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A New Approach to Compare the Strong Convergence of the Milstein Scheme with the Approximate Coupling Method

Yousef Alnafisah

Department of Mathematics, College of Science, Qassim University, P.O. Box 6644, Buraydah 51452, Saudi Arabia; nfiesh@qu.edu.sa

Abstract: Milstein and approximate coupling approaches are compared for the pathwise numerical solutions to stochastic differential equations (SDE) driven by Brownian motion. These methods attain an order one convergence under the nondegeneracy assumption of the diffusion term for the approximate coupling method. We use MATLAB to simulate these methods by applying them to a particular two-dimensional SDE. Then, we analyze the performance of both methods and the amount of time required to obtain the result. This comparison is essential in several areas, such as stochastic analysis, financial mathematics, and some biological applications.

Keywords: stochastic differential equation; approximate coupling method; Milstein scheme

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1. Introduction

It is observed in the literature that the research studies investigating methods for solving stochastic differential equations (SDEs) are progressing rapidly and are attracting the interest of many researchers working in this field. Recently, numerical solutions to stochastic differential equations have become popular with computing simulations. The solution of SDEs has potential applications in many fields, such as economics, finance, and physics [1–3]. Some studies have been conducted to find strong solutions to stochastic differential equations to obtain approximations of an order greater than $\frac{1}{2}$. In [1,4,5], the authors developed new methods and used the truncation of the related transforms of the stochastic process to approximate double integrals in higher dimensions. However, these methods required significant computational time. In [6], Fournier used the quadratic Wasserstein metric approach to approximate the Euler scheme. In [7], Davie described the application of the Wasserstein bound to approximate the solutions of SDE and used a version of the method in [2] to obtain order one approximation under some assumptions. Yang et al. [8] used the Itô-Taylor expansion with a specific condition to approximate the densities of multivariate. Under some conditions, Alfonsi et al. [9,10] developed the Wasserstein convergence for the Euler–Maruyama scheme and proved an $O(h^{(\frac{2}{3}-\epsilon)})$ convergence for one-dimensional diffusions. Gaines and Lyons [11] developed a new method for two-dimensional SDEs using the rectangle-wedge-tail method. A new method for solving two-dimensional SDEs using the condition on the endpoints was presented in [12]. In [13], the coupling method was used to establish the bounds of an approximate pathwise solution in a given probability space. Some simulation methods for the stochastic differential equation have been studied in [14]. The MATLAB implementation for the Euler and Milstein methods in one- and two-dimensional SDEs was introduced in [15]. Readers interested in knowing more about the simulation of stochastic differential equations can refer to [16,17]. Recently, Kerimkulov et al. [18] proposed a modification to the MSA method based on meticulous estimates for the backward stochastic differential equation. For broad stochastic control problems with control in both the drift and diffusion coefficients, this improved MSA is demonstrated to converge. In [19], the rate of convergence results for a new class of explicit Euler schemes that approximate SDEs with superlinearly rising drift coefficients that meet a certain form of strong monotonicity are described.

There are many applications for finding numerical solutions to stochastic differential equations using several innovative methods, see [20–28].

Based on the Milstein method, references [29–32] provide approximate solutions to some stochastic differential equations. In [29], the infinite-dimensional version of Milstein's approach for finite-dimensional stochastic ordinary differential equations is investigated. Guo et al. [30] suggested the truncated Milstein approach, which was inspired by Mao's [33] truncated Euler–Maruyama method. Alnafisah [31] showed how the Milstein approach may be utilized to simulate a two-dimensional SDE using the Fourier series expansion of the Wiener process. To numerically solve the system that consists of replication of several reacting species using activated monomers and inactivated residues, Zahri [32] considered a generalized Milstein method for multi-dimensional SDEs.

In this paper, two numerical methods, Milstein and approximate coupling, are used and compared. This comparison is based on the first-order Milstein method using a Wasserstein matrix with the condition that SDE has invertible diffusion. We show the MATLAB implementation for both methods and compare the result as well as the computational time. We used (MATLAB ver. R2017b) software to obtain the implementation and approximation results. The importance of these comparisons is that they determine the most appropriate method for use in many vital applications in stochastic analysis, financial mathematics, and some biological applications.

The remainder of this article is structured in the following manner. In Section 2, we review various findings concerning SDEs and the Davie method [34]. In Section 4, the comparison between the Milstein and approximate coupling methods is presented. In addition, we provide a numerical implementation to illustrate the convergence behavior in two-dimensional SDEs.

2. Preliminaries

In this section, we provide background information relevant to this study, see [1,3]. Throughout this paper, N denotes the normal distribution, and E represents the expectation.

A standard Brownian motion (Weiner process) over an interval $[0,T]$ is a random variable $\psi(t)$, which depends continuously on a time $t \in [0,T)$, if the following conditions are satisfied:

(i) $\psi(0) = 0$ (with probability one).
(ii) The random variable is given by increment $\psi(u) - \psi(\nu)$, for $0 \leq \nu < u \leq T$, is normally distributed with mean zero and variance $u - \nu$. Equivalently, $\psi(u) - \psi(\nu) \approx \mathrm{N}(0, u-\nu)$.
(iii) The increments $\psi(u) - \psi(\nu)$ and $\psi(t) - \psi(s)$ for $0 \leq \nu < u < s < t \leq T$.

The stochastic process $\psi = \psi(u)$, which is considered in this work, can be described by stochastic differential equations

$$\mathrm{d}\psi(u) = \eta(u, \psi(u))\mathrm{d}u + \mu(u, \psi(u))\mathrm{d}\varphi(u), \quad u \in [0,T], \tag{1}$$

where $\{\varphi(u)\}_{u \geq 0}$ is a m-dimensional standard Brownian motion with probability space (Ω, F, P) equipped with a filtration $F = (F_u)_{u \geq 0}$, $\eta = \eta(u,\nu)$ is a m-dimensional vector function, and $\mu = \mu(u,\nu)$ is a $m \times m$-matrix function. These functions are called the drift coefficient and diffusion coefficient, respectively.

A $\mathcal{F}_u$-adapted stochastic process $\psi = (\psi(u))_{u \geq 0}$ is called a solution of Equation (1), if

$$\psi(u) = \psi(0) + \int_0^u \eta(\varrho, \psi(\varrho))\mathrm{d}\varrho + \int_0^u \mu(\varrho, \psi(\varrho))\mathrm{d}\varphi(\varrho), \tag{2}$$

holds, where the initial condition $\psi(0) = \nu$ is an $\mathcal{F}_0$-measurable random vector in $\mathbb{R}^m$. We note that the integral processes

$$\int_0^u \eta(\varrho, \psi(\varrho))\mathrm{d}\varrho \quad \textit{and} \quad \int_0^u \mu(\varrho, \psi(\varrho))\mathrm{d}\varphi(\varrho),$$

must be well-defined in order for (2) to be satisfied. The functions $\eta(\varrho, \psi(\varrho))$ and $\mu(\varrho, \psi(\varrho))$ must satisfy the following conditions:

$$E \int_0^u \mu^2(\varrho, \psi(\varrho)) \mathrm{d}\varrho < \infty,$$

and almost surely for all $u \geq 0$,

$$\int_0^u |\eta(\varrho, \psi(\varrho))| \mathrm{d}\varrho < \infty.$$

One property for the stochastic integral is

$$\int_0^u \varphi(\varrho) \mathrm{d}\varphi(\varrho) = \frac{1}{2} \int_0^u \mathrm{d}(\varphi^2(\varrho)) - \frac{1}{2} \int_0^u \mathrm{d}\varrho = \frac{1}{2} \varphi^2(u) - \frac{u}{2}.$$

For more details on the stochastic integral, the interested reader is referred to [1].

To present the existence and uniqueness theorems, we need the following conditions:

1. Measurability: Let $\eta : [0, \infty) \times \mathbb{R}^m \to \mathbb{R}^m$ and $\mu : [0, \infty) \times \mathbb{R}^m \to \mathbb{R}^{m \times m}$ be jointly Borel measurable in $[u_0, T] \times \mathbb{R}^m$.
2. Lipschitz condition: There is a positive constant $C > 0$ such that $|\eta(u, \nu) - \eta(u, y)| \leq C|\nu - y|$, and $|\mu(u, \nu) - \mu(u, y)| \leq C|\nu - y|$, for all $u \in [u_0, T]$ and $\nu, y \in \mathbb{R}$.
3. Growth condition: There is a constant $K > 0$ such that $|\eta(u, \nu)|^2 \leq K^2(1 + |\nu|^2)$, and $|\mu(u, \nu)|^2 \leq K^2(1 + |\nu|^2)$, for all $u \in [u_0, T]$ and $\nu, y \in \mathbb{R}$.

Theorem 1 ([1], Theorem 4.5.3)**.** *Under the previous conditions (1)–(3), the stochastic differential Equation (1) has a unique solution $\psi(u) \in [u_0, T]$ with*

$$\sup_{u_0 \leq u \leq u} E(|\psi(u)|^2) < \infty.$$

2.1. Approximation Schemes

In this subsection, we briefly review the schemes of the Euler–Maruyama, Milstein and Davie methods. Consider the stochastic differential equation

$$\mathrm{d}\psi_i(u) = \eta_i(u, \psi(u)) \mathrm{d}u + \sum_{k=1}^{m} \mu_{ik}(u, \psi(u)) \mathrm{d}\varphi_k(u), \ \psi_i(0) = \psi_i^{(0)}, \tag{3}$$

where $u \in [0, T]$, $\psi(u)$ is an m-dimensional vector, and $\varphi(u)$ is an m-dimensional driving Brownian path. Moreover, the coefficients $\mu_{ik}(u, \psi(u))$ satisfy the global Lipschitz conditions

$$|\eta(u, \nu) - \eta(u, y)| \leq A|\nu - y|$$

and

$$|\mu(u, \nu) - \mu(u, y)| \leq A|\nu - y|,$$

for all $u \in [u_0, T]$ and $\nu, y \in \mathbb{R}$, where $A > 0$ is a constant. If η_i and μ_i are continuous in u, for each ψ, then the Equation (3) has a unique solution $\psi(u)$. This is a process adapted to the filtration induced by Brownian motion. Under these conditions, the solution satisfies $E(|\psi(u)|^p) < \infty$ for each $p \in [1, \infty]$ and $u \in [0, T]$. The standard method for the pathwise approximation of the solution of Equation (3) is to divide $[0, T]$ into a finite number of N of equal intervals with length $h = T/N$. The simplest form of such approximation for the SDE by using only the linear term in the Taylor expansion, gives the following Euler–Maruyama scheme

$$\nu_i^{(j+1)} = \nu_i^{(j)} + \sum_{k=1}^{m} \mu_{ik}(\nu^{(j)}) \Delta\varphi_k^{(j)}, \tag{4}$$

where $\Delta\varphi_k^{(j)} = \varphi_k((j+1)h) - \varphi_k(jh)$. Now, we represent a scheme that is proposed by Milstein and gives an order one strong Taylor scheme.

$$\nu_i^{(j+1)} = \nu_i^{(j)} + \eta_i(jh, \nu^{(j)})h + \sum_{k=1}^{m} \mu_{ik}(jh, \nu^{(j)})\Delta\varphi_k^{(j)} + \sum_{k,l=1}^{m} \rho_{ikl}(jh, \nu^{(j)})M_{kl}^{(j)}, \tag{5}$$

where

$$\begin{aligned} \Delta\varphi_k^{(j)} &: = \varphi_k((j+1)h) - \varphi_k(jh), \\ M_{kl}^{(j)} &: = \int_{jh}^{(j+1)h} \{\varphi_k(u) - \varphi_k(jh)\} \mathrm{d}\varphi_l(u), \end{aligned}$$

and

$$\rho_{ikl}(u, \nu) := \sum_{m=1}^{q} \mu_{mk}(u, \nu) \frac{\partial \mu_{il}}{\partial \nu_m}(u, \nu).$$

If the following condition

$$\rho_{ikl}(u, \nu) = \rho_{ilk}(u, \nu), \tag{6}$$

for all $\nu \in \mathbb{R}^m$, $u \in [0, T]$ and all i, k, l holds, then the Milstein scheme reduces to

$$\nu_i^{(j+1)} = \nu_i^{(j)} + \eta_i(jh, \nu^{(j)})h + \sum_{k=1}^{m} \mu_{ik}(jh, \nu^{(j)})\Delta\varphi_k^{(j)} + \sum_{k,l=1}^{m} \rho_{ikl}(jh, \nu^{(j)})B_{kl}^{(j)}. \tag{7}$$

This is dependent on the generation of the Brownian motion $\Delta\varphi_k^{(j)}$. It can be implemented for the Milstein method using Brownian motion $\Delta\varphi_k^{(j)}$ and a unique set of equations. This comes from the observation that $M_{kl}^{(j)} + A_{lk}^{(j)} = 2B_{kl}^{(j)}$ where $B_{kl}^{(j)} = \frac{1}{2}\Delta\varphi_k^{(j)}\Delta\varphi_l^{(j)}$, for $k \neq l$ and $B_{kk}^{(j)} = \frac{1}{2}\{(\Delta\varphi_k^{(j)})^2 - h\}$. Scheme (7) achieves an order of 1, for $m = 1$. However, for the dimension $m > 1$, we obtain the order $\frac{1}{2}$. According to Davie's approximate coupling method, we could modify the previous scheme (7). This gives order one under invertible diffusion conditions.

One can implement the Milstein scheme by separately generating the random variables $\Delta\varphi_k^{(j)}$ and $M_{kl}^{(j)}$ and combining them to obtain the RHS of the scheme (7). According to Davie's (approximate coupling) method, we attempt to directly generate the following:

$$Y := \sum \mu_{ik}(jh, \nu^{(j)})\Delta\varphi_k^{(j)} + \sum \rho_{ikl}(jh, \nu^{(j)})M_{kl}^{(j)},$$

By replacing $\Delta\varphi_k^{(j)}$ with $\psi_k^{(j)}$, and not assuming $\Delta\varphi_k^{(j)} = \psi_k^{(j)}$, the following scheme

$$\nu_i^{(j+1)} = \nu_i^{(j)} + \eta_i(jh, \nu^{(j)})h + \sum \mu_{ik}(jh, \nu^{(j)})\psi_k^{(j)} + \sum \rho_{ikl}(jh, \nu^{(j)})(\psi_k^{(j)}\psi_l^{(j)} - h\delta_{kl}), \tag{8}$$

is the same as (7) with the increment $\psi_k^{(j)}$ being independent and $N(0, h)$ being random variables.

2.2. Strong Order of Convergence

A discrete-time approximation ν_S with step-size S converges strongly with order γ at time $u = NS$ to the solution $\psi(u)$, if

$$E|\nu_S - \psi(u)| \leq CS^{\gamma}, \qquad S \in (0, 1).$$

where S is the step size, which divides the interval $[0, T]$ into equal length $S = \frac{u}{N}$ and $\psi(u)$ is the solution to the stochastic differential equation. C is a positive constant and independent of S

Theorem 2. *Assume that μ_{ik} is a twice differentiable invertible matrix, and that μ_{ik}, μ''_{ik} and the inverse of the matrix μ_{ik} are bounded. Then*

$$(E|\nu_i^{(j+1,2r)} - \nu_i^{(j,r)}|^p)^{2/p} \leq k_2 h^2 e^{TL},$$

where $\nu_i^{(j,r)}$ and $\nu_i^{(j+1,2r)}$ are defined as

$$\nu_i^{(j,r+1)} = \nu_i^{(j,r)} + \sum_{k=1}^{d} \mu_{ik}(\nu^{(j,r)})\nu_k^{(j,r)} + \frac{1}{2}\sum_{k,l=1}^{d} \rho_{ikl}(\nu^{(j,r)})(\nu_k^{(j,r)}\nu_l^{(j,r)} - h^{(j)}\delta_{kl}),$$

$$\begin{aligned}\nu_i^{(j+1,2r+1)} &= \nu_i^{(j+1,2r)} + \sum_{k=1}^{d} \mu_{ik}(\nu^{(j+1,2r)})\nu_k^{(j+1,2r)} \\ &+\frac{1}{2}\sum_{k,l=1}^{d} \rho_{ikl}(\nu^{(j+1,2r)})(\nu_k^{(j+1,2r)}\nu_l^{(j+1,2r)} - h^{(j+1)}\delta_{kl}),\end{aligned}$$

and

$$\begin{aligned}\nu_i^{(j+1,2r+2)} &= \nu_i^{(j+1,2r+1)} + \sum_{k=1}^{d} \mu_{ik}(\nu^{(j+1,2r+1)})\nu_k^{(j+1,2r+1)} \\ &+\frac{1}{2}\sum_{k,l=1}^{d} \rho_{ikl}(\nu^{(j+1,2r+1)})(\nu_k^{(j+1,2r+1)}\nu_l^{(j+1,2r+1)} - h^{(j+1)}\delta_{kl}).\end{aligned}$$

Proof. Suppose that

$$\max_i (E(|\nu_i^{(j+1,2r)} - \nu_i^{(j,r)}|^p))^{2/p} = e_r.$$

Hence,

$$\begin{aligned}(E|\nu_i^{(j+1,2r+2)} - \nu_i^{(j,r+1)}|^p)^{2/p} &= (E|(y - \nu_i^{(j,r+1)}) + (\nu_i^{(j+1,2r+2)} - y)|^p)^{2/p} \\ &= (E|(\nu_i^{(j+1,2r)} - \nu_i^{(j,r)}) + (y - \nu_i^{(j+1,2r)}) \\ &- (\nu_i^{(j,r+1)} - \nu_i^{(j,r)}) + (\nu_i^{(j+1,2r+2)} - y)|^p)^{2/p} \\ &\leq e_r + a_1[|(E(\nu_i^{(j+1,2r)} - \nu_i^{(j,r)})|\nu_i^{(j+1,2r)} - \nu_i^{(j,r)}|^{(p-2)} \\ &(y - \nu_i^{(j+1,2r)}) - (\nu_i^{(j,r+1)} - \nu_i^{(j,r)}) + (\nu_i^{(j+1,2r+2)} - y))|]^{2/p} \\ &+ a_2[(E|(y - \nu_i^{(j+1,2r)}) - (\nu_i^{(j,r+1)} - \nu_i^{(j,r)}) \\ &+ (\nu_i^{(j+1,2r+2)} - y)|^p)]^{2/p}.\end{aligned}$$

It follows from Lemma 3.1 in [35] with $\nu = (\nu_i^{(j+1,2r)} - \nu_i^{(j,r)})$ that

$$\begin{aligned}Y &= (y - \nu_i^{(j+1,2r)}) - (\nu_i^{(j,r+1)} - \nu_i^{(j,r)}) + (\nu_i^{(j+1,2r+2)} - y) \\ &= (\sum_{k=1}^{d} \mu_{ik}(\nu^{(j+1,2r)})\nu_k^{(j,r)} + \frac{1}{2}\sum_{k,l=1}^{d} \rho_{ikl}(\nu^{(j+1,2r)})(\nu_k^{(j,r)}\nu_l^{(j,r)} - h^{(j)}\delta_{kl}))) \\ &- ((\sum_{k=1}^{d} \mu_{ik}(\nu^{(j,r)})\nu_k^{(j,r)} + \frac{1}{2}\sum_{k,l=1}^{d} \rho_{ikl}(\nu^{(j,r)})(\nu_k^{(j,r)}\nu_l^{(j,r)} - h^{(j)}\delta_{kl})) + (\nu_i^{(j+1,2r+2)} - y),\end{aligned}$$

and so

$$
\begin{aligned}
E(Y|\nu) = E[(&\sum_{k=1}^{d} \mu_{ik}(\nu^{(j+1,2r)})\nu_k^{(j,r)} + \frac{1}{2}\sum_{k,l=1}^{d} \rho_{ikl}(\nu^{(j+1,2r)})(\nu_k^{(j,r)}\nu_l^{(j,r)} - h^{(j)}\delta_{kl}))) \\
&- ((\sum_{k=1}^{d} \mu_{ik}(\nu^{(j,r)})\nu_k^{(j,r)} + \frac{1}{2}\sum_{k,l=1}^{d} \rho_{ikl}(\nu^{(j,r)})(\nu_k^{(j,r)}\nu_l^{(j,r)} - h^{(j)}\delta_{kl})) \\
&+ (\nu_i^{(j+1,2r+2)} - y)|(\nu_i^{(j+1,2r)} - \nu_i^{(j,r)})] = 0.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
(E|\nu_i^{(j+1,2r+2)} - \nu_i^{(j,r+1)}|^p)^{2/p} &\le e_r + a_2[(E|(y - \nu_i^{(j+1,2r)}) - (\nu_i^{(j,r+1)} - \nu_i^{(j,r)}) \\
&\quad + (\nu_i^{(j+1,2r+2)} - y)|^p)]^{2/p} \\
&\le e_r + a_3[(E|(y - \nu_i^{(j+1,2r)}) - (\nu_i^{(j,r+1)} - \nu_i^{(j,r)})|^p]^{2/p} \\
&\quad + a_4 E[|(\nu_i^{(j+1,2r+2)} - y)|^p)]^{2/p},
\end{aligned}
$$

and then

$$
\begin{aligned}
(E|\nu_i^{(j+1,2r+2)} - \nu_i^{(j,r+1)}|^p)^{2/p} &= e_r + a_3[(E|(\sum_{k=1}^{d} \mu_{ik}(\nu^{(j+1,2r)})\nu_k^{(j,r)} - (\sum_{k=1}^{d} \mu_{ik}(\nu^{(j,r)})\nu_k^{(j,r)} \\
&\quad + \frac{1}{2}\sum_{k,l=1}^{d} \rho_{ikl}(\nu^{(j+1,2r)})(\nu_k^{(j,r)}\nu_l^{(j,r)} - h^{(j)}\delta_{kl})) \\
&\quad + \frac{1}{2}\sum_{k,l=1}^{d} \rho_{ikl}(\nu^{(j,r)})(\nu_k^{(j,r)}\nu_l^{(j,r)} - h^{(j)}\delta_{kl}))|^p]^{2/p} \\
&\quad + a_4 E[|(\nu_i^{(j+1,2r+2)} - y)|^p)]^{2/p}.
\end{aligned}
$$

Therefore, we arrive at

$$
\begin{aligned}
(E|\nu_i^{(j+1,2r+2)} - \nu_i^{(j,r+1)}|^p)^{2/p} &\le e_r + a_5[E|\sum_{k=1}^{d}(\mu_{ik}(\nu^{(j,r)}) - \mu_{ik}(\nu^{(j+1,2r)}))\nu_k^{(j,r)}|^p]^{2/p} \\
&\quad + a_6[E|\frac{1}{2}\sum_{k,l=1}^{d}(\rho_{ikl}(\nu^{(j,r)}) - \rho_{ikl}(\nu^{(j+1,2r)})) \\
&\quad (\nu_k^{(j,r)}\nu_l^{(j,r)} - h^{(j)}\delta_{kl})|^p]^{2/p} + a_4 E[|(\nu_i^{(j+1,2r+2)} - y)|^p)]^{2/p},
\end{aligned}
$$

where a_1, a_2, a_3, and a_4 are constants that depend on p. We have that the Lipschitz condition is satisfied because μ_{ik} is twice differentiable and μ''_{ik} is bounded. Thus, there exists a constant $C > 0$ such that

$$|\mu_{ik}(\nu) - \mu_{ik}(y)| \le C|\nu - y|$$

and

$$\left|\mu_{ik}(\nu)\frac{\partial \mu_{ik}(\nu)}{\partial \nu} - \mu_{ik}(y)\frac{\partial \mu_{ik}(y)}{\partial y}\right| \le C|\nu - y|,$$

for all $t \in [t_0, slT]$ and $\nu, y \in \mathbb{R}$. Hence,

$$a_3[E|\sum_{k=1}^{d}(\mu_{ik}(\nu^{(j,r)}) - \mu_{ik}(\nu^{(j+1,2r)}))\nu_k^{(j,r)}|^p]^{2/p} \le L^2 h e_r$$

and

$$a_3[E|\frac{1}{2}\sum_{k,l=1}^{d}(\rho_{ikl}(\nu^{(j,r)})-\rho_{ikl}(\nu^{(j+1,2r)}))(\nu_k^{(j,r)}\nu_l^{(j,r)}-h^{(j)}\delta_{kl})|^p]^{2/p}\leq L_1^2h^2e_r.$$

It follows from Lemma 4.2 in [35] that

$$(E|\nu_i^{(j+1,2r+2)}-y|^p)^{2/p}\leq a_p\eta^4h^3.$$

Now, we assume that $|\eta|^4$ is bounded by a constant a_1. Hence,

$$(E|\nu_i^{(j+1,2r+2)}-\nu_i^{(j,r+1)}|^p)^{2/p}\leq e_r+hL^2e_r+L_1^2h^2e_r+a_p\eta^4h^3.$$

Therefore, we obtain

$$\begin{aligned}e_{r+1}&\leq e_r+hL^2e_r+L_1^2h^2e_r+a_p\eta^4h^3\\&\leq e_r+hL^2e_r+L_1^2he_r+a_p\eta^4h^3\\&\leq e_r+hLe_r+k_1h^3\\&\leq(1+hL)e_r+j,\end{aligned}$$

where $j=k_1h^3$. Since $(r+1)h\leq T$ for $r<N$ and $e_0=0$, we obtain

$$\begin{aligned}e_r&\leq j\sum_{k=0}^{r-1}(1+hL)^k\\&\leq j\sum_{k=0}^{N-1}(1+hL)^k\\&=j\frac{(1+hL)^N-1}{hL}\\&=(k_1h^3)(\frac{(1+hL)^N-1}{hL})\\&\leq k_2h^2e^{TL}.\end{aligned}$$

Thus, the proof is complete. □

3. Comparison between Milstein and Approximate Coupling Methods

In this section, we present a useful comparison between two methods for solving SDE. Time-consuming and accurate solutions can be an effective procedure for obtaining the approximate solution for different types of methods. To give a clear overview of the methodology as a numerical implementation, we consider a two-dimensional SDE with invertible diffusion. We apply the Milstein and the approximate coupling methods on a particular SDE, so that the comparisons are made numerically. For the Milstein method, we truncate the Fourier series with specific terms, which is enough to give an accurate result. For the approximate coupling method, the diffusion is nondegenerate.

For comparison purposes, we consider the following two-dimensional SDE:

$$\begin{aligned}d\chi(t)&=(\sin(Y(t)))^2d\psi(t)-\frac{1}{1+\chi^2(t)}dV(t),\\dY(t)&=\frac{1}{1+Y^4(t)}d\psi(t)+(\cos(\chi(t)))^2dV(t),\\&\text{for } 0\leq t\leq 1,\ \text{with } \chi(0)=2 \quad\text{and}\quad Y(0)=0.\end{aligned}\tag{9}$$

$\varphi(u)$ and $V(u)$ are both independent standard Brownian motions. Because solutions for SDEs cannot typically be known explicitly, we uitlize approximate solutions to compare the two different methods. We use the absolute error for the different number of steps to calculate the approximation error for each method. We use the same number of simulations for both methods (R = 10,000). We compute R = 10,000 different Brownian paths over the interval $[0, 1]$ with different step sizes. The experimental error and the elapsed time for the Milstein method are presented in Table 1.

Table 1. Implementation of the Milstein scheme.

	Steps	Step-Size	Absolute Error	Elapsed Time (Hour)
1	400	0.0025	0.0692	0.126
2	800	0.0013	0.0353	0.433
3	1600	0.0006	0.0176	21.839
4	3200	0.00030	0.0091	102.817
5	6400	0.000150	0.0046	261.888

Figure 1 displays the log plot of the absolute error with respect to the five different time steps. We can see that the Milstein scheme converges strongly with order one. We use five different step-sizes $(0.0025, 0.0013, 0.0006, 0.0003, 0.00015)$ for both methods. It is clear from Table 1 and Figure 1 that the strong approximation error decreases as the step size decreases.

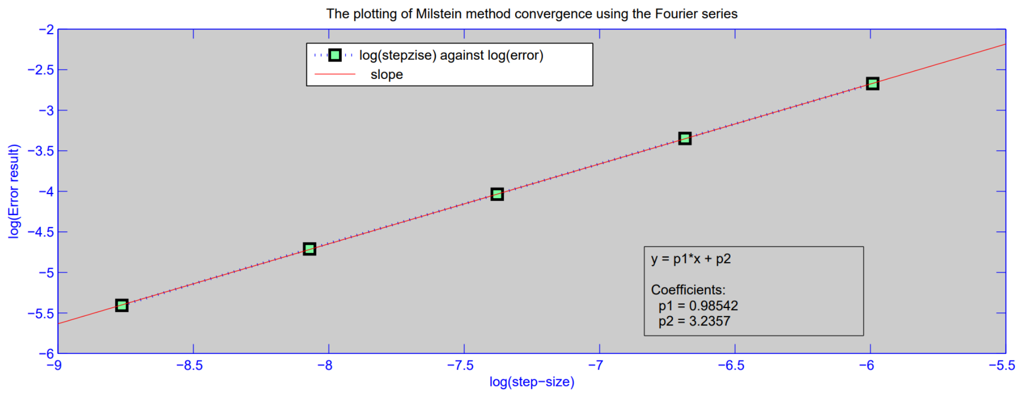

Figure 1. Plot of the Milstein scheme.

The strong convergence for the approximate scheme should be an order one convergence as described in Davie's paper. We run the following MATLAB code with different step sizes over a large number of paths R as follows:

```
[Error for approximate coupling]
S=[ 400, 800, 1600, 3200, 6400];
Error1=zeros(1,length1(S));
for i=1:length1(S)
Error1(1,i)=
log(approximat2022('YA',[1; 0],1,S(1,i)));
end
h=1./S;
fad1=log(h)
plot(log(h), Error1)
```

The approximate coupling method is an alternative to the previous Milstein method. The command Error1(1,i) = log(approximat2022('YA',[1; 0],1,S(1,i))) calculates the absolute value of the difference between the approximate solution ν_h and the solution $\psi(u)$ of the SDE with different step sizes. Table 2 provides the experimental error for each of the five time steps and the elapsed time for the approximate coupling method.

Table 2. Implementation result of approximate coupling method.

	Steps	Step-Size	Absolute Error	Elapsed Time (Hour)
1	400	0.0025	0.0029	0.05805
2	800	0.0013	0.0015	0.01163
3	1600	0.0006	0.00075	0.2325
4	3200	0.0003	0.00036	0.4664
5	6400	0.00015	0.00018	0.9344

Figure 2 displays the log plot of the absolute error for each of the five time steps. The plot indicates a strong convergence between the approximate coupling method and order one.

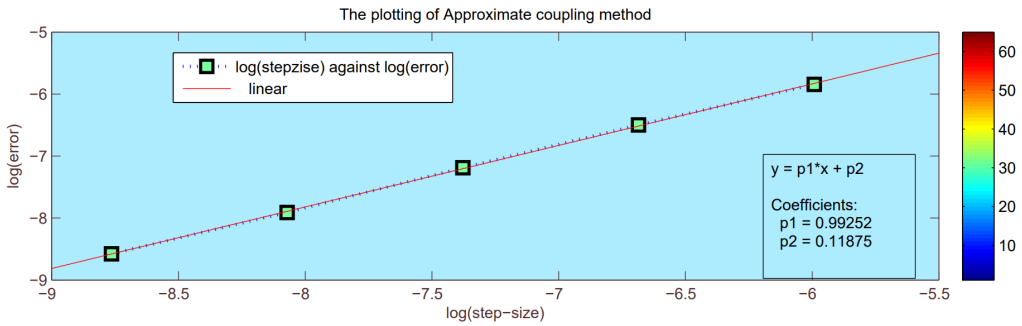

Figure 2. Plot of approximate coupling.

Comparing the results in Tables 1 and 2, we observe in both methods that as step size decreases, the estimate of the absolute error also decreases. We can also observe in the previous tables and plots that the Milstein and approximate coupling methods strongly converge with order one. We emphasize that we applied these methods over the same number of Brownian paths (R = 10,000) for the same step sizes. It can also be seen that using the approximate coupling method can reduce the total computational time. We see from the tables that there is a significant difference between the elapsed time. The Milstein code takes more than two weeks to obtain the result, but the approximate coupling code takes a few hours.

4. Conclusions

Generally, the solution of the stochastic differential equation cannot be known explicitly. Therefore, we use a simulation to find the approximate solution and the convergence behavior. In this paper, we simulated Milstein and the approximate coupling methods in MATLAB to find the approximate solution of the SDE. Both of these methods give an order one convergence. We then implemented these schemes on a stochastic differential equation to compare the Milstein and the approximate coupling methods to each other while illustrating efficiency. Additionally, we calculated error values for the Milstein and the approximate coupling methods to compare the strong order and computation time. According to our results, we can say that the approximate coupling method is faster for the solution of invertible two-dimensional SDEs than the Milstein method. However, the disadvantage of this method is that we should assume the nondegeneracy condition for the diffusion term. The advantage of the Milstein method is that there is no need for this condition, but it involves a significant computational cost. Therefore, we may conclude that the approximate coupling method is more effective than the Milstein method for invertible SDEs. It is interesting to extend the comparisons to include other new methods or equations of higher order.

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