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On the Global Well-Posedness of Rotating Magnetohydrodynamics Equations with Fractional Dissipation

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Abstract: This work considers the three-dimensional incompressible rotating magnetohydrodynamics equation spaces with fractional dissipation $(-\Delta)^{\wp}$ for $\frac{1}{2} < \wp \leq 1$. Furthermore, we use the Littlewood–Paley decomposition and frequency localization techniques to establish the global well-posedness of fractional rotating magnetohydrodynamics equations in a more generalized Besov spaces characterized by the time evolution semigroup related to the generalized linear Stokes–Coriolis operator.

Keywords: fractional dissipation; rotating magnetohydrodynamics equations; well-posedness

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1. Introduction

In this paper, we study the following three-dimensional fractional rotating magnetohydrodynamics (MHD) equations:

$$\begin{cases} u_t + (u\cdot\nabla)u + \mu(-\Delta)^{\wp}u + \Im e_3 \times u + \nabla P = (B\cdot\nabla)B & \text{in } \Re^3\times\Re_+, \\ B_t + (u\cdot\nabla)B + \gamma(-\Delta)^{\wp}B = (B\cdot\nabla)u & \text{in } \Re^3\times\Re_+, \\ \operatorname{div} u = 0, \operatorname{div} B = 0 & \text{in } \Re^3\times\Re_+, \\ u|_{t=0} = u_0,\ B|_{t=0} = B_0 & \text{in } \Re^3, \end{cases} \tag{1}$$

where u is the incompressible velocity field, $\Im$ denotes the speed of rotation around a vertical unit vector $e_3 = (0,0,1)$, $P = p + \frac{1}{2}|B|^2$ in which B is the magnetic field and p is pressure, and μ and γ represent the viscosity coefficient and diffusion of the magnetic field, respectively. For convenience, we use $\mu = \gamma = 1$.

For $\wp = 1$, Equation (1) explains why the Earth has a nonzero large-scale magnetic field, the polarity of which appears to change over several hundred centuries.

When $\Im = 0$, Equation (1) becomes the following fractional MHD equations:

$$\begin{cases} u_t + (u\cdot\nabla)u + (-\Delta)^{\wp}u + \nabla P = (B\cdot\nabla)B & \text{in } \Re^3\times\Re_+, \\ B_t + (u\cdot\nabla)B + (-\Delta)^{\wp}B = (B\cdot\nabla)u & \text{in } \Re^3\times\Re_+, \\ \operatorname{div} u = 0, \operatorname{div} B = 0 & \text{in } \Re^3\times\Re_+, \\ u|_{t=0} = u_0,\ B|_{t=0} = B_0 & \text{in } \Re^3. \end{cases} \tag{2}$$

Equation (2) studies the magnetic characteristics of electrically conducted fluids. Since Duvaut and Lions [1] developed a globally Leray–Hopf weak solution and a locally strong solution to the 3D inviscid MHD equations, the MHD equations remain a challenging open problem in terms of determining whether there is always a globally smooth solution for smooth initial data. Sermange and Temam further investigated the properties and conditions of these solutions [2]. Melo and Santos [3] proved the existence and decay rates of a unique global asymptotic solution for Equation (2) in Sobolev–Gevrey spaces, where Zhao and Li [4] obtained the time decay rate of weak solutions to (2) in $\mathbb{R}^2$. Liu et al. [5] obtained the existence of attractors for (2) with damping and overcame the main difficulty in dealing with the nonlinear term. For the detailed study related to the existence of a solution for fractional MHD equations, we refer the readers to [6–10].

When $B = 0, \Im \neq 0$ and $\wp = 1$, Equation (1) corresponds to the following Navier–Stokes equations with Coriolis force:

$$\begin{cases} u_t + (u \cdot \nabla)u - \Delta u + \Im e_3 \times u + \nabla P = 0 & \text{in } \Re^3 \times \Re_+, \\ \operatorname{div} u = 0, \operatorname{div} B = 0 & \text{in } \Re^3 \times \Re_+, \\ u|_{t=0} = u_0, & \text{in } \Re^3. \end{cases} \tag{3}$$

Equation (3) has received considerable attention due to its importance in geophysical flows. The existence of a solution for Equation (3) starts with the work of Babin et al. [11–13], who established the global existence and regularity of solutions for Equation (3) with periodic initial velocity, when the rotation speed $\Im$ is sufficiently large. Under the smallness condition of initial data u_0, Giga et al. [14] proved the uniform mild solution in $FM_0^{-1}(\Re^3)$. The uniform global well-posedness was established by Hieber and Shibata [15] in $H^{\frac{1}{2}}(\Re^3)$. More recently, for the fractional case $\frac{1}{2} < \wp \leq 1$, Wang and Wu [16] obtained the global well-posedness for small initial data belonging to Lei–Lin spaces $\mathcal{X}^s$.

When $B = 0, \Im = 0$ and $\wp = 1$, Equation (1) becomes the following classical Navier–Stokes equations:

$$\begin{cases} u_t + (u \cdot \nabla)u - \Delta u + \nabla P = 0 & \text{in } \Re^3 \times \Re_+, \\ \operatorname{div} u = 0, \operatorname{div} B = 0 & \text{in } \Re^3 \times \Re_+, \\ u|_{t=0} = u_0, & \text{in } \Re^3. \end{cases} \tag{4}$$

Equation (4) is one of the most fundamental mathematical model in fluid dynamics. Numerous publications have addressed the global well-posedness of Equation (4) in critical spaces. This work originates with the work of Fujita and Kato [17], who established the existence of a strong solution to the Cauchy problem of Equation (4) under the smallness condition of u_0 in $H^{-1+\frac{n}{2}}(\Re^n)$. In [18], they rewrote Equation (4) to an integral form and obtained the local well-posedness in some Sobolev and Lebesgue spaces. These results led to an intensive study of well-posedness for Navier–Stokes equations in various function spaces in recent years, such as [19–22]. As far as the fractional case is concerned, many authors have studied the existence of solutions. For reference we refer the readers to [23–25].

This paper considers the global well-posedness of Equation (1) in some function space characterized by the time evolution semigroup $T_{\Im,\wp}(t)$. We obtain the global well-posedness of Equation (1) in scaling subcritical spaces $X_{\Im}^{s,p,\pi}(\Re^3)$ for $\frac{1}{2} < \wp \leq 1$, $\frac{2\wp}{\pi} + \frac{3}{p} < s + (2\wp - 1)$ and the critical spaces $X_{\Im}^{p}(\Re^3)$ for $\frac{7}{8} < \wp \leq \frac{5}{4}$, $\frac{3}{2\wp-1} < p \leq 4$. The following are the main results of the paper.

Theorem 1. *Let $\wp, s, p,$ and π satisfy*

$$\frac{1}{2} < \wp \leq 1, 3 - 3\wp < s < \frac{3(15 - 4\wp)}{2(9 + 8\wp)},$$
$$\frac{1}{3} + \frac{s}{9} \leq \frac{1}{p} < \min\left\{\frac{1}{4} + \frac{5}{16\wp} - \frac{s}{8\wp}, \frac{2\wp - 1}{3} + \frac{s}{3}\right\},$$
$$\max\left\{0, \tfrac{2}{p} - \tfrac{1}{2\wp}\left(1 + \tfrac{3}{p} - s\right)\right\} < \tfrac{1}{\pi} < \min\left\{\tfrac{1}{2}, 1 - \tfrac{1}{2\wp}\left(1 + \tfrac{3}{p} - s\right), \tfrac{1}{2} + \tfrac{1}{8\wp} - \tfrac{3}{2\wp p} + \tfrac{s}{4\wp}\right\}.$$

Then, for $u_0, B_0 \in X_{\Im}^{s,p,\pi}(\Re^3)^3 \cap \dot{H}^s(\Re^3)^3$ with $\operatorname{div} u_0 = 0, \operatorname{div} B_0 = 0$ and

$$\|u_0\|_{X_{\Im}^{s,p,\pi}} + \|B_0\|_{X_{\Im}^{s,p,\pi}} \lesssim |\Im|^{1 - \frac{1}{2\wp}\left(1 + \frac{3}{p} - s\right) - \frac{1}{\pi}},$$

Equation (1) has a unique global solution

$$u, B \in L^{\pi}\left(0,\infty; \dot{W}^{s,p}\left(\Re^3\right)^3\right) \cap C\left([0,\infty); \dot{H}^s\left(\Re^3\right)^3\right)$$

such that $\operatorname{div} u = 0$ *and* $\operatorname{div} B = 0$.

Theorem 2. *Let $\wp, s, p$ satisfy*

$$\frac{7}{8} < \wp \leq \frac{5}{4}, \quad 0 \leq s < 2\wp - 1, \quad \max\left\{\frac{1}{4}, \frac{s}{3}\right\} \leq \frac{1}{p} < \frac{2\wp-1}{3}$$

Then, for $u_0, B_0 \in X^p_{\Im}(\Re^3)^3 \cap \dot{H}^s(\Re^3)^3$ *with* $\operatorname{div} u_0 = 0$, $B_0 = 0$ *and*

$$\|u_0\|_{X^p_{\Im}} + \|B_0\|_{X^p_{\Im}} \leq \delta,$$

Equation (1) *has a unique global solution*

$$u, B \in C\left([0,\infty); \dot{H}^s\left(\Re^3\right)^3\right)$$

such that

$$\sup_{t>0} t^{\frac{1}{2\wp}\left(1-\frac{3}{p}\right)} \|u(t)\|_{L^p} + \sup_{t>0} t^{\frac{1}{2\wp}\left(1-\frac{3}{p}\right)} \|B(t)\|_{L^p} \leq 2\left(\|u_0\|_{X^p_{\Im}} + \|B_0\|_{X^p_{\Im}}\right),$$

$\operatorname{div} u = 0$ *and* $\operatorname{div} B = 0$.

Throughout this paper we write $f \lesssim g$ to denote $f \leq Kg$, where K is positive constant.

2. Preliminaries

This section gives a brief review of important definitions and useful lemmas. By substituting $B = 0$ into Equation (1), we get the fractional Navier–Stokes equations with Coriolis force. We need to define an equivalent fractional Stokes–Coriolis semigroup $T_{\Im,\wp}$. Therefore, we consider the following fractional linear problem:

$$\begin{cases} u_t + (-\Delta)^{\wp} u + \Im e_3 \times u + \nabla p = 0 & \text{in } \Re^3 \times \Re_+ \\ \operatorname{div} u = 0 & \text{in } \Re^3 \times \Re_+ \\ u|_{t=0} = u_0 & \text{in } \Re^3 \end{cases} \tag{5}$$

The solution of Equation (5) can be obtained using the fractional Stokes–Coriolis semigroup $T_{\Im,\wp}$, which has an explicit representation of the following form [15,16]:

$$T_{\Im,\wp}(t)u = \mathscr{F}^{-1}\left[\cos\left(\Im\frac{\eta_3}{|\eta|}t\right)I + \sin\left(\Im\frac{\eta_3}{|\eta|}t\right)R(\eta)\right] * \left(\mathrm{e}^{-(-\Delta)^{\wp}t}u\right), \tag{6}$$

where I denotes the unit matrix in $M_{3\times3}(\Re)$ and $N(\eta)$ denotes the skew-symmetric matrix given by

$$R(\eta) := \frac{1}{|\eta|}\begin{pmatrix} 0 & \eta_3 & -\eta_2 \\ -\eta_3 & 0 & \eta_1 \\ \eta_2 & -\eta_1 & 0 \end{pmatrix}, \quad \eta \in \Re^3 \backslash \{0\}.$$

Therefore, we can write the semigroup as

$$\mathscr{A}_{\Im,\wp}(t) = \begin{pmatrix} T_{\Im,\wp}(t) & 0 \\ 0 & S_{\wp}(t) \end{pmatrix},$$

where $S_{\wp}(t) := \mathrm{e}^{-(-\Delta)^{\wp}t} = \mathcal{F}^{-1}\left(\mathrm{e}^{-|\eta|^{2\wp}t}\right)$. Using the semigroup $\mathscr{A}_{\Im,\wp}(t)$, we can transform Equation (1) into the following integral form:

$$\begin{pmatrix} u \\ B \end{pmatrix} = \mathscr{A}_{\Im,\wp}(t)\begin{pmatrix} u_0 \\ B_0 \end{pmatrix} - \int_0^t \mathscr{A}_{\Im,\wp}(t-y)\aleph\begin{pmatrix} \mathrm{div}(u\otimes u - B\otimes B) \\ \mathrm{div}(u\otimes B - B\otimes u) \end{pmatrix}(y)dy, \tag{7}$$

where $\aleph = I - \nabla(-\Delta)^{-1}\mathrm{div}$ is the Leray–Hopf projection.

If (u, B) satisfies Equation (7) in a suitable function space, then (u, B) is a solution to Equation (1).

Now, we give the definitions of the function spaces $X^{s,p,\pi}_{\Im}(\Re^3)$ and $X^{p}_{\Im}(\Re^3)$ of Besov type, which are generated by the linear semigroup $T_{\Im,\wp}$. The set of all tempered distributions is denoted by $\mathscr{S}'(\Re^3)$. Let $-\infty < s, \Im < \infty$ and the function spaces $X^{s,p,\pi}_{\Im}(\Re^3)$ and $X^{p}_{\Im}(\Re^3)$ are defined as

$$X^{s,p,\pi}_{\Im}(\Re^3) = \{u \in \mathscr{S}' | \|u\|_{X^{s,p,\pi}_{\Im}} < \infty\},$$

$$\|u\|_{X^{s,p,\pi}_{\Im}} = \|T_{\Im}(t)u\|_{L^{\pi}_t(0,\infty;\dot{W}^{s,p}_x(\Re^3))}$$

and

$$X^{p}_{\Im}(\Re^3) = \{u \in \mathscr{S}' | \|u\|_{X^{p}_{\Im}} < \infty\},$$

$$\|u\|_{X^{p}_{\Im}} = \sup_{t>0} t^{\frac{1}{2\wp}\left(1-\frac{3}{p}\right)}\|T_{\Im}(t)u\|_{L^p}.$$

When $\Im = 0$, $T_{\Im,\wp}(t)$ becomes $T_{0,\wp}(t)u = e^{-t(-\Delta)^{\wp}}(t)u$. By [26], for $1 \le \pi < \infty$, we have

$$\|u\|_{X^{s,p,\pi}_0} = \|t^{\frac{1}{\pi}}\|(-\Delta)^{\frac{s}{2}}e^{-t(-\Delta)^{\wp}}u\|_{L^p}\|_{L^{\pi}(0,\infty;\frac{dt}{t})} \simeq \|(-\Delta)^{\frac{s}{2}}u\|_{\dot{B}^{-\frac{2\wp}{\pi}}_{p,\pi}} = \|u\|_{\dot{B}^{s-\frac{2\wp}{\pi}}_{p,\pi}}. \tag{8}$$

For $3 < p \le \infty$, we see that

$$\|u\|_{X^{p}_0} = \|t^{\frac{1}{2\wp}(1-\frac{3}{p})}\|e^{-t(-\Delta)^{\wp}}u\|_{L^p}\|_{L^{\pi}(0,\infty;\frac{dt}{t})} \simeq \|u\|_{\dot{B}^{-1+\frac{3}{p}}_{p,\pi}}. \tag{9}$$

Hence, the function spaces and $X^{s,p,\pi}_{\Im}(\Re^3)$ and $X^{p}_{\Im}(\Re^3)$ can be considered as a generalization of the Besov spaces $\dot{B}^{s-\frac{2\wp}{\pi}}_{p,\pi}$ and $\dot{B}^{-1+\frac{3}{p}}_{p,\pi}$, respectively.

Next, we recall the Littlewood–Paley decomposition tool. Let $\mathscr{S}(\Re^3)$ be a Schwartz space and let $\psi \in \mathscr{S}(\Re^3)$ satisfy the following conditions:

$$0 \le \hat{\psi}_0(\eta) \le 1 \quad \text{for all } \eta \in \Re^3,$$

$$\mathrm{supp}\,\hat{\psi}_0 \subset := \left\{\eta \in \Re^3 : \frac{1}{2} \le |\eta| \le 2\right\}$$

and

$$\sum_{j\in\mathbb{Z}} \hat{\psi}_0\left(2^{-j}\eta\right) = 1 \quad \text{for all } \eta \in \Re^3\backslash\{0\},$$

where $\psi_j(x) := 2^{3j}\psi_0(2^j x)$ and $\Delta_j u := \psi_j * u$ for $u \in \mathscr{S}'(\Re^3)$ and $j \in \mathbb{Z}$.

Now, we define the homogeneous Besov space $\dot{B}^s_{p,q}(\Re^3)$. For $s \in \Re$ and $p, q \in [1, \infty]$

$$\dot{B}^s_{p,q}\left(\Re^3\right) := \left\{u \in \mathscr{S}'\left(\Re^3\right) \mid \|u\|_{\dot{B}^s_{p,q}} < +\infty\right\},$$

$$\|u\|_{\dot{B}^s_{p,q}} := \left(\sum_{j\in\mathbb{Z}} 2^{jsq}\|\Delta_j u\|^q_{L^p}\right)^{\frac{1}{q}}.$$

Further, we write the operator $\mathcal{G}$ as

$$\mathcal{G}_{\pm}(t)u(x) := e^{\pm it\frac{D_3}{|D|}}u(x) := \int_{\Re^3} e^{ix\cdot\eta^2 \pm it\frac{\eta_3}{|\eta|}}\hat{u}(\eta)d\eta \quad \text{for } x \in \Re^3 \text{ and } t \in \Re.$$

So the operator $T_{\Im,\wp}$ becomes

$$T_{\Im,\wp}(t)u = \frac{1}{2}\mathcal{G}+(\Im t)\left[e^{-t(-\Delta)^{\wp}}(I+J)u\right] + \frac{1}{2}\mathcal{G}-(\Im t)\left[e^{-t(-\Delta)^{\wp}}(I-J)u\right],$$

where J is the singular integral operator matrix defined as

$$J := \begin{pmatrix} 0 & R_3 & -R_2 \\ -R_3 & 0 & R_1 \\ R_2 & -R_1 & 0 \end{pmatrix},$$

where R_k for $k = 1,2,3$ is the Riesz transforms in $\Re^3$. Next, we give the following lemmas as important tools to prove our main results.

Lemma 1 ([26])**.** *Let $u \in L^r(\Re^n)$ and $1 \le r \le p \le \infty$. Then, for $\wp, \beta > 0$ we have*

$$\left\| e^{-t(-\Delta)^{\wp}}u \right\|_{L^p} \lesssim t^{-\frac{n}{2\wp}\left(\frac{1}{r}-\frac{1}{p}\right)}\|u\|_{L^r},$$

and

$$\left\| (-\Delta)^{\frac{\beta}{2}} e^{-t(-\Delta)^{\wp}}u \right\|_{L^p} \lesssim t^{-\frac{\beta}{2\wp}-\frac{n}{2\wp}\left(\frac{1}{r}-\frac{1}{p}\right)}\|u\|_{L^r}.$$

Lemma 2 ([27])**.** *Let $1 \le p,q \le \infty$ and $-\infty < s_0 \le s_1 < \infty$. Then,*

$$\left\| e^{-t(-\Delta)^{\wp}}u \right\|_{\dot{B}^{s_1}_{p,q}} \lesssim t^{-\frac{1}{2\wp}(s_1-s_0)}\|u\|_{\dot{B}^{s_0}_{p,q}}.$$

Lemma 3 ([28])**.** *Let $1 \le p_0 \le p_1 \le \infty, 1 \le q \le \infty$ and $-\infty < s_0 \le s_1 < \infty$. Then,*

$$\left\| e^{-t(-\Delta)^{\wp}}u \right\|_{\dot{B}^{s_1}_{p_1,q}} \lesssim t^{-\frac{1}{2\wp}(s_1-s_0)-\frac{3}{2\wp}\left(\frac{1}{p_0}-\frac{1}{p_1}\right)}\|u\|_{\dot{B}^{s_0}_{p_0,\mathcal{A}}}.$$

Lemma 4 is used to obtain the dispersive estimate for the operator $\mathcal{G}_{\pm}(y)$.

Lemma 4 ([29])**.** *Let $y, s \in \Re$, $2 \le p \le \infty$ and $1 \le q \le \infty$. Then,*

$$\|\mathcal{G}_{\pm}(y)u\|_{\dot{B}^{s}_{p,q}} \lesssim (1+|y|)^{-\left(1-\frac{2}{p}\right)}\|u\|_{\dot{B}^{s+3(1-2)}_{p',q}}$$

where $\frac{1}{p}+\frac{1}{p'} = 1$.

Lemma 5. *Let $1 \le p_0 \le 2 \le p_1 \le \infty$, $1 \le q \le \infty$. and $-\infty < s_0 \le s_1 < \infty$. Then,*

$$\|T_{\Im,\wp}(t)u\|_{\dot{B}^{s_1}_{p_1,q}} \lesssim t^{-\frac{1}{2\wp}(s_1-s_0)-\frac{3}{2a}\left(\frac{1}{p_0}-\frac{1}{p_1}\right)}\|u\|_{\dot{B}^{s_0}_{p_0,q}}.$$

Proof. According to Plancherel's theorem, $\mathcal{R}$ is bounded in $L^2(\Re^3)$, and using the semi-group estimate $e^{-t(-\Delta)^{\wp}}$ in $L^{p_0} - L^2$ and Lemmas 3, 4 we have

$$\begin{aligned}\left\| e^{-\frac{t}{2}(-\Delta)^{\wp}}\mathcal{G}_{\pm}(\Im t)\left[e^{-\frac{t}{2}(-\Delta)^{\wp}}(I\pm\mathcal{R})u\right]\right\|_{\dot{B}^{s_1}_{p_1,q}} &\lesssim t^{-\frac{1}{2\wp}(s_1-s_0)-\frac{3}{2\wp}\left(\frac{1}{2}-\frac{1}{p_1}\right)}\left\|\mathcal{G}_{\pm}(\Im t)\left[e^{-\frac{t}{2}(-\Delta)^{\wp}}u\right]\right\|_{\dot{B}^{s_0}_{2,q}} \\ &\lesssim t^{-\frac{1}{2a}(s_1-s_0)-\frac{3}{2\wp}\left(\frac{1}{2}-\frac{1}{p_1}\right)}\left\|e^{-\frac{t}{2}(-\Delta)^{\wp}}u\right\|_{\dot{B}^{s_0}_{2,q}} \\ &\lesssim t^{-\frac{1}{2\wp}(s_1-s_0)-\frac{3}{2\wp}\left(\frac{1}{p_0}-\frac{1}{p_1}\right)}\|u\|_{\dot{B}^{s_0}_{p_0,q}}.\end{aligned}$$

□

Lemma 6. *Let* $1 < p_0 \leq 2 \leq p_1 < \infty$ *and* $0 \leq s < \infty$. *Then,*

$$\left\| \partial_x^\beta T_{\Im,\wp}(t) u \right\|_{H^s} \lesssim t^{-\frac{1}{2\wp}(s+|\beta|) - \frac{3}{2a}\left(\frac{1}{p_0} - \frac{1}{2}\right)} \|u\|_{L^{p_0}},$$

and

$$\left\| \partial_x^\beta T_{\Im,\wp}(t) u \right\|_{L^{p_1}} \lesssim t^{-\frac{\beta}{2\wp} - \frac{3}{2\wp}\left(\frac{1}{p_0} - \frac{1}{p_1}\right)} \|u\|_{L^{p_0}}.$$

Proof. By using Lemma 4 and the embeddings

$$L^p\left(\Re^3\right) \hookrightarrow \dot{B}^0_{p,2}\left(\Re^3\right),$$

$$\dot{B}^0_{p,2}\left(\Re^3\right) \hookrightarrow L^p\left(\Re^3\right), \text{ for } p \in (1,2]$$

and

$$\dot{H}^s\left(\Re^3\right) = \dot{B}^s_{2,2}, \text{ for } p \in [2,\infty),$$

we can easily obtain the result. □

Lemma 7 ([27])**.** *Let* $t > 0, \Im \in \Re$ *and* $s \in \Re$. *Then,*

$$\left\| T_{\Im,\wp}(t) u \right\|_{\dot{H}^s} \lesssim \|u\|_{\dot{H}^s}.$$

Lemma 8. *Let* $\wp, p, q$ *and* π *satisfy* $\frac{1}{2} < \wp < \frac{5}{4}, 2 < p < \frac{3}{2-\wp}$ *and* $1 - \frac{1}{p} \leq \frac{1}{q} < \frac{2\wp-1}{3} + \frac{1}{p}$, $\max\left\{0, 1 - \frac{1}{2\wp} - \frac{3}{2\wp}\left(\frac{1}{q} - \frac{1}{p}\right) - \left(1 - \frac{2}{p}\right)\right\} < \frac{1}{\pi} \leq \min\left\{\frac{1}{2}, 1 - \frac{1}{2\wp} - \frac{3}{2\wp}\left(\frac{1}{q} - \frac{1}{p}\right)\right\}$. *Then,*

$$\left\| \int_0^t T_\Im(t-y) \aleph \nabla u(y) dy \right\|_{L^\pi(0,\infty;L^p)} \lesssim |\Im|^{-\left\{1 - \frac{1}{2a} - \frac{3}{2a}\left(\frac{1}{q} - \frac{1}{p}\right) - \frac{1}{\pi}\right\}} \|u\|_{L^{\frac{\pi}{2}}(0,\infty;L^q)} \tag{10}$$

for $u, B \in L^{\frac{\pi}{2}}(0,\infty;L^q)\left(\Re^3\right)$.

Proof. By considering the boundedness of $\aleph$ in $L^q\left(\Re^3\right)$, using the embedding $\dot{B}^0_{p,2}\left(\Re^3\right) \hookrightarrow L^p\left(\Re^3\right)$ for $p \in [2,\infty)$, applying Lemmas 1, 3, and 4, we get

$$\left\| \int_0^t T_\Im(t-y) \aleph \nabla u(y) dy \right\|_{L^\pi(0,\infty;L^p)} \lesssim \left\| \int_0^t k_\Im(t-y) \|u(y)\|_{L^q} dy \right\|_{L^\pi_t(0,\infty)'}$$

where

$$k_\Im(t) := \{1 + |\Im| t\}^{-\left(1 - \frac{2}{p}\right)} t^{-\frac{3}{2a}\left(\frac{1}{q} - \frac{1}{p}\right) - \frac{1}{2a}}$$

If $\frac{1}{\pi} < 1 - \frac{1}{2\wp} - \frac{3}{2\wp}\left(\frac{1}{q} - \frac{1}{p}\right)$, we have $\|k_\Im\|_{L'} \lesssim |\Im|^{-\left\{1 - \frac{1}{2\wp} - \frac{3}{2\wp}\left(\frac{1}{q} - \frac{1}{p}\right) - \frac{1}{\pi}\right\}}$. By using Young's inequality, $\frac{1}{\pi} = \frac{1}{\pi'} + \frac{2}{\pi} - 1$, we obtain inequality (10). □

In order to prove the bilinear term in Equation (7), the following lemma is significant.

Lemma 9 ([29])**.** *Let* p, q, *and* s *satisfy the conditions* $\frac{s}{3} < \frac{1}{p} < \frac{1}{2} + \frac{s}{6}, \frac{1}{q} = \frac{2}{p} - \frac{s}{3}$ *and* $0 \leq s < 3$. *Then,*

$$\|uv\|_{\dot{W}^{s,q}} \lesssim \|u\|_{\dot{W}^{s,p}} \|v\|_{\dot{W}^{s,p}},$$

where $u, v \in \dot{W}^{s,p}\left(\Re^3\right)$.

Proof of Theorem 1: Let us define $\|\cdot\|_{Z_1} := \|\cdot\|_{L^\pi\left(0,\infty;W^{s,p}\left(\Re^3\right)^3\right)}$. It is easy to get

$$\left\| \mathscr{A}_{\Im,\wp}(t) \begin{pmatrix} u_0 \\ B_0 \end{pmatrix} \right\|_{Z_1} \leq \|T_{\Im,\wp}(t)u_0\|_{Z_1} + \|S_\wp(t)B_0\|_{Z_1} = \|u_0\|_{X^{s,p,\pi}_{\Im}} + \|B_0\|_{X^{s,p,\pi}_{\Im}}. \tag{11}$$

Next, we write a complete metric space $(\mathscr{X}_1, d_1)$ with mapping $\mathscr{J}$ as:

$$\mathscr{X}_1 := \left\{ u, B \in L^\pi\left(0,\infty; \dot{W}^{s,p}\left(\Re^3\right)^3\right) \mid \|u\|_{Z_1} \leqslant 2\|u_0\|_{X^{s,p,\pi}_{\Im}}, \|B\|_{Z_1} \leqslant 2\|B_0\|_{X^{s,p,\pi}_{\Im}} \right\},$$

$$d_1\left(\begin{pmatrix} u \\ B \end{pmatrix}, \begin{pmatrix} v \\ b \end{pmatrix}\right) := \|u-v\|_{Z_1} + \|B-b\|_{Z_1},$$

$$\mathscr{J}\begin{pmatrix} u \\ B \end{pmatrix}(t) := \mathscr{A}_{\Im,\wp}\begin{pmatrix} u_0 \\ B_0 \end{pmatrix}(t) - B\left(\begin{pmatrix} u \\ B \end{pmatrix}, \begin{pmatrix} u \\ B \end{pmatrix}\right),$$

where

$$B\left(\begin{pmatrix} u \\ B \end{pmatrix}, \begin{pmatrix} u \\ B \end{pmatrix}\right) = \int_0^t \mathscr{A}_{\Im,\wp}(t-y)\aleph \begin{pmatrix} \operatorname{div}(u\otimes u - B\otimes B) \\ \operatorname{div}(u\otimes B - B\otimes u) \end{pmatrix} y dy. \tag{12}$$

We can write

$$\left\| \mathscr{J}\begin{pmatrix} u \\ B \end{pmatrix}(t) \right\|_{Z_1} \leq \left\| \mathscr{A}_{\Im,\wp}\begin{pmatrix} u_0 \\ B_0 \end{pmatrix}(t) \right\|_{Z_1} + \left\| B\left(\begin{pmatrix} u \\ B \end{pmatrix}, \begin{pmatrix} u \\ B \end{pmatrix}\right) \right\|_{Z_1} = I_1 + I_2. \tag{13}$$

I_1 is estimated in inequality (11). To estimate I_2, let $\frac{1}{q} = \frac{2}{p} - \frac{s}{3}$, using Lemmas 7, 8, and Holder's inequality, it is easy to obtain that

$$\begin{aligned} \left\| \int_0^t T_{\Im,\wp}(t-y)\aleph \operatorname{div}(u\otimes u)(y)dy \right\|_{Z_1} &\lesssim |\Im|^{-\left\{1-\frac{1}{2\wp}\left(1+\frac{3}{p}-s\right)-\frac{1}{\pi}\right\}} \|u\|^2_{Z_1} \\ &\lesssim 4|\Im|^{-\left\{1-\frac{1}{2\wp}\left(1+\frac{3}{p}-s\right)-\frac{1}{\pi}\right\}} \|u_0\|^2_{X^{s,p,\pi}_{\Im}}. \end{aligned} \tag{14}$$

Similarly, we have

$$\left\| \int_0^t T_{\Im,\wp}(t-y)\aleph \operatorname{div}(B\otimes B)(y)dy \right\|_{Z_1} \lesssim 4|\Im|^{-\left\{1-\frac{1}{2\wp}\left(1+\frac{3}{p}-s\right)-\frac{1}{\pi}\right\}} \|B_0\|^2_{X^{s,p,\pi}_{\Im}}, \tag{15}$$

$$\left\| \int_0^t S_\wp(t-y)\aleph \operatorname{div}(u\otimes B)(y)dy \right\|_{Z_1} \lesssim 4|\Im|^{-\left\{1-\frac{1}{2\wp}\left(1+\frac{3}{p}-s\right)-\frac{1}{\pi}\right\}} \|u_0\|_{X^{s,p,\pi}_{\Im}} \|B_0\|_{X^{s,p,\pi}_{\Im}}, \tag{16}$$

and

$$\left\| \int_0^t S_\wp(t-y)\aleph \operatorname{div}(B\otimes u)(y)dy \right\|_{Z_1} \lesssim 4|\Im|^{-\left\{1-\frac{1}{2\wp}\left(1+\frac{3}{p}-s\right)-\frac{1}{\pi}\right\}} \|B_0\|_{X^{s,p,\pi}_{\Im}} \|u_0\|_{X^{s,p,\pi}_{\Im}}. \tag{17}$$

So from Equation (13), we have

$$\begin{aligned} \left\| \mathscr{J}\begin{pmatrix} u \\ B \end{pmatrix}(t) \right\|_{Z_1} &\lesssim \|u_0\|_{X^{s,p,\pi}_{\Im}} + \|B_0\|_{X^{s,p,\pi}_{\Im}} + 4|\Im|^{-\left\{1-\frac{1}{2\wp}\left(1+\frac{3}{p}-s\right)-\frac{1}{\pi}\right\}} \|u_0\|^2_{X^{s,p,\pi}_{\Im}} \\ &+ 4|\Im|^{-\left\{1-\frac{1}{2\wp}\left(1+\frac{3}{p}-s\right)-\frac{1}{\pi}\right\}} \|B_0\|^2_{X^{s,p,\pi}_{\Im}} \\ &+ 4|\Im|^{-\left\{1-\frac{1}{2\wp}\left(1+\frac{3}{p}-s\right)-\frac{1}{\pi}\right\}} \|u_0\|_{X^{s,p,\pi}_{\Im}} \|B_0\|_{X^{s,p,\pi}_{\Im}} \\ &+ 4|\Im|^{-\left\{1-\frac{1}{2\wp}\left(1+\frac{3}{p}-s\right)-\frac{1}{\pi}\right\}} \|B_0\|_{X^{s,p,\pi}_{\Im}} \|u_0\|_{X^{s,p,\pi}_{\Im}} \\ &= \|u_0\|_{X^{s,p,\pi}_{\Im}} \left(1 + 4|\Im|^{-\left\{1-\frac{1}{2\wp}\left(1+\frac{3}{p}-s\right)-\frac{1}{\pi}\right\}} \left(\|u_0\|_{X^{s,p,\pi}_{\Im}} + \|B_0\|_{X^{s,p,\pi}_{\Im}}\right)\right) \\ &+ \|B_0\|_{X^{s,p,\pi}_{\Im}} \left(1 + 4|\Im|^{-\left\{1-\frac{1}{2\wp}\left(1+\frac{3}{p}-s\right)-\frac{1}{\pi}\right\}} \left(\|B_0\|_{X^{s,p,\pi}_{\Im}} + \|u_0\|_{X^{s,p,\pi}_{\Im}}\right)\right). \end{aligned} \tag{18}$$

We can also write

$$
\begin{aligned}
&\left\| \mathscr{J}\begin{pmatrix} u \\ B \end{pmatrix}(t) - \mathscr{J}\begin{pmatrix} v \\ b \end{pmatrix}(t) \right\|_{Z_1} \\
&= \left\| B\left(\begin{pmatrix} u \\ B \end{pmatrix}, \begin{pmatrix} u-v \\ B-b \end{pmatrix} \right) + B\left(\begin{pmatrix} u-v \\ B-b \end{pmatrix}, \begin{pmatrix} u \\ B \end{pmatrix} \right) \right\|_{Z_1} \\
&\le |\Im|^{-\left\{1-\frac{1}{2\wp}\left(1+\frac{3}{p}-s\right)-\frac{1}{\pi}\right\}}(\|u\|_{Z_1} + \|v\|_{Z_1} + \|B\|_{Z_1} + \|b\|_{Z_1})(\|u-v\|_{Z_1} + \|B-b\|_{Z_1}) \\
&\lesssim 4|\Im|^{-\left\{1-\frac{1}{2\wp}\left(1+\frac{3}{p}-s\right)-\frac{1}{\pi}\right\}}\left(\|u_0\|_{X_{\Im}^{s,p,\pi}} + \|B_0\|_{X_{\Im}^{s,p,\pi}}\right)(\|u-v\|_{Z_1} + \|B-b\|_{Z_1}).
\end{aligned} \tag{19}
$$

Let us consider that $u_0, B_0 \in X_{\Im}^{s,p,\pi}(\Re^3)^3 \cap \dot{H}^s(\Re^3)^3$ satisfies

$$
\sup_{\Im \in \Re \setminus \{0\}} |\Im|^{-\left\{1-\frac{1}{2\wp}\left(1+\frac{3}{p}-s\right)-\frac{1}{\pi}\right\}}\left(\|u_0\|_{X_{\Im}^{s,p,\pi}} + \|u_0\|_{X_{\Im}^{s,p,\pi}}\right) \le \min\left\{\frac{1}{8K_1}, \frac{1}{4K_2}\right\}. \tag{20}
$$

Then, in the view of Equations (18) and (19), for all $\begin{pmatrix} u \\ B \end{pmatrix}, \begin{pmatrix} v \\ b \end{pmatrix} \in X_1$ we can obtain that

$$
\left\| \mathscr{J}\begin{pmatrix} u \\ B \end{pmatrix} \right\|_{Z_1} \le 2\left(\|u_0\|_{X_{\Im}^{s,p,\pi}} + \|B_0\|_{X_{\Im}^{s,p,\pi}}\right),
$$

$$
\left\| \mathscr{J}\begin{pmatrix} u \\ B \end{pmatrix} - \mathscr{J}\begin{pmatrix} v \\ b \end{pmatrix} \right\|_{Z_1} \le \frac{1}{2}(\|u-v\|_{Z_1} + \|B-b\|_{Z_1}).
$$

Therefore, by using Banach's contraction principle, there exists a unique solution $u, B \in X_1$ satisfying (7). Next, we need to show that the solution $u, B \in X_1$ also belongs to $C([0,\infty); H^s(R^3)^3)$.

$$
\left\| \begin{pmatrix} u \\ B \end{pmatrix}(t) \right\|_{\dot{H}^s} \le \left\| \mathscr{A}_{\Im,\wp}\begin{pmatrix} u_0 \\ B_0 \end{pmatrix}(t) \right\|_{\dot{H}^s} + \left\| B\left(\begin{pmatrix} u \\ B \end{pmatrix}, \begin{pmatrix} u \\ B \end{pmatrix} \right) \right\|_{\dot{H}^s} = I_3 + I_4.
$$

To obtain I_1, we have $I_3 \lesssim \|u_0\|_{\dot{H}^s} + \|B_0\|_{\dot{H}^s}$. For I_4, let $\frac{1}{q} = \frac{2}{p} - \frac{3}{s}$ with $q \in (1,2]$, applying Lemmas 6, 9 and Holder's inequality, we have the following

$$
\begin{aligned}
I_4 &\lesssim \int \frac{1}{(t-y)^{\left(\frac{3}{p\wp}-\frac{s}{2\wp}-\frac{3}{4\wp}+\frac{1}{2\wp}\right)}}\|u\otimes u\|_{L^p}dy + \int \frac{1}{(t-y)^{\left(\frac{3}{p\wp}-\frac{s}{2\wp}-\frac{3}{4\wp}+\frac{1}{2\wp}\right)}}\|B\otimes B\|_{L^p}dy \\
&+ \int \frac{1}{(t-y)^{\left(\frac{3}{p\wp}-\frac{s}{2\wp}-\frac{3}{4\wp}+\frac{1}{2\wp}\right)}}\|u\otimes B\|_{L^p}dy + \int \frac{1}{(t-y)^{\left(\frac{3}{p\wp}-\frac{s}{2\wp}-\frac{3}{4\wp}+\frac{1}{2\wp}\right)}}\|B\otimes u\|_{L^p}dy \\
&\lesssim \left(\int_0^t \frac{1}{(t-y)^{\left(\frac{3}{p\wp}-\frac{s}{2\wp}-\frac{3}{4\wp}+\frac{1}{2\wp}\right)\left(\frac{\pi}{2}\right)'}}dy \right)^{\frac{1}{\left(\frac{\pi}{2}\right)'}} \|u\|^2_{L^\pi(0,\infty;\dot{W}^{s,p})} \\
&+ \left(\int_0^t \frac{1}{(t-y)^{\left(\frac{3}{p\wp}-\frac{s}{2\wp}-\frac{3}{4\wp}+\frac{1}{2\wp}\right)\left(\frac{\pi}{2}\right)'}}dy \right)^{\frac{1}{\left(\frac{\pi}{2}\right)'}} \|u\|^2_{L^\pi(0,\infty;\dot{W}^{s,p})} \\
&+ \left(\int_0^t \frac{1}{(t-y)^{\left(\frac{3}{p\wp}-\frac{s}{2\wp}-\frac{3}{4\wp}+\frac{1}{2\wp}\right)\left(\frac{\pi}{2}\right)'}}dy \right)^{\frac{1}{\left(\frac{\pi}{2}\right)'}} \|u\|^2_{L^\pi(0,\infty;\dot{W}^{s,p})}\|B\|^2_{L^\pi(0,\infty;\dot{W}^{s,p})} \\
&+ \left(\int_0^t \frac{1}{(t-y)^{\left(\frac{3}{p\wp}-\frac{s}{2\wp}-\frac{3}{4\wp}+\frac{1}{2\wp}\right)\left(\frac{\pi}{2}\right)'}}dy \right)^{\frac{1}{\left(\frac{\pi}{2}\right)'}} \|B\|^2_{L^\pi(0,\infty;\dot{W}^{s,p})}\|u\|^2_{L^\pi(0,\infty;\dot{W}^{s,p})}.
\end{aligned} \tag{21}
$$

Since $\frac{1}{\pi} < \frac{1}{2} + \frac{1}{8\wp} + \frac{s}{4\wp} - \frac{3}{2\wp p}$, the time integral on the right hand side in inequality (21) converges and

$$\left(\int_0^t \frac{1}{(t-y)^{\left(\frac{3}{p\wp} - \frac{s}{2\wp} - \frac{3}{4\wp} + \frac{1}{2\wp}\left(\frac{\pi}{2}\right)'\right)}} dy \right)^{\frac{1}{\left(\frac{\pi}{2}\right)'}} \lesssim t^{2\left(\frac{1}{2} + \frac{1}{8\wp} + \frac{s}{4\wp} - \frac{3}{2\wp p} - \frac{1}{\pi}\right)}.$$

This implies that for all $t \geq 0$, $u, B \in \dot{H}^s(\Re^3)^3$. Similarly, it is easy to get that $u, B \in C([0,\infty); \dot{H}^s(\Re^3)^3)$, which finishes the proof of Theorem 1. $\square$

Remark 1. *By substituting $B = 0$ and $\wp = 1$ in Theorem 1, we get Theorem 1.3, [30].*

Proof of Theorem 2: Let us define

$$\|u\|_{Y_2} := \sup_{t>0} \|u(t)\|_{\dot{H}^s} \text{and} \quad \|u\|_{Z_2} := \sup_{t>0} t^{\frac{1}{2}\left(1-\frac{3}{p}\right)} \|u(t)\|_{L^p}. \tag{22}$$

It is easy to get

$$\left\| \mathscr{A}_{\Im,\wp}(t) \begin{pmatrix} u_0 \\ B_0 \end{pmatrix} \right\|_{Y_1} \leq \|T_{\Im,\wp}(t)u_0\|_{Y_1} + \|S_\wp(t)B_0\|_{Y_1} \lesssim \|u_0\|_{\dot{H}^s} + \|B_0\|_{\dot{H}^s} \tag{23}$$

and

$$\left\| \mathscr{A}_{\Im,\wp}(t) \begin{pmatrix} u_0 \\ B_0 \end{pmatrix} \right\|_{Z_2} \leq \|T_{\Im,\wp}(t)u_0\|_{Z_2} + \|S_\wp(t)B_0\|_{Z_2} = \|u_0\|_{X^p_\Im} + \|B_0\|_{X^p_\Im}. \tag{24}$$

Next, we write a complete metric space $(\mathscr{X}_2, d_2)$ with mapping $\mathscr{J}$ as:

$$\mathscr{X}_2 := \left\{ u, B \in C\left([0,\infty); \dot{H}^s\left(\Re^3\right)^3 \right) \right\}$$

such that $\|u\|_{Y_1} \lesssim 2\|u_0\|_{\dot{H}^s}, \|B\|_{Y_1} \lesssim 2\|B_0\|_{\dot{H}^s}, \|u\|_{Z_2} \leq 2\|u_0\|_{X^p_\Im}, \|B\|_{Z_2} \leqslant 2\|B_0\|_{X^p_\Im},$

$$d_1\left(\begin{pmatrix} u \\ B \end{pmatrix}, \begin{pmatrix} v \\ b \end{pmatrix} \right) := \|u-v\|_{Y_1} + \|B-b\|_{Y_1},$$
$$\mathscr{J}\begin{pmatrix} u \\ B \end{pmatrix}(t) := \mathscr{A}_{\Im,\wp}\begin{pmatrix} u_0 \\ B_0 \end{pmatrix}(t) - B\left(\begin{pmatrix} u \\ B \end{pmatrix}, \begin{pmatrix} u \\ B \end{pmatrix} \right),$$

where the bilinear term $B\left(\begin{pmatrix} u \\ B \end{pmatrix}, \begin{pmatrix} u \\ B \end{pmatrix} \right)$ is defined in Equation (12). We can write

$$\left\| \mathscr{J}\begin{pmatrix} u \\ B \end{pmatrix}(t) \right\|_{Y_1} \leq \left\| \mathscr{A}_{\Im,\wp}\begin{pmatrix} u_0 \\ B_0 \end{pmatrix}(t) \right\|_{Y_1} + \left\| B\left(\begin{pmatrix} u \\ B \end{pmatrix}, \begin{pmatrix} u \\ B \end{pmatrix} \right) \right\|_{Y_1} = I_5 + I_6. \tag{25}$$

I_5 is estimated in inequality (23). For I_6, let $1 < r \leq 2$ and set $\frac{1}{r} = \frac{1}{s'} + \frac{1}{p}$, where $\frac{1}{s'} = \frac{1}{2} - \frac{s}{3}$ with $2 \leq s' < \frac{6}{5-4\wp}$. Then, by Lemma 6, the boundedness of $\aleph$ in $L^2(\Re^n)$, and Holder's inequality, we have

$$\begin{aligned}
&\left\|B\left(\begin{pmatrix} u \\ B \end{pmatrix}, \begin{pmatrix} u \\ B \end{pmatrix}\right)\right\|_{\dot{H}^s} \\
&\leq \int_0^t \|T_{\Im}(t-y)\aleph\nabla[u(y)\otimes u(y)]\|_{\dot{H}^s}dy + \int_0^t \|T_{\Im}(t-y)\aleph\nabla[B(y)\otimes B(y)]\|_{\dot{H}^s}dy \\
&+ \int_0^t \|T_{\Im}(t-y)\aleph\nabla[u(y)\otimes B(y)]\|_{\dot{H}^s}dy + \int_0^t \|T_{\Im}(t-y)\aleph\nabla[B(y)\otimes u(y)]\|_{\dot{H}^s}dy \\
&\lesssim \int_0^t \frac{1}{(t-y)^{\frac{1}{2\wp}+\frac{3}{2\wp p}}}\|u(y)\otimes u(y)\|_{L^r}dy + \int_0^t \frac{1}{(t-y)^{\frac{1}{2\wp}+\frac{3}{2\wp p}}}\|B(y)\otimes B(y)\|_{L^r}dy \\
&+ \int_0^t \frac{1}{(t-y)^{\frac{1}{2\wp}+\frac{3}{2\wp p}}}\|u(y)\otimes B(y)\|_{L^r}dy + \int_0^t \frac{1}{(t-y)^{\frac{1}{2\wp}+\frac{3}{2\wp p}}}\|B(y)\otimes u(y)\|_{L^r}dy \\
&\lesssim \int_0^t \frac{1}{(t-y)^{\frac{1}{2\wp}+\frac{3}{2\wp p}}}\|u(y)\|_{L^{s'}}\|u(y)\|_{L^p}dy + \int_0^t \frac{1}{(t-y)^{\frac{1}{2\wp}+\frac{3}{2\wp p}}}\|B(y)\|_{L^{s'}}\|B(y)\|_{L^p}dy \\
&+ \int_0^t \frac{1}{(t-y)^{\frac{1}{2\wp}+\frac{3}{2\wp p}}}\|u(y)\|_{L^{s'}}\|B(y)\|_{L^p}dy + \int_0^t \frac{1}{(t-y)^{\frac{1}{2\wp}+\frac{3}{2\wp p}}}\|B(y)\|_{L^{s'}}\|u(y)\|_{L^p}dy.
\end{aligned}$$

Using $\|\cdot\|_{Y_1}, \|\cdot\|_{Z_2}$ defined in Equation (22) with $\frac{1}{2\wp}\left(1-\frac{3}{p}\right) < 1$ and $\dot{H}^s(\Re^3) \hookrightarrow L^{s'}(\Re^3)$, we have

$$\begin{aligned}
&\left\|B\left(\begin{pmatrix} u \\ B \end{pmatrix}, \begin{pmatrix} u \\ B \end{pmatrix}\right)\right\|_{\dot{H}^s} \\
&\lesssim \|u\|_{Y_1}\|u\|_{Z_2}\int_0^t \frac{1}{(t-y)^{\frac{1}{2\wp}+\frac{3}{2\wp p}}y^{\frac{1}{2\wp}\left(1-\frac{3}{p}\right)}}dy + \|B\|_{Y_1}\|B\|_{Z_2}\int_0^t \frac{1}{(t-y)^{\frac{1}{2\wp}+\frac{3}{2\wp p}}y^{\frac{1}{2\wp}\left(1-\frac{3}{p}\right)}}dy \\
&+ \|u\|_{Y_1}\|B\|_{Z_2}\int_0^t \frac{1}{(t-y)^{\frac{1}{2\wp}+\frac{3}{2\wp p}}y^{\frac{1}{2\wp}\left(1-\frac{3}{p}\right)}}dy + C\|B\|_{Y_1}\|u\|_{Z_2}\int_0^t \frac{1}{(t-y)^{\frac{1}{2\wp}+\frac{3}{2\wp p}}y^{\frac{1}{2\wp}\left(1-\frac{3}{p}\right)}}dy \\
&\lesssim \|u\|_{Y_1}\|u\|_{Z_2} + \|B\|_{Y_1}\|B\|_{Z_2} + \|u\|_{Y_1}\|B\|_{Z_2} + \|B\|_{Y_1}\|u\|_{Z_2}.
\end{aligned}$$

Therefore, from Equation (25), we have

$$\begin{aligned}
\left\|\mathscr{J}\begin{pmatrix} u \\ B \end{pmatrix}(t)\right\|_{Y_1} &\lesssim \|u_0\|_{\dot{H}^s} + \|B_0\|_{\dot{H}^s} \\
&+ 4\left(\|u_0\|_{\dot{H}^s}\|u_0\|_{X^p_{\Im}} + \|B_0\|_{\dot{H}^s}\|B_0\|_{X^p_{\Im}} + \|u_0\|_{\dot{H}^s}\|B_0\|_{X^p_{\Im}} + \|B_0\|_{\dot{H}^s}\|u_0\|_{X^p_{\Im}}\right) \\
&\lesssim \left(\|u_0\|_{\dot{H}^s} + \|B_0\|_{\dot{H}^s}\right)\left(1 + 4\left(\|u_0\|_{X^p_{\Im}} + \|B_0\|_{X^p_{\Im}}\right)\right).
\end{aligned} \tag{26}$$

Similarly, we estimate the following

$$\left\|\mathscr{J}\begin{pmatrix} u \\ B \end{pmatrix}(t)\right\|_{Z_2} \leq \left\|\mathscr{A}_{\Im,\wp}\begin{pmatrix} u_0 \\ B_0 \end{pmatrix}(t)\right\|_{Z_2} + \left\|B\left(\begin{pmatrix} u \\ B \end{pmatrix}, \begin{pmatrix} u \\ B \end{pmatrix}\right)\right\|_{Z_2} = I_7 + I_8. \tag{27}$$

I_7 is estimated in inequality (24) and we have to estimate I_8. By using Lemma 6, Holder's inequality, and the definition of Z_2 defined in (22), we have

$$
\begin{aligned}
& t^{\frac{1}{2\wp}\left(1-\frac{3}{p}\right)}\left\|B\left(\left(\begin{array}{c} u \\ B \end{array}\right),\left(\begin{array}{c} u \\ B \end{array}\right)\right)\right\|_{L^p} \\
& \leq t^{\frac{1}{2\wp}\left(1-\frac{3}{p}\right)}\left(\int_0^t\|T_{\Im}(t-y)\aleph\nabla[u(y)\otimes u(y)]\|_{L^p}dy+\int_0^t\|T_{\Im}(t-y)\aleph\nabla[B(y)\otimes B(y)]\|_{L^p}dy\right) \\
& +t^{\frac{1}{2\wp}\left(1-\frac{3}{p}\right)}\left(\int_0^t\|T_{\Im}(t-y)\aleph\nabla[u(y)\otimes B(y)]\|_{L^p}dy+\int_0^t\|T_{\Im}(t-y)\aleph\nabla[B(y)\otimes u(y)]\|_{L^p}dy\right) \\
& \lesssim t^{\frac{1}{2\wp}\left(1-\frac{3}{p}\right)}\left(\int_0^t \frac{1}{(t-y)^{\frac{1}{2\wp}+\frac{3}{2\wp p}}\|u(y)\otimes v(y)\|_{L^{\frac{p}{2}}}dy}+\int_0^t \frac{1}{(t-y)^{\frac{1}{2\wp}+\frac{3}{2\wp p}}\|u(y)\otimes v(y)\|_{L^{\frac{p}{2}}}dy}\right) \\
& +t^{\frac{1}{2\wp}\left(1-\frac{3}{p}\right)}\left(\int_0^t \frac{1}{(t-y)^{\frac{1}{2\wp}+\frac{3}{2\wp p}}\|u(y)\otimes v(y)\|_{L^{\frac{p}{2}}}dy}+\int_0^t \frac{1}{(t-y)^{\frac{1}{2\wp}+\frac{3}{2\wp p}}\|u(y)\otimes v(y)\|_{L^{\frac{p}{2}}}dy}\right) \\
& \lesssim t^{\frac{1}{2\wp}\left(1-\frac{3}{p}\right)}\left(\|u\|_{Z_2}\|u\|_{Z_2}+\|B\|_{Z_2}\|B\|_{Z_2}+2\|u\|_{Z_2}\|B\|_{Z_2}\right)\int_0^t \frac{1}{(t-y)^{\frac{1}{2\wp}+\frac{3}{2\wp p}}y^{\frac{1}{\wp}\left(1-\frac{3}{p}\right)}}dy \\
& \lesssim \|u\|_{Z_2}\|u\|_{Z_2}+\|B\|_{Z_2}\|B\|_{Z_2}+2\|u\|_{Z_2}\|B\|_{Z_2}.
\end{aligned}
$$

Here, we notice that $\frac{1}{2\wp}+\frac{3}{2\wp p}<1$ as $p>\frac{3}{2\wp-1}$. As a result, from Equation (27), we have

$$
\begin{aligned}
\left\|\mathscr{J}\left(\begin{array}{c} u \\ B \end{array}\right)(t)\right\|_{Z_2} & \lesssim \|u_0\|_{X^p_{\Im}}+\|B_0\|_{X^p_{\Im}}+4\left(\|u_0\|^2_{X^p_{\Im}}+\|B_0\|^2_{X^p_{\Im}}+2\|u_0\|_{X^p_{\Im}}\|B_0\|_{X^p_{\Im}}\right) \\
& \lesssim \left(\|u_0\|_{X^p_{\Im}}+\|B_0\|_{X^p_{\Im}}\right)\left(1+4\left(\|u_0\|_{X^p_{\Im}}+\|B_0\|_{X^p_{\Im}}\right)\right).
\end{aligned} \tag{28}
$$

Furthermore, we can write

$$
\begin{aligned}
& \left\|\mathscr{J}\left(\begin{array}{c} u \\ B \end{array}\right)(t)-\mathscr{J}\left(\begin{array}{c} v \\ b \end{array}\right)(t)\right\|_{Y_1} \\
& =\left\|B\left(\left(\begin{array}{c} u \\ B \end{array}\right),\left(\begin{array}{c} u-v \\ B-b \end{array}\right)\right)+B\left(\left(\begin{array}{c} u-v \\ B-b \end{array}\right),\left(\begin{array}{c} u \\ B \end{array}\right)\right)\right\|_{Y_1} \\
& \lesssim \left(\|u\|_{Y_1}+\|v\|_{Y_1}+\|B\|_{Y_1}+\|b\|_{Y_1}\right)\left(\|u-v\|_{Y_1}+\|B-b\|_{Y_1}\right) \\
& \lesssim 4\left(\|u_0\|_{X^p_{\Im}}+\|B_0\|_{X^p_{\Im}}\right)\left(\|u-v\|_{Y_1}+\|B-b\|_{Y_1}\right).
\end{aligned} \tag{29}
$$

Let us consider that $u_0, B_0 \in X^p_{\Im}(\Re^3)^3 \cap \dot{H}^s(\Re^3)^3$ satisfies

$$
\sup_{\Im\in\Re}\left(\|u_0\|_{X^p_{\Im}}+\|u_0\|_{X^p_{\Im}}\right)\leq \min\left\{\frac{1}{8K_1},\frac{1}{4K_2}\frac{1}{4K_3}\right\}. \tag{30}
$$

Then, from Equations (19), (26) and (28), for all $\left(\begin{array}{c} u \\ B \end{array}\right),\left(\begin{array}{c} v \\ b \end{array}\right)\in X_1$ we can easily obtain that

$$
\left\|\mathscr{J}\left(\begin{array}{c} u \\ B \end{array}\right)\right\|_{Y_1}\lesssim 2\left(\|u_0\|_{\dot{H}^s}+\|B_0\|_{\dot{H}^s}\right),
$$

$$
\left\|\mathscr{J}\left(\begin{array}{c} u \\ B \end{array}\right)\right\|_{Z_2}\leq 2\left(\|u_0\|_{X^p_{\Im}}+\|B_0\|_{X^p_{\Im}}\right),
$$

$$
\left\|\mathscr{J}\left(\begin{array}{c} u \\ B \end{array}\right)-\mathscr{J}\left(\begin{array}{c} v \\ b \end{array}\right)\right\|_{Y_1}\leq \frac{1}{2}\left(\|u-v\|_{Y_1}+\|B-b\|_{Z_1}\right).
$$

As a result, by using Banach's contraction principle, there exists a unique solution $u, B \in X_2$ that satisfies (7), which finishes the proof of Theorem 2. $\square$

Remark 2. *By substituting* $B = 0$ *and* $\wp = 1$ *in Theorem 1, we get Theorem 1.5, [30].*

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