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Variable Step Hybrid Block Method for the Approximation of Kepler Problem

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Abstract: In this article, a variable step size strategy is adopted in formulating a new variable step hybrid block method (VSHBM) for the solution of the Kepler problem, which is known to be a rigid and stiff differential equation. To derive the VSHBM, the step size ratio r is left the same, halved, or doubled in order to optimize the total number of steps, minimize the number of formulae stored in the code, and ensure that the method is zero-stable. The method is formulated by integrating the Lagrange polynomial with limits of integration selected at special points. The article further analyzed the stability, order, consistency, and convergence properties of the VSHBM. The stability regions of the VSHBM at different values of the step size ratios were also plotted and plots showed that the method is fit for solving the Kepler problem. The results generated were then compared with some existing methods, including the MATLAB inbuilt stiff solver (ode 15 s), with respect to total number of failure steps, total number of steps, total function calls, maximum error, and computation time.

Keywords: hybrid method; Kepler's equation; Lagrange polynomial; stiff; variable step size

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1. Introduction

Step size selection is an important criterion required in solving stiff differential equations using the integration method, [1]. It is however important to state that too small or too large a step size affects the efficiency of any integration method. A variable step size strategy is one approach that has been employed in choosing the correct step size required for the integration of differential equations.

The Kepler equation first derived in 1609 by Johannes Kepler is an equation in mechanics that establishes the relationship among geometric properties of orbits with respect to central force. The equation plays a prominent role in mathematics and physics, most especially in celestial mechanics. The Kepler equation has various forms, which largely depends on the type of orbit.

The Kepler standard equation (which we shall consider in article) is employed for elliptic orbits, $(0 \le e < 1)$, where e is called the orbital eccentricity. This is a stiff second order differential equation which can be transformed to the following system of first order differential equations,

$$\overline{y}' = f(x, \overline{y}), \overline{y}(a) = \overline{\tau}, a \le x \le b \tag{1}$$

where $\overline{y}^T = (y_1, y_2, \dots, y_m)$ and $\overline{\tau}^T = (\tau_1, \tau_2, \dots, \tau_m)$. We assume the functions $\overline{y}(x)$ and $f(x, \overline{y})$ are sufficiently smooth and also satisfy the existence and uniqueness theorem stated in Theorem 1.

Other forms of the Kepler equation include the radial Kepler equation, which is applied for radial or linear trajectories $(e = 1)$, the Barker's equation applied parabolic trajectories ($e = 1$), and the hyperbolic Kepler equation applied for hyperbolic trajectories ($e > 1$). For $e = 0$, the orbit becomes circular. Equation (1) is assumed to satisfy Theorem 1, which establishes the uniqueness and existence of a solution.

Theorem 1 ([2]). *Let the functions $f_1(x, y_1, y_2, \ldots, y_m)$, $f_2(x, y_1, y_2, \ldots, y_m)$, $\ldots$, $f_m(x, y_1, y_2, \ldots, y_m)$ and their corresponding partial derivatives $\frac{\partial f_1}{\partial x_1}, \frac{\partial f_2}{\partial x_2}, \ldots, \frac{\partial f_m}{\partial x_m}$ be continuous in a region R containing the points $(x, y_1, y_2, \ldots, y_m)$. Then, the initial value problem*

$$\left.\begin{array}{c} y_1{}' = f_1(x, y_1, y_2, \ldots, y_m), y_1(x_0) = t_1 \\ y_2{}' = f_2(x, y_1, y_2, \ldots, y_m), y_2(x_0) = t_2 \\ \cdot \\ \cdot \\ \cdot \\ y_m{}' = f_m(x, y_1, y_2, \ldots, y_m), y_m(x_0) = t_m \end{array}\right\} \tag{2}$$

has a unique solution of the form,

$$\left.\begin{array}{c} y_1 = \varphi_1(x) \\ y_2 = \varphi_2(x) \\ \cdot \\ \cdot \\ \cdot \\ y_m = \varphi_m(x) \end{array}\right\} \tag{3}$$

on the interval I containing $x = x_0$.

Definition 1 ([3]). *The general k-step linear multistep method (LMM) is defined as,*

$$\sum_{j=1}^{k} \alpha_j y_{n+j} = h \sum_{j=1}^{k} \beta_j f_{n+j} \tag{4}$$

where $\alpha_j{}'s$ and $\beta_j{}'s$ are real constant coefficients and μ is the differential equation's order. Equation (4) is implicit if $\beta_k \neq 0$ and explicit if $\beta_k = 0$.

Definition 2 ([4]). *A differential equation is stiff if it satisfies any or all of the following conditions:*

(i) stability requirements in contrast to accuracy constrain the step length,
(ii) some solution components decay much more slowly or rapidly compared to others,
(iii) it has time scales that vary widely, and/or
(iv) all its eigenvalues have negative real parts with large stiffness ratio.

The Kepler problem, which is stiff in nature, satisfies all the conditions stated in Definition 2. Historically, the study of the motion of springs led to the discovery of stiff differential equations. These equations occur frequently in science and engineering. A lot of numerical techniques have been derived for approximating stiff differential equations ranging from trigonometrically fitted methods, nonstandard finite difference methods, and others. See the works of [5–13]. All these methods are constant step methods where the step length is fixed. However, in this research article, emphasis shall be laid on variable step method. Quite a number of researchers have developed different variable step techniques for solving stiff differential equations including the Kepler equations. The authors in [1] proposed variable step methods for solving some differential equations. The authors went further to prove the efficacy of their methods by solving some standard problems, e.g., the Kepler, Van der Pol, and Lokta–Volterra problems. Ref. [14] derived a two-point variable step predictor-corrector block method for the solution of ordinary differential equations. The method developed was in the form of Adams Bashforth-Moulton. The authors developed the method using the step size ratios $r = 1$, $r = 2$, and $r = 1/2$. They went further to plot the stability regions of the method at different step ratios and also applied the method on some ODEs. Ref. [15] also developed a variable step size sixth order Adams block method for the solution of differential equations. The method approximates the solution in each of the steps with the aid of three points simultaneously. Ref. [16] also formulated a variable step, variable order method for solving stiff ODEs. The idea employed in their work is the combination of divided difference and Newton's interpolation formulas as the basis function in the design of the method. Other authors that derived variable step size methods include [17–29].

2. Formulation of the VSHBM

The formulation of the VSHBM is discussed in this section, where the interval $[a, b]$ is subdivided into blocks with interpolation points (x_{n-2}, y_{n-2}), (x_{n-1}, y_{n-1}), (x_n, y_n), (x_{n+1}, y_{n+1}), $(x_{n+3/2}, y_{n+3/2})$ and (x_{n+2}, y_{n+2}); see Figure 1. The approximations y_{n+1}, $y_{n+3/2}$, and y_{n+2} are concurrently determined using three previous values at x_{n-2}, x_{n-1}, and x_n of the previous two steps each with step size rh.

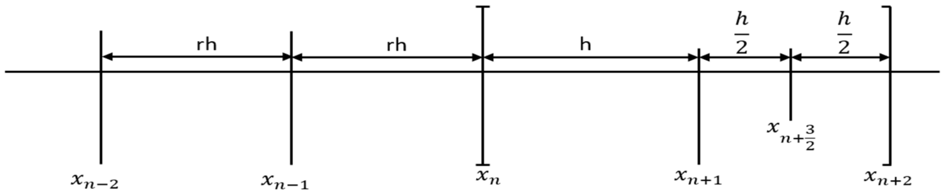

Figure 1. VSHBM showing interpolation points.

In order to optimize the number of steps taken, ensure zero-stability, and reduce the total number of formulae in the code, the step size ratio r is maintained $(r = 1)$, halved $(r = 2)$, or doubled $(r = 1/2)$. This approach is sometimes called the Milne device [30]. This strategy was first suggested by [31,32].

The VSHBM is formulated at the points $x_{n+r}, r = 1, \frac{3}{2}$, and 2 by integrating Equation (1) in the interval (x_n, x_{n+r}),

$$\int_{x_n}^{x_{n+r}} \overline{y}' dx = \int_{x_n}^{x_{n+r}} f(x, \overline{y}) dx \tag{5}$$

The function $f(x, \overline{y})$ in (1) is approximated using Lagrange polynomial $P_q(x)$ of the form

$$P_q(x) = \sum_{j=0}^{k} L_{q,j}(x) f(x_{n+2-j}) \tag{6}$$

where

$$L_{q,j}(x) = \prod_{\substack{i=0 \\ i \neq j}}^{k-1} \frac{x - x_{n+2-i}}{x_{n+2-j} - x_{n+2-i}}, j = 0, \frac{1}{2}, 1, 2, \ldots, k$$

The Lagrange polynomial at the points (x_{n-2}, y_{n-2}), (x_{n-1}, y_{n-1}), (x_n, y_n), (x_{n+1}, y_{n+1}), $(x_{n+3/2}, y_{n+3/2})$, and (x_{n+2}, y_{n+2}) given by

$$\begin{aligned}
P_2(x) = {} & \left[\frac{(x-x_{n-2})(x-x_{n-1})(x-x_n)(x-x_{n+1})(x-x_{n+3/2})}{(x_{n+2}-x_{n-2})(x_{n+2}-x_{n-1})(x_{n+2}-x_n)(x_{n+2}-x_{n+1})(x_{n+2}-x_{n+3/2})} \right] f(x_{n+2}) \\
& + \left[\frac{(x-x_{n-2})(x-x_{n-1})(x-x_n)(x-x_{n+1})(x-x_{n+2})}{(x_{n+3/2}-x_{n-2})(x_{n+3/2}-x_{n-1})(x_{n+3/2}-x_n)(x_{n+3/2}-x_{n+1})(x_{n+3/2}-x_{n+2})} \right] f(x_{n+3/2}) \\
& + \left[\frac{(x-x_{n-2})(x-x_{n-1})(x-x_n)(x-x_{n+3/2})(x-x_{n+2})}{(x_{n+1}-x_{n-2})(x_{n+1}-x_{n-1})(x_{n+1}-x_n)(x_{n+1}-x_{n+3/2})(x_{n+1}-x_{n+2})} \right] f(x_{n+1}) \\
& + \left[\frac{(x-x_{n-2})(x-x_{n-1})(x-x_{n+1})(x-x_{n+3/2})(x-x_{n+2})}{(x_n-x_{n-2})(x_n-x_{n-1})(x_n-x_{n+1})(x_n-x_{n+3/2})(x_n-x_{n+2})} \right] f(x_n) \\
& + \left[\frac{(x-x_{n-2})(x-x_n)(x-x_{n+1})(x-x_{n+3/2})(x-x_{n+2})}{(x_{n-1}-x_{n-2})(x_{n-1}-x_n)(x_{n-1}-x_{n+1})(x_{n-1}-x_{n+3/2})(x_{n-1}-x_{n+2})} \right] f(x_{n-1}) \\
& + \left[\frac{(x-x_{n-1})(x-x_n)(x-x_{n+1})(x-x_{n+3/2})(x-x_{n+2})}{(x_{n-2}-x_{n-1})(x_{n-2}-x_n)(x_{n-2}-x_{n+1})(x_{n-2}-x_{n+3/2})(x_{n-2}-x_{n+2})} \right] f(x_{n-2})
\end{aligned} \tag{7}$$

is used in determining the corrector formulae for y_{n+1}, $y_{n+3/2}$, and y_{n+2}.

The VSHBM for the corrector at y_{n+1}, $y_{n+3/2}$, and y_{n+2} are derived by integrating (1) with respect to s, $s = \frac{(x-x_{n+2})}{h}$, substituting hds for dx and taking the limits of integration at $(-2,-1)$, $(-2,-1/2)$, and $(-2,0)$ respectively. This gives,

$$\begin{aligned} y_{n+1} = y_n &+ \left[\left(\frac{h}{240r^2}\right)\left(\frac{31r+13}{8r^3+18r^2+13r+3}\right)\right]f_{n-2} - \left[\left(\frac{h}{60r^2}\right)\left(\frac{62r+13}{2r^3+9r^2+13r+6}\right)\right]f_{n-1} \\ &+ \left[\left(\frac{h}{720r^2}\right)(240r^2+93r+13)\right]f_n \\ &+ \left[\left(\frac{h}{60}\right)\left(\frac{140r^2+117r+26}{2r^2+3r+1}\right)\right]f_{n+1} - \left[\left(\frac{8h}{45}\right)\left(\frac{30r^2+21r+4}{8r^2+18r+9}\right)\right]f_{n+\frac{3}{2}} + \left[\left(\frac{h}{240}\right)\left(\frac{40r^2+27r+5}{r^2+3r+2}\right)\right]f_{n+2} \end{aligned} \tag{8}$$

$$\begin{aligned} y_{n+\frac{3}{2}} = y_n &+ \left[\left(\frac{9h}{1280r^2}\right)\left(\frac{17r+6}{8r^3+18r^2+13r+3}\right)\right]f_{n-2} - \left[\left(\frac{9h}{160r^2}\right)\left(\frac{17r+3}{2r^3+9r^2+13r+6}\right)\right]f_{n-1} \\ &+ \left[\left(\frac{3h}{1280r^2}\right)(140r^2+51r+6)\right]f_n \\ &+ \left[\left(\frac{9h}{320}\right)\left(\frac{100r^2+99r+27}{2r^2+3r+1}\right)\right]f_{n+1} + \left[\left(\frac{3h}{20}\right)\left(\frac{-20r^2+6r+9}{8r^2+18r+9}\right)\right]f_{n+\frac{3}{2}} + \left[\left(\frac{9h}{1280r^2}\right)\left(\frac{20r+9}{r^2+3r+2}\right)\right]f_{n+2} \end{aligned} \tag{9}$$

$$\begin{aligned} y_{n+2} = y_n &+ \left[\left(\frac{h}{15r^2}\right)\left(\frac{1}{4r^2+7r+3}\right)\right]f_{n-2} - \left[\left(\frac{4h}{15r^2}\right)\left(\frac{4r+1}{2r^3+9r^2+13r+6}\right)\right]f_{n-1} \\ &+ \left[\left(\frac{h}{45r^2}\right)(15r^2+6r+1)\right]f_n \\ &+ \left[\left(\frac{4h}{15}\right)\left(\frac{5r+2}{r+1}\right)\right]f_{n+1} + \left[\left(\frac{128h}{45}\right)\left(\frac{3r+2}{8r^2+18r+9}\right)\right]f_{n+\frac{3}{2}} + \left[\left(\frac{h}{15}\right)\left(\frac{5r^2+9r+5}{r^2+3r+2}\right)\right]f_{n+2} \end{aligned} \tag{10}$$

On the substitution of $r=1$, $r=2$, and $r=1/2$, Equations (8)–(10) give the VSHBM presented in Table 1.

Table 1. VSHBM Formulae at different step size ratios.

Step-Size Ratio	Formulae
$r=1$	$y_{n+1} = y_n + h\left(\frac{11}{2520}f_{n-2} - \frac{1}{24}f_{n-1} + \frac{173}{360}f_n + \frac{283}{360}f_{n+1} - \frac{88}{315}f_{n+\frac{3}{2}} + \frac{1}{20}f_{n+2}\right)$
	$y_{n+\frac{3}{2}} = y_n + h\left(\frac{69}{17920}f_{n-2} - \frac{3}{80}f_{n-1} + \frac{591}{1280}f_n + \frac{339}{320}f_{n+1} - \frac{3}{140}f_{n+\frac{3}{2}} + \frac{87}{2560}f_{n+2}\right)$
	$y_{n+2} = y_n + h\left(\frac{1}{210}f_{n-2} - \frac{2}{45}f_{n-1} + \frac{22}{45}f_n + \frac{14}{15}f_{n+1} + \frac{128}{315}f_{n+\frac{3}{2}} + \frac{19}{90}f_{n+2}\right)$
$r=2$	$y_{n+1} = y_n + h\left(\frac{1}{2112}f_{n-2} - \frac{137}{20160}f_{n-1} + \frac{1159}{2880}f_n + \frac{41}{45}f_{n+1} - \frac{1328}{3465}f_{n+\frac{3}{2}} + \frac{73}{960}f_{n+2}\right)$
	$y_{n+\frac{3}{2}} = y_n + h\left(\frac{3}{7040}f_{n-2} - \frac{111}{17920}f_{n-1} + \frac{501}{1280}f_n + \frac{75}{64}f_{n+1} - \frac{177}{1540}f_{n+\frac{3}{2}} + \frac{147}{2560}f_{n+2}\right)$
	$y_{n+2} = y_n + h\left(\frac{1}{1980}f_{n-2} - \frac{1}{140}f_{n-1} + \frac{73}{180}f_n + \frac{16}{15}f_{n+1} + \frac{1024}{3465}f_{n+\frac{3}{2}} + \frac{43}{180}f_{n+2}\right)$
$r=\frac{1}{2}$	$y_{n+1} = y_n + h\left(\frac{19}{600}f_{n-2} - \frac{44}{225}f_{n-1} + \frac{239}{360}f_n + \frac{239}{360}f_{n+1} - \frac{44}{225}f_{n+\frac{3}{2}} + \frac{19}{600}f_{n+2}\right)$
	$y_{n+\frac{3}{2}} = y_n + h\left(\frac{87}{3200}f_{n-2} - \frac{69}{400}f_{n-1} + \frac{399}{640}f_n + \frac{609}{640}f_{n+1} + \frac{21}{400}f_{n+\frac{3}{2}} + \frac{57}{3200}f_{n+2}\right)$
	$y_{n+2} = y_n + h\left(\frac{8}{225}f_{n-2} - \frac{16}{75}f_{n-1} + \frac{31}{45}f_n + \frac{4}{5}f_{n+1} + \frac{112}{225}f_{n+\frac{3}{2}} + \frac{43}{225}f_{n+2}\right)$

Since the proposed VSHBM is a predictor-corrector method, the predictor formulae were also formulated using the same procedure above at the interpolation points (x_{n-2}, y_{n-2}), (x_{n-1}, y_{n-1}), and (x_n, y_n). This gives

$$y_{n+1} = y_n + \left[\left(\frac{h}{12}\right)\left(\frac{3r+2}{r^2}\right)\right]f_{n-2} - \left[\left(\frac{h}{3}\right)\left(\frac{3r+1}{r^2}\right)\right]f_{n-1} + \left[\left(\frac{h}{12}\right)\left(\frac{12r^2+9r+2}{r^2}\right)\right]f_n \tag{11}$$

$$y_{n+\frac{3}{2}} = y_n + \left[\left(\frac{3h}{16}\right)\left(\frac{8r^2+9r+3}{r^2}\right)\right]f_{n-2} - \left[\left(\frac{9h}{8}\right)\left(\frac{2r+1}{r^2}\right)\right]f_{n-1} + \left[\left(\frac{9h}{16}\right)\left(\frac{r+1}{r^2}\right)\right]f_n \tag{12}$$

$$y_{n+2} = y_n + \left[\left(\frac{h}{3}\right)\left(\frac{3r+4}{r^2}\right)\right]f_{n-2} - \left[\left(\frac{4h}{3}\right)\left(\frac{3r+2}{r^2}\right)\right]f_{n-1} + \left[\left(\frac{h}{3}\right)\left(\frac{6r^2+9r+4}{r^2}\right)\right]f_n \tag{13}$$

At $r = 1$, $r = 2$, and $r = 1/2$, Equations (11)–(13) produce the predictor formulae for the VSHBM as shown in Table 2.

Table 2. Predictor formulae for the VSHBM at different step size ratios.

Step-Size Ratio	Formulae
$r = 1$	$y^p_{n+1} = y_n + h\left(\frac{5}{12}f_{n-2} - \frac{4}{3}f_{n-1} + \frac{23}{12}f_n\right)$
	$y^p_{n+\frac{3}{2}} = y_n + h\left(\frac{9}{8}f_{n-2} - \frac{27}{8}f_{n-1} + \frac{15}{4}f_n\right)$
	$y^p_{n+2} = y_n + h\left(\frac{7}{3}f_{n-2} - \frac{20}{3}f_{n-1} + \frac{19}{3}f_n\right)$
$r = 2$	$y^p_{n+1} = y_n + h\left(\frac{1}{6}f_{n-2} - \frac{7}{12}f_{n-1} + \frac{17}{12}f_n\right)$
	$y^p_{n+\frac{3}{2}} = y_n + h\left(\frac{27}{64}f_{n-2} - \frac{45}{32}f_{n-1} + \frac{159}{64}f_n\right)$
	$y^p_{n+2} = y_n + h\left(\frac{5}{6}f_{n-2} - \frac{8}{3}f_{n-1} + \frac{23}{6}f_n\right)$
$r = \frac{1}{2}$	$y^p_{n+1} = y_n + h\left(\frac{7}{6}f_{n-2} - \frac{10}{3}f_{n-1} + \frac{19}{6}f_n\right)$
	$y^p_{n+\frac{3}{2}} = y_n + h\left(\frac{27}{8}f_{n-2} - 9f_{n-1} + \frac{57}{8}f_n\right)$
	$y^p_{n+2} = y_n + h\left(\frac{22}{3}f_{n-2} - \frac{56}{3}f_{n-1} + \frac{40}{3}f_n\right)$

3. Order, Stability, Consistency and Convergence Analysis of the VSHBM

The order, stability, consistence, and convergence analysis of the VSHBM shall be carried out in this section.

3.1. Order of the VSHBM

Definition 3 ([3]). *The LMM (4) and its associated difference operator L given by*

$$L\{y(x);h\} = \sum_{j=0}^{k}\left[\alpha_j y(x+jh) - h\beta_j y'(x+jh)\right] \tag{14}$$

are called order p if $c_0 = c_1 = c_2 = \ldots = c_p = 0, c_{p+1} \neq 0$.

The component $c_{p+1} \neq 0$ is the method's error constant. The general form for the constant c_p is

$$\left.\begin{aligned}
c_0 &= \sum_{j=0}^{k}\alpha_j \\
c_1 &= \sum_{j=0}^{k}\left(j\alpha_j - \beta_j\right) \\
&\;. \\
&\;. \\
&\;. \\
c_p &= \sum_{j=0}^{k}\left[\frac{1}{p!}j^p\alpha_j - \frac{1}{(p-1)!}j^{p-1}\beta_j\right], p = 2,3,\ldots,q+1
\end{aligned}\right\} \tag{15}$$

The application of Equation (15) on the VSHBM at $r = 1$ gives

$$c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = \begin{bmatrix}0 & 0 & 0\end{bmatrix}^T \tag{16}$$

while

$$c_7 = \begin{bmatrix}\frac{-151}{120960} & \frac{-151}{143360} & \frac{-11}{7560}\end{bmatrix}^T \tag{17}$$

Therefore, the VSHBM is of the uniform sixth order with its error constant given by Equation (17). The same procedure is applied for the VSHBM at r = 2 and r = 1/2.

3.2. Stability of the VSHBM

3.2.1. Zero-Stability of the VSHBM

Definition 4 ([4]). *If no root of the characteristic polynomial has a modulus greater than one and every root with modulus one is simple, then such a method is called zero-stable.*

The scalar test to ascertain the zero-stability of a method was first proposed by [33]. Therefore, substituting

$$y' = f = \lambda y \tag{18}$$

into the VSHBM at $r = 1$ gives

$$\left.\begin{aligned} y_{n+1} &= y_n + h\left(\tfrac{11}{2520}\lambda y_{n-2} - \tfrac{1}{24}\lambda y_{n-1} + \tfrac{173}{360}\lambda y_n + \tfrac{283}{360}\lambda y_{n+1} - \tfrac{88}{315}\lambda y_{n+\frac{3}{2}} + \tfrac{1}{20}\lambda y_{n+2}\right) \\ y_{n+\frac{3}{2}} &= y_n + h\left(\tfrac{69}{17920}\lambda y_{n-2} - \tfrac{3}{80}\lambda y_{n-1} + \tfrac{591}{1280}\lambda y_n + \tfrac{339}{320}\lambda y_{n+1} - \tfrac{3}{140}\lambda y_{n+\frac{3}{2}} + \tfrac{87}{2560}\lambda y_{n+2}\right) \\ y_{n+2} &= y_n + h\left(\tfrac{1}{210}\lambda y_{n-2} - \tfrac{2}{45}\lambda y_{n-1} + \tfrac{22}{45}\lambda y_n + \tfrac{14}{15}\lambda y_{n+1} + \tfrac{128}{315}\lambda y_{n+\frac{3}{2}} + \tfrac{19}{90}\lambda y_{n+2}\right) \end{aligned}\right\} \tag{19}$$

Equation (19) is compactly written in matrix form as

$$\begin{aligned} &\begin{bmatrix} 1-\frac{283}{360}h\lambda & \frac{88}{315}h\lambda & -\frac{1}{20}h\lambda \\ \frac{-339}{320}h\lambda & 1+\frac{3}{140}h\lambda & \frac{-87}{2560}h\lambda \\ \frac{-14}{15}h\lambda & \frac{-128}{315}h\lambda & 1-\frac{19}{90}h\lambda \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} \\ &+h \begin{bmatrix} \frac{11}{2520}\lambda & \frac{-1}{24}\lambda & \frac{173}{360}\lambda \\ \frac{69}{17920}\lambda & \frac{-3}{80}\lambda & \frac{591}{1280}\lambda \\ \frac{1}{210}\lambda & \frac{-2}{45}\lambda & \frac{22}{45}\lambda \end{bmatrix} \begin{bmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} \end{aligned} \tag{20}$$

Equation (20) is rewritten as

$$AY_m = (B + Ch)Y_{m-1} \tag{21}$$

where

$$A = \begin{bmatrix} 1-\frac{283}{360}h\lambda & \frac{88}{315}h\lambda & -\frac{1}{20}h\lambda \\ \frac{-339}{320}h\lambda & 1+\frac{3}{140}h\lambda & \frac{-87}{2560}h\lambda \\ \frac{-14}{15}h\lambda & \frac{-128}{315}h\lambda & 1-\frac{19}{90}h\lambda \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} \frac{11}{2520}\lambda & \frac{-1}{24}\lambda & \frac{173}{360}\lambda \\ \frac{69}{17920}\lambda & \frac{-3}{80}\lambda & \frac{591}{1280}\lambda \\ \frac{1}{210}\lambda & \frac{-2}{45}\lambda & \frac{22}{45}\lambda \end{bmatrix}$$

$$Y_m = \begin{bmatrix} y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{bmatrix} \text{ and } Y_{m-1} = \begin{bmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix}$$

Let $H = h\lambda$. Then, the stability polynomial at $r = 1$ denoted by $R_1(t, H)$ is given by,

$$\begin{aligned} R_1(t,H) = &-t^3\left(\tfrac{18667}{302400}H^3 - \tfrac{172397}{453600}H^2 + \tfrac{2459}{2520}H - 1\right) - t^2\left(\tfrac{15697}{179200}H^3 + \tfrac{2915951}{7257600}H^2 + \tfrac{1039}{1008}H + 1\right) \\ &+t\left(\tfrac{3971}{6451200}H^3 + \tfrac{31997}{8294400}H^2 + \tfrac{11}{1680}H\right) + \tfrac{1}{19353600}H^3 + \tfrac{13}{58060800}H^2 \end{aligned} \tag{22}$$

Applying the same procedure above for the step size ratios $r = 2$ and $r = 1/2$, we obtain the stability polynomials,

$$\begin{aligned} R_2(t,H) = &-t^3\left(\tfrac{8471}{54000}H^3 - \tfrac{106703}{226800}H^2 + \tfrac{797}{770}H - 1\right) - t^2\left(\tfrac{3380029}{53222400}H^3 + \tfrac{1084}{30412800}H^2 + \tfrac{1713499}{1774080}H + 1\right) \\ &+t\left(\tfrac{2699}{38016000}H^3 + \tfrac{846649}{1703116800}H^2 + \tfrac{1627}{1774080}H\right) + \tfrac{1}{1703116800}H^3 + \tfrac{13}{5109350400}H^2 \end{aligned} \tag{23}$$

$$
\begin{aligned}
R_{\frac{1}{2}}(t,H) = & -t^3\left(\tfrac{20399}{432000}H^3 - \tfrac{419447}{1296000}H^2 + \tfrac{363}{400}H - 1\right) - t^2\left(\tfrac{63881}{432000}H^3 + \tfrac{198427}{432000}H^2 + \tfrac{2033}{1800}H + 1\right) \\
& + t\left(\tfrac{2039}{43200}H^3 + \tfrac{10607}{432000}H^2 + \tfrac{133}{3600}H\right) + \tfrac{1}{432000}H^3 + \tfrac{13}{1296000}H^2
\end{aligned}
\tag{24}
$$

The zero-stability for the VSHBM is determined by substituting $H = 0$ in Equations (22)–(24) to obtain

$$R_1(t,H) = R_2(t,H) = R_{\frac{1}{2}}(t,H) = t^3 - t^2 \tag{25}$$

The zeros for the variable step sizes at $r = 1$, $r = 2$, and $r = 1/2$ are presented in Table 3.

Table 3. Zeros for the VSHBM.

Step-Size Ratio	Zeros
$r = 1$	$t_1 = t_2 = 0, t_3 = 1$
$r = 2$	$t_1 = t_2 = 0, t_3 = 1$
$r = 1/2$	$t_1 = t_2 = 0, t_3 = 1$

Since all the zeros satisfy $|t| \leq 1$ as described by Definition 4, it implies that the VSHBM is zero-stable at the step size ratios $r = 1$, $r = 2$, and $r = 1/2$.

3.2.2. Stability Regions of the VSHBM

Definition 5 ([34])**.** *If the stability region of a method contains the whole left half-plane* $Re(h\lambda) < 0$*, such a method is referred to as A-stable.*

The stability regions for the VSHBM are presented in Figure 2.

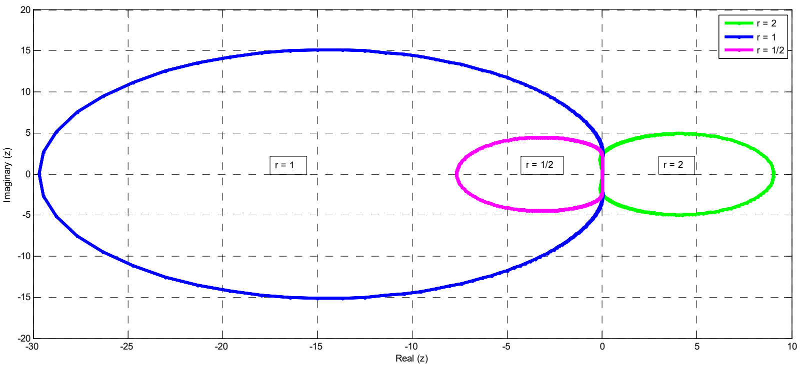

Figure 2. Stability regions of the VSHBM.

The stability region of the VSHBM at $r = 1$ is the interior of the blue contour while that of the VSHBM at $r = \frac{1}{2}$ is the interior of the magenta contour. For the VSHBM at $r = 2$, the stability region is the outer region of the green contour. Thus, in terms of size, the VSHBM at $r = 2$ has the largest stability region followed by the VSHBM at $r = 1$, then the VSHBM at $r = 1/2$.

3.3. Consistency of the VSHBM

Definition 6 ([3]). *The LMM (4) is called consistent if it is of order* $p \geq 1$.

Since the VSHBM is of order 6, it is consistent.

3.4. Convergence of the VSHBM

Theorem 2 ([4])**.** *The necessary and sufficient conditions for the LMM (4) to be convergent are that*

(i) it must be consistent, and
(ii) it must be zero-stable".

See [35] for the proof.

Thus, the proposed VSHBM is convergent since it satisfies the conditions for consistency and zero-stability.

4. Algorithm and Choice of Step Size for the VSHBM

4.1. Algorithm

The implementation algorithm for the VSHBM is presented in this subsection. The first step is the determination of initial points of the method's starting block. The step ratio r in Figure 1 is taken as one. The truncated Taylor series method is used to determine the three back values x_{n-2}, x_{n-1} and x_n since the values of h are small. The values of y_{n+1}, $y_{n+3/2}$, and y_{n+2} are approximated using predictor-corrector methods. Suppose s is the number of iterations required, the sequence of calculations at any point is $(PE)(CE)^1(CE)^2\ldots(CE)^s$, where P is the predictor formulae, C is the corrector formulae, and E represents the function evaluation f of the problem. The corrector VSHBM is iterated to converge with the convergence test employed as $\left|y_{n+2}^{(s+1)} - y_{n+2}^{(s)}\right| < 0.1 * TOL$. The algorithm for the implementation of the VSHBM in code is explicitly stated below:

Step 1: Set data input, such as tolerance level (TOL), initial condition x_0, $y_{0,}$ and step length h

Step 2: Set $y'(x) = f(x, y(x))$

Step 3: Set Taylor series $y_i = y_{i-1} + hy'_{i-1} + \frac{h^2}{2!}y''_{i-1} + \frac{h^3}{3!}y'''_{i-1}$

Step 4: Set the predictor equations

$$P : y_{n+1}^p = \sum_{i=0}^{2} \alpha_{n+i} f_{n-i} + \alpha_{n+3/2} f_{n+3/2}$$

$$y_{n+3/2}^p = \sum_{i=0}^{2} \beta_{n+i} f_{n-i} + \beta_{n+3/2} f_{n+3/2}$$

$$y_{n+2}^p = \sum_{i=0}^{2} \gamma_{n+i} f_{n-i} + \gamma_{n+3/2} f_{n+3/2}$$

$$E : f_{n+1}^p = \left(x_{n+1}, y_{n+1}^p\right)$$

$$f_{n+3/2}^p = \left(x_{n+3/2}, y_{n+3/2}^p\right)$$

$$f_{n+2}^p = \left(x_{n+2}, y_{n+2}^p\right)$$

Step 5: Set the corrector equations

$$C : y_{n+1}^c = \sum_{i=0}^{4} \delta_{n+i} f_{n+2-i} + \delta_{n+3/2} f_{n+3/2}$$

$$y_{n+3/2}^c = \sum_{i=0}^{4} \varepsilon_{n+i} f_{n+2-i} + \varepsilon_{n+3/2} f_{n+3/2}$$

$$y_{n+2}^c = \sum_{i=0}^{4} \sigma_{n+i} f_{n+2-i} + \sigma_{n+3/2} f_{n+3/2}$$

$$E : f_{n+1}^c = (x_{n+1}, y_{n+1}^c)$$

$$f_{n+3/2}^c = (x_{n+3/2}, y_{n+3/2}^c)$$

$$f^c_{n+2} = \left(x_{n+2}, y^c_{n+2}\right)$$

Step 6: Compute the LTE, that is local truncation error
Step 7: If the LTE<TOL, then the solution is acceptable. Maintain $(r = 1)$ or double $(r = 1/2)$ the previous step size and then proceed to Step 9.
Step 8: If LTE>TOL, halve the previous step size $(r = 2)$ and then go back to Step 5.
Step 9: Stop

The flowchart for the VSHBM is shown in Figure 3.

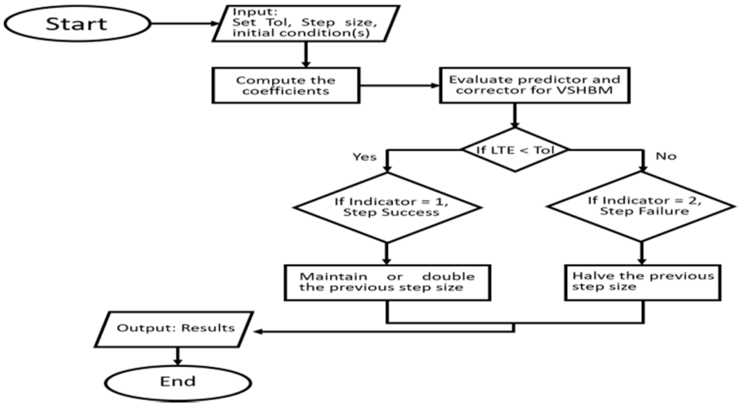

Figure 3. Flowchart for the proposed VSHBM.

4.2. Choice of Step Size

The authors in [36–43] highlighted the importance of the selection of step size in Adams methods for the integration of differential equations. When step size is properly chosen, it minimizes the total number of iterations as well as computation time. The values of approximations $y_{n+1}, y_{n+3/2}$, and y_{n+2} are acceptable if LTE < TOL. The LTE is determined by,

$$LTE = y^{(p)}_{n+2} - y^{(p-1)}_{n+2} \tag{26}$$

where $y^{(p)}_{n+2}$ and $y^{(p-1)}_{n+2}$ are the pth and $(p-1)$th orders of the method. If, on the other hand, LTE is greater than the tolerance level, then $y_{n+1}, y_{n+3/2}$, and y_{n+2} are rejected. This implies that the step is carried out again by taking the step ratio $r = 2$. The step size increment after a successful step (LTE < TOL) is given by

$$h_{new} = u * h_{old} * \left(\frac{TOL}{LTE}\right)^{\frac{1}{p}} \tag{27}$$

where p is the method's order, h_{old} is the previous block's step size, and h_{new} is the current block's step size. The parameter u, called the safety factor, ensures that the failure steps are minimized to the barest minimum.

5. The Kepler Problem

The Kepler problem is a renowned two-body problem that describes planetary motion in an orbit. The center of the coordinate system is represented by one of the bodies while the position of the second body at time t is given by two coordinates $k_1(t)$ and $k_2(t)$. The Kepler problem is given by,

$$\left.\begin{array}{l} k_1''(t) = -\frac{k_1(t)}{\left(k_1^2(t)+k_2^2(t)\right)^{\frac{3}{2}}}, k_1(0) = 1-e, k_1'(0) = 0 \\ k_2''(t) = -\frac{k_2(t)}{\left(k_1^2(t)+k_2^2(t)\right)^{\frac{3}{2}}}, k_2(0) = 0, k_2'(0) = \sqrt{1-e^2} \end{array}\right\} \tag{28}$$

where e defined by the constraint $0 \leq e < 1$ is the orbital eccentricity. The exact solution of the Kepler problem (28) is given by,

$$\left.\begin{array}{l} k_1(t) = \cos(t) - e \\ k_2(t) = \sqrt{1-e^2}\sin(t) \end{array}\right\} \tag{29}$$

It is important to state that Equation (28) is equivalent to the Hamiltonian system

$$\left.\begin{array}{l} k_i' = s_i, i = 1,2 \\ H(s_1, s_2, k_1, k_2) = \frac{(s_1^2+s_2^2)}{2} - \frac{1}{\sqrt{k_1^2+k_2^2}} \end{array}\right\} \tag{30}$$

where $H(s_1, s_2, k_1, k_2)$ denotes the Hamiltonian of the system in Equation (28). To effectively apply the proposed VSHBM (which is in the form of LMM in Equation (4)) on the Kepler problem, we transform (28) to its equivalent first order system of equations. This is achieved by letting $y_1 = k_1$, $y_2 = k_2$, $y_3 = k_1'$, and $y_4 = k_2'$. Equation (28) is thus given by the following system of equations

$$\left.\begin{array}{l} y_1' = y_3, y_1(0) = 1 - e \\ y_2' = y_4, y_2(0) = 0 \\ y_3' = -\frac{y_1}{(y_1^2+y_2^2)^{3/2}}, y_3(0) = 0 \\ y_4' = -\frac{y_2}{(y_1^2+y_2^2)^{3/2}}, y_4(0) = 1 - e \end{array}\right\} \tag{31}$$

where e is the orbital eccentricity.

6. Results and Discussion

The newly derived VSHBM shall be applied on the Kepler problem to test its accuracy, efficiency, and computational reliability in terms of parameters, such as the number of steps, number of failure/rejected steps, number of function calls, maximum error, and computation time.

Absolute error (AbsE) is defined as

$$\text{AbsE} = |y(x) - y_n(x)|$$

Maximum error (MaxE) is defined as

$$\text{MaxE} = \max_{0 \leq n \leq TS} |y(x) - y_n(x)|$$

where $y(x)$ is theoretical/exact solution while $y_n(x)$ is computed/approximate solution.

The proposed VSHBM was employed in approximating the Kepler problem. The results obtained were compared with that of variable step predictor-corrector method (2PVSPCM) developed by [14] at eccentricity $1 - e = 0.0000001$ and run time $t \epsilon [0, 20]$. From Table 4, it is obvious that the VSHBM performed better than that of [14] using different indicators. The maximum error (MaxE) of the VSHBM is by far less than those of [14]. It was also observed that fewer steps (TS) were taken in generating the results in contrast to the method of [14]. This in turn translates to the faster generation of results (see the ComT column) using the VSHBM. From Table 4, it is also obvious that no failure steps (FS) were recorded. The total function call (FCN) column also showed that the proposed VSHBM has more function calls than the variable step method developed by [14].

Table 4. Performance of the VSHBM on Kepler problem with eccentricity $1 - e = 0.0000001$ and run time $t\epsilon[0, 20]$.

TOL	Method	TS	FS	MaxE	FCN	ComT
10^{-2}	VSHBM	18	0	4.2456×10^{-3}	411	452
	2PVSPCM	30	0	9.3294×10^{-2}	309	944
10^{-4}	VSHBM	37	0	1.2477×10^{-5}	1021	1021
	2PVSPCM	61	0	1.4804×10^{-3}	513	1322
10^{-6}	VSHBM	93	0	3.1129×10^{-8}	2478	1892
	2PVSPCM	137	0	1.9884×10^{-5}	1121	2904
10^{-8}	VSHBM	242	0	6.7189×10^{-9}	4781	3622
	2PVSPCM	322	0	2.0891×10^{-7}	2001	5742
10^{-10}	VSHBM	502	0	4.3321×10^{-11}	6719	9876
	2PVSPCM	781	0	2.2126×10^{-9}	4767	13906

In Figure 4, the efficiency curves of the Kepler problem were plotted in terms of time versus maximum error, while in Figure 5 the efficiency curves were plotted with respect to the number of steps versus the maximum error. From the two figures, it is clear that the VSHBM performed better than the 2PVSPCM developed by [14] at the run time [0, 20]. This is because the VSHBM takes less time (Figure 4) and fewer steps (Figure 5) to generate result than the 2PVSPCM developed by [14].

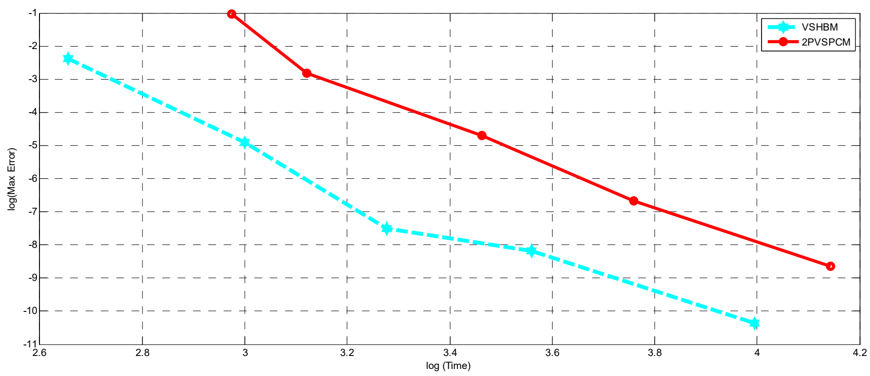

Figure 4. Efficiency curves for Kepler problem in terms of time versus maximum error.

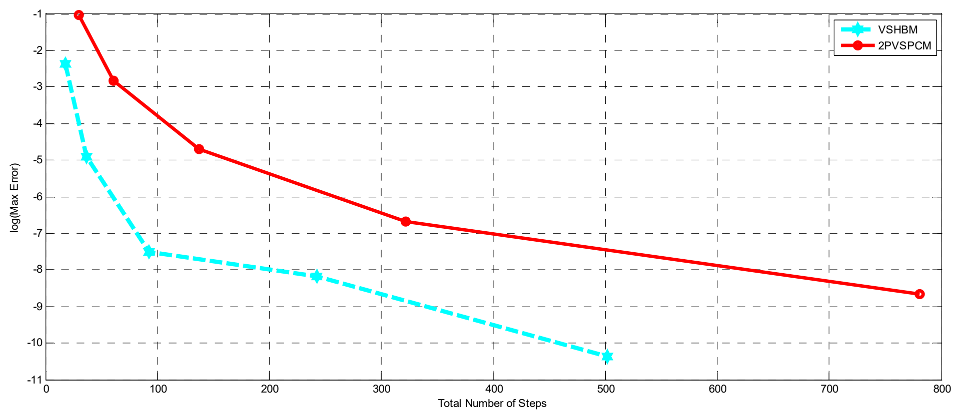

Figure 5. Efficiency curves for Kepler problem in terms of number of steps versus maximum error.

Table 5 clearly shows the performance of the VSHBM on the Kepler problem at different eccentricities $1 - e = 0.1$, 0.001 and 0.00001. Different tolerance levels were considered in the computation. The results generated show that no step failed (i.e., FS = 0) and the accuracy of the VSHBM increases as $1 - e$ reduces or tends to zero.

Table 5. Performance of the VSHBM on Kepler problem at different eccentricities and run time $t \in \left[0,\ 10^8\right]$.

TOL	TS	FS	Eccentricity	MaxE
10^{-2}	18	0	$1 - e = 0.1$	6.6112×10^{-2}
		0	$1 - e = 0.001$	1.5571×10^{-2}
		0	$1 - e = 0.00001$	9.3294×10^{-3}
10^{-6}	93	0	$1 - e = 0.1$	8.4519×10^{-5}
		0	$1 - e = 0.001$	4.1562×10^{-5}
		0	$1 - e = 0.00001$	3.9945×10^{-7}
10^{-10}	502	0	$1 - e = 0.1$	5.1903×10^{-6}
		0	$1 - e = 0.001$	5.1984×10^{-8}
		0	$1 - e = 0.00001$	6.5527×10^{-10}

The Kepler problem (31) was also integrated using the new VSHBM at constant step in the interval $t \in [0, 20]$. The VSHBM was used to calculate the stage value y_1 and the results obtained were compared with those of the Matlab inbuilt solver, ode 15 s. It is obvious that at constant step size, h, the method performed slightly better than the inbuilt ode 15 s in terms of computation time and maximum error; see Table 6 and Figures 6 and 7. However, if variable step strategy was solely adopted in the generation of the results, the VSHBM would have performed better, as clearly seen in Table 4.

Table 6. Computation time and maximum error of the VSHBM using constant step h and eccentricity $1 - e = 0.0000001$.

h	**0.001**	**0.01**	**0.1**
ComT (ode 15 s)	316.84	126.73	15.69
MaxE (ode 15 s)	0.79×10^{-5}	1.14×10^{-3}	10.41×10^{-2}
ComT (VSHBM)	212.48	73.41	12.23
MaxE (VSHBM)	0.59×10^{-5}	1.00×10^{-3}	9.89×10^{-2}

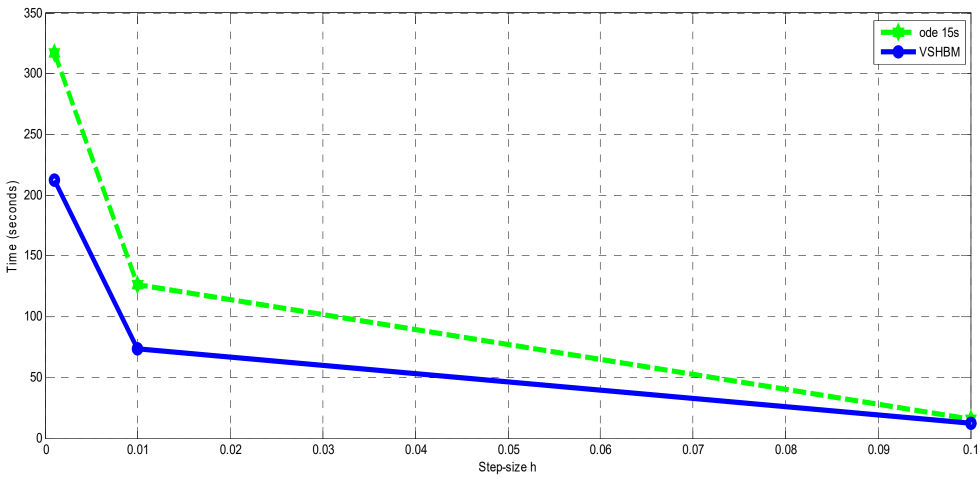

Figure 6. Comparison of computation time for constant step size h.

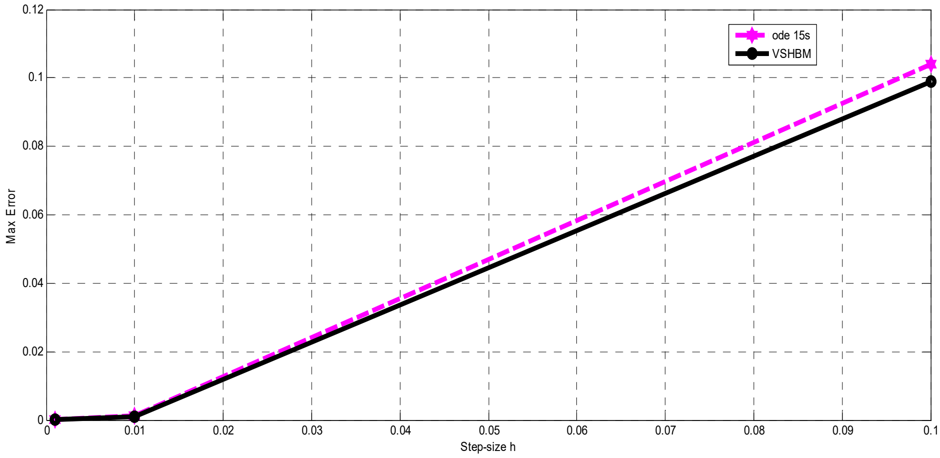

Figure 7. Maximum error comparison for with constant step size h.

7. Conclusions

A VSHBM was formulated for the approximation of the Kepler problem at different eccentricities. From the numerical and graphical results generated, it is clear that the method is computationally reliable. The order, zero-stability, consistence, and convergence of the method were verified. The stability regions of the VSHBM at different values of the step size ratios were also analysed. The results obtained show that the method is convergent, consistent, and also exhibited zero stability. The results further showed that the VSHBM performed better than some existing methods, including the inbuilt MATLAB solver, ode 15 s. It is also important to state that the proposed VSHBM can approximate any stiff differential equation of the form (1).

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Abbreviations

The following abbreviations are used in Tables 4–6 and Figures 4–7.

h	Step size
r	Step size ratio
TS	Total number of steps taken
FS	Total number of failure/rejected steps
FCN	Total function calls
ComT	Computation or execution time in microseconds
AbsE	Absolute error
MaxE	Maximum error
Ode 15s	MATLAB inbuilt stiff solver
2PVSPCM	2-point variable step predictor-corrector block method developed by [14]
VSHBM	Newly formulated variable step hybrid block method

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