeast University, Nanjing 210096, China; jun.he@monash.edu
2 Center for Energy and Environment, Southeast University-Monash University Joint Graduate School and Joint Research Institute, Suzhou 215123, China
3 Department of Electronic Science and Applied Physics, Kunming University of Science and Technology, Kunming 650500, China
4 School of Mathematics, Southeast University, Nanjing 210096, China; jdcao@seu.edu.cn
5 Yonsei Frontier Lab, Yonsei University, Seoul 03722, Korea
* Correspondence: yangping@mail.ustc.edu.cn; Tel.:+86-158-1210-3309

Abstract: This paper investigates an environmental protection expenses model, which considers the relations between the visitors to the protected areas V, the quality of the environmental resource E, and the capital stock K. In this model, the total tourism income is used partly to increase the capital stock or as the environmental protection expenses. Two time delays are introduced into the number of visitors, since the visitors need time to respond the changes of the environment, and the environment will take time to respond to the input of money. Stability crossing curves in the plane of delays (τ_1, τ_2) are used to obtain the stable region of equilibrium. Numerical simulations represent the mutual transformation of the supercritical bifurcation and the subcritical bifurcation. Our model shows that under some parameter conditions, the share of tourism income η is related closely to the delay τ_1, while the capital stock and the environmental quality can be maintained persistently if the delay τ_1 is not too large.

Keywords: environmental protection expenses; two delayed model; dynamical behaviors; stability crossing curves

Citation: He, J.; Yang, P.; Cao, J. Dynamical Behaviors of an Environmental Protection Expenses Model in Protected Areas with Two Delays. *Fractal Fract.* **2022**, *6*, 323. https://doi.org/10.3390/fractalfract6060323

Academic Editor: Carlo Cattani

Received: 25 March 2022
Accepted: 3 June 2022
Published: 8 June 2022

1. Introduction

Protected areas are now expected to achieve an increasing conservation for social and economic objectives. The protection of Protected Areas (PAs) is always a project for researchers [1–3]. Recently, the models of environmental protection expenditures have attracted new attention because of the deterioration of environment in some PAs [4–8]. Russu [4] studied a three-variable model among visitors V, quality of environmental resource E and the capital stock K in the PAs:

$$\begin{cases} \dot{V}(t) = m_1 E(t) + m_2 K(t) - aV^2(t) \\ \dot{E}(t) = r(\bar{P} - E(t)) - (b - c\eta)V(t-\tau) \\ \dot{K}(t) = (1-\eta)V(t-\tau) - \delta K(t), \end{cases} \tag{1}$$

where $a, r, \bar{P}, b, c, \delta, m_1$, and m_2 are strictly positive constants. $a > 0$ represents the crowding influence; this means that the PA becomes less attractive when the number of tourists visiting the PA increases. $\bar{P}$ is the pollution stock of maximum tolerance, while b means the waste generated by every visitor, $0 < r < 1$ is a constant proportion of the pollution, c determines how much the environmental expenses increase the quality of the environment. Visitors impact negatively on the environmental resource, but environment and infrastructures are attractive for visitors; therefore, the manager of PA uses a part η of total tourism income to protect the environmental resource and the remaining part $1-\eta$ to increase the capital stock, where $0 \leq \eta < 1$. The depreciate rate of capital stock is δ. m_1, m_2 mean that the number of visitor is proportional to E and K , $\tau > 0$ means the delay from visitors for increasing capital stock and the quality of the environment. The parameters involved in this topic can also see [5–7].

In Reference [8], Caraballo et al. suggested a modified version of Russu's model as:

$$\begin{cases} \dot{V}(t) = m_1 E(t) + m_2 K(t) - aV^2(t) \\ \dot{E}(t) = r(\bar{P} - E(t)) - (b - c\eta)V(t-\tau)E(t) \\ \dot{K}(t) = (1-\eta)V(t-\tau) - \delta K(t). \end{cases} \quad (2)$$

The authors gave some remarks for Russu's model; they pointed out that there was something wrong in Russu [4].

In many subjects such as biology, epidemiology, ecology, chemistry, and physics, numerous engineering problems delays always occur. The models that have multiple delays are of great interest mathematically and scientifically [9–16].

The stability crossing curve is an effective tool to understand the stable region for a system with multiple delays and to comprehend the bifurcation behaviors. For instance, Hale and Huang [17] investigated the stable region for the two delay differential equations

$$\dot{x}(t) + ax(t) + bx(t-r) + cx(t-\sigma) = 0. \quad (3)$$

The authors described the stable region on the (r, σ) plane and pointed out that the stable region could be unbounded. Gu et al. [18] studied the stability crossing curve carefully for a special case of characteristic equation. Lin and Wang [19] used a different approach to extend the results of Gu et al. result to a general case. An et al. [20] studied the stability switching properties of a model with delay dependent parameters. Matsumto and Szidarovszky [21] considered a delayed Lotka–Volterra competition model with two delays and some symmetries; the stability crossing curves on which stability is switched to instability were investigated.

In this paper, we suggest a modified model of Caraballo et al. Considering that the public praise will affect the amount of visitors, but with a delay, $r_1 V(t-\tau_1)$ is introduced to the model. The crowding effect is considered as $aV^2(t-\tau_1)$. $m_1 E(t)$ in the first equation of (2) is changed into $bV(t-\tau_1)E(t)$ by considering the visitor's effect on this term, and the influence of the capital stock $m_2 K(t)$ to the visitors is not adopted in our model. In the second equation of (2), we consider that the environment resource has self-purification ability, so a term $r_2 E(t)$ is added. We hope that the pollution is not tolerable, so we let $\bar{P} = 0$. The term $\eta V(t-\tau)E(t)$ is changed into $\eta V(t-\tau)K(t)$, since we think in here the capital stock is more important to the change of the environment. So, we obtain the following model:

$$\begin{cases} \dot{V}(t) = r_1 V(t-\tau_1) - aV^2(t-\tau_1) + bV(t-\tau_1)E(t) \\ \dot{E}(t) = r_2 E(t) - cV(t-\tau_1)E(t) + \eta V(t-\tau_2)K(t) \\ \dot{K}(t) = (1-\eta)V(t-\tau_2) - \delta K(t), \end{cases} \quad (4)$$

where r_1 is the rate of effect of the public praise; visitors will increase if r_1 increases. r_2 is the self-purification ability of the environment. $a > 0$ represents the crowding effect, b is the rate of the visitors affected by environment resource, c means the waste generated by every visitor which is affected by environment resource, and $0 \leq \eta < 1$ means a share η of total revenues is used to protect the environment. Capital stock is depreciated at the rate δ. The increment of the visitors relies on the visitors at the time $t-\tau_1$ for their spread of public praise and the crowding effect, while the dynamic evolution of the environment and the capital stock rely on the contribution of visitors at the time $t-\tau_2$, where $\tau_1 \geq 0$ and $\tau_2 \geq 0$. We name τ_1 as the spread delay and τ_2 as the protecting delay.

We give the initial conditions of system (4) as:

$$V(\theta) = v_1(\theta), E(\theta) = v_2(\theta), K(\theta) = v_3(\theta), v_i(\theta) \geq 0, i = 1, 2, 3, -\tau \leq \theta \leq 0,$$

where $(v_1(\theta), v_2(\theta), v_3(\theta)) \in C([-\tau, 0], \mathbf{R}_+^3)$, $\tau = max[\tau_1, \tau_2]$. Then, according to the theorem on functional differential equations [22], system (4) has one and only one solution $(V(t), E(t), K(t))$ that satisfies the initial conditions. In this paper, we provide the theory of the stability crossing curves and apply it to model (4). By means of numerical simulations, we obtain the stable region of the equilibrium in the $\tau_1 - \tau_2$ plane. Bifurcation directions of the periodic solutions are determined by using the normal form and the center manifold theorem. Numerical simulations show how the equilibrium changes from stable to unstable and how the bifurcation direction changes from supercritical to subcritical and vice versa. Through the research of model (4), we find that the spread delay τ_1 we introduced to the model (4) is more important than τ_2 which is τ in [4,8] for the stability of equilibrium, since τ_1 needs to be on the left of the stability crossing curves. We find that the share η of the tourism user fees and the spread delay τ_1 are very important parameters in our discussion.

2. Equilibria and Stability Crossing Curves

By straightforward computation, system (4) has equilibrium $S_0 = (0, 0, 0)$, which is unstable, since there are always positive eigenvalues $\lambda_1 = r_1$ and $\lambda_2 = r_2$, while the characteristic equation of (4) at S_0 has no relation with delays τ_1, τ_2.

If

Hypothesis 1 (H1). $b\eta(1-\eta) - a\delta c > 0, \qquad r_2 > \frac{r_1 b\eta(1-\eta)}{(1-a^2)\delta},$

system (4) has $S_* = (V_*, E_*, K_*)$ as a positive equilibrium:

$$V_* = \frac{-\delta(ar_2 + cr_1) + \sqrt{\delta^2(ar_2 + cr_1)^2 + 4\delta r_1 r_2 (b\eta(1-\eta) - ac\delta)}}{2(b\eta(1-\eta) - ac\delta)},$$

$$E_* = \frac{aV_* - r_1}{b}, K_* = \frac{1-\eta}{\delta} V_*.$$

If

Hypothesis 2 (H2). $b\eta(1-\eta) - a\delta c < 0, \qquad \delta^2(ar_2 + cr_1)^2 + 4\delta r_1 r_2 (b\eta(1-\eta) - ac\delta) > 0,$

system (4) has $S_* = (V_*, E_*, K_*)$ as a positive equilibrium:

$$V_* = \frac{-\delta(ar_2 + cr_1) - \sqrt{\delta^2(ar_2 + cr_1)^2 + 4\delta r_1 r_2 (b\eta(1-\eta) - ac\delta)}}{2(b\eta(1-\eta) - ac\delta)},$$

$$E_* = \frac{aV_* - r_1}{b}, K_* = \frac{1-\eta}{\delta} V_*.$$

If $b\eta(1-\eta) - a\delta c = 0$, there is no positive equilibrium.

Let $u_1(t) = V(t) - V_*, u_2(t) = E(t) - E_*, u_3(t) = K(t) - K_*$, then (4) becomes

$$\begin{cases} \dot{u}_1(t) = & -aV_* u_1(t-\tau_1) + bV_* u_2(t) - au_1^2(t-\tau_1) + bu_1(t-\tau_1)u_2(t) \\ \dot{u}_2(t) = & -cE_* u_1(t-\tau_1) + \eta K_* u_1(t-\tau_2) + (r_2 - cV_*)u_2(t) + \eta V_* u_3(t) \\ & -cu_1(t-\tau_1)u_2(t) + \eta u_1(t-\tau_2)u_3(t) \\ \dot{u}_3(t) = & (1-\eta)u_1(t-\tau_2) - \delta u_3(t). \end{cases} \tag{5}$$

The corresponding characteristic equation of system (5) can be written by

$$\lambda^3 + \alpha_1\lambda^2 + \alpha_2\lambda + (\alpha_3\lambda^2 + \alpha_4\lambda + \alpha_5)e^{-\lambda\tau_1} + (\alpha_6\lambda + \alpha_7)e^{-\lambda\tau_2} = 0, \tag{6}$$

where

$$\begin{aligned} \alpha_1 &= cV_* - r_2 + \delta, \\ \alpha_2 &= (cV_* - r_2)\delta, \\ \alpha_3 &= a\,V_*, \\ \alpha_4 &= a\,V_*(cV_* - r_2) + aV_*\delta + bcV_*E_*, \\ \alpha_5 &= a\delta V_*(cV_* - r_2) + bc\delta V_*E_* \\ \alpha_6 &= -b\eta V_*K_*, \\ \alpha_7 &= -b\eta\delta V_*K_* - b\eta(1-\eta)V_*^2. \end{aligned} \tag{7}$$

When $\tau_1 = \tau_2 = 0$, by the criterion of Routh–Hurwitz, we have

Theorem 1. *Assume $\tau_1 = \tau_2 = 0$, (H1) (or (H2)) hold. If*

Hypothesis 3 (H3). $\alpha_1 + \alpha_3 > 0, \quad \alpha_5 + \alpha_7 > 0, \quad (\alpha_1 + \alpha_3)(\alpha_2 + \alpha_4 + \alpha_6) > \alpha_5 + \alpha_7$

is satisfied, then S_* is asymptotically locally stable.

Next, we investigate the distribution of the roots of (6). From Rouche theorem [23], as (τ_1, τ_2) vary continuously in $\mathbf{R}^2_+$, the roots of Equation (6) vary continuously, and the roots (counting multiplicity) can change their symbols of real parts if and only if they cross the imaginary axis [24].

We consider two situations: (I) $\tau_1 = \tau_2 = \tau > 0$, or (II) $\tau_1 > 0, \tau_2 > 0, \tau_1 \neq \tau_2$.

Case (I) $\tau_1 = \tau_2 = \tau > 0$.

Suppose that on the imaginary axis, system (6) has a root $i\omega(\omega > 0)$. Substituting it into (6), separating the imaginary part and the real parts, we have

$$\begin{cases} \alpha_1\omega^2 = \alpha_4\omega\sin\omega\tau + (-\alpha_3\omega^2 + \alpha_5)\cos\omega\tau, \\ \alpha_2\omega - \omega^3 = (-\alpha_3\omega^2 + \alpha_5)\sin\omega\tau - \alpha_4\omega\cos\omega\tau. \end{cases} \tag{8}$$

Then, we obtain

$$\omega^6 + (\alpha_1^2 - 2\alpha_2 - \alpha_3^2)\omega^4 + (\alpha_2^2 + 2\alpha_3\alpha_5 - \alpha_4^2)\omega^2 - \alpha_5^2 = 0. \tag{9}$$

Let $z = \omega^2, p = \alpha_1^2 - 2\alpha_2 - \alpha_3^2, q = \alpha_2^2 + 2\alpha_3\alpha_5 - \alpha_4^2, r = -\alpha_5^2$; then, (9) becomes

$$h(z) = z^3 + pz^2 + qz + r = 0. \tag{10}$$

We see from (10) that since $r < 0$, there is at least one positive root. We assume that there are three positive roots for generality, defined by z_1, z_2 and z_3. According to (8), we have

$$\tau_k^{(j)} = \frac{1}{\omega_k}\{\cos^{-1}[\frac{(\alpha_1\omega_k^2(\alpha_5 - \alpha_3\omega_k^2) + \alpha_4\omega_k^2(\omega_k^2 - \alpha_2)}{(\alpha_5 - \alpha_3\omega_k)^2 + \alpha_4^2\omega_k^2}] + 2j\pi\}, \tag{11}$$

where $j = 0, 1, \cdots$ and $k = 1, 2, 3$. Denote

$$\tau_0 = \tau_{k_0}^{(0)} = \min_{k\in\{1,2,3\}}\{\tau_k^{(0)}\}, \omega_0 = \omega_{k_0}. \tag{12}$$

Then, we know

Lemma 1. *When (H1) (or (H2)), (H3) hold, all roots of (6) have strictly negative real parts when $\tau \in [0, \tau_0)$. (6) has simple purely imaginary roots when $\tau = \tau_0$.*

Near $\tau = \tau_k^{(j)}$, consider $\lambda(\tau) = \sigma(\tau) + i\omega(\tau)$ as the root of (6) where $\sigma(\tau_k^{(j)}) = 0, \omega(\tau_k^{(j)}) = \omega_k, j = 0, 1, 2 \ldots, k = 1, 2, 3$. For the transversality, we know that

Lemma 2. *Suppose $z_k = \omega_k^2, k = 1, 2, 3$ and $h'(z_k) \neq 0$, then $[\frac{d\mathrm{Re}(\lambda)}{d\tau}]_{\tau=\tau_k^{(j)}} \neq 0$.*

According to Lemma 1 and Lemma 2, we have

Theorem 2. *When $\tau_1 = \tau_2 = \tau$, let τ_0 be denoted by (12), and assume that $(H1)(or(H2))$, $(H3)$ are satisfied,*

(i) If $\tau \in [0, \tau_0)$, then system (4) has a locally asymptotically stable positive equilibrium S_*.
(ii) If $\tau > \tau_0$ and $h'(z_k) \neq 0$, then for system (4), Hopf bifurcation will occur at S_* when $\tau = \tau_0$.

Case (II) $\tau_1 > 0, \tau_2 > 0, \tau_1 \neq \tau_2$
Characteristic Equation (6) can be rewritten as

$$P(\lambda, \tau_1, \tau_2) \equiv P_0(\lambda) + P_1(\lambda)e^{-\lambda\tau_1} + P_2(\lambda)e^{-\lambda\tau_2} = 0, \tag{13}$$

where

$$P_0(\lambda) = \lambda^3 + \alpha_1\lambda^2 + \alpha_2\lambda,$$
$$P_1(\lambda) = \alpha_3\lambda^2 + \alpha_4\lambda + \alpha_5,$$
$$P_2(\lambda) = \alpha_6\lambda + \alpha_7.$$

We can easily confirm that Equation (13) satisfies:

(I) $deg(P_0(\lambda)) \geq max\{deg(P_1(\lambda)), deg(P_2(\lambda))\}$;
(II) $P_0(0) + P_1(0) + P_2(0) = \alpha_5 + \alpha_7 \neq 0, \quad if (H3) holds$;
(III) $P_0(\lambda), P_1(\lambda), P_2(\lambda)$ have no common zeros;
(IV) $\lim_{\lambda\to\infty}(|P_1(\lambda)/P_0(\lambda)| + |P_2(\lambda)/P_0(\lambda)|) < 1$.

Lemma 3. *For each $\omega > 0, P_0(i\omega) \neq 0$, $P(\lambda, \tau_1, \tau_2) = 0$ has $\lambda = i\omega$ as its root for some $(\tau_1, \tau_2) \in R_+^2$ if and only if*

Hypothesis 4 (H4).

$$|P_0(i\omega)| \leq |P_1(i\omega)| + |P_2(i\omega)|, \tag{14}$$

and

$$-|P_0(i\omega)| \leq |P_1(i\omega)| - |P_2(i\omega)| \leq |P_0(i\omega)|.) \tag{15}$$

The proof of Lemma 3 can be found in [18].
Let

$$G1(\omega) = -|P_0(i\omega)| + |P_1(i\omega)| - |P_2(i\omega)|,$$

$$G2(\omega) = -|P_0(i\omega)| - |P_1(i\omega)| + |P_2(i\omega)|,$$

$$G3(\omega) = |P_0(i\omega)| - |P_1(i\omega)| - |P_2(i\omega)|,$$

then we know that $P(\lambda, \tau_1, \tau_2) = 0$ has $\lambda = i\omega$ as its solution if and only if $G1(\omega) \leq 0, G2(\omega) \leq 0, G3(\omega) \leq 0$ simultaneously.

Denote the crossing set Ω of all ω, which satisfy (14) and (15). For given $\omega \in \Omega, P_k(i\omega) \neq 0, k = 0, 1, 2$. From (14) and (15), we can find all of $(\tau_1(\omega), \tau_2(\omega))$:

$$\tau_1 = \tau_1^{m\pm}(\omega) = (\arg \frac{P_1(i\omega)}{P_0(i\omega)} + (2m-1)\pi \pm \psi_1)/\omega, m = m_0^{\pm}, m_0^{\pm}+1, \ldots, \quad (16)$$

$$\tau_2 = \tau_2^{n\pm}(\omega) = (\arg \frac{P_2(i\omega)}{P_0(i\omega)} + (2n-1)\pi \mp \psi_2)/\omega, n = n_0^{\pm}, n_0^{\pm}+1, \ldots, \quad (17)$$

where $\psi_1, \psi_2 \in [0, \pi]$ can be calculated as

$$\psi_1 = \cos^{-1}\{\frac{|P_0(i\omega)|^2 + |P_1(i\omega)|^2 - |P_2(i\omega)|^2}{2|P_0(i\omega)||P_1(i\omega)|}\}, \quad (18)$$

$$\psi_2 = \cos^{-1}\{\frac{|P_0(i\omega)|^2 - |P_1(i\omega)|^2 + |P_2(i\omega)|^2}{2|P_0(i\omega)||P_2(i\omega)|}\}, \quad (19)$$

m_0^+, m_0^-, n_0^+ and n_0^- are the smallest integers such that the corresponding $\tau_1^{m_0^+ +}, \tau_1^{m_0^- -}, \tau_2^{n_0^+ +}$, and $\tau_2^{n_0^- -}$ are non-negative.

Let $\omega_0 \in \Omega$, we can obtain $\tau_{10} = \tau_{10}(\omega_0), \tau_{20} = \tau_{20}(\omega_0)$ from (16) and (17).

Next, we discuss the transversality. Choose τ_2 to be the bifurcating parameter, and take the derivative of $\lambda(\tau_2)$ in (6); then, we obtain

$$\begin{aligned}
&\{3\lambda^2 + 2\alpha_1\lambda + \alpha_2 + [2\alpha_3\lambda + \alpha_4 - (\alpha_3\lambda^2 + \alpha_4\lambda + \alpha_5)\tau_1]e^{-\lambda\tau_1} \\
&+[\alpha_6 - (\alpha_6\lambda + \alpha_7)\tau_2]e^{-\lambda\tau_2}\}\tfrac{d\lambda}{d\tau_2} \\
&= \lambda(\alpha_6\lambda + \alpha_7)e^{-\lambda\tau_2},
\end{aligned}$$

Therefore,

$$\begin{aligned}
&[\tfrac{d\lambda}{d\tau_2}]^{-1} = \tfrac{(3\lambda^2+2\alpha_1\lambda+\alpha_2)e^{\lambda\tau_2}}{\lambda(\alpha_6\lambda+\alpha_7)} + \tfrac{[2\alpha_3\lambda+\alpha_4-(\alpha_3\lambda^2+\alpha_4\lambda+\alpha_5)\tau_1]e^{\lambda(\tau_2-\tau_1)}}{\lambda(\alpha_6\lambda+\alpha_7)} \\
&+\tfrac{\alpha_6}{\lambda(\alpha_6\lambda+\alpha_7)} - \tfrac{\tau_2}{\lambda}.
\end{aligned} \quad (20)$$

When $\tau_2 = \tau_{20}$, $\lambda = i\omega_0$, then

$$\begin{aligned}
&((3\lambda^2 + 2\alpha_1\lambda + \alpha_2)e^{\lambda\tau_2})_{\tau_2=\tau_{20}} = ((\alpha_2 - 3\omega_0^2)\cos(\omega_0\tau_{20}) - 2\alpha_1\omega_0\sin(\omega_0\tau_{20})) \\
&+i(2\alpha_1\omega_0\cos(\omega_0\tau_{20}) + (\alpha_2 - 3\omega_0^2)\sin(\omega_0\tau_{20})), \\
&([2\alpha_3\lambda + \alpha_4 - (\alpha_3\lambda^2 + \alpha_4\lambda + \alpha_5)\tau_1]e^{\lambda(\tau_2-\tau_1)})_{\tau_2=\tau_{20}} \\
&= (\alpha_4 + \tau_{10}\alpha_3\omega_0^2 - \tau_{10}\alpha_5)\cos(\omega(\tau_{20} - \tau_{10})) - (2\alpha_3 - \tau_{10}\alpha_4)\omega_0\sin(\omega_0(\tau_{20} - \tau_{10})) \\
&+i[(2\alpha_3 - \tau_{10}\alpha_4)\omega_0\cos(\omega_0(\tau_{20} - \tau_{10})) + (\alpha_4 + \tau_{10}\alpha_3\omega_0^2 - \tau_{10}\alpha_5)\sin(\omega_0(\tau_{20} - \tau_{10}))]. \\
&\lambda(\alpha_6\lambda + \alpha_7)_{\tau_2=\tau_{20}} = -\alpha_6\omega_0^2 + i\alpha_7\omega_0.
\end{aligned} \quad (21)$$

We have

$$Re[\tfrac{d\lambda}{d\tau_2}]^{-1}_{\tau_2=\tau_{20}} = \tfrac{M_1N_1+M_2N_2}{\alpha_6^2\omega_0^3+\alpha_7^2\omega_0}, \quad (22)$$

where

$$\begin{aligned}
M_1 = &\ -\alpha_6\omega_0, \quad M_2 = \alpha_7, \\
N_1 = &\ (\alpha_2 - 3\omega_0^2)\cos(\omega_0\tau_{20}) - 2\alpha_1\omega_0\sin(\omega_0\tau_{20}) + \alpha_6 + (\alpha_4 + \tau_{10}\alpha_3\omega_0^2 - \tau_{10}\alpha_5)\cos(\omega_0(\tau_{20} - \tau_{10})) \\
&-(2\alpha_3 - \tau_{10}\alpha_4)\omega_0\sin(\omega_0(\tau_{20} - \tau_{10})),
\end{aligned}$$

$$\begin{aligned}
N_2 = &\ 2\alpha_1\omega_0\cos(\omega_0\tau_{20}) + (\alpha_2 - 3\omega_0^2)\sin(\omega_0\tau_{20}) + (2\alpha_3 - \tau_{10}\alpha_4)\omega_0\cos(\omega_0(\tau_{20} - \tau_{10})) \\
&+(\alpha_4 + \tau_{10}\alpha_3\omega_0^2 - \tau_{10}\alpha_5)\sin(\omega_0(\tau_{20} - \tau_{10})).
\end{aligned}$$

Let

Hypothesis 5 (H5). $$M_1N_1 + M_2N_2 \neq 0.$$

Denote $T^j = \{(\tau_{10}^j(\omega), \tau_{20}^j(\omega)), \omega \in \Omega\}, j = 1, 2, \dots, p$ are p sections of continuous curves defined on Ω, T° is the internal region surrounded by $T = \bigcup_{j=1}^p T^j$ with coordinate axis $\tau_1 = 0$ and $\tau_2 = 0$. Then, we can obtain the following:

Theorem 3. *Assume that $(H1)$ (or $(H2)$), $(H3)$, $(H4)$ hold,*

(I) *If $(\tau_1, \tau_2) \in T^\circ$, then system (4) has a positive equilibrium S_* which is locally asymptotically stable.*
(II) *If (τ_1, τ_2) crossing T and $(H5)$ holds, then $(\tau_1, \tau_2) = (\tau_{10}, \tau_{20}) \in T$ is a critical point, system (4) has Hopf bifurcation at S_**

We denote continuous curve $T^j, j = 1, 2, \dots, p$ as stability crossing curves.

3. Hopf Bifurcation Direction and the Stability of Periodic Solution

In this section, we suppose $0 < \tau_1 < \tau_2$. If $0 < \tau_2 < \tau_1$; we can discuss it using the same method.

Let $\tau_1 = \tau_{10}, \tau_2 = \tau_{20} + \mu$, then system (5) has $\mu = 0$ as its bifurcation value. Let $t = \tau\bar{t}, u_i(\tau\bar{t}) = \bar{u_i}(\bar{t}), i = 1, 2, 3$ and omit "$-$" above, (5) can be expressed as

$$\dot{u}(t) = L_\mu u_t + f(\mu, u_t), \tag{23}$$

where $u(t) = (u_1(t), u_2(t), u_3(t))^\top \in \mathbf{R}^3$, $u_t(\theta) = u(t+\theta),\ \theta \in [-1, 0]$.

$$L_\mu\phi = (\tau_{20} + \mu)(A\phi(0) + B\phi(-\frac{\tau_1}{\tau_2}) + C\phi(-1)),\quad \phi(\theta) = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta))^\top, \tag{24}$$

$$\begin{aligned} f(\mu, \phi) &= (\tau_{20} + \mu)(f_1, f_2, f_3)^\top \\ &= (\tau_{20} + \mu)(-a\phi_1^2(-\tfrac{\tau_1}{\tau_2}) + b\phi_1(-\tfrac{\tau_1}{\tau_2})\phi_2(0), -c\phi_1(-\tfrac{\tau_1}{\tau_2})\phi_2(0) + \eta\phi_1(-1)\phi_3(0), 0)^\top, \end{aligned}$$

where

$$A = \begin{pmatrix} 0 & bV_* & 0 \\ 0 & r_2 - cV_* & \eta V_* \\ 0 & 0 & -\delta \end{pmatrix}, B = \begin{pmatrix} -aV_* & 0 & 0 \\ -cE_* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 & 0 \\ \eta K_* & 0 & 0 \\ 1 - \eta & 0 & 0 \end{pmatrix}.$$

By the Riesz representation theorem, for $\theta \in [-1, 0]$, we write

$$L_\mu\phi = \int_{-1}^0 d\rho(\theta, \mu)\phi(\theta), \tag{25}$$

where

$$\rho(\theta, \mu) = (\tau_{20} + \mu)(A\delta(\theta) + B\delta(\theta + \frac{\tau_1}{\tau_2}) + C\delta(\theta + 1)). \tag{26}$$

$\delta(\theta)$ satisfies $\delta(\theta) = 0$ if $\theta \neq 0$, and $\delta(0) = 1$.

Define

$$\mathcal{A}(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \theta \in [-1, 0), \\ \int_{-1}^0 d\rho(s, \mu)\phi(s), & \theta = 0, \end{cases}$$

and

$$R(\mu)\phi = \begin{cases} 0, & \theta \in [-1, 0), \\ f(\mu, \phi), & \theta = 0. \end{cases}$$

Then, (23) can be expressed as

$$\dot{u}_t = \mathcal{A}(\mu)u_t + R(\mu)u_t, \tag{27}$$

it is a differential equation in functional space $C^1([-1,0],R^3)$.

Denote $\mathcal{A} = \mathcal{A}(0)$,

$$\mathcal{A}^*\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0,1], \\ \int_{-1}^{0} d\rho^\top(t,0)\psi(-t), & s = 0, \end{cases}$$

and

$$\langle \psi(s), \phi(\theta) \rangle = \bar{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{\psi}(\xi-\theta)d\rho(\theta,0)\phi(\xi)d\xi, \tag{28}$$

where $\psi \in C^{1*}([0,1],(R^3)^*)$. Then, $\mathcal{A}^*$ is an adjoint operator of $\mathcal{A}$.

Suppose $q(\theta) = (1,\beta,\gamma)^\top e^{i\omega_0\tau_{20}\theta}$ satisfies $\mathcal{A}q(\theta) = i\omega_0\tau_{20}q(\theta)$, which means $q(\theta)$ is an eigenvector of $\mathcal{A}$. We obtain that

$$\beta = \frac{(i\omega_0+\delta)(\eta K_* e^{-i\omega_0\tau_{20}} - cE_* e^{-i\omega_0\tau_{10}}) + \eta(1-\eta)V_* e^{-i\omega_0\tau_{20}}}{(i\omega_0+\delta)(i\omega_0+cV_*-r_2)}, \quad \gamma = \frac{(1-\eta)e^{-i\omega_0\tau_{20}}}{i\omega_0+\delta}.$$

Let $-i\omega_0\tau_{20}$ be an eigenvalue of $\mathcal{A}^*$, and $q^*(s) = D(1,\beta^*,\gamma^*)e^{i\omega_0\tau_{20}s}$ is an eigenvector; then, we have

$$\beta^* = \frac{bV_*}{-i\omega_0+cV_*-r_2}, \quad \gamma^* = \frac{b\eta V_*^2}{(-i\omega_0+\delta)(-i\omega_0+cV_*-r_2)}.$$

By (28),

$$\begin{aligned}\langle q^*(s), q(\theta) \rangle &= \bar{D}(1+\beta\overline{\beta^*}+\gamma\overline{\gamma^*} - \textstyle\int_{-1}^{0}(1,\overline{\beta^*},\overline{\gamma^*})\theta e^{i\theta\omega_0\tau_{20}}d\eta(\theta,0)(1,\beta,\gamma)^\top) \\ &= \bar{D}(1+\beta\overline{\beta^*}+\gamma\overline{\gamma^*}+\tau_{20}e^{-i\omega_0\tau_{20}}(\eta\overline{\beta^*}K_*+(1-\eta)\overline{\gamma^*})+\tau_{10}e^{-i\omega_0\tau_{10}}(-aV_*-c\overline{\beta^*}E_*)).\end{aligned}$$

For $\langle q^*(s), q(\theta)\rangle = 1$, $\langle q^*(s), \bar{q}(\theta)\rangle = 0$, we choose

$$\bar{D} = \frac{1}{1+\beta\overline{\beta^*}+\gamma\overline{\gamma^*}+\tau_{20}e^{-i\omega_0\tau_{20}}(\eta\overline{\beta^*}K_*+(1-\eta)\overline{\gamma^*})+\tau_{10}e^{-i\omega_0\tau_{10}}(-aV_*-c\overline{\beta^*}E_*)} \tag{29}$$

Define

$$z(t) = \langle q^*, u_t \rangle, \quad u_t(\theta) = W(t,\theta) + 2Re\{z(t)q(\theta)\} \tag{30}$$

when $\mu = 0$. Then, (27) becomes

$$\begin{cases} \dot{z} = i\omega_0 z + \frac{g_{20}}{2}z^2 + g_{11}z\bar{z} + \frac{g_{02}}{2}\bar{z}^2 + g_{21}z^2\bar{z} + \cdots, \\ \dot{W} = \mathcal{A}W + H(z,\bar{z},0). \end{cases} \tag{31}$$

Here,

$$\begin{aligned} g_{20} &= 2\tau_{20}\bar{D}((-ae^{-i\omega_0\tau_{10}} + (b-c\overline{\beta^*})\beta e^{-i\omega_0\tau_{10}} + \eta\gamma\overline{\beta^*}e^{-i\omega_0\tau_{20}}), \\ g_{11} &= \tau_{20}\bar{D}(-2a + (b-c\overline{\beta^*})\bar{\beta}e^{-i\omega_0\tau_{10}} + (b-c\overline{\beta^*})\beta e^{i\omega_0\tau_{10}} + \eta\bar{\gamma}\overline{\beta^*}e^{-i\omega_0\tau_{20}} + \eta\gamma\overline{\beta^*}e^{i\omega_0\tau_{20}}), \\ g_{02} &= 2\tau_{20}\bar{D}((-ae^{i\omega_0\tau_{10}} + (b-c\overline{\beta^*})\bar{\beta})e^{i\omega_0\tau_{10}} + \eta\bar{\gamma}\overline{\beta^*}e^{i\omega_0\tau_{20}}), \\ g_{21} &= \tau_{20}\bar{D}\{\tfrac{1}{2}(-2ae^{i\omega_0\tau_{10}} + (b-c\overline{\beta^*})\bar{\beta})W_{20}^1(-\tfrac{\tau_{10}}{\tau_{20}}) + \tfrac{1}{2}(b-c\overline{\beta^*})e^{i\omega_0\tau_{10}}W_{20}^2(0) \\ &\quad + \tfrac{1}{2}\eta\overline{\beta^*}e^{i\omega_0\tau_{20}}W_{20}^3(0) + \tfrac{1}{2}\eta\overline{\beta^*}\bar{\gamma}W_{20}^1(-1) + (-2ae^{-i\omega_0\tau_{10}} + (b-c\overline{\beta^*})\beta)W_{11}^1(-\tfrac{\tau_{10}}{\tau_{20}}) \\ &\quad + (b-c\overline{\beta^*})e^{-i\omega_0\tau_{10}}W_{11}^2(0) + \eta\overline{\beta^*}e^{-i\omega_0\tau_{20}}W_{11}^3(0) + \eta\gamma\overline{\beta^*}W_{11}^1(-1)\}. \end{aligned}$$

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_0\tau_{20}}q(0)e^{i\theta\omega_0\tau_{20}} + \frac{i\bar{g}_{02}}{3\omega_0\tau_{20}}\bar{q}(0)e^{-i\theta\omega_0\tau_{20}} + C_1e^{2i\theta\omega_0\tau_{20}},$$

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega_0\tau_{20}}q(0)e^{i\theta\omega_0\tau_{20}} + \frac{i\bar{g}_{11}}{\omega_0\tau_{20}}\bar{q}(0)e^{-i\theta\omega_0\tau_{20}} + C_2.$$

Here, $C_1, C_2 \in R^3$ satisfy

$$A_1C_1 = f_0(20),$$

$$A_1 = \begin{pmatrix} 2i\omega_0 + aV_*e^{-2i\omega_0\tau_{10}} & -bV_* & 0 \\ cE_*e^{-2i\omega_0\tau_{10}} - \eta K_*e^{-2i\omega_0\tau_{20}} & 2i\omega_0 - r_2 + cV_* & -\eta V_* \\ -(1-\eta)e^{-2i\omega_0\tau_{20}} & 0 & 2i\omega_0 + \delta \end{pmatrix},$$

$$f_0(20) = 2\begin{pmatrix} (-ae^{-i\omega_0\tau_{10}} + b\beta)e^{-i\omega_0\tau_{10}} \\ -c\beta e^{-i\omega_0\tau_{10}} + \eta\gamma e^{-i\omega_0\tau_{20}} \\ 0 \end{pmatrix},$$

and

$$A_2C_2 = f_0(11),$$

$$A_2 = \begin{pmatrix} -aV_* & bV_* & 0 \\ -cE_* + \eta K_* & r_2 - cV_* & \eta V_* \\ 1-\eta & 0 & -\delta \end{pmatrix},$$

$$f_0(11) = 2\begin{pmatrix} -a + bR_e(\beta e^{i\omega_0\tau_0}) \\ -cR_e(\beta e^{i\omega_0\tau_0}) + \eta R_e(\gamma e^{i\omega_0\tau_0}) \\ 0 \end{pmatrix}.$$

From Hassard's method [25], we have

$$l_1(0) = \frac{i}{2\omega_0\tau_{20}}(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2) + g_{21}. \tag{32}$$

$\mu_2 = -\frac{Re\{l_1(0)\}}{Re\{\lambda'(\tau_{20})\}}$ confirms the Hopf bifurcation direction: if $Re\{\lambda'(\tau_{20})\} > 0$, $\mu_2 > 0$ (or if $\mu_2 < 0$), the Hopf bifurcation periodic solution exists for $\tau_2 > \tau_{20}$ (or $\tau_2 < \tau_{20}$), the bifurcation is supercritical (or subcritical). If $Re\{\lambda'(\tau_{20})\} < 0$, the bifurcation is on the opposite direction.

$\nu_2 = 2Re\{l_1(0)\}$ confirms the stability of the Hopf bifurcation periodic solutions: the solution is stable if $\nu_2 < 0$ or it is unstable if $\nu_2 > 0$.

4. Numerical Simulations of the System

We consider some numerical results with different values of τ_1, τ_2. Let the parameters of system (4) be $r_1 = 1, r_2 = 0.1, a = 0.09, b = 0.2, c = 4, \eta = 0.8, \delta = 0.5$, then condition $(H2)$ holds. The unique positive equilibrium is $S_* = (13.5189, 1.0835, 5.4076)$.

The corresponding characteristic equation of system (4) at S_* is

$$\lambda^3 + 54.4757\lambda^2 + 26.9879\lambda + (1.2167\lambda^2 + 77.9992\lambda + 38.6954)e^{-\lambda\tau_1} + (-11.6967\lambda - 11.6967)e^{-\lambda\tau_2} = 0. \tag{33}$$

When $\tau_1 = 0, \tau_2 = 0$, the roots of Equation (33) are $-53.9732, -1.3481, -0.3710$. Thus, S_* is asymptotically stable.

Next, we just consider $\tau_1 \neq \tau_2$. From Lemma 3, we obtain that if and only if $\omega \in [1.173, 1.676]$, $P(\lambda, \tau_1, \tau_2) = 0$ has a solution $\lambda = i\omega$ (see Figure 1).

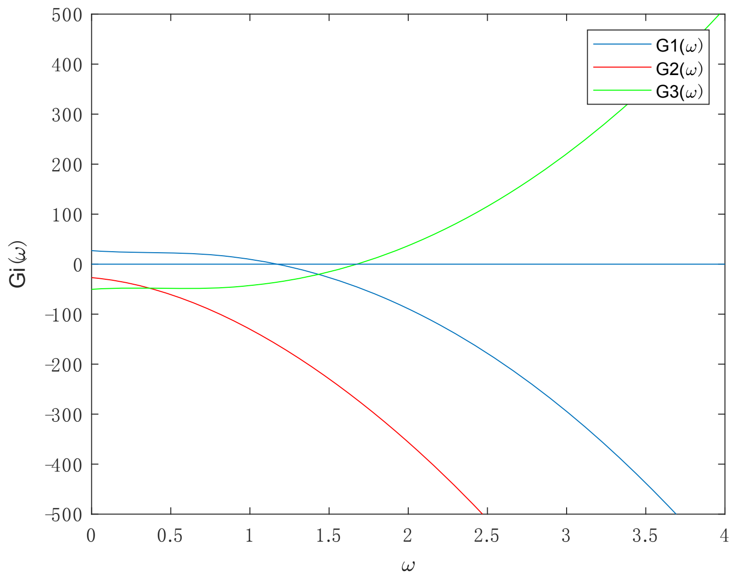

Figure 1. For $r_1 = 1, r_2 = 0.1, a = 0.09, b = 0.2, c = 4, \eta = 0.8, \delta = 0.5$, $G1(\omega), G2(\omega), G3(\omega)$ are all less then or equal to 0 if $\omega \in [1.173, 1.676]$.

From Theorem 3, we know that if $(\tau_1, \tau_2) \in T^\circ$, which is surrounded by $T = \bigcup_{j=1}^{5} T^j$, coordinate axis $\tau_1 = 0$, $\tau_2 = 0$ and $\tau_2 = 7$ (see Figure 2, here, τ_2 can be larger); thus, S_* is asymptotically stable, where $T^1 = ((\tau_{10}^{1-}(\omega), \tau_{20}^{2-}(\omega)), T^2 = ((\tau_{10}^{1-}(\omega), \tau_{20}^{2+}(\omega)), T^3 = ((\tau_{10}^{1-}(\omega), \tau_{20}^{1-}(\omega)), T^4 = ((\tau_{10}^{1-}(\omega), \tau_{20}^{1+}(\omega)), T^5 = ((\tau_{10}^{1-}(\omega), \tau_{20}^{0-}(\omega))$ from (16) and (17).

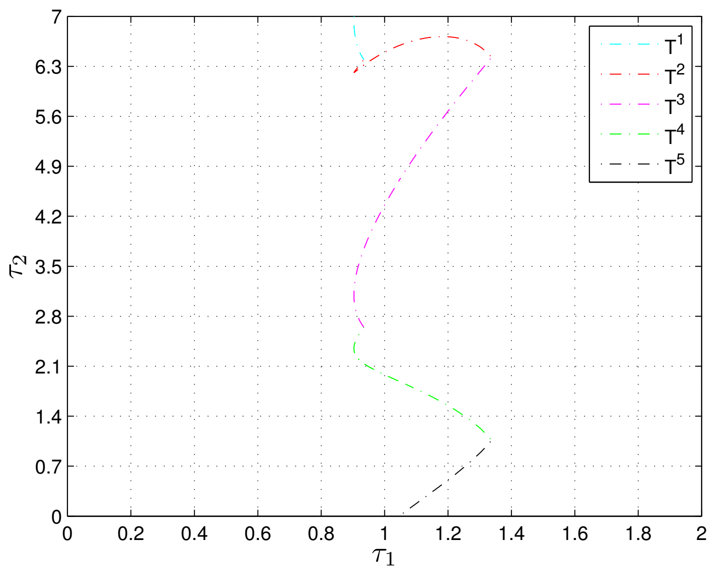

Figure 2. For $r_1 = 1, r_2 = 0.1, a = 0.09, b = 0.2, c = 4, \eta = 0.8, \delta = 0.5$, $T^1 - T^5$ are stability crossing curves. The stable region is on the left of the stability crossing curves.

When (τ_1, τ_2) crossing T and $(H5)$ holds, there are periodic solutions bifurcating from S_*. We choose some points (τ_1, τ_2) to illustrate the result.

We know from Figure 2 that if $\tau_1 < 0.9034$, then S_* is always stable. Let $\tau_1 = 1.0003$; we see from Figure 2 that there are three critical points on the stability crossing curves, $(\tau_{10}^4, \tau_{20}^4) \doteq (1.0003, 1.9807) \in T^4$, $(\tau_{10}^3, \tau_{20}^3) \doteq (1.0003, 4.3577) \in T^3$, $(\tau_{10}^2, \tau_{20}^2) \doteq (1.0003, 6.4977) \in T^2$. For each point, we calculate $Re\{\lambda'(\tau_{20})\}, \mu_2, \nu_2$.

When delay $\tau_1 = 1.0003$ remains unchanged, $\tau_{20} < 1.9807$, since (τ_1, τ_2) is in the left of T, we know that S_* is asymptotically stable (see Figure 3).

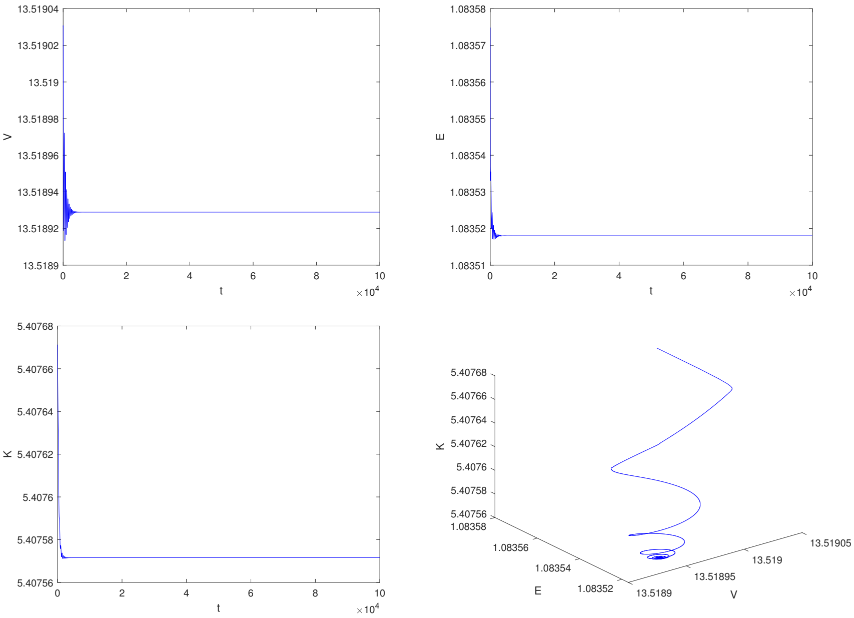

Figure 3. The phase graph and the trajectories of system (4) with $\tau_1 = 1.00, \tau_2 = 1.80$, S_* is asymptotically stable.

Let the delays (τ_1, τ_2) increase and pass through the critical point $(\tau_{10}, \tau_{20}) \doteq (1.0003, 1.9807)$, where $\omega = 1.372$. We obtain that at $(\tau_{10}, \tau_{20}) = (1.0003, 1.9807)$, $Re\{\lambda'(\tau_{20})\} = 0.2762 > 0$, $Re\{l_1(0)\} = -0.0565 < 0$, $\mu_2 = 0.2047 > 0$, and $\nu_2 = -0.1131 < 0$. Therefore, the bifurcation is supercritical; the bifurcation periodic solution is in the direction of $\tau_{20} > 1.9807$ and is stable (see Figure 4).

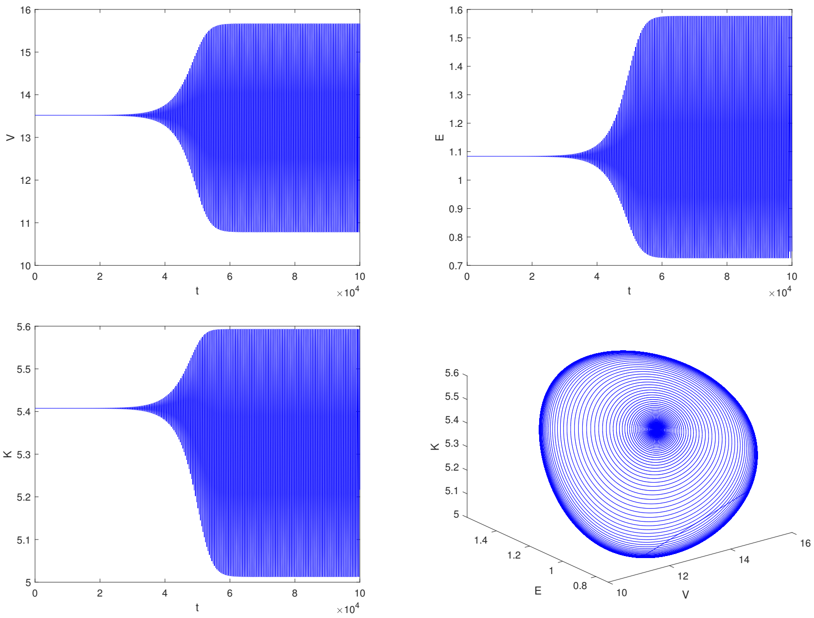

Figure 4. The phase graph and the trajectories of system (4) with $\tau_1 = 1.00, \tau_2 = 2.55$, S_* is unstable, from S_* bifurcates a stable periodic solution.

Let the delays (τ_1, τ_2) increase further to arrive at the critical point $(\tau_{10}, \tau_{20}) \doteq (1.0003, 4.3577)$, where $\omega = 1.372$. We obtain $Re\{\lambda'(\tau_{20})\} = -0.0935 < 0$, $Re\{l_1(0)\} = -0.0580 < 0$, $\mu_2 = -0.6206 < 0$, and $\nu_2 = -0.1160 < 0$. Therefore, the bifurcation is subcritical, and the bifurcation periodic solution is stable, which is on the side less than $\tau_{20} = 4.3577$. When τ_{20} increases crossing 4.3577, the delay (τ_1, τ_2) enters the area of T°, and S_* becomes stable again (see Figure 5).

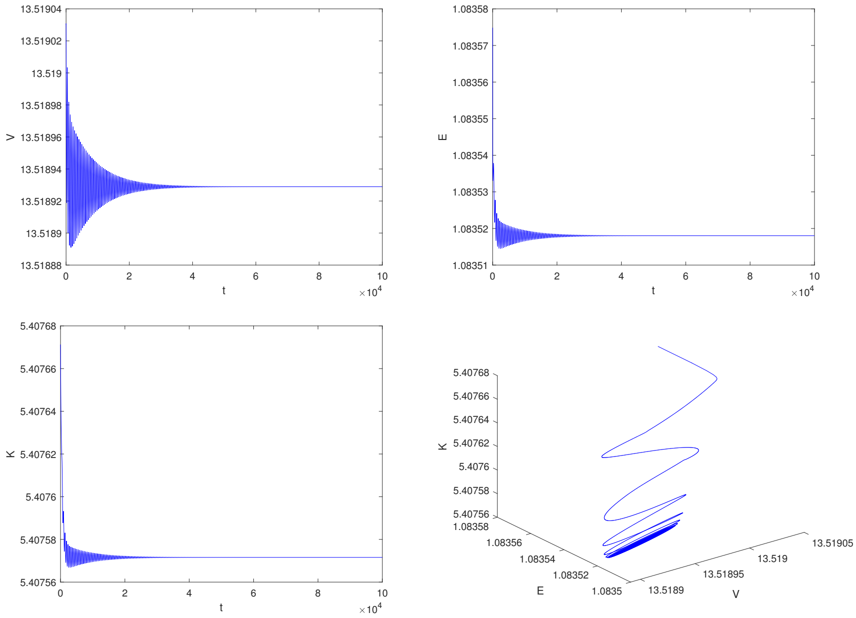

Figure 5. The phase graph and the trajectories of system (4) with $\tau_1 = 1.00, \tau_2 = 4.50$, S_* is stable again.

Let the delays (τ_1, τ_2) increase further to arrive at the critical point $(\tau_{10}, \tau_{20}) \doteq (1.0003, 6.4977)$, where $\omega = 1.372$. We obtain $Re\{\lambda'(\tau_{20})\} = 0.2771 > 0$, $Re\{l_1(0)\} = -0.0538 < 0$, $\mu_2 = 0.1941 > 0$, and $\nu_2 = -0.1076 < 0$. Therefore, the bifurcation is supercritical; when τ_{20} increases crossing 6.4977, S_* becomes unstable, and a stable periodic solution bifurcates from S_* (see Figure 6).

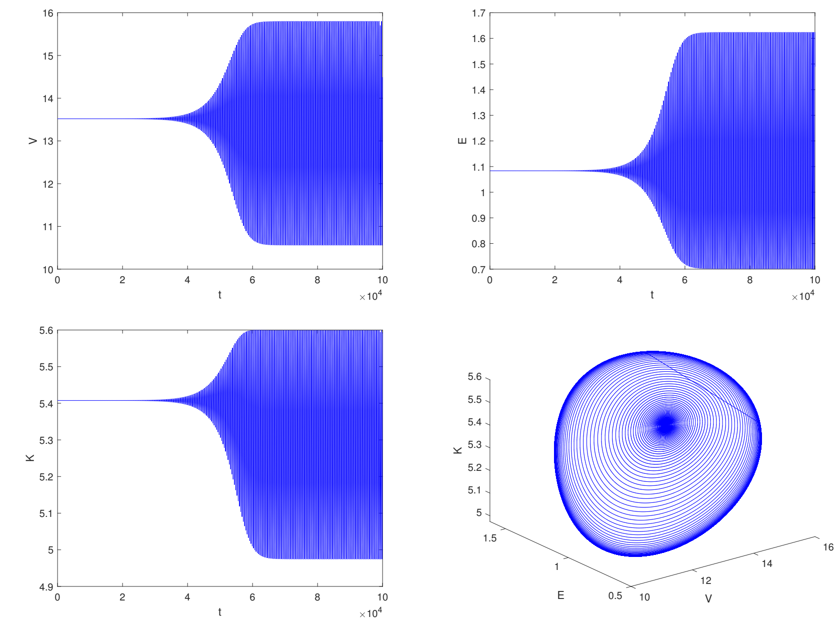

Figure 6. The trajectories and phase graph of system (4) with $\tau_1 = 1.00, \tau_2 = 6.50$, S_* is unstable and a stable periodic solution bifurcates from S_*.

In this example, we can see that if the Hopf bifurcation is supercritical for one point $(\tau_{10}^{m_1}, \tau_{20}^{n_1})$ on the stability crossing curve, then for another point $(\tau_{10}^{m_2}, \tau_{20}^{n_2})$ on the adjacent stability crossing curve, the Hopf bifurcation is subcritical and vice versa. The Hopf bifurcation will be supercritical and subcritical alternately. From Figure 7, we understand that η is closely related to the delay τ_1, with a smaller η, a smaller τ_1 is needed for the equilibrium's stability.

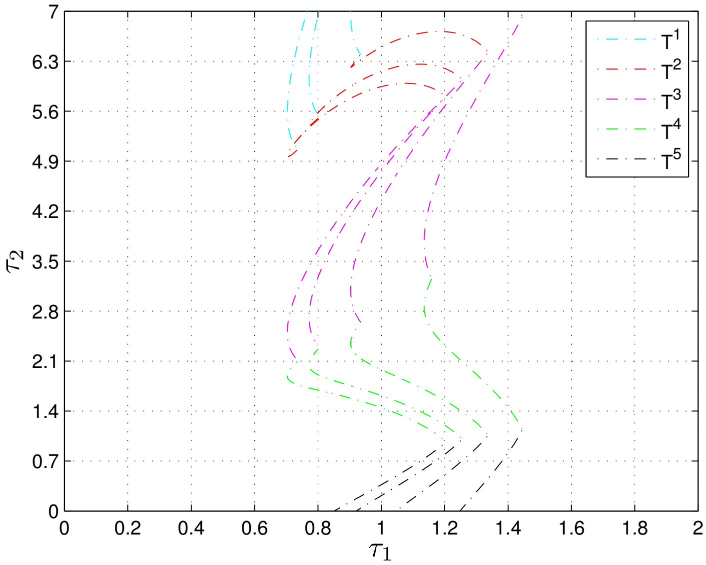

Figure 7. Stability crossing curves for different η. From the left to the right are the stability crossing curves of $\eta = 0.6$, $\eta = 0.7$, $\eta = 0.8$, and $\eta = 0.9$.

5. Conclusions

In this paper, we proposed a delayed environmental protection expenditures model with two delays. We discussed the existence of equilibrium and bifurcation using delays τ_1 and τ_2 as the bifurcation parameters. We depicted the stability crossing curves and obtained the stability of the equilibrium S_*; the direction of bifurcation was also considered.

Since the financial support for the protection of the environment is only depending on the share η of the tourism user fees in this model, the stability of the equilibrium relies closely on η with delay τ_1, the lower the share η, the smaller the spread delay τ_1 needs to be. The external capital support for sustaining the PAs has not been considered in the model; it will be an interesting problem in the management of PAs. We leave this subject for future work.

Author Contributions: Conceptualization, J.H.; methodology, J.H. and P.Y.; software, J.H. and P.Y.; validation, J.H. and P.Y.; formal analysis, J.H. and P.Y.; investigation, J.H. and P.Y.; resources, J.C.; data curation, J.H.; writing—original draft preparation, J.H.; writing—review and editing, J.H. and P.Y.; visualization, J.H.; supervision, J.C.; project administration, J.C.; funding acquisition, J.C. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the National Natural Science Foundation of China grant number 11761040.

Conflicts of Interest: The authors declare no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

References

1. Daskin, J.H.; Pringle, R.M. Warfare and wildlife declines in Africa's protected areas. *Nature* **2018**, *553*, 328–332. [CrossRef] [PubMed]
2. Pringle, R.M. Upgrading protected areas to conserve wild biodiversity. *Nature* **2017**, *546*, 91–99. [CrossRef] [PubMed]
3. Watson, J.E.M.; Dudley, N.; Segan, D.B.; Hockings, M. The performance and potential of protected areas. *Nature* **2014**, *515* , 67–73. [CrossRef]

4. Russu, P. Hopf bifurcation in a environmental defensive expenditures model with time delay. *Chaos Solit. Fract.* **2009**, *42*, 3147–3159. [CrossRef]
5. Becker, R. Intergenerational equity: The capital-environment trade-off. *J. Environ. Econ. Manag.* **1982**, *9*, 165–185. [CrossRef]
6. Cazzavillan, G.; Musu, I. Transitional dynamics and uniqueness of the balanced-growth path in a simple model of endogenous growth with an environmental asset. *FEEM Work.* **2001**, *65*, 1–12. [CrossRef]
7. Eagles, P.F.J.; McCool, S.F.; Haynes, C.D. *Sustainable Tourism in Protected Areas: Guidelines for Planning and Management*; IUCN: Gland, Switzerland; Cambridge, UK, 2002.
8. Caraballo, T.; Colucci, R.; Guerrini, L. Same remarks on an enviromental defensive expenditures model. *Discrete Cont. Dyn. B* **2019**, *24* ,1079–1093.
9. Berezansky, L.; Braverman, E. A note on the stability of Mackey-Glass equations with two delays. *J. Math. Anal. Appl.* **2017**, *450*, 1208–1228. [CrossRef]
10. Campbell, S.A.; Ncube, I. Stability in a scalar differential equation with multiple, distributed time delays. *J. Math. Anal. Appl.* **2017**, *450*, 1104–1122. [CrossRef]
11. El-Morshedy, H.A.; Ruiz-Herrera, A. Criteria of global attraction in systems of delay differential equations with mixed monotonicity. *J. Differ. Equ.* **2020**, *268*, 5945–5968. [CrossRef]
12. Zhang, Z.; Kundub, S.; Tripathi, J.P.; Bugalia, S. Stability and Hopf bifurcation analysis of an SVEIR epidemic model with vaccination and multiple time delays. *Chaos Solit. Fract.* **2020**, *131*, 109483. [CrossRef]
13. Pawelek, K.A.; Liu, S.; Pahlevani, F.; Rong, L. A model of HIV-1 infection with two time delays: Mathematical analysis and comparison with patient data. *Math. Biosci.* **2012**, *235*, 98–109. [CrossRef] [PubMed]
14. Li, Q.; Dai, Y.; Guo, X.; Zhang, X. Hopf bifurcation analysis for a model of plant virus propagation with two delays. *Adv. Differ. Equ.* **2018**, *259*, 1–22. [CrossRef]
15. Arafa, A.A.; Xu, Y.; Mahmoud, G.M. Chaos Suppression via Integrative Time Delay Control. *Int. J. Bifurc. Chaos* **2020**, *30* , 205–208. [CrossRef]
16. Su, H.; Xu, J. Time-Delayed Sampled-Data Feedback Control of Differential Systems Undergoing Hopf Bifurcation. *Int. J. Bifurc. Chaos* **2021**, *31*, 2150004. [CrossRef]
17. Hale, J.K.; Huang, W. Global geometry of the stable regions for two delay differential equations. *J. Math. Anal. Appl.* **1993**, *178*, 344–362. [CrossRef]
18. Gu, K.; Niculescu, S.I.; Chen, J. On stability crossing curves for general systems with two delays. *J. Math. Anal. Appl.* **2005**, *311*, 231–253. [CrossRef]
19. Lin, X.; Wang, H. Stability analysis of delay differential equations with two discrete delays. *Can. Appl. Math. Q.* **2012**, *20*, 519–533.
20. An, Q.; Beretta, E.; Kuang, Y.; Wang, C.; Wang, H. Geometric stability switch criteria in delay differential equations with two delays and delay dependent parameters. *J. Differ. Equ.* **2019**, *266*, 7073–7100. [CrossRef]
21. Matsumotoa, A.; Szidarovszky, F. Stability switching curves in a Lotka-Volterra competition system with two delays. *Math. Comput. Simulat.* **2020**, *178*, 422–438. [CrossRef]
22. Hale, J.K.; Lunel, S.M.V. *Introduction to Functional Differential Equations*; Springer: New York, NY, USA, 1993.
23. Levinson, N.; Redheffer, R.M. *Complex Variables*; Holden-Day: Baltimore, MD, USA, 1970.
24. Ruan, S.; Wei, J. On the zeros of transcendental functions with applications to stability of delay differential equations with two delays. *Dynam. Cont. Dis. Ser.* **2003**, *10*, 863–874.
25. Hassard, B.; Kazarinoff, D.; Wan, Y. *Theory and Applications of Hopf bifurcation*; Cambridge University Press: Cambridge, UK, 1981.