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Analytical and Numerical Solutions for a Kind of High-Dimensional Fractional Order Equation

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Abstract: In this paper, a (4+1)-dimensional nonlinear integrable Fokas equation is studied. It is rarely studied because the order of the highest-order derivative term of this equation is higher than the common generalized (4+1)-dimensional Fokas equation. Firstly, the (4+1)-dimensional time-fractional Fokas equation with the Riemann–Liouville fractional derivative is derived by the semi-inverse method and variational method. Further, the symmetry of the time-fractional equation is obtained by the fractional Lie symmetry analysis method. Based on the symmetry, the conservation laws of the time fractional equation are constructed by the new conservation theorem. Then, the $\left(\frac{G'}{G}\right)$ -expansion method is used here to solve the equation and obtain the exact traveling wave solutions. Finally, the spectral method in the spatial direction and the Grünwald–Letnikov method in the time direction are considered to obtain the numerical solutions of the time-fractional equation. The numerical solutions are compared with the exact solutions, and the error results confirm the effectiveness of the proposed numerical method.

Keywords: (4+1)-dimensional time-fractional Fokas equation; conservation laws; $\left(\frac{G'}{G}\right)$ -expansion method; spectral method



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1. Introduction

Due to the dimensions of equations increasing, which causes difficulties in the analysis and calculation of high-dimensional partial differential equations, there are more studies on low-dimensional problems than on high-dimensional problems. However, the real world is (3+1)-dimensional, thus high-dimensional equations have important applications in real-world problems [1–3]. For the past few years, more and more scholars have begun to pay attention to high-dimensional integrable equations in the integrable field. These equations are mathematical models that can describe some complex physical phenomena in physics, ocean process, engineering, biology, chemistry, nonlinear optics and so on. Now, the research of high-dimensional problems mainly focuses on the integer order. Many scholars are devoted to the study of the exact solutions of integer-order high-dimensional problems. These exact solutions have rich physical meanings and can describe some physical phenomena that cannot be observed easily. In recent years, the theory of fractional calculus has developed rapidly [4–6], and it has been applied to various fields, such as viscoelastic mechanics, biology, signal and image processing, machinery, physics and other fields. The fractional calculus has great significance for us to observe and study these practical problems because the fractional-order equations can describe complex problems that integer order equations cannot. So it is necessary to study some useful properties and solutions of the fractional order high-dimensional equation. Motivated by the above considerations, in this paper, a (4+1)-dimensional time fractional Fokas equation is derived and studied.

A popular (4+1)-dimensional research issue is a common generalized Fokas equation. It is generated by extending the Lax pair of the low-dimensional Kadomtsev–Petviashvili equation and Davey–Stewartson equation to higher dimensions [7]. It is mainly used

to describe the interaction between elastic and inelastic internal waves. The common generalized Fokas equation is as follows:

$$(4u_t - u_{x_1x_1x_2} + u_{x_2x_2x_2} + 12uu_{x_2})_{x_1} - 6u_{y_1y_2} = 0. \quad (1)$$

This is a nonlinear partial differential equation with four variables x_1, x_2, y_1, y_2 in space direction, one variable t in time direction and a fourth derivative term. A similar equation to be introduced in this paper is produced together with the common generalized Fokas equation, but it is rarely studied because it is more complex, and its form is as follows:

$$16u_{tx_2} - u_{x_1x_1x_1x_1} - u_{x_2x_2x_2x_2} + 6u_{x_1x_1x_2x_2x_2} - 12(u^2)_{x_2x_2} + 12(u^2)_{x_1x_1} - 24u_{y_1y_1} + 24u_{y_2y_2} = 0, \quad (2)$$

where $x_1, x_2, y_1, y_2 \in \mathbb{R}^4, t \in [0, T]$. This is a fifth-order nonlinear integrable partial differential equation with four variables x_1, x_2, y_1, y_2 in space direction and one variable t in time direction. Compared with Equation (1), Equation (2) is more complicated, so it is more difficult to study Equation (2).

As far as we all know, Equation (2) has hardly been studied and has not been extended to the time-fractional form. In this paper, the time-fractional form of Equation (2) is studied; the time-fractional form of Equation (2) can be derived for the first time by using the semi-inverse method and the variational method [8]. Because the Kadomtsev–Petviashvili equation and Davey–Stewartson equation have important physical applications in describing surface waves and internal waves in channels with different depths and widths, respectively, this (4+1)-dimensional time fractional Fokas equation can be used to represent many complex nonlinear phenomena in ocean engineering, shallow water waves, fluid mechanics, plasma physics and other fields. We mainly study the symmetry, conservation laws, exact solutions and numerical solutions of this equation.

The symmetry and conservation laws of partial differential equations play an important role in the study of integrability and solutions of partial differential equations [9,10], and it can explain various physical phenomena described by partial differential equations. At first, the Noether theorem [11] provides us with a method of constructing the conservation laws of an integer order partial differential equation. This method establishes the relationship between conservation laws and the symmetry of partial differential equations. However, this approach requires that they are Euler–Lagrange equations. For fractional partial differential equations, we can use the extended Noether theorem to construct the conservation laws [12–14], but this method also requires the fractional partial differential equations to satisfy the Lagrangian. Recently, some research has been conducted, where, based on the new conservation law [15], the fractional generalization of Noether operators can be used to obtain the conservation laws of the time-fractional partial differential equation, which does not require the fractional partial differential equation to have the Lagrangian [16]. In this paper, the symmetry property of the time fractional (4+1)-dimensional partial differential equation can be obtained by using Lie symmetry analysis, and the new conservation theorem (Lie points symmetry combined adjoint equations) are considered to obtain the conservation laws of the (4+1)-dimensional time-fractional Fokas equation.

The exact solutions of Equation (1) have been studied by many scholars using different methods. The main methods include the exponential function method, modified tanh-coth method, extended Jacobian elliptic function method, $\left(\frac{G'}{G}\right)$ -expansion method, extended F-expansion method, extended simplest function method, modified simplest function method, simplified Hirota method and Lie group method. Using the above methods, traveling wave solutions, multi-soliton solutions, soliton solutions, periodic wave solutions, rouge wave and lump wave solutions, V-type solitary wave solutions, breather wave solutions and rational solutions in the determinant form can be obtained [17–23]. Influenced by the analysis of the exact solutions of the Equation (1), for the exact solutions of the (4+1)-dimensional time-fractional equation, we adopt the $\left(\frac{G'}{G}\right)$ -expansion

method [24,25]. We can obtain several exact traveling wave solutions, which we hope to enrich the dynamic behaviors of high-dimensional nonlinear evolution equations. This method is mainly used to find traveling wave solutions of nonlinear evolution equations. Its advantage is that it can transform partial differential equations into ordinary differential equations for solving equations, simplifying the calculation. This method is very effective for solving high-dimensional nonlinear problems in mathematical physics. Some research can be studied from these articles [26–30].

At present, there are few numerical solutions for (4+1)-dimensional partial differential equations. Bi et al. numerically solved the (4+1)-dimensional anomalous diffusion equation by using the alternating direction method [31]. However, this method is suitable for parabolic equations, but not for other types of partial differential equations. In addition, the finite difference method, finite element method and iterative algorithms for stochastic systems [32] need a lot of calculation to deal with high-dimensional problems, which brings great difficulties to the computer. For the numerical solutions of (3+1)-dimensional seismic waves, as early as 1998, Takashi et al. proposed a numerical method [33,34], which is the spectral method in the spatial direction and the finite difference method in the time direction. Recently, Sun and Wang et al. used this method to numerically calculate the (3+1)-dimensional seismic wave, and compared it with the finite difference method [35–37]. They found that the accuracy is improved. The advantage of this method is that it can calculate the equations in each spatial direction and time direction discretely. So it can greatly reduce the amount of calculation compared with the classical finite difference method and finite element method. Inspired by this, we apply it to the (4+1)-dimensional time fractional Fokas equation. The numerical solutions obtained by the considered method is satisfactory and this method may provide a new technique for numerically solving such (4+1)-dimensional high-dimensional equations.

The rest of this article is organized as follows. In Section 2, the semi-inverse method and the variational approach are used to derive the (4+1)-dimensional time fractional Fokas equation. In Section 3, we make use of Lie symmetry analysis to study the symmetry of the time-fractional equation, and we construct the conservation laws of the equation by the new conservation theorem [38–44]. In Section 4, the exact solutions of the time fractional equation are given by using the $\left(\frac{G'}{G}\right)$ -expansion method. In Section 5, the time-fractional equation is numerically solved by the spectral method in the spatial direction and the Grünwald–Letnikov method [45,46] in the time direction. Then, we have some discussions between the exact solutions and numerical solutions. Finally, some conclusions are given in Section 6.

2. Derivation of the (4+1)-Dimensional Time Fractional Fokas Equation

In this section, a (4+1)-dimensional time-fractional Fokas equation in the sense of the Riemann–Liouville fractional derivative is derived by the semi-inverse method and the variational approach. An integer-order (4+1)-dimensional Fokas equation is

$$16u_{tx_2} - u_{x_1x_1x_1x_1} - u_{x_2x_2x_2x_2} + 6u_{x_1x_1x_2x_2x_2} - 12(u^2)_{x_2x_2} + 12(u^2)_{x_1x_1} - 24u_{y_1y_1} + 24u_{y_2y_2} = 0. \quad (3)$$

The derivation of the (4+1)-dimensional time-fractional Fokas equation is as follows.

Introducing a potential function $v = v(x_1, x_2, y_1, y_2, t)$ and $u = v_{x_2}$, the subscript of v represents the partial derivatives of v with respect to the variable x_2 , so the potential equation of Equation (3) can be written as

$$16v_{x_2tx_2} - v_{x_2x_1x_1x_1x_1} - v_{x_2x_2x_2x_2x_2} + 6v_{x_2x_1x_1x_2x_2x_2} - 12(u^2)_{x_2x_2} + 12(u^2)_{x_1x_1} - 24v_{x_2y_1y_1} + 24v_{x_2y_2y_2} = 0. \quad (4)$$

The functional form of Equation (4) is as follows:

$$\begin{aligned}
 J(v) = & \int_R dx_2 \int_{X_1} dx_1 \int_{Y_1} dy_1 \int_{Y_2} dy_2 \int_T dt \{v[16c_1 v_{x_2 t x_2} - c_2 v_{x_2 x_1 x_1 x_1} \\
 & - c_3 v_{x_2 x_2 x_2 x_2} + 6c_4 v_{x_2 x_1 x_1 x_2 x_2} - 12c_5 (u^2)_{x_2 x_2} + 12c_6 (u^2)_{x_1 x_1} \\
 & - 24c_7 v_{x_2 y_1 y_1} + 24c_8 v_{x_2 y_2 y_2}]\}, \tag{5}
 \end{aligned}$$

where the coefficients $c_i (i = 1, 2, \dots, 8)$ are the undetermined Lagrange multipliers. A new integral is obtained by integrating by parts of the above equation with considering the conditions that $(u^2)_{x_2 x_2}, (u^2)_{x_1 x_1}$ are fixed functions and $v_{x_2}|_R = v_{x_2}|_{X_1} = v_{x_2}|_{Y_1} = v_{x_2}|_{Y_2} = v_{x_2}|_T = 0$. Then using the variation of the new integral, we integrate by parts for each term of the new integral under the variational optimal condition $\delta J(v) = 0$, so we have

$$\begin{aligned}
 & 32c_1 v_{x_2 t x_2} - 2c_2 v_{x_2 x_1 x_1 x_1} - 2c_3 v_{x_2 x_2 x_2 x_2} + 12c_4 v_{x_2 x_1 x_1 x_2 x_2} \\
 & - 12c_5 (u^2)_{x_2 x_2} + 12c_6 (u^2)_{x_1 x_1} - 48c_7 v_{x_2 y_1 y_1} + 48c_8 v_{x_2 y_2 y_2} = 0. \tag{6}
 \end{aligned}$$

We know Equation (6) is equal to Equation (4), so by comparing these two equations, we obtain the constant multipliers $c_1 = c_2 = c_3 = c_4 = c_7 = c_8 = \frac{1}{2}, c_5 = c_6 = 1$. Substituting the values of $c_i (i = 1, 2, \dots, 8)$ into Equation (5), we obtain the Lagrangian form of Equation (3):

$$\begin{aligned}
 & L(v, v_{x_1}, v_{x_2}, v_{y_1}, v_{y_2}, v_t, v_{x_2 x_2}, v_{x_2 y_1}, v_{x_2 y_2}, v_{x_2 x_1 x_1}, v_{x_2 x_2 x_2}, v_{x_2 x_1 x_2 x_2}) \\
 & = -8v_t v_{x_2 x_2} + \frac{1}{2}v_{x_1} v_{x_2 x_1 x_1} + \frac{1}{2}v_{x_2} v_{x_2 x_2 x_2} - 3v_{x_2} v_{x_2 x_1 x_1 x_2} \\
 & - 12(u^2)_{x_2 x_2} v + 12(u^2)_{x_1 x_1} v + 12v_{y_1} v_{x_2 y_1} - 12v_{y_2} v_{x_2 y_2}. \tag{7}
 \end{aligned}$$

Similarly, the time-fractional form of Equation (3) has the following Lagrangian form:

$$\begin{aligned}
 & F(v, v_{x_1}, v_{x_2}, v_{y_1}, v_{y_2}, D_t^\alpha v, v_{x_2 x_2}, v_{x_2 y_1}, v_{x_2 y_2}, v_{x_2 x_1 x_1}, v_{x_2 x_2 x_2}, v_{x_2 x_1 x_2 x_2}) \\
 & = -8D_t^\alpha v v_{x_2 x_2} + \frac{1}{2}v_{x_1} v_{x_2 x_1 x_1} + \frac{1}{2}v_{x_2} v_{x_2 x_2 x_2} - 3v_{x_2} v_{x_2 x_1 x_1 x_2} \\
 & - 12(u^2)_{x_2 x_2} v + 12(u^2)_{x_1 x_1} v + 12v_{y_1} v_{x_2 y_1} - 12v_{y_2} v_{x_2 y_2}, \tag{8}
 \end{aligned}$$

where D_t^α is the Riemann–Liouville fractional derivative operator [40].

Consequently, the functional form of (4+1)-dimensional time-fractional Fokas equation is as follows:

$$\begin{aligned}
 J(v) = & \int_R dx_2 \int_{X_1} dx_1 \int_{Y_1} dy_1 \int_{Y_2} dy_2 \int_T (dt)^\alpha F(v, v_{x_1}, v_{x_2}, v_{y_1}, v_{y_2}, D_t^\alpha v, \\
 & v_{x_2 x_2}, v_{x_2 y_1}, v_{x_2 y_2}, v_{x_2 x_1 x_1}, v_{x_2 x_2 x_2}, v_{x_2 x_1 x_2 x_2}), \tag{9}
 \end{aligned}$$

where $\int_a^t (d\tau)^\alpha f(\tau) = \alpha \int_a^t d\tau (t - \tau)^{\alpha-1} f(\tau)$.

Giving the relation of integrating by parts [47]:

$$\int_a^b (d\tau)^\alpha f(t) D_t^\alpha g(t) = \Gamma(1 + \alpha) \left[g(t) f(t) \Big|_a^b - \int_a^b (dt)^\alpha g(t) D_t^\alpha f(t) \right], f(t), g(t) \in [a, b].$$

Integrating by parts for Equation (9), making use of the above relation and variational optimal condition $\delta J(v) = 0$, the Euler–Lagrangian equation of (4+1)-dimensional time-fractional Fokas equation can be given as follows:

$$\begin{aligned} & \left(\frac{\partial F}{\partial v}\right) \cdot v + \left(\frac{\partial F}{\partial v_{x_1}}\right) \cdot v_{x_1} + \left(\frac{\partial F}{\partial v_{x_2}}\right) \cdot v_{x_2} + \left(\frac{\partial F}{\partial v_{y_1}}\right) \cdot v_{y_1} + \left(\frac{\partial F}{\partial v_{y_2}}\right) \cdot v_{y_2} \\ & + \left(\frac{\partial F}{\partial D_t^\alpha v}\right) \cdot D_t^\alpha v + \left(\frac{\partial F}{\partial v_{x_2 x_2}}\right) \cdot v_{x_2 x_2} + \left(\frac{\partial F}{\partial v_{x_2 y_1}}\right) \cdot v_{x_2 y_1} \\ & + \left(\frac{\partial F}{\partial v_{x_2 y_2}}\right) \cdot v_{x_2 y_2} + \left(\frac{\partial F}{\partial v_{x_2 x_1 x_1}}\right) \cdot v_{x_2 x_1 x_1} + \left(\frac{\partial F}{\partial v_{x_2 x_2 x_2}}\right) \cdot v_{x_2 x_2 x_2} \\ & + \left(\frac{\partial F}{\partial v_{x_2 x_1 x_1 x_2}}\right) \cdot v_{x_2 x_1 x_1 x_2} = 0. \end{aligned} \tag{10}$$

By substituting Equation (8) into Equation (10) and making use of the fractional potential function $D_{x_2}^\alpha v(x_1, x_2, y_1, y_2, t) = u(x_1, x_2, y_1, y_2, t)$, the (4+1)-dimensional time-fractional Fokas equation can be obtained:

$$\begin{aligned} & 16D_t^\alpha u_{x_2} - u_{x_1 x_1 x_1} - u_{x_2 x_2 x_2} + 6u_{x_1 x_1 x_2 x_2} - 12(u^2)_{x_2 x_2} \\ & + 12(u^2)_{x_1 x_1} - 24u_{y_1 y_1} + 24u_{y_2 y_2} = 0. \end{aligned} \tag{11}$$

Equation (11) is the (4+1)-dimensional time-fractional Fokas equation in the sense of the Riemann–Liouville derivative. So $D_t^\alpha u$ can be defined as [40]

$$D_t^\alpha u = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t (t-s)^{n-\alpha-1} u(x_1, x_2, y_1, y_2, s) ds, & n-1 < \alpha < n, \\ \frac{\partial^n u}{\partial t^n}, & \alpha = n. \end{cases}$$

where $\Gamma(x)$ is a gamma function.

3. Analysis of the Symmetry and Conservation Laws for the (4+1)-Dimensional Time-Fractional Fokas Equation

In this section, the Lie symmetry analysis and the new conservation theorem are used to study the symmetry and conservation laws of the (4+1)-dimensional time-fractional Fokas equation.

3.1. Analysis of the Lie Symmetry for the (4+1)-Dimensional Time-Fractional Fokas Equation

The (4+1)-dimensional time-fractional Fokas equation here has five variables x_1, x_2, y_1, y_2, t , giving some infinitesimal transformations as follows:

Infinitesimal transformation of each variable:

$$\begin{aligned} x_1^* &= x_1 + \epsilon \zeta_1(x_1, x_2, y_1, y_2, t, u) + o(\epsilon^2), \\ x_2^* &= x_2 + \epsilon \zeta_2(x_1, x_2, y_1, y_2, t, u) + o(\epsilon^2), \\ y_1^* &= y_1 + \epsilon \zeta_3(x_1, x_2, y_1, y_2, t, u) + o(\epsilon^2), \\ y_2^* &= y_2 + \epsilon \zeta_4(x_1, x_2, y_1, y_2, t, u) + o(\epsilon^2), \\ t^* &= t + \epsilon \tau(x_1, x_2, y_1, y_2, t, u) + o(\epsilon^2), \\ u^* &= u + \epsilon \eta(x_1, x_2, y_1, y_2, t, u) + o(\epsilon^2), \end{aligned} \tag{12}$$

where $\epsilon \ll 1$ is a group parameter, $\xi_1, \xi_2, \xi_3, \xi_4, \tau, \eta$ are infinitesimal parameters. The infinitesimal transformations of the partial derivatives of u with respect to different variables are

$$\begin{aligned}
 \frac{\partial u^*}{\partial x_1^*} &= \frac{\partial u}{\partial x_1} + \epsilon \eta^{x_1}(x_1, x_2, y_1, y_2, t, u) + o(\epsilon^2), \\
 \frac{\partial u^*}{\partial x_2^*} &= \frac{\partial u}{\partial x_2} + \epsilon \eta^{x_2}(x_1, x_2, y_1, y_2, t, u) + o(\epsilon^2), \\
 \frac{\partial u^*}{\partial t^{*\alpha}} &= \frac{\partial^\alpha u_{x_2}}{\partial t^\alpha} + \epsilon \eta^{\alpha,t}(x_1, x_2, y_1, y_2, t, u) + o(\epsilon^2), \\
 \frac{\partial u^*}{\partial x_1^{*2}} &= \frac{\partial u}{\partial x_1^2} + \epsilon \eta^{x_1 x_1}(x_1, x_2, y_1, y_2, t, u) + o(\epsilon^2), \\
 \frac{\partial u^*}{\partial x_2^{*2}} &= \frac{\partial u}{\partial x_2^2} + \epsilon \eta^{x_2 x_2}(x_1, x_2, y_1, y_2, t, u) + o(\epsilon^2), \\
 \frac{\partial u^*}{\partial y_1^{*2}} &= \frac{\partial u}{\partial y_1^2} + \epsilon \eta^{y_1 y_1}(x_1, x_2, y_1, y_2, t, u) + o(\epsilon^2), \\
 \frac{\partial u^*}{\partial y_2^{*2}} &= \frac{\partial u}{\partial y_2^2} + \epsilon \eta^{y_2 y_2}(x_1, x_2, y_1, y_2, t, u) + o(\epsilon^2), \\
 \frac{\partial u^*}{\partial x_1^{*4}} &= \frac{\partial u}{\partial x_1^4} + \epsilon \eta^{x_1 x_1 x_1 x_1}(x_1, x_2, y_1, y_2, t, u) + o(\epsilon^2), \\
 \frac{\partial u^*}{\partial x_2^{*4}} &= \frac{\partial u}{\partial x_2^4} + \epsilon \eta^{x_2 x_2 x_2 x_2}(x_1, x_2, y_1, y_2, t, u) + o(\epsilon^2), \\
 \frac{\partial u^*}{\partial x_1^{*2} \partial x_2^{*3}} &= \frac{\partial u}{\partial x_1^2 \partial x_2^3} + \epsilon \eta^{x_1 x_1 x_2 x_2 x_2}(x_1, x_2, y_1, y_2, t, u) + o(\epsilon^2),
 \end{aligned}
 \tag{13}$$

where $\epsilon \ll 1$ is the group parameter, $\eta^{x_1}, \eta^{x_2}, \eta^{x_1 x_1}, \eta^{x_2 x_2}, \eta^{y_1 y_1}, \eta^{y_2 y_2}, \eta^{x_1 x_1 x_1 x_1}, \eta^{x_2 x_2 x_2 x_2}, \eta^{x_1 x_1 x_2 x_2 x_2}, \eta^{\alpha,t}$ are extended infinitesimal parameters. According to the previous study [40], $\eta^{x_1}, \eta^{x_2}, \eta^{x_1 x_1}, \eta^{x_2 x_2}, \eta^{y_1 y_1}, \eta^{y_2 y_2}, \eta^{x_1 x_1 x_1 x_1}, \eta^{x_2 x_2 x_2 x_2}, \eta^{x_1 x_1 x_2 x_2 x_2}, \eta^{\alpha,t}$ can be defined as the following form.

The extended infinitesimal transformations of the first-order partial derivative of u and fractional derivative of u are as follows:

$$\begin{aligned}
 \eta^{x_1} &= D_{x_1}(\eta) - u_{x_1} D_{x_1}(\xi_1) - u_{x_2} D_{x_1}(\xi_2) - u_{y_1} D_{x_1}(\xi_3) - u_{y_2} D_{x_1}(\xi_4) - u_t D_{x_1}(\tau), \\
 \eta^{x_2} &= D_{x_2}(\eta) - u_{x_1} D_{x_2}(\xi_1) - u_{x_2} D_{x_2}(\xi_2) - u_{y_1} D_{x_2}(\xi_3) - u_{y_2} D_{x_2}(\xi_4) - u_t D_{x_2}(\tau), \\
 \eta^{y_1} &= D_{y_1}(\eta) - u_{x_1} D_{y_1}(\xi_1) - u_{x_2} D_{y_1}(\xi_2) - u_{y_1} D_{y_1}(\xi_3) - u_{y_2} D_{y_1}(\xi_4) - u_t D_{y_1}(\tau), \\
 \eta^{y_2} &= D_{y_2}(\eta) - u_{x_1} D_{y_2}(\xi_1) - u_{x_2} D_{y_2}(\xi_2) - u_{y_1} D_{y_2}(\xi_3) - u_{y_2} D_{y_2}(\xi_4) - u_t D_{y_2}(\tau), \\
 \eta^{\alpha,t} &= D_t^\alpha(\eta^{x_2}) + \xi_1 D_t^\alpha(u_{x_2 x_1}) - D_t^\alpha(\xi_1 u_{x_2 x_1}) + \xi_2 D_t^\alpha(u_{x_2 x_2}) - D_t^\alpha(\xi_2 u_{x_2 x_2}) \\
 &\quad + \xi_3 D_t^\alpha(u_{x_2 y_1}) - D_t^\alpha(\xi_3 u_{x_2 y_1}) + \xi_4 D_t^\alpha(u_{x_2 y_2}) - D_t^\alpha(\xi_4 u_{x_2 y_2}) + D_t^\alpha(D_t(\tau) u_{x_2}) \\
 &\quad - D_t^{\alpha+1}(\tau u_{x_2}) + \tau D_t^{\alpha+1}(u_{x_2}),
 \end{aligned}
 \tag{14}$$

the extended infinitesimal transformation of the second-order partial derivative of u is

$$\begin{aligned}
 \eta^{x_1 x_1} &= D_{x_1}(\eta^{x_1}) - u_{x_1 x_1} D_{x_1}(\xi_1) - u_{x_2 x_1} D_{x_1}(\xi_2) - u_{y_1 x_1} D_{x_1}(\xi_3) - u_{y_2 x_1} D_{x_1}(\xi_4) \\
 &\quad - u_{t x_1} D_{x_1}(\tau), \\
 \eta^{x_2 x_2} &= D_{x_2}(\eta^{x_2}) - u_{x_1 x_2} D_{x_2}(\xi_1) - u_{x_2 x_2} D_{x_2}(\xi_2) - u_{y_1 x_2} D_{x_2}(\xi_3) - u_{y_2 x_2} D_{x_2}(\xi_4) \\
 &\quad - u_{t x_2} D_{x_2}(\tau), \\
 \eta^{y_1 y_1} &= D_{y_1}(\eta^{y_1}) - u_{x_1 y_1} D_{y_1}(\xi_1) - u_{x_2 y_1} D_{y_1}(\xi_2) - u_{y_1 y_1} D_{y_1}(\xi_3) - u_{y_2 y_1} D_{y_1}(\xi_4) \\
 &\quad - u_{t y_1} D_{y_1}(\tau), \\
 \eta^{y_2 y_2} &= D_{y_2}(\eta^{y_2}) - u_{x_1 y_2} D_{y_2}(\xi_1) - u_{x_2 y_2} D_{y_2}(\xi_2) - u_{y_1 y_2} D_{y_2}(\xi_3) - u_{y_2 y_2} D_{y_2}(\xi_4) \\
 &\quad - u_{t y_2} D_{y_2}(\tau),
 \end{aligned}
 \tag{15}$$

the extended infinitesimal transformation of the third-order partial derivative of u is:

$$\begin{aligned} \eta^{x_1x_1x_1} &= D_{x_1}(\eta^{x_1x_1}) - u_{x_1x_1x_1}D_{x_1}(\xi_1) - u_{x_2x_1x_1}D_{x_1}(\xi_2) - u_{y_1x_1x_1}D_{x_1}(\xi_3) \\ &\quad - u_{y_2x_1x_1}D_{x_1}(\xi_4) - u_{tx_1x_1}D_{x_1}(\tau), \\ \eta^{x_2x_2x_2} &= D_{x_2}(\eta^{x_2x_2}) - u_{x_1x_2x_2}D_{x_2}(\xi_1) - u_{x_2x_2x_2}D_{x_2}(\xi_2) - u_{y_1x_2x_2}D_{x_2}(\xi_3) \\ &\quad - u_{y_2x_2x_2}D_{x_2}(\xi_4) - u_{tx_2x_2}D_{x_2}(\tau), \end{aligned} \tag{16}$$

the extended infinitesimal transformations of the fourth-order partial derivative of u and fifth-order partial derivative of u can be written as

$$\begin{aligned} \eta^{x_1x_1x_1x_1} &= D_{x_1}(\eta^{x_1x_1x_1}) - u_{x_1x_1x_1x_1}D_{x_1}(\xi_1) - u_{x_2x_1x_1x_1}D_{x_1}(\xi_2) \\ &\quad - u_{y_1x_1x_1x_1}D_{x_1}(\xi_3) - u_{y_2x_1x_1x_1}D_{x_1}(\xi_4) - u_{tx_1x_1x_1}D_{x_1}(\tau), \\ \eta^{x_2x_2x_2x_2} &= D_{x_2}(\eta^{x_2x_2x_2}) - u_{x_1x_2x_2x_2}D_{x_2}(\xi_1) - u_{x_2x_2x_2x_2}D_{x_2}(\xi_2) \\ &\quad - u_{y_1x_2x_2x_2}D_{x_2}(\xi_3) - u_{y_2x_2x_2x_2}D_{x_2}(\xi_4) - u_{tx_2x_2x_2}D_{x_2}(\tau), \\ \eta^{x_1x_1x_2x_2x_2} &= D_{x_2}(\eta^{x_1x_1x_2x_2}) - u_{x_1x_1x_2x_2x_2}D_{x_2}(\xi_1) - u_{x_1x_2x_2x_2x_2}D_{x_2}(\xi_2) \\ &\quad - u_{x_1y_1x_2x_2x_2}D_{x_2}(\xi_3) - u_{x_1y_2x_2x_2x_2}D_{x_2}(\xi_4) - u_{x_1tx_2x_2x_2}D_{x_2}(\tau), \end{aligned} \tag{17}$$

in which $D_{x_1}, D_{x_2}, D_{y_1}, D_{y_2}$ and D_t are total derivative operators:

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx_1} \frac{\partial}{\partial u_{x_1}} + u_{tx_2} \frac{\partial}{\partial u_{x_2}} + u_{ty_1} \frac{\partial}{\partial u_{y_1}} + u_{ty_2} \frac{\partial}{\partial u_{y_2}} + \dots, \\ D_{x_1} &= \frac{\partial}{\partial x_1} + u_{x_1} \frac{\partial}{\partial u} + u_{x_1x_1} \frac{\partial}{\partial u_{x_1}} + u_{x_1x_2} \frac{\partial}{\partial u_{x_2}} + u_{x_1y_1} \frac{\partial}{\partial u_{y_1}} + u_{x_1y_2} \frac{\partial}{\partial u_{y_2}} + u_{x_1t} \frac{\partial}{\partial t} + \dots, \\ D_{x_2} &= \frac{\partial}{\partial x_2} + u_{x_2} \frac{\partial}{\partial u} + u_{x_2x_2} \frac{\partial}{\partial u_{x_2}} + u_{x_2x_1} \frac{\partial}{\partial u_{x_1}} + u_{x_2y_1} \frac{\partial}{\partial u_{y_1}} + u_{x_2y_2} \frac{\partial}{\partial u_{y_2}} + u_{x_2t} \frac{\partial}{\partial t} + \dots, \\ D_{y_1} &= \frac{\partial}{\partial y_1} + u_{y_1} \frac{\partial}{\partial u} + u_{y_1y_1} \frac{\partial}{\partial u_{y_1}} + u_{y_1x_1} \frac{\partial}{\partial u_{x_1}} + u_{y_1x_2} \frac{\partial}{\partial u_{x_2}} + u_{y_1y_2} \frac{\partial}{\partial u_{y_2}} + u_{y_1t} \frac{\partial}{\partial t} + \dots, \\ D_{y_2} &= \frac{\partial}{\partial y_2} + u_{y_2} \frac{\partial}{\partial u} + u_{y_2y_2} \frac{\partial}{\partial u_{y_2}} + u_{y_2x_1} \frac{\partial}{\partial u_{x_1}} + u_{y_2x_2} \frac{\partial}{\partial u_{x_2}} + u_{y_2y_1} \frac{\partial}{\partial u_{y_1}} + u_{y_2t} \frac{\partial}{\partial t} + \dots \end{aligned} \tag{18}$$

Consider the structure of the fractional derivative under the transformations of Equations (12) and (13). Noting that the lower limit of integral of Equation (11) is fixed, it is supposed to be invariant under transformations Equations (12) and (13). The invariant condition yields $\tau(x_1, x_2, y_1, y_2, t, u)|_{t=0} = 0$.

In addition, the generalized Leibnitz rule [38] and generalized chain rule [48] are defined as

$$D_t^\alpha(f(t)g(t)) = \sum_{n=0}^\infty \binom{\alpha}{n} D_t^{\alpha-n}(f(t))D_t^n(g(t)), \alpha > 0, \tag{19}$$

$$\frac{d^m g(y(t))}{dt^m} = \sum_{k=0}^m \sum_{r=0}^k \binom{k}{r} \frac{1}{k!} [-y(t)]^r \frac{d^m}{dt^m} [(y(t))^{k-r}] \frac{d^k g(y)}{dy^k}, \tag{20}$$

where $\binom{\alpha}{n} = \frac{-1^{(n-1)}\alpha\Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)}$ and $\Gamma(x)$ is a gamma function.

Now, make use of Equations (14)–(19) and Equation (20) with $f(t) = 1$, which leads to the specific expression of the extended infinitesimal parameters. Taking $\eta^{x_1}, \eta^{x_2}, \eta^{y_1 y_1}, \eta^{\alpha, t}$, for example,

$$\begin{aligned}
 \eta^{x_1} &= \eta_{x_1} + u_{x_1} \eta_u - u_{x_1}^2 \xi_{1u} - u_{x_2} \xi_{2x_1} - u_{x_2} u_{x_1} \xi_{2u} - u_{y_1} \xi_{3x_1} \\
 &\quad - u_{y_1} u_{x_1} \xi_{3u} - u_t \tau_{x_1} - u_t u_{x_1} \tau_u - u_{y_2} \xi_{4x_1} - u_{y_2} u_{x_1} \xi_{4u}, \\
 \eta^{x_2} &= \eta_{x_2} + u_{x_2} \eta_u - u_{x_1} \xi_{1x_2} - u_{x_1 x_2} \xi_{1u} - u_{x_2} \xi_{2x_2} - u_{x_2}^2 \xi_{2u} - u_{y_1} \xi_{3x_2} \\
 &\quad - u_{y_1} u_{x_2} \xi_{3u} - u_t \tau_{x_2} - u_t u_{x_2} \tau_u - u_{y_2} \xi_{4x_2} - u_{y_2} u_{x_2} \xi_{4u}, \\
 \eta^{y_1 y_1} &= \eta_{y_1 y_1} + u_{y_1 y_1} \eta_u + 2u_{y_1} \eta_{u y_1} - 2u_{x_1 y_1} \xi_{1y_1} - u_{x_1} \xi_{y_1 y_1} - 2u_{x_1 y_1} u_{y_1} \xi_{1u} \\
 &\quad - u_{x_1} u_{y_1 y_1} \xi_{1u} - 2u_{x_1} u_{y_1} \xi_{1u y_1} - 2u_{x_2 y_1} \xi_{2y_1} - u_{x_2} \xi_{2y_1 y_1} - 2u_{x_2 y_1} u_{y_1} \xi_{2u} \\
 &\quad - u_{x_2} u_{y_1 y_1} \xi_{2u} - 2u_{x_2} u_{y_1} \xi_{2u y_1} - 2u_{y_1 y_1} \xi_{3y_1} - u_{y_1} \xi_{3y_1 y_1} - 3u_{y_1} u_{y_1 y_1} \xi_{3u} \\
 &\quad - 3u_{y_1}^2 \xi_{3u y_1} - 2u_{t y_1} \tau_{y_1} - u_t \tau_{y_1 y_1} - 2u_{t y_1} u_{y_1} \tau_u - u_t u_{y_1 y_1} \tau_u - 2u_t u_{y_1} \tau_{u y_1} \\
 &\quad - 2u_{y_2 y_1} \xi_{4y_1} - u_{y_2} \xi_{4y_1 y_1} - 2u_{y_2 y_1} u_{y_1} \xi_{4u} - u_{y_2} u_{y_1 y_1} \xi_{4u} - 2u_{y_2} u_{y_1} \xi_{4u y_1} \\
 &\quad + u_{y_1}^2 \eta_{uu} - u_{x_1} u_{y_1}^2 \xi_{1uu} - u_{x_2} u_{y_1}^2 \xi_{2uu} - u_{y_1}^3 \xi_{3uu} - u_t u_{y_1}^2 \tau_{uu} - u_{y_2} u_{y_1}^2 \xi_{4uu}, \\
 \eta^{\alpha, t} &= \partial_t^\alpha (\eta^{x_2}) + [(\eta^{x_2})_u - \alpha D_t(\tau)] \partial_t^\alpha u - u \partial_t^\alpha (\eta^{x_2})_u + \mu \\
 &\quad + \sum_{n=1}^\infty \left[\binom{\alpha}{n} \partial_t^n (\eta^{x_2})_u - \binom{\alpha}{n+1} D_t^{n+1}(\tau) \right] \partial_t^{\alpha-n} u \\
 &\quad - \sum_{n=1}^\infty \binom{\alpha}{n} D_t^n(\xi_1) \partial_t^{\alpha-n} (u_{x_2 x_1}) - \sum_{n=1}^\infty \binom{\alpha}{n} D_t^n(\xi_2) \partial_t^{\alpha-n} (u_{x_2 x_2}) \\
 &\quad - \sum_{n=1}^\infty \binom{\alpha}{n} D_t^n(\xi_3) \partial_t^{\alpha-n} (u_{x_2 y_1}) - \sum_{n=1}^\infty \binom{\alpha}{n} D_t^n(\xi_4) \partial_t^{\alpha-n} (u_{x_2 y_2}).
 \end{aligned} \tag{21}$$

where

$$\mu = \sum_{n=2}^\infty \sum_{m=2}^n \sum_{k=2}^m \sum_{r=0}^{k-1} \binom{\alpha}{n} \binom{n}{m} \binom{k}{r} \frac{1}{k!} \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)} [-u]^r \frac{\partial^m}{\partial t^m} [u^{k-r}] \frac{\partial^{n-m+k}}{\partial t^{n-m} \partial u^k}.$$

Considering the infinitesimal invariant criterion [40] of Equation (12),

$$pr^\alpha X(\Delta)|_{\Delta=0} = 0, \tag{22}$$

where X is the infinitesimal generator of one parameter Lie group transformation, which has the following form:

$$\begin{aligned}
 X &= \xi_1(x_1, x_2, y_1, y_2, t, u) \frac{\partial}{\partial x_1} + \xi_2(x_1, x_2, y_1, y_2, t, u) \frac{\partial}{\partial x_2} \\
 &\quad + \xi_3(x_1, x_2, y_1, y_2, t, u) \frac{\partial}{\partial y_1} + \xi_4(x_1, x_2, y_1, y_2, t, u) \frac{\partial}{\partial y_2} \\
 &\quad + \tau(x_1, x_2, y_1, y_2, t, u) \frac{\partial}{\partial t} + \eta(x_1, x_2, y_1, y_2, t, u) \frac{\partial}{\partial u},
 \end{aligned} \tag{23}$$

and

$$\begin{aligned}
 \Delta &= 16D_t^\alpha u_{x_2} - u_{x_1 x_1 x_1 x_1} - u_{x_2 x_2 x_2 x_2} + 6u_{x_1 x_1 x_2 x_2 x_2} - 24(u_{x_2})^2 \\
 &\quad - 24uu_{x_2 x_2} + 24(u_{x_1})^2 + 24uu_{x_1 x_1} - 24u_{y_1 y_1} + 24u_{y_2 y_2} = 0,
 \end{aligned} \tag{24}$$

the prolongation operator $pr^\alpha X$ is

$$\begin{aligned}
 pr^\alpha X = & X + \eta^{x_1 x_1 x_1 x_1} \frac{\partial}{\partial u_{x_1 x_1 x_1 x_1}} + \eta^{x_2 x_2 x_2 x_2} \frac{\partial}{\partial u_{x_2 x_2 x_2 x_2}} \\
 & + \eta^{x_1 x_1 x_2 x_2 x_2} \frac{\partial}{\partial u_{x_1 x_1 x_2 x_2 x_2}} + \eta^{\alpha, t} \frac{\partial}{\partial D_t^\alpha u_{x_2}} \\
 & + \eta^{x_1} \frac{\partial}{\partial u_{x_1}} + \eta^{x_1 x_1} \frac{\partial}{\partial u_{x_1 x_1}} + \eta^{x_2} \frac{\partial}{\partial u_{x_2}} \\
 & + \eta^{x_2 x_2} \frac{\partial}{\partial u_{x_2 x_2}} + \eta^{y_1 y_1} \frac{\partial}{\partial u_{y_1 y_1}} + \eta^{y_2 y_2} \frac{\partial}{\partial u_{y_2 y_2}}.
 \end{aligned} \tag{25}$$

According to the specific expression of $pr^\alpha X$ in Equation (25), substituting Equation (24) into Equation (22), and then after the calculation, the following formula can be obtained:

$$\begin{aligned}
 16\eta^{\alpha, t} - \eta^{x_1 x_1 x_1 x_1} - \eta^{x_2 x_2 x_2 x_2} + 6\eta^{x_1 x_1 x_2 x_2 x_2} + 24\eta(u_{x_1 x_1} - u_{x_2 x_2}) \\
 + 24u(\eta^{x_1 x_1} - \eta^{x_2 x_2}) + 48(u_{x_1} \eta^{x_1} - u_{x_2} \eta^{x_2}) + 24(\eta^{y_2 y_2} - \eta^{y_1 y_1}) = 0.
 \end{aligned} \tag{26}$$

Substituting the specific expression of the extended infinitesimal parameters into Equation (26), the following determining equations can be obtained by equaling the coefficients of the partial derivatives of u of different orders to zero:

$$\begin{aligned}
 \zeta_{1uuuu} = \zeta_{2uuuu} = 4\zeta_{1x_1} = 4\tau_{x_1} = 4\zeta_{4x_1} = 48\tau_{y_1} = 24\zeta_{3uu} = -48\tau_{y_2} = -24\zeta_{4uu} \\
 = 4\tau_{x_2} = -36\tau_{x_2 x_2 x_1} = -18\tau_{x_2 x_2} = 4\zeta_{2x_2} = -18\zeta_{3x_2 x_2} = -36\zeta_{3x_2 x_2} = -18\zeta_{4x_2 x_1} \\
 = -18\zeta_{4x_2 x_2} = -36\zeta_{4x_2 x_2 x_1} = 4\zeta_{3x_2} = 4\zeta_{2x_2} = 0, \\
 48\eta_u - 48\zeta_{1x_1} + 4\zeta_{1ux_1 x_1 x_1} + 6\eta_{uux_1} - 6\zeta_{1x_2 x_2 x_2 u} = 0, \\
 -6\eta_{ux_1 x_1} + 4\zeta_{1x_1 x_1 x_1} - 12\zeta_{1x_2 x_2 x_2 x_1} + 6\eta_{x_2 x_2 x_2 u} + 24\eta = 0, \\
 -48\zeta_{2u} + 6\zeta_{2uux_2 x_2} - 4\eta_{uuux_2} - 18\zeta_{2uux_2 x_1} + 6\eta_{uuux_1 x_1} = 0, \\
 -48\zeta_{1u} + 6\zeta_{1uux_1 x_1} - 4\eta_{uuux_1} = 0, \\
 48\zeta_{2y_1} + 4\zeta_{3x_2 x_2 x_2} - 18\zeta_{3x_2 x_2 x_1 x_1} = 0, \\
 4\zeta_{3x_1 x_1 x_1} + 48\zeta_{1y_1} - 12\zeta_{3x_2 x_2 x_2 x_1} = 0, \\
 4\zeta_{4x_1 x_1 x_1 x_1} - 12\zeta_{4x_2 x_2 x_2 x_2 x_1} - 48\zeta_{1y_2} = 0, \\
 48\zeta_{2y_2} + 4\zeta_{4x_2 x_2 x_2} - 18\zeta_{4x_2 x_2 x_1 x_1} = 0, \\
 4\zeta_{1x_2} + 12\eta_{ux_1} - 36\zeta_{2x_1} - 6\zeta_{2x_1 x_1} = 0, \\
 12\zeta_{1ux_1} - 3\eta_{uu} = 0, \\
 48\zeta_{4y_1} - 48\zeta_{3y_2} = 0, \\
 48\zeta_{4y_1} - 48\zeta_{3y_2} = 0, \\
 -18\zeta_{3x_2} - 6\zeta_{3x_1} = 0, \\
 -18\zeta_{4x_2} - 6\zeta_{4x_1} = 0, \\
 -24\eta_u - 48\zeta_{3y_1} = 0, \\
 24\eta_u + 48\zeta_{4y_2} = 0, \\
 \binom{\alpha}{n} \partial_t^n (\eta^{x_2})_u - \binom{\alpha}{n+1} D_t^{\alpha+1}(\tau) = 0.
 \end{aligned} \tag{27}$$

Solving the above equations, we can obtain

$$\begin{aligned}
 \eta = 0, \zeta_1 = \frac{A_1}{9} x_2 + c_1, \zeta_2 = A_1 x_1 + c_2, \zeta_3 = A_2 y_2 + c_3, \\
 \zeta_4 = A_2 y_1 + c_4, \tau = A_3 t + A_4 u + c_5,
 \end{aligned} \tag{28}$$

where $A_i, c_j (i = 1, 2, 3, 4, j = 1, 2, 3, 4, 5)$ are arbitrary constants.

According to Equation (23), the corresponding infinitesimal generator has the following form:

$$X = (A_1x_2 + c_1) \frac{\partial}{\partial x_1} + \left(\frac{A_1}{9}x_1 + c_2\right) \frac{\partial}{\partial x_2} + (A_2y_2 + c_3) \frac{\partial}{\partial y_1} + (A_2y_1 + c_4) \frac{\partial}{\partial y_2} + (A_3t + A_4u + c_5) \frac{\partial}{\partial t}, \tag{29}$$

thus, we can obtain the corresponding Lie algebra that can be spanned by the following six vectors:

$$X_1 = x_2 \frac{\partial}{\partial x_1}, X_2 = \frac{A_1}{9} x_1 \frac{\partial}{\partial x_2}, X_3 = y_2 \frac{\partial}{\partial y_1}, X_4 = y_1 \frac{\partial}{\partial y_2}, X_5 = t \frac{\partial}{\partial t}, X_6 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} + (1 + u) \frac{\partial}{\partial t}. \tag{30}$$

3.2. Conservation Laws of the (4+1)-Dimensional Time-Fractional Fokas Equation

In this section, the conservation laws of (4+1)-dimensional time-fractional Fokas equation are constructed by the new conservation laws theorem. First, recalling some basic definitions.

Definition 1. The time-fractional integral with order $n - \alpha$ can be defined as [42]

$$I_T^{n-\alpha} f(t, x) = \frac{1}{\Gamma(n - \alpha)} \int_t^T \frac{f(\tau, x)}{(\tau - t)^{1+\alpha-n}} d\tau, \quad n = [\alpha] + 1, \tag{31}$$

where $\Gamma(x)$ is a gamma function.

Definition 2. A conservation laws for Equation (11) is defined as [42]

$$D_t(C^t) + D_{x_1}(C^{x_1}) + D_{x_2}(C^{x_2}) + D_{y_1}(C^{y_1}) + D_{y_2}(C^{y_2}) = 0, \tag{32}$$

where $C = (C^t, C^{x_1}, C^{x_2}, C^{y_1}, C^{y_2})$ is a conserved vector. According to the Noether operators, we can obtain the components $C^t, C^{x_1}, C^{x_2}, C^{y_1}$ and C^{y_2} of conserved vectors C :

$$C^t = \tau \mathcal{L} + \sum_{k=0}^{n-1} (-1)^k D_t^{\alpha-1-k}(W) D_t^k \left(\frac{\partial \mathcal{L}}{\partial (D_t^\alpha u)} \right) - (-1)^n J \left(W, D_t^n \left(\frac{\partial \mathcal{L}}{\partial (D_t^\alpha u)} \right) \right), \tag{33}$$

and $C^i (i$ stands for $x_1, x_2, y_1, y_2)$ can be defined as

$$C^i = \xi^i \mathcal{L} + W_\beta \left[\frac{\partial \mathcal{L}}{\partial u_i^\beta} - D_j \left(\frac{\partial \mathcal{L}}{\partial u_{ij}^\beta} \right) + D_j D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\beta} \right) - \dots \right] + D_j (W_\beta) \left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\beta} - D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\beta} \right) + \dots \right] + D_j D_k (W_\beta) \left[\frac{\partial \mathcal{L}}{\partial u_{ijk}^\beta} - \dots \right] + \dots, \tag{34}$$

where \mathcal{L} is a formal Lagrangian for Equation (11), $n = [\alpha] + 1$, $W = \eta - \xi_1 u_{x_1} - \xi_2 u_{x_2} - \xi_3 u_{y_1} - \xi_4 u_{y_2} - \tau u_t$ is a Lie characteristic function of $X = \xi_1 \partial_{x_1} + \xi_2 \partial_{x_2} + \xi_3 \partial_{y_1} + \xi_4 \partial_{y_2} + \tau \partial_t + \eta \partial_u$, and J is defined as follows:

$$J(f, g) = \frac{1}{\Gamma(n - \beta)} \int_0^x \int_x^p \frac{f(x, s)g(x, r)}{(r - s)^{\beta+1-n}} dr ds. \tag{35}$$

Definition 3. The definition of the Euler operator is as follows [44]:

$$\begin{aligned} \frac{\delta}{\delta u} = & \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - D_x \frac{\partial}{\partial u_x} - D_y \frac{\partial}{\partial u_y} + D_t^2 \frac{\partial}{\partial u_{tt}} \\ & + D_x^2 \frac{\partial}{\partial u_{xx}} + D_y^2 \frac{\partial}{\partial u_{yy}} - D_x^2 D_t \frac{\partial}{\partial u_{xx} \partial u_t} - D_y^2 D_x \frac{\partial}{\partial u_{yy} \partial u_x} + \dots, \end{aligned} \tag{36}$$

where D_t, D_x and D_y are the total derivative operators introduced in Equation (18).

Now, we start to construct the conservation laws of Equation (11) by using Lie point symmetry. A formal Lagrangian for Equation (11) is given in the following form:

$$\begin{aligned} \mathcal{L} = & q(x_1, x_2, y_1, y_2, t)(16D_t^\alpha u_{x_2} - u_{x_1 x_1 x_1 x_1} - u_{x_2 x_2 x_2 x_2} \\ & + 6u_{x_1 x_1 x_2 x_2 x_2} - 12(u^2)_{x_2 x_2} + 12(u^2)_{x_1 x_1} - 24u_{y_1 y_1} + 24u_{y_2 y_2}), \end{aligned} \tag{37}$$

where $q(x_1, x_2, y_1, y_2, t)$ is a new dependent variable. Next, we integrate the above equation and take advantage of the Agrawal fractional variational method [49] under the assumption that variable q is constant. Then, we can obtain the Euler–Lagrange operator about u ,

$$\begin{aligned} \frac{\delta}{\delta u} = & \frac{\partial}{\partial u} + (D_t^\alpha)^* D_{x_2} \frac{\partial}{\partial D_t^\alpha u_{x_2}} - D_{x_1} \frac{\partial}{\partial u_{x_1}} - D_{x_2} \frac{\partial}{\partial u_{x_2}} + D_{x_1 x_1} \frac{\partial}{\partial u_{x_1 x_1}} + D_{x_2 x_2} \frac{\partial}{\partial u_{x_2 x_2}} \\ & + D_{y_1 y_1} \frac{\partial}{\partial u_{y_1 y_1}} + D_{y_2 y_2} \frac{\partial}{\partial u_{y_2 y_2}} + D_{x_1 x_1 x_1 x_1} \frac{\partial}{\partial u_{x_1 x_1 x_1 x_1}} + D_{x_2 x_2 x_2 x_2} \frac{\partial}{\partial u_{x_2 x_2 x_2 x_2}} \\ & - D_{x_1 x_1 x_2 x_2 x_2} \frac{\partial}{\partial u_{x_1 x_1 x_2 x_2 x_2}}, \end{aligned} \tag{38}$$

where $(D_t^\alpha)^*$ is the adjoint operator of D_t^α

$$(D_t^\alpha)^* = (-1)^n I_T^{n-\alpha} (D_t^n) = {}^C D_T^\alpha. \tag{39}$$

The adjoint equation of Equation (11) can be given as

$$F^* = \frac{\delta \mathcal{L}}{\delta u} = 0. \tag{40}$$

So according to Equations (38) and (40), the adjoint equation of Equation (11) can be written as

$$\begin{aligned} F^* = & 16(D_t^\alpha)^* q_{x_2} - q_{x_1 x_1 x_1 x_1} - q_{x_2 x_2 x_2 x_2} - 6q_{x_1 x_1 x_2 x_2 x_2} \\ & + 24u(q_{x_1 x_1} - q_{x_2 x_2}) - 24(q_{y_1 y_1} - q_{y_2 y_2}). \end{aligned} \tag{41}$$

In Equation (32), we give the definition of conservation laws for Equation (11). Then, according to the definition of the Lie characteristic function and Equation (28), we obtain the following Lie characteristic function:

$$\begin{aligned} W_1 = & -x_2 u_{x_1}, W_2 = -\frac{x_1}{9} u_{x_2}, W_3 = -y_2 u_{y_2}, W_4 = -y_1 u_{y_1}, W_5 = -t u_t, \\ W_6 = & -u_{x_1} - u_{x_2} - u_{y_1} - u_{y_2} - (1 + u) u_t. \end{aligned} \tag{42}$$

Take an example of W_6 to obtain the conservation laws for Equation (11). Substituting W_6 into Equations (33) and (34), the conserved components with respect to x_1, x_2, y_1, y_2, t of conserved vector C are as follows, and the conserved component C^t is the following form:

$$\begin{aligned}
 C^t &= \tau \mathcal{L} + D_t^{\alpha-1}(W_6) \frac{\partial \mathcal{L}}{\partial D_t^\alpha u_{x_2}} + J \left(W_6, \frac{\partial \mathcal{L}}{\partial D_t^\alpha u_{x_2}} \right) \\
 &= 16q D_t^{\alpha-1} [-u_{x_1} - u_{x_2} - u_{y_1} - u_{y_2} - (1+u)u_t] \\
 &\quad + J [(-u_{x_1} - u_{x_2} - u_{y_1} - u_{y_2} - (1+u)u_t), 16q_t],
 \end{aligned}
 \tag{43}$$

the conserved component C^{x_1} is the following form:

$$\begin{aligned}
 C^{x_1} &= \xi_1 \mathcal{L} + W_6 \left[\frac{\partial \mathcal{L}}{\partial u_{x_1}} - D_{x_1} \frac{\partial \mathcal{L}}{\partial u_{x_1 x_1}} - D_{x_1} D_{x_1} D_{x_1} \frac{\partial \mathcal{L}}{\partial u_{x_1 x_1 x_1 x_1}} + D_{x_1} D_{x_2} D_{x_2} D_{x_2} \frac{\partial \mathcal{L}}{\partial u_{x_1 x_1 x_2 x_2 x_2}} \right] \\
 &\quad + D_{x_1}(W_6) \left[\frac{\partial \mathcal{L}}{\partial u_{x_1 x_1}} + D_{x_1} D_{x_1} \frac{\partial \mathcal{L}}{\partial u_{x_1 x_1 x_1 x_1}} - D_{x_2} D_{x_2} D_{x_2} \frac{\partial \mathcal{L}}{\partial u_{x_1 x_1 x_2 x_2 x_2}} \right] \\
 &\quad + D_{x_2}(W_6) \left[-D_{x_1} D_{x_2} D_{x_2} \frac{\partial \mathcal{L}}{\partial u_{x_1 x_1 x_2 x_2 x_2}} \right] + D_{x_1} D_{x_1}(W_6) \left[-D_{x_1} \frac{\partial \mathcal{L}}{\partial u_{x_1 x_1 x_1 x_1}} \right] \\
 &\quad + D_{x_1} D_{x_2}(W_6) \left[D_{x_2} D_{x_2} \frac{\partial \mathcal{L}}{\partial u_{x_1 x_1 x_2 x_2 x_2}} \right] + D_{x_2} D_{x_2}(W_6) \left[D_{x_1} D_{x_2} \frac{\partial \mathcal{L}}{\partial u_{x_1 x_1 x_2 x_2 x_2}} \right] \\
 &\quad + D_{x_1} D_{x_1} D_{x_1}(W_6) \left[\frac{\partial \mathcal{L}}{\partial u_{x_1 x_1 x_1 x_1}} \right] + D_{x_1} D_{x_2} D_{x_2}(W_6) \left[-D_{x_2} \frac{\partial \mathcal{L}}{\partial u_{x_1 x_1 x_2 x_2 x_2}} \right] \\
 &\quad + D_{x_2} D_{x_2} D_{x_2}(W_6) \left[-D_{x_1} \frac{\partial \mathcal{L}}{\partial u_{x_1 x_1 x_2 x_2 x_2}} \right] + D_{x_1} D_{x_2} D_{x_2} D_{x_2}(W_6) \left[\frac{\partial \mathcal{L}}{\partial u_{x_1 x_1 x_2 x_2 x_2}} \right] \\
 &= [-u_{x_1} - u_{x_2} - u_{y_1} - u_{y_2} - (1+u)u_t] (24u_{x_1}q - 24uq_{x_1} + q_{x_1 x_1 x_1} + 6q_{x_1 x_2 x_2 x_2}) \\
 &\quad + [-u_{x_1} - u_{x_2} - u_{y_1} - u_{y_2} - (1+u)u_t]_{x_1} (24uq - q_{x_1 x_1} - 6q_{x_2 x_2 x_2}) \\
 &\quad - 6q_{x_1 x_2 x_2} [-u_{x_1} - u_{x_2} - u_{y_1} - u_{y_2} - (1+u)u_t]_{x_2} \\
 &\quad + q_{x_1} [-u_{x_1} - u_{x_2} - u_{y_1} - u_{y_2} - (1+u)u_t]_{x_1 x_1} \\
 &\quad + 6q_{x_2 x_2} [-u_{x_1} - u_{x_2} - u_{y_1} - u_{y_2} - (1+u)u_t]_{x_1 x_2} \\
 &\quad + 6q_{x_1 x_2} [-u_{x_1} - u_{x_2} - u_{y_1} - u_{y_2} - (1+u)u_t]_{x_2 x_2} \\
 &\quad - q [-u_{x_1} - u_{x_2} - u_{y_1} - u_{y_2} - (1+u)u_t]_{x_1 x_1 x_1} \\
 &\quad - 6q_{x_2} [-u_{x_1} - u_{x_2} - u_{y_1} - u_{y_2} - (1+u)u_t]_{x_1 x_2 x_2} \\
 &\quad - 6q_{x_1} [-u_{x_1} - u_{x_2} - u_{y_1} - u_{y_2} - (1+u)u_t]_{x_2 x_2 x_2} \\
 &\quad + 6q [-u_{x_1} - u_{x_2} - u_{y_1} - u_{y_2} - (1+u)u_t]_{x_1 x_2 x_2 x_2}'
 \end{aligned}
 \tag{44}$$

the conserved component C^{x_2} is the following form:

$$\begin{aligned}
 C^{x_2} = & \zeta_2 \mathcal{L} + W_6 \left[\frac{\partial \mathcal{L}}{\partial u_{x_2}} - D_{x_2} \frac{\partial \mathcal{L}}{\partial u_{x_2 x_2}} - D_{x_2} D_{x_2} D_{x_2} \frac{\partial \mathcal{L}}{\partial u_{x_2 x_2 x_2}} + D_{x_2} D_{x_2} D_{x_1} D_{x_1} \frac{\partial \mathcal{L}}{\partial u_{x_1 x_1 x_2 x_2}} \right] \\
 & + D_{x_2} (W_6) \left[\frac{\partial \mathcal{L}}{\partial u_{x_2 x_2}} + D_{x_2} D_{x_2} \frac{\partial \mathcal{L}}{\partial u_{x_2 x_2 x_2}} - D_{x_2} D_{x_1} D_{x_1} \frac{\partial \mathcal{L}}{\partial u_{x_1 x_1 x_2 x_2}} \right] \\
 & + D_{x_1} (W_6) \left[-D_{x_2} D_{x_2} D_{x_1} \frac{\partial \mathcal{L}}{\partial u_{x_1 x_1 x_2 x_2}} \right] + D_{x_2} D_{x_1} (W_6) \left[D_{x_2} D_{x_1} \frac{\partial \mathcal{L}}{\partial u_{x_1 x_1 x_2 x_2}} \right] \\
 & + D_{x_2} D_{x_2} (W_6) \left[-D_{x_2} \frac{\partial \mathcal{L}}{\partial u_{x_2 x_2 x_2}} + D_{x_1} D_{x_1} \frac{\partial \mathcal{L}}{\partial u_{x_1 x_1 x_2 x_2}} \right] \\
 & + D_{x_1} D_{x_1} (W_6) \left[D_{x_2} D_{x_2} \frac{\partial \mathcal{L}}{\partial u_{x_1 x_1 x_2 x_2}} \right] + D_{x_2} D_{x_2} D_{x_2} (W_6) \left[\frac{\partial \mathcal{L}}{\partial u_{x_2 x_2 x_2}} \right] \\
 & + D_{x_2} D_{x_1} D_{x_1} (W_6) \left[-D_{x_2} \frac{\partial \mathcal{L}}{\partial u_{x_1 x_1 x_2 x_2}} \right] + D_{x_2} D_{x_2} D_{x_1} (W_6) \left[-D_{x_1} \frac{\partial \mathcal{L}}{\partial u_{x_1 x_1 x_2 x_2}} \right] \\
 & + D_{x_1} D_{x_1} D_{x_2} D_{x_2} (W_6) \left[\frac{\partial \mathcal{L}}{\partial u_{x_1 x_1 x_2 x_2}} \right] \\
 = & [-u_{x_1} - u_{x_2} - u_{y_1} - u_{y_2} - (1 + u)u_t] (-24u_{x_2}q + 24uq_{x_2} + q_{x_2 x_2 x_2} + 6q_{x_2 x_2 x_1 x_1}) \\
 & + [-u_{x_1} - u_{x_2} - u_{y_1} - u_{y_2} - (1 + u)u_t]_{x_2} (-24uq - q_{x_2 x_2} - 6q_{x_2 x_1 x_1}) \\
 & - 6q_{x_2 x_2 x_1} [-u_{x_1} - u_{x_2} - u_{y_1} - u_{y_2} - (1 + u)u_t]_{x_1} \\
 & + 6q_{x_2 x_1} [-u_{x_1} - u_{x_2} - u_{y_1} - u_{y_2} - (1 + u)u_t]_{x_2 x_1} \\
 & + [-u_{x_1} - u_{x_2} - u_{y_1} - u_{y_2} - (1 + u)u_t]_{x_2 x_2} (q_{x_2} + 6q_{x_1 x_1}) \\
 & + 6q_{x_2 x_2} [-u_{x_1} - u_{x_2} - u_{y_1} - u_{y_2} - (1 + u)u_t]_{x_1 x_1} \\
 & - q [-u_{x_1} - u_{x_2} - u_{y_1} - u_{y_2} - (1 + u)u_t]_{x_2 x_2 x_2} \\
 & - 6q_{x_2} [-u_{x_1} - u_{x_2} - u_{y_1} - u_{y_2} - (1 + u)u_t]_{x_2 x_1 x_1} \\
 & - 6q_{x_1} [-u_{x_1} - u_{x_2} - u_{y_1} - u_{y_2} - (1 + u)u_t]_{x_2 x_2 x_1} \\
 & + 6q [-u_{x_1} - u_{x_2} - u_{y_1} - u_{y_2} - (1 + u)u_t]_{x_1 x_1 x_2 x_2'}
 \end{aligned} \tag{45}$$

the conserved components C^{y_1} and C^{y_2} are the following forms:

$$\begin{aligned}
 C^{y_1} = & \zeta_3 \mathcal{L} + W_6 \left[-D_{y_1} \frac{\partial \mathcal{L}}{\partial u_{y_1 y_1}} \right] + D_{y_1} (W_6) \left[\frac{\partial \mathcal{L}}{\partial u_{y_1 y_1}} \right] \\
 = & 24q_{y_1} [-u_{x_1} - u_{x_2} - u_{y_1} - u_{y_2} - (1 + u)u_t] \\
 & - 24q [-u_{x_1} - u_{x_2} - u_{y_1} - u_{y_2} - (1 + u)u_t]_{y_1}, \\
 C^{y_2} = & \zeta_4 \mathcal{L} + W_6 \left[-D_{y_2} \frac{\partial \mathcal{L}}{\partial u_{y_2 y_2}} \right] + D_{y_2} (W_6) \left[\frac{\partial \mathcal{L}}{\partial u_{y_2 y_2}} \right] \\
 = & -24q_{y_2} [-u_{x_1} - u_{x_2} - u_{y_1} - u_{y_2} - (1 + u)u_t] \\
 & + 24q [-u_{x_1} - u_{x_2} - u_{y_1} - u_{y_2} - (1 + u)u_t]_{y_2}.
 \end{aligned} \tag{46}$$

4. Analytical Solutions for the (4+1)-Dimensional Time-Fractional Fokas Equation

In this section, the $\left(\frac{G'}{G}\right)$ -expansion method is given to obtain the analytical solutions of this (4+1)-dimensional time-fractional Fokas equation. The process of solving Equation (11) is as follows.

The complex fractional transformation can be introduced:

$$u(x_1, x_2, y_1, y_2, t) = U(\zeta), \zeta = \alpha_1 x_1 + \beta x_2 + \gamma y_1 + \delta y_2 + \frac{\varepsilon t^\alpha}{\Gamma(\alpha + 1)}, \tag{47}$$

where $\alpha_1, \beta, \gamma, \delta, \varepsilon$ are constants to be determined later and α is a fractional order.

So Equation (11) can be transformed into the following ordinary differential equation:

$$(16\beta\varepsilon - 24\gamma^2 + 24\delta^2)U_{\zeta\zeta} - (\alpha_1^4 + \beta^4)U_{\zeta\zeta\zeta} + 6\alpha_1^2\beta^3U_{\zeta\zeta\zeta\zeta} + 24(\alpha_1^2 - \beta^2)(U_\zeta)^2 + 24(\alpha_1^2 - \beta^2)UU_{\zeta\zeta} = 0. \tag{48}$$

Integrating Equation (48) twice with respect to ζ and considering the constant of integration to be zero, we have

$$(16\beta\varepsilon - 24\gamma^2 + 24\delta^2)U - (\alpha_1^4 + \beta^4)U_{\zeta\zeta} + 6\alpha_1^2\beta^3U_{\zeta\zeta\zeta} + 12(\alpha_1^2 - \beta^2)U^2 = 0. \tag{49}$$

Balancing $U_{\zeta\zeta\zeta}$ and U^2 in Equation (49), we have $n + 3 = 2n + 0$, which leads to $n = 3$. So assuming that the solutions of Equation (49) can be denoted by a polynomial in $\left(\frac{G'}{G}\right)$ as follows:

$$U(\zeta) = a_3\left(\frac{G'}{G}\right)^3 + a_2\left(\frac{G'}{G}\right)^2 + a_1\left(\frac{G'}{G}\right) + a_0, \quad a_3 \neq 0, \tag{50}$$

where $G = G(\zeta)$, and G satisfies the second-order ordinary differential equation:

$$G'' + \lambda G' + \mu G = 0, \tag{51}$$

in which $\lambda, \mu, a_0, a_1, a_2, a_3$ are constants to be determined later.

Substituting Equation (50) into Equation (49) and putting together items of the same order of $\left(\frac{G'}{G}\right)$, the left-hand side of Equation (49) is going to be a polynomial about $\left(\frac{G'}{G}\right)$. Let the coefficients of the polynomial be equal to zero, which leads to a set of algebraic equations. These algebraic equations are solved, and we have a set of nontrivial solutions as follows:

$$a_3 = \frac{18\alpha_1^2\beta^3}{(\alpha_1^2 - \beta^3)}, a_2 = \frac{3(\alpha_1^4 + \beta^4) + 234\alpha_1^2\beta^3\lambda}{4(\alpha_1^2 - \beta^2)}, \tag{52}$$

$$a_1 = -\frac{4500\alpha_1^4\beta^6\lambda^2 - 6048\alpha_1^4\beta^6\mu + 36\alpha_1^6\beta^3\lambda + 36\alpha_1^2\beta^7\lambda + \alpha_1^8 + 2\alpha_1^4\beta^4 + \beta^8}{208\alpha_1^3\beta^3(\alpha_1^2 - \beta^2)},$$

$$a_0 = a_0, \varepsilon = \varepsilon,$$

where $\alpha_1, \beta, \gamma, \delta, \lambda$ and μ are arbitrary constants.

According to the above results, we can obtain a set of exact solutions of Equation (11). For brevity, the specific expression for a_1 is not shown in the following exact solutions.

When $\lambda^2 - 4\mu > 0$, we have the hyperbolic function solution of the following form:

$$U(\zeta) = \frac{18\alpha_1^2\beta^3}{8(\alpha_1^2 - \beta^2)}A^3 + \left[\frac{126\alpha_1^2\beta^3\lambda + 3\alpha_1^4 + 3\beta^4}{16(\alpha_1^2 - \beta^2)}\right]A^2 - \left[\frac{180\alpha_1^2\beta^3\lambda^2 + 3\lambda\alpha_1^4 + 3\lambda\beta^4}{8(\alpha_1^2 - \beta^2)} - \frac{a_1}{2}\right]A + \frac{3\lambda^2(66\alpha_1^2\beta^3\lambda + \alpha_1^4 + \beta^4)}{16(\alpha_1^2 - \beta^2)} + a_0 - \frac{\lambda a_1}{2}, \tag{53}$$

where $A = \frac{\sqrt{\lambda^2 - 4\mu}(C_1 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\zeta + C_2 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\zeta)}{C_1 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\zeta + C_2 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\zeta}$, and c_1, c_2 are arbitrary constants.

When $\lambda^2 - 4\mu < 0$, we have the trigonometric function solution

$$\begin{aligned}
 U(\zeta) = & \frac{18\alpha_1^2\beta^3}{8(\alpha_1^2 - \beta^2)}B^3 + \left[\frac{126\alpha_1^2\beta^3\lambda + 3\alpha_1^4 + 3\beta^4}{16(\alpha_1^2 - \beta^2)} \right] B^2 \\
 & - \left[\frac{180\alpha_1^2\beta^3\lambda^2 + 3\lambda\alpha_1^4 + 3\lambda\beta^4}{8(\alpha_1^2 - \beta^2)} - \frac{a_1}{2} \right] B \\
 & + \frac{3\lambda^2(66\alpha_1^2\beta^3\lambda + \alpha_1^4 + \beta^4)}{16(\alpha_1^2 - \beta^2)} + a_0 - \frac{\lambda a_1}{2},
 \end{aligned} \tag{54}$$

where $B = \frac{\sqrt{4\mu - \lambda^2}(-C_1 \sin \frac{1}{2}\sqrt{4\mu - \lambda^2}\zeta + C_2 \cos \frac{1}{2}\sqrt{4\mu - \lambda^2}\zeta)}{C_1 \cos \frac{1}{2}\sqrt{4\mu - \lambda^2}\zeta + C_2 \sin \frac{1}{2}\sqrt{4\mu - \lambda^2}\zeta}$, and c_1, c_2 are arbitrary constants.

When $\lambda^2 - 4\mu = 0$, we have the following solution:

$$\begin{aligned}
 U(\zeta) = & \frac{18\alpha_1^2\beta^3}{\alpha_1^2 - \beta^2} \left(\frac{c_2}{c_1 + c_2\zeta} \right)^3 + \left[\frac{126\alpha_1^2\beta^3\lambda + 3\alpha_1^4 + 3\beta^4}{4(\alpha_1^2 - \beta^2)} \right] \left(\frac{c_2}{c_1 + c_2\zeta} \right)^2 \\
 & - \left[\frac{180\alpha_1^2\beta^3\lambda^2 + 3\lambda\alpha_1^4 + 3\lambda\beta^4}{4(\alpha_1^2 - \beta^2)} - a_1 \right] \left(\frac{c_2}{c_1 + c_2\zeta} \right) \\
 & + \frac{3\lambda^2(66\alpha_1^2\beta^2\lambda + \alpha_1^4 + \beta^4)}{16(\alpha_1^2 - \beta^4)} + a_0 - \frac{\lambda a_1}{2},
 \end{aligned} \tag{55}$$

where c_1, c_2 are arbitrary constants.

Taking solutions in Equation (53) as an example, we give the following images of the exact solutions. It is evident from Figure 1a–c that when the time changes slightly, both the amplitude and the shape of the waves vary markedly in the $x_1 - y_1$ plane with fixed $x_2 = y_2 = 9.7, \alpha = 0.9$. In addition, it can be seen from Figure 2a–c that the exact solutions can be influenced by fractional-order α of the (4+1)-dimensional time-fractional Fokas equation. As α increases, the waveform becomes regular, the amplitude increases, and the wavelength becomes longer. Therefore, it can be inferred that the nonlinear phenomena of the (4+1)-dimensional time-fractional Fokas equations studied in this paper are very complex and changeable in practical problems. Additionally, the study of Equation (11) is very necessary and meaningful.

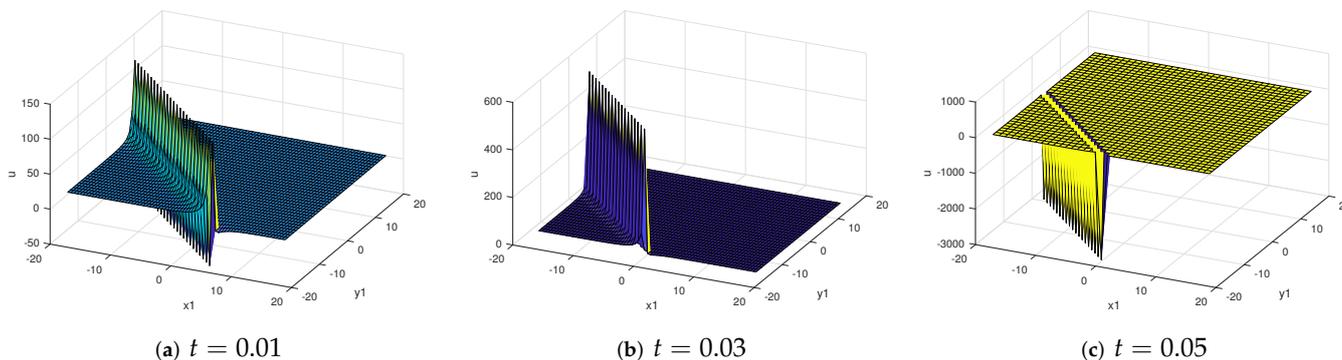


Figure 1. Exact solutions in Equation (53) at different time with $\alpha = 0.9, \alpha_1 = 1, \beta = 0.5, \gamma = 1, \delta = 0.5, \lambda = 1.5, \mu = 0.5, \epsilon = 85.746, a_0 = 1, c_1 = 1, c_2 = 2, x_2 = 9.7, y_2 = 9.7$.

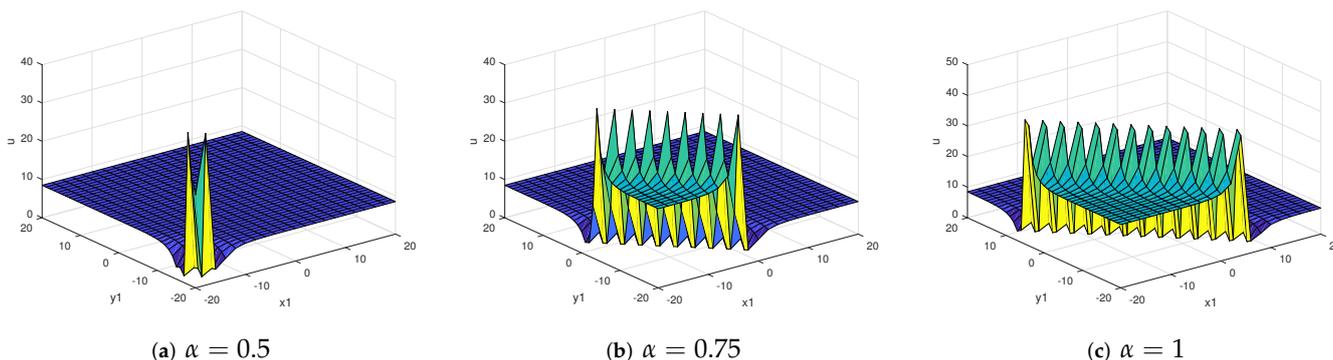


Figure 2. Exact solutions in Equation (53) for different α with $\alpha_1 = 1, \beta = 0.5, \gamma = 1, \delta = 0.5, \lambda = 1.5, \mu = 0.5, \varepsilon = 85.746, a_0 = 1, c_1 = 1, c_2 = 2, x_2 = 4, y_2 = 4, t = 0.1$.

5. Numerical Solutions

In this section, the Grünwald–Letnikov method for the Riemann–Liouville time-fractional derivative and Fourier spectral method for spatial derivative are proposed to obtain the numerical solutions. Recently, Sun et al. numerically solved three-dimensional seismic waves; the spectral method was used for the numerical calculation in the spatial direction [35]. The method proposed by them is different from the traditional spectral method, where the difference mainly lies in the discrete calculation for each spatial direction, fast Fourier transform and inverse fast Fourier transform operations for each spatial direction, respectively. This saves a lot of calculations, and it is very beneficial to the numerical calculation for high-dimensional problems. This makes it easier for some high-dimensional problems to be solved numerically, without the difficulty of the numerical calculation due to the increase in dimension. Inspired by this, we apply it to the (4+1)-dimensional time-fractional equation that we study in this paper.

Considering the (4+1)-dimensional time-fractional Fokas equation as

$$16D_t^\alpha u_{x_2} - u_{x_1 x_1 x_1 x_1} - u_{x_2 x_2 x_2 x_2} + 6u_{x_1 x_1 x_2 x_2 x_2} - 12(u^2)_{x_2 x_2} + 12(u^2)_{x_1 x_1} - 24u_{y_1 y_1} + 24u_{y_2 y_2} = 0, \quad (x_1, x_2, y_1, y_2) \in \Omega \subset R^4, t \in (0, T], \tag{56}$$

$$u(x_1, x_2, y_1, y_2, 0) = u_0(x_1, x_2, y_1, y_2), \quad (x_1, x_2, y_1, y_2) \in \partial\Omega \cup \Omega, \tag{57}$$

$$u(x_1, x_2, y_1, y_2, t) = \phi(x_1, x_2, y_1, y_2, t), \quad (x_1, x_2, y_1, y_2) \in \partial\Omega, t \in (0, T]. \tag{58}$$

5.1. Time Discretization

Take time-step $\tau = \frac{T}{N}$ with N as a positive integer and denote time points $t = n\tau (0 \leq n \leq N)$. The grid function can be given as $p = \{p^n | 0 \leq n \leq N\}$. Using the Grünwald–Letnikov method for the Riemann–Liouville time-fractional derivative operator D_t^α leads to

$$D_t^\alpha p^n \approx \tau^{-\alpha} \sum_{s=0}^n w_s^{(\alpha)} p^{n-s}, \tag{59}$$

where $w_s^{(\alpha)} = (-1)^s \binom{\alpha}{s}$, and $\binom{\alpha}{s} = \frac{\Gamma(\alpha+1)}{\Gamma(s+1)\Gamma(\alpha-s+1)}$.

5.2. Space Discretization

In space, we suppose the space domain $\Omega = [0, a] \times [0, b] \times [0, c] \times [0, d]$ and spatial mesh size $h_1 = \frac{a}{M_1}, h_2 = \frac{b}{M_2}, h_3 = \frac{c}{M_3}, h_4 = \frac{d}{M_4}$, where $h_1 = h_2 = h_3 = h_4$, and M_1, M_2, M_3, M_4 are both integral powers of 2. The grid points can be given as

$x_{1j} = jh_1(0 \leq j \leq M_1)$, $x_{2k} = kh_2(0 \leq k \leq M_2)$, $y_{1l} = lh_3(0 \leq l \leq M_3)$, $y_{2m} = mh_4(0 \leq m \leq M_4)$. Denote the index sets as

$$\begin{aligned} \hbar &= \{(j, k, l, m) | 0 < j < M_1, 0 < k < M_2, 0 < l < M_3, 0 < m < M_4\}, \\ \ell &= \{(j, k, l, m) | 0 \leq j \leq M_1, 0 \leq k \leq M_2, 0 \leq l \leq M_3, 0 \leq m \leq M_4\}, \\ \mathcal{L} &= \{j, k, l, m | j = 0, \text{ or } j = M_1; \text{ or } k = 0, \text{ or } k = M_2; \text{ or } l = 0, \text{ or } l = M_3; \text{ or } m = 0, \text{ or } m = M_4\}. \end{aligned} \tag{60}$$

So each grid point can be represented by its coordinate (j, k, l, m) . The grid function can be given as $v = \{v_{jklm} | (j, k, l, m) \in \ell\}$.

5.3. The Numerical Scheme

Taking the x_1 direction as an example, the numerical method of the space directions is illustrated. Denoting $k_{x_1} = \frac{2\pi r_1}{a}$, where $r_1 = -\frac{M_1}{2}, -\frac{M_1}{2} + 1, \dots, \frac{M_1}{2} - 1$, and there are M_1 grid points in the x_1 direction when fixed x_2, y_1, y_2 and t .

Step 1: Take out the value of u at each grid node in the x_1 direction (there are $M_2 * M_3 * M_4$ columns in totals, one column has M_1 values) and take the fast Fourier transform for each column of data. We know that when x_2, y_1, y_2 and t are fixed, the $u(x_1, x_{2k}, y_{1l}, y_{2m}, t_n)$ is a one-dimensional function of x_1 . So the fast Fourier transform for $u(x_1, x_{2k}, y_{1l}, y_{2m}, t_n)$ can be given as

$$F_{x_1}[u_j] = \sum_{i=0}^{M_1-1} u_j e^{-ik_{x_1} x_{1j}}, \tag{61}$$

where $k_{x_1} = \frac{2\pi r_1}{a}, -\frac{M_1}{2} \leq r_1 \leq \frac{M_1}{2} - 1$.

When taking the fast Fourier transform of each column of data, the fast Fourier transform of $u(x_1, x_2, y_1, y_2, t)$ in the x_1 direction is completed, called $F_{x_1}[u]$.

Step 2: The derivative of the Fourier transform for $u(x_1, x_{2k}, y_{1l}, y_{2m}, t_n)$ is as follows:

$$F_{x_1}[(u_{x_1 x_1 x_1 x_1})_j] = k_{x_1}^4 F_{x_1}[u_j], \tag{62}$$

so, we have

$$F_{x_1}[u_{x_1 x_1 x_1 x_1}] = k_{x_1}^4 F_{x_1}[u]. \tag{63}$$

Step 3: Inverse fast Fourier transform of $u(x_1, x_{2j}, y_{1l}, y_{2m}, t_n)$ can be given as

$$F_{x_1}^{-1}[u_j] = \frac{1}{M_1} \sum_{r_1=-\frac{M_1}{2}}^{\frac{M_1}{2}-1} F_{x_1}[u_j] e^{ik_{x_1} x_{1j}}, 0 \leq j \leq M_1 - 1, \tag{64}$$

where $k_{x_1} = \frac{2\pi r_1}{a}, -\frac{M_1}{2} \leq r_1 \leq \frac{M_1}{2} - 1$.

Similarly, when taking the inverse fast Fourier transform of each column of data, the inverse fast Fourier transform of $u(x_1, x_2, y_1, y_2, t)$ in the x_1 direction is completed, called $F_{x_1}^{-1}[u]$. So, for $u_{x_1 x_1 x_1 x_1}(x_1, x_2, y_1, y_2, t)$, we have $u_{x_1 x_1 x_1 x_1} = F_{x_1}^{-1}\{k_{x_1}^4 F_{x_1}[u]\}$.

Similarly, we have

$$\begin{aligned} u_{x_2 x_2 x_2 x_2} &= F_{x_2}^{-1}\{k_{x_2}^4 F_{x_2}[u]\}, u_{x_1 x_1 x_2 x_2 x_2} = F_{x_2}^{-1}\{ik_{x_2}^3 F_{x_2}\{F_{x_1}^{-1}\{-k_{x_1}^2 F_{x_1}[u]\}\}\}, \\ (u^2)_{x_1 x_1} &= F_{x_1}^{-1}\{-k_{x_1}^2 F_{x_1}[(u^2)]\}, (u^2)_{x_2 x_2} = F_{x_2}^{-1}\{-k_{x_2}^2 F_{x_2}[(u^2)]\}, \\ u_{y_1 y_1} &= F_{y_1}^{-1}\{-k_{y_1}^2 F_{y_1}[u]\}, u_{y_2 y_2} = F_{y_2}^{-1}\{-k_{y_2}^2 F_{y_2}[u]\}, \end{aligned} \tag{65}$$

where $k_{x_2} = \frac{2\pi r_2}{b}, k_{y_1} = \frac{2\pi r_3}{c}, k_{y_2} = \frac{2\pi r_4}{d}, r_2 = -\frac{M_2}{2}, -\frac{M_2}{2} + 1, \dots, \frac{M_2}{2} - 1, r_3 = -\frac{M_3}{2}, -\frac{M_3}{2} + 1, \dots, \frac{M_3}{2} - 1, r_4 = -\frac{M_4}{2}, -\frac{M_4}{2} + 1, \dots, \frac{M_4}{2} - 1$. $F_{x_2}[u], F_{y_1}[u]$ and $F_{y_2}[u]$ are Fast Fourier transform of $u(x_1, x_2, y_1, y_2, t)$ in x_2 direction, y_1 direction and y_2 direction, respectively.

Consider Equations (56)–(58) at the point $(x_{1j}, x_{2k}, y_{1l}, y_{2m}, t_n)$. Denote grid function $\{U_{jklm}^n = u(x_{1j}, x_{2k}, y_{1l}, y_{2m}, t_n) | (j, k, l, m) \in \ell, 0 \leq n \leq N\}$, and taking $v = D_t^\alpha u, V_{jklm}^n = D_t^\alpha U_{jklm}^n, (V_{x_2})_{jklm}^n = D_t^\alpha (U_{x_2})_{jklm}^n$, we have

$$\begin{aligned}
 16(V_{x_2})_{jklm}^n &= F_{x_1}^{-1}\{k_{x_1}^4 F_{x_1}[U_{jklm}^n]\} + F_{x_2}^{-1}\{k_{x_2}^4 F_{x_2}[U_{jklm}^n]\} \\
 &\quad - 6F_{x_2}^{-1}\{ik_{x_2}^3 F_{x_2}\{F_{x_1}^{-1}\{-k_{x_1}^2 F_{x_1}[U_{jklm}^n]\}\}\} \\
 &\quad + 12F_{x_2}^{-1}\{-k_{x_2}^2 F_{x_2}[(U_{jklm}^n)^2]\} - 12F_{x_1}^{-1}\{-k_{x_1}^2 F_{x_1}[(U_{jklm}^n)^2]\} \\
 &\quad + 24F_{y_1}^{-1}\{-k_{y_1}^2 F_{y_1}[U_{jklm}^n]\} - 24F_{y_2}^{-1}\{-k_{y_2}^2 F_{y_2}[U_{jklm}^n]\}, \\
 (j, k, l, m) &\in \mathfrak{h}, 1 \leq n \leq N,
 \end{aligned} \tag{66}$$

$$U_{jklm}^0 = u_0(x_{1j}, x_{2k}, y_{1l}, y_{2m}), \quad (j, k, l, m) \in \ell, \tag{67}$$

$$U_{jklm}^n = \phi(x_{1j}, x_{2k}, y_{1l}, y_{2m}, t_n), \quad (j, k, l, m) \in \mathcal{L}, 0 \leq n \leq N. \tag{68}$$

For the sake of simplicity, we have

$$\begin{aligned}
 A^n &= F_{x_1}^{-1}\{k_{x_1}^4 F_{x_1}[U_{jklm}^n]\} + F_{x_2}^{-1}\{k_{x_2}^4 F_{x_2}[U_{jklm}^n]\} \\
 &\quad - 6F_{x_2}^{-1}\{ik_{x_2}^3 F_{x_2}\{F_{x_1}^{-1}\{-k_{x_1}^2 F_{x_1}[U_{jklm}^n]\}\}\} \\
 &\quad + 12F_{x_2}^{-1}\{-k_{x_2}^2 F_{x_2}[(U_{jklm}^n)^2]\} - 12F_{x_1}^{-1}\{-k_{x_1}^2 F_{x_1}[(U_{jklm}^n)^2]\} \\
 &\quad + 24F_{y_1}^{-1}\{-k_{y_1}^2 F_{y_1}[U_{jklm}^n]\} - 24F_{y_2}^{-1}\{-k_{y_2}^2 F_{y_2}[U_{jklm}^n]\}, \\
 (j, k, l, m) &\in \mathfrak{h}, 1 \leq n \leq N.
 \end{aligned} \tag{69}$$

So

$$16(V_{x_2})_{jklm}^n = A^n, \quad (j, k, l, m) \in \mathfrak{h}, 1 \leq n \leq N, \tag{70}$$

$$U_{jklm}^0 = u_0(x_{1j}, x_{2k}, y_{1l}, y_{2m}), \quad (j, k, l, m) \in \ell, \tag{71}$$

$$U_{jklm}^n = \phi(x_{1j}, x_{2k}, y_{1l}, y_{2m}, t_n), \quad (j, k, l, m) \in \mathcal{L}, 0 \leq n \leq N. \tag{72}$$

Taking the Fast Fourier transform and the Inverse Fourier transform both sides of Equation (70) with respect to the x_2 direction leads to

$$-16ik_{x_2} F_{x_2}[V_{jklm}^n] = F_{x_2}[A^n], \quad (j, k, l, m) \in \mathfrak{h}, 1 \leq n \leq N, \tag{73}$$

$$V_{jklm}^n = F_{x_2}^{-1}\left\{\frac{F_{x_2}[A^n]}{-16ik_{x_2}}\right\}, \quad (j, k, l, m) \in \mathfrak{h}, 1 \leq n \leq N, \tag{74}$$

$$U_{jklm}^n \approx \left(\tau^\alpha V_{jklm}^n - \sum_{s=1}^n w_s^{(\alpha)} U_{jklm}^{n-s}\right) / w_0^{(\alpha)}, \quad (j, k, l, m) \in \mathfrak{h}, 1 \leq n \leq N, \tag{75}$$

$$U_{jklm}^0 = u_0(x_{1j}, x_{2k}, y_{1l}, y_{2m}), \quad (j, k, l, m) \in \ell, \tag{76}$$

$$U_{jklm}^n = \phi(x_{1j}, x_{2k}, y_{1l}, y_{2m}, t_n), j = 0, \quad (j, k, l, m) \in \mathcal{L}, 0 \leq n \leq N. \tag{77}$$

Replacing U_{jklm}^n with u_{jklm}^n and replacing V_{jklm}^n with v_{jklm}^n , we have

$$-16ik_{x_2} F_{x_2}[v_{jklm}^n] = F_{x_2}[A^n], \quad (j, k, l, m) \in \mathfrak{h}, 1 \leq n \leq N, \tag{78}$$

$$v_{jklm}^n = F_{x_2}^{-1}\left\{\frac{F_{x_2}[A^n]}{-16ik_{x_2}}\right\}, \quad (j, k, l, m) \in \mathfrak{h}, 1 \leq n \leq N, \tag{79}$$

$$u_{jklm}^n = \left(\tau^\alpha v_{jklm}^n - \sum_{s=1}^n w_s^{(\alpha)} u_{jklm}^{n-s}\right) / w_0^{(\alpha)}, \quad (j, k, l, m) \in \mathfrak{h}, 1 \leq n \leq N, \tag{80}$$

$$u_{jklm}^0 = u_0(x_{1j}, x_{2k}, y_{1l}, y_{2m}), \quad (j, k, l, m) \in \ell, \tag{81}$$

$$u_{jklm}^n = \phi(x_{1j}, x_{2k}, y_{1l}, y_{2m}, t_n), \quad (j, k, l, m) \in \ell, 0 \leq n \leq N. \tag{82}$$

5.4. Numerical Results

Periodic boundary conditions need to be considered when using the Fourier spectral method. In order to satisfy the periodic boundary condition, we consider the following two examples to illustrate the effectiveness of our numerical method proposed in Section 5.3.

Example 1. An unbounded (4+1)-dimensional time-fractional Fokas equation based on Equation (11) can be described as follows:

$$16D_t^\alpha u_{x_2} - u_{x_1x_1x_1x_1} - u_{x_2x_2x_2x_2} + 6u_{x_1x_1x_2x_2x_2} - 12(u^2)_{x_2x_2} + 12(u^2)_{x_1x_1} - 24u_{y_1y_1} + 24u_{y_2y_2} = f(x_1, x_2, y_1, y_2, t), \tag{83}$$

where $(x_1, x_2, y_1, y_2) \in R^4, t \in (0, T]$ and the exact solution of Equation (83) is

$$u(x_1, x_2, y_1, y_2, t) = \operatorname{sech}^2\left(x_1 + x_2 + y_1 + y_2 + \frac{2t^\alpha}{\Gamma(1 + \alpha)} - 6\right), \tag{84}$$

where $\Gamma(x)$ is gamma function.

The initial condition, the boundary conditions and $f(x_1, x_2, y_1, y_2, t)$ are determined by exact solutions in Equation (84). In order to illustrate the validity of the numerical method proposed above, we compare the numerical solutions with the exact solutions in Equation (84). When $\alpha = 0.75, \alpha = 0.8, \alpha = 0.9$ and $\alpha = 1$, the maximum absolute errors of the time-fractional (4+1)-dimensional partial differential Fokas equation are shown in Table 1. It can be seen from the table that the numerical results given by the Fourier spectral method and Grünwald–Letnikov method are satisfactory, which also shows the validity of this numerical method. In addition, some images of the comparison between numerical solutions and exact solutions for different fractional order α are given: $\alpha = 0.75$ in Figure 3a, $\alpha = 0.8$ in Figure 3b, $\alpha = 0.9$ in Figure 3c, and $\alpha = 1$ in Figure 3d. From these images, we can see that as α increases, the curves of the exact solutions and numerical solutions become closer. Additionally, the curve of the numerical solutions and the curve of the exact solutions are almost coincident with $\alpha = 1$.

Table 1. The maximum absolute errors between the numerical solutions and the exact solutions in Equation (84) for different fractional order α .

α	Errors	α	Errors
0.75	5.8927×10^{-5}	0.9	3.1545×10^{-5}
0.8	5.2142×10^{-5}	1	8.6031×10^{-11}

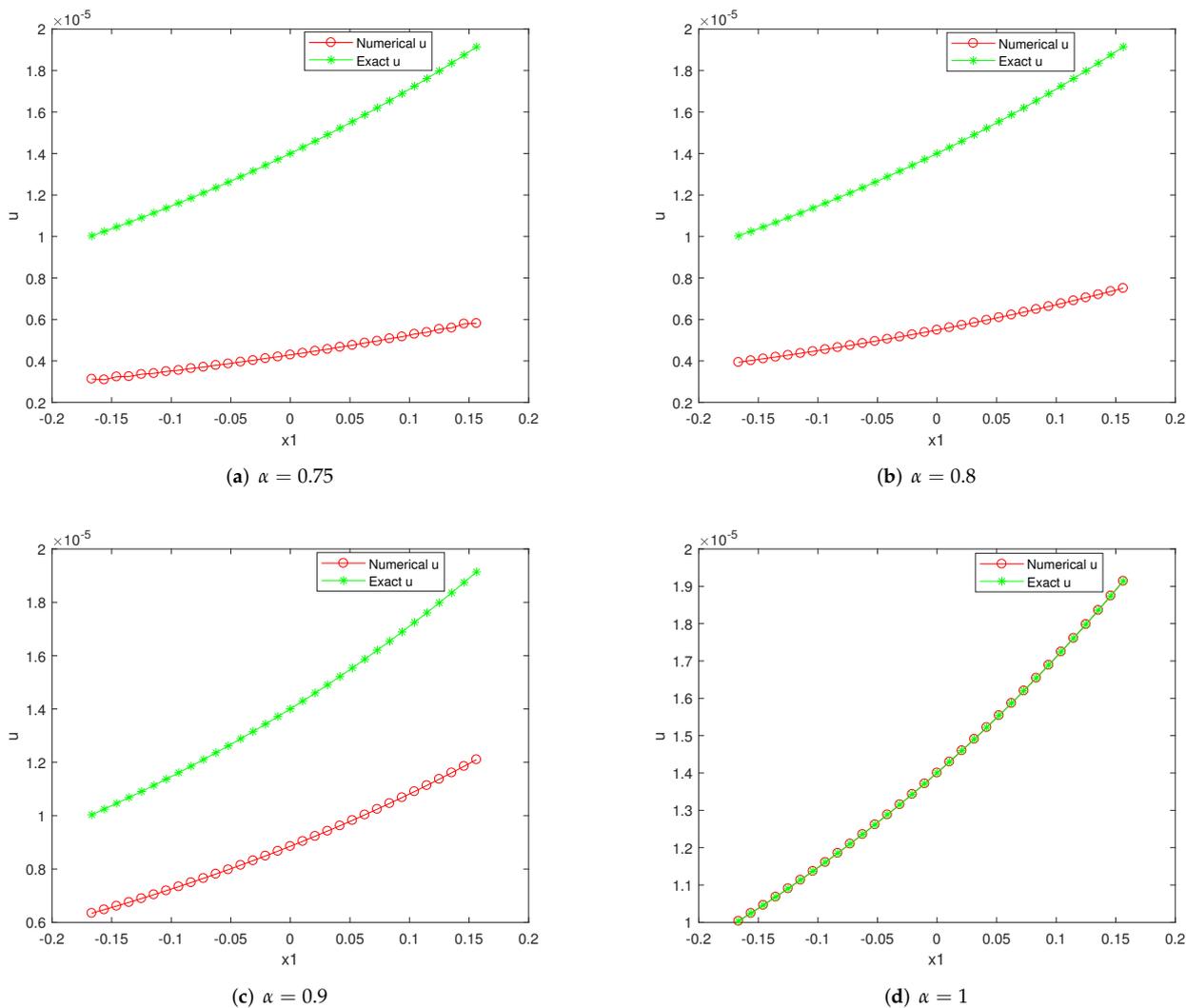


Figure 3. Example 1. Comparison of the numerical solutions and the exact solutions in Equation (84) at the end time for different fractional order α with $x_2 = 1/24, y_1 = 1/12, y_2 = 3/24$.

Example 2. A bounded periodic (4+1)-dimensional time-fractional Fokas equation based on Equation (11) can be given as

$$16D_t^\alpha u_{x_2} - u_{x_1 x_1 x_1 x_1} - u_{x_2 x_2 x_2 x_2} + 6u_{x_1 x_1 x_2 x_2 x_2} - 12(u^2)_{x_2 x_2} + 12(u^2)_{x_1 x_1} - 24u_{y_1 y_1} + 24u_{y_2 y_2} = g(x_1, x_2, y_1, y_2, t), \tag{85}$$

where $(x_1, x_2, y_1, y_2) \in [0, 2\pi] \times [0, 2\pi] \times [0, 2\pi] \times [0, 2\pi], t \in (0, t]$ and the exact solution of Equation (85) is

$$u(x_1, x_2, y_1, y_2, t) = e^{-\frac{t^\alpha}{\Gamma(1+\alpha)}} \sin(x_1 + x_2 + y_1 + y_2), \tag{86}$$

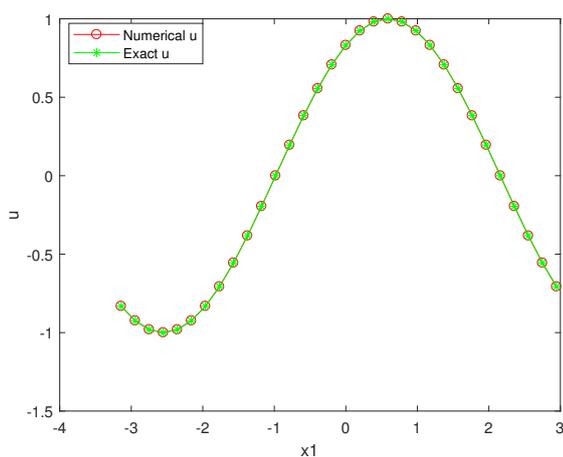
where $\Gamma(x)$ is gamma function.

The initial condition, the boundary conditions and $g(x_1, x_2, y_1, y_2, t)$ are determined by exact solutions in Equation (86). As in Example 1, for different fractional order α , the maximum absolute errors between the numerical solutions and the exact solutions in Equation (86) are given in Table 2, and the comparison images between the numerical solutions and exact solutions in Equation (86) are given in Figure 4. We can see from the table that the error results under different fractional order α are acceptable, and it is obvious

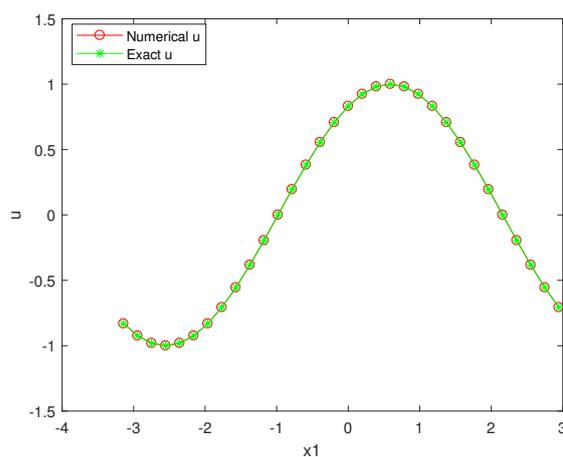
that the curve-fitting results in the figure are satisfactory. The feasibility of the proposed numerical method can be also judged from the error results and curve-fitting results.

Table 2. The maximum absolute errors between the numerical solutions and the exact solutions in Equation (86) for different fractional order α .

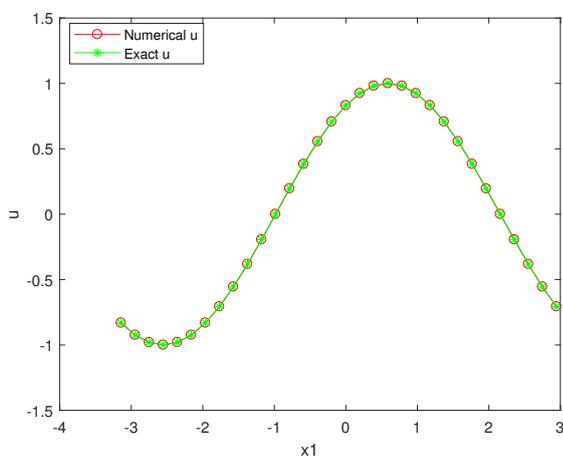
α	Errors	α	Errors
0.75	1.5015×10^{-5}	0.9	1.6518×10^{-8}
0.8	1.3549×10^{-7}	1	2.0000×10^{-9}



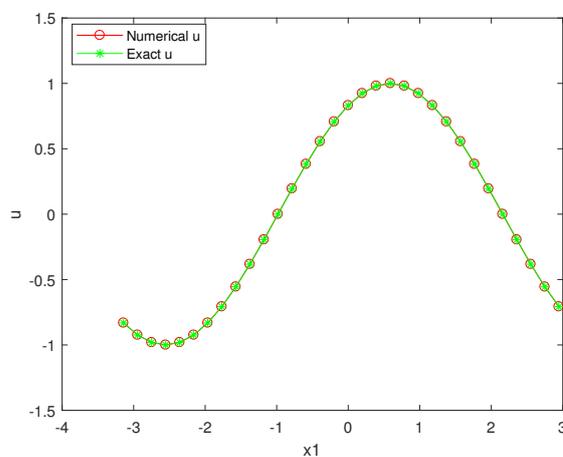
(a) $\alpha = 0.75$



(b) $\alpha = 0.8$



(c) $\alpha = 0.9$



(d) $\alpha = 1$

Figure 4. Example 2. Comparison of the numerical solutions and the exact solutions in Equation (86) at the end time for different fractional order α with $x_2 = 1/24, y_1 = 1/12, y_2 = 3/24$.

6. Conclusions

In this paper, we studied an uncommon Fokas equation. The (4+1)-dimensional time-fractional Fokas equation in the sense of the Riemann–Liouville fractional derivative was derived in detail for the first time. For the (4+1)-dimensional time-fractional Fokas equation, the Lie symmetry analysis method was used to investigate the symmetry of the equation, and at the same time, we discussed the conservational laws of the equation. In addition, several exact traveling wave solutions were obtained by using the $\left(\frac{G'}{G}\right)$ -expansion method, and numerical solutions were obtained by using the Fourier spectral method

and the Grünwald–Letnikov method. The error results between numerical solutions and exact solutions showed the effectiveness of the numerical method considered in this paper, and the numerical method may be helpful for the study of this kind of high-dimensional problem.

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