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Diamond Alpha Hilbert-Type Inequalities on Time Scales

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Abstract: In this article, we will prove some new diamond alpha Hilbert-type dynamic inequalities on time scales which are defined as a linear combination of the nabla and delta integrals. These inequalities extend some known dynamic inequalities on time scales, and unify and extend some continuous inequalities and their corresponding discrete analogues. Our results will be proven by using some algebraic inequalities, diamond alpha Hölder inequality, and diamond alpha Jensen's inequality on time scales.

Keywords: Hilbert's inequality; dynamic inequality; time scale; diamond-α calculus

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1. Introduction

Over the past decade, a great number of dynamic Hilbert-type inequalities on time scales has been established by many researchers who were motivated by various applications; see the papers [1–4].

For example, Pachpatte [5] proved that if $\{a_m\}$, $\{b_n\}$ are two non-negative sequences of real numbers defined for $m = 1, \dots, k$, and $n = 1, \dots, r$ with $a_0 = b_0 = 0$, and $\{p_m\}$, $\{q_n\}$, are two positive sequences of real numbers defined for $m = 1, \dots, k$ and $n = 1, \dots, r$ where k, r are natural numbers. Further $P_m = \sum_{s=1}^{m} p_s$ and $Q_n = \sum_{t=1}^{n} q_t$, and Φ and Ψ are two real-valued non-negative, convex, and submultiplicative functions defined on $[0, \infty)$, then

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{\Phi(a_m)\Psi(b_n)}{m+n} \leqslant M(k,r)\left(\sum_{m=1}^{k}(k-m+1)\left(p_m\Phi\left(\frac{\nabla a_m}{p_m}\right)\right)^2\right)^{\frac{1}{2}} \times\left(\sum_{n=1}^{r}(r-n+1)\left(q_n\Psi\left(\frac{\nabla b_n}{q_n}\right)\right)^2\right)^{\frac{1}{2}}, \tag{1}$$

where

$$M(k,r) = \frac{1}{2}\left(\sum_{m=1}^{k}\left(\frac{\Phi(P_m)}{p_m}\right)^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{r}\left(\frac{\Psi(Q_n)}{Q_n}\right)^2\right)^{\frac{1}{2}}.$$

Additionally, in the same paper [5], Pachpatte proved that if $\digamma \in C^1[[0,\vartheta],\mathbb{R}^+]$, $g \in C^1[[0,\varsigma],\mathbb{R}^+]$ with $\digamma(0) = g(0) = 0$ and $p(\xi)$, $q(\xi)$ are two positive functions defined for $\xi \in [0,\vartheta)$ and $\tau \in [0,\varsigma)$, $P(\xi) = \int_0^{\xi} p(\xi)d\xi$ and $Q(\xi) = \int_0^{\xi} p(\tau)d\tau$ for $s \in [0,\vartheta)$ and $t \in [0,\varsigma)$ where ϑ, ς are positive real numbers; thus

$$\begin{aligned}\int_0^{\vartheta}\int_0^{\varsigma}\frac{\Phi(F(s))\Psi(g(\Im))}{s+\Im}dsd\Im \leqslant L(\vartheta,\varsigma)\left(\int_0^{\vartheta}(\vartheta-s)\left(p(s)\Phi\left(\frac{F'(s)}{p(s)}\right)^2 ds\right)^{\frac{1}{2}}\right. \\ \times\left(\int_0^{\varsigma}(\varsigma-\Im)\left(q(\Im)\Psi\left(\frac{g'(\Im)}{q(\Im)}\right)^2 d\Im\right)^{\frac{1}{2}},\end{aligned} \tag{2}$$

where

$$L(\vartheta,\varsigma) = \frac{1}{2}\left(\int_0^{\vartheta}\left(\frac{\Phi(P(s))}{P(s)}\right)^2 ds\right)^{\frac{1}{2}}\left(\int_0^{\varsigma}\left(\frac{\Psi(Q(\Im))}{Q(\Im)}\right)^2 d\Im\right)^{\frac{1}{2}}.$$

Under the same conditions as seen above, with few modifications, Handley et al. [6] extended (1) and (2) as follows:

$$\sum_{m_1=1}^{k_1}\cdots\sum_{m_n=1}^{k_n}\frac{\prod_{\ell=1}^{n}\Phi_\ell(a_{\ell,m_\ell})}{\left(\sum_{\ell=1}^{n}\gamma'_\ell m_\ell\right)^{\gamma'}} \leqslant M(k_1,\ldots,k_n)\prod_{\ell=1}^{n}\left(\sum_{m_\ell=1}^{k_\ell}(k_\ell-m_\ell+1)\left(p_{\ell,m_\ell}\Phi_\ell\left(\frac{\nabla a_{\ell,m_\ell}}{p_{\ell,m_\ell}}\right)^{\frac{1}{\gamma_\ell}}\right)^{\gamma_\ell}\right., \tag{3}$$

$$M(k_1,\ldots,k_n) = \frac{1}{(\gamma')^{\gamma'}}\prod_{\ell=1}^{n}\left(\sum_{m_\ell=1}^{k_\ell}\left(\frac{\Phi_\ell(P_{\ell,m_\ell})}{P_{\ell,m_\ell}}\right)^{\frac{1}{\gamma'_\ell}}\right)^{\gamma'_\ell},$$

and

$$\begin{aligned}&\int_0^{\vartheta_1}\cdots\int_0^{\vartheta_n}\frac{\prod_{\ell=1}^{n}\Phi_\ell(F(s_\ell))}{\left(\sum_{\ell=1}^{n}\gamma'_\ell s_\ell\right)^{\gamma'}}ds_1\ldots ds_n \\ &\leqslant L(\vartheta_1,\ldots,\vartheta_n)\prod_{\ell=1}^{n}\left(\int_0^{\vartheta_\ell}(\vartheta_\ell-s_\ell)\left(p_\ell(s_\ell)\Phi_\ell\left(\frac{F'(s_\ell)}{p(s_\ell)}\right)^{\frac{1}{\gamma_\ell}}ds_\ell\right)^{\gamma_\ell}\right.,\end{aligned} \tag{4}$$

$$L(\vartheta_1,\ldots,\vartheta_n) = \frac{1}{(\gamma')^{\gamma'}}\prod_{\ell=1}^{n}\left(\int_0^{\vartheta_\ell}\left(\frac{\Phi_\ell(P_\ell(s_\ell))}{P_\ell(s_\ell)}\right)^{\frac{1}{\gamma'_\ell}}ds_\ell\right)^{\gamma'_\ell},$$

where $\gamma_i \in (0.1)$, $\gamma'_i = 1-\gamma$, $\gamma = \sum_{i=1}^{n}\gamma_i$, and $\gamma'_i = n-\gamma$.

In [7], Pachpatte established the following Hilbert-type integral inequalities under the following conditions: If $h \geqslant 1$, $l \geqslant 1$, and $f(\xi) \geqslant 0$, $g(\xi) \geqslant 0$, for $\xi \in (0,\vartheta)$ and $\tau \in (0,\varsigma)$, where ϑ and ς are positive real numbers and define $F(s) = \int_0^s f(\xi)d\xi$ and $G(t) = \int_0^t g(\tau)d\tau$, for $s \in (0,\vartheta)$ and $t \in (0,\varsigma)$, then

$$\begin{aligned}\int_0^{\vartheta}\int_0^{\varsigma}\frac{F^h(s)G^l(\Im)}{s+\Im}dsd\Im \leqslant \tfrac{1}{2}hl(\vartheta\varsigma)^{\frac{1}{2}}\left(\int_0^{\vartheta}(\vartheta-s)\left(F^{h-1}(s)F(s)\right)^2 ds\right)^{\frac{1}{2}} \\ \times\left(\int_0^{\varsigma}(\varsigma-\Im)\left(G^{l-1}g(\Im)\right)^2 d\Im\right)^{\frac{1}{2}},\end{aligned} \tag{5}$$

and

$$\int_0^{\vartheta}\int_0^{\varsigma}\frac{\Phi(F(s))\Psi(G(\Im))}{s+\Im}dsd\Im \leqslant L(\vartheta,\varsigma)\left(\int_0^{\vartheta}(\vartheta-s)\left(p(s)\Phi\left(\frac{F(s)}{p(s)}\right)\right)^2 ds\right)^{\frac{1}{2}} \times\left(\int_0^{\varsigma}(\varsigma-\Im)\left(q(\Im)\Psi\left(\frac{g(\Im)}{q(\Im)}\right)\right)^2 d\Im\right)^{\frac{1}{2}} \tag{6}$$

where

$$L(\vartheta,\varsigma)=\frac{1}{2}\left(\int_0^{\vartheta}\left(\frac{\Phi(P(s))}{P(s)}\right)^2 ds\right)^{\frac{1}{2}}\left(\int_0^{\varsigma}\left(\frac{\Psi(Q(\Im))}{Q(\Im)}\right)^2 d\Im\right)^{\frac{1}{2}},$$

and

$$\int_0^{\vartheta}\int_0^{\varsigma}\frac{P(s)Q(\Im)\Phi(F(s))\Psi(G(\Im))}{s+\Im}dsd\Im \leqslant \frac{1}{2}(s\Im)^{\frac{1}{2}}\left(\int_0^{\vartheta}(\vartheta-s)\Big(p(s)\Phi\Big(F(s)\Big)\Big)^2 ds\right)^{\frac{1}{2}} \times\left(\int_0^{\varsigma}(\varsigma-\Im)\Big(q(\Im)\Psi\Big(g(\Im)\Big)\Big)^2 d\Im\right)^{\frac{1}{2}}. \tag{7}$$

$$\int_0^{\vartheta_1}\int_0^{\varsigma_1}\cdots\int_0^{\vartheta_n}\int_0^{\varsigma_n}\frac{\prod\limits_{\ell=1}^{n}\Phi_\ell(F_\ell(s_\ell,\Im_\ell))}{\left(\gamma\sum\limits_{\ell=1}^{n}\frac{1}{\gamma_\ell}(s_\ell)(\Im_\ell)\right)^{\frac{1}{\gamma}}}ds_1d\Im_1\ldots ds_nd\Im_n \geqslant L(\vartheta_1\varsigma_1,\ldots,\vartheta_n\varsigma_n) \times\prod_{\ell=1}^{n}\left(\int_0^{\vartheta_\ell}\int_0^{\varsigma_\ell}(\vartheta_\ell-s_\ell)(\varsigma_\ell-\Im_\ell)\left(p_\ell(s_\ell,\Im_\ell)\Phi_\ell\left(\frac{F_\ell(s_\ell,\Im_\ell)}{p_\ell(s_\ell,\Im_\ell)}\right)\right)^{\beta_\ell}ds_\ell d\Im_\ell\right)^{\frac{1}{\beta_\ell}}. \tag{8}$$

where

$$L(\vartheta_1\varsigma_1,\ldots,\vartheta_n\varsigma_n)=\prod_{\ell=1}^{n}\left(\int_0^{\vartheta_\ell}\int_0^{\varsigma_\ell}\left(\frac{\Phi_\ell(P_\ell(s_\ell,\Im_\ell))}{P_\ell(s_\ell,\Im_\ell)}\right)^{\gamma_\ell}ds_\ell d\Im_\ell\right)^{\frac{1}{\gamma_\ell}}.$$

A time scale $\mathbb{T}$ is an arbitrary, non-empty, closed subset of the set of real numbers $\mathbb{R}$. Throughout the article, we assume that $\mathbb{T}$ has the topology that it inherits from the standard topology on $\mathbb{R}$. We define the forward jump operator $\sigma:\mathbb{T}\to\mathbb{T}$ for any $\zeta\in\mathbb{T}$ by

$$\sigma(\zeta):=\inf\{s\in\mathbb{T}:s>\zeta\},$$

and the backward jump operator $\rho:\mathbb{T}\to\mathbb{T}$ for any $\zeta\in\mathbb{T}$ by

$$\rho(\zeta):=\sup\{s\in\mathbb{T}:s<\zeta\}.$$

In the preceding two definitions, we set $\inf\emptyset=\sup\mathbb{T}$ (i.e., if ζ is the maximum of $\mathbb{T}$, then $\sigma(\zeta)=\zeta$) and $\sup\emptyset=\inf\mathbb{T}$ (i.e., if ζ is the minimum of $\mathbb{T}$, then $\rho(\zeta)=\zeta$), where $\emptyset$ denotes the empty set.

A point $\zeta\in\mathbb{T}$ with $\inf\mathbb{T}<\zeta<\sup\mathbb{T}$ is said to be right-scattered if $\sigma(\zeta)>\zeta$, right-dense if $\sigma(\zeta)=\zeta$, left-scattered if $\rho(\zeta)<\zeta$, and left-dense if $\rho(\zeta)=\zeta$. Points that are simultaneously right-dense and left-dense are said to be dense points, whereas points that are simultaneously right-scattered and left-scattered are said to be isolated points.

The forward graininess function $\mu:\mathbb{T}\to[0,\infty)$ is defined for any $\zeta\in\mathbb{T}$ by $\mu(\zeta):=\sigma(\zeta)-\zeta$.

If $F:\mathbb{T}\to\mathbb{R}$ is a function, then the function $F^\sigma:\mathbb{T}\to\mathbb{R}$ is defined by $F^\sigma(\zeta)=F(\sigma(\zeta)),\forall\zeta\in\mathbb{T}$, that is $F^\sigma=F\circ\sigma$. Similarly, the function $F^\rho:\mathbb{T}\to\mathbb{R}$ is defined by $F^\rho(\zeta)=g(\rho(\zeta)),\forall\zeta\in\mathbb{T}$; that is, $F^\rho=F\circ\rho$.

The sets $\mathbb{T}^\kappa$, $\mathbb{T}_\kappa$ and $\mathbb{T}^\kappa_\kappa$ are introduced as follows: if $\mathbb{T}$ has a left-scattered maximum ζ_1, then $\mathbb{T}^\kappa=\mathbb{T}-\{\zeta_1\}$, otherwise $\mathbb{T}^\kappa=\mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum ζ_2, then $\mathbb{T}^\kappa=\mathbb{T}-\{\zeta_2\}$, otherwise $\mathbb{T}_\kappa=\mathbb{T}$. Finally, we have $\mathbb{T}^\kappa_\kappa=\mathbb{T}^\kappa\cap\mathbb{T}_\kappa$.

The interval $[a, b]$ in $\mathbb{T}$ is defined by

$$[a, b]_{\mathbb{T}} = \{\zeta \in \mathbb{T} : a \leq \zeta \leq b\}.$$

We define the open intervals and half-closed intervals similarly.

Assume $F : \mathbb{T} \to \mathbb{R}$ is a function and $\zeta \in \mathbb{T}^{\kappa}$. Then $F^{\Delta}(\zeta) \in \mathbb{R}$ is said to be the delta derivative of F at ζ if for any $\varepsilon > 0$ there exists a neighborhood U of ζ such that, for every $s \in U$, we have

$$\left|F(\sigma(\zeta)) - F(s)] - F^{\Delta}(\zeta)[\sigma(\zeta) - s]\right| \leq \varepsilon|\sigma(\zeta) - s|.$$

Moreover, F is said to be delta differentiable on $\mathbb{T}^{\kappa}$ if it is delta differentiable at every $\zeta \in \mathbb{T}^{\kappa}$.

Similarly, we say that $F^{\nabla}(\zeta) \in \mathbb{R}$ is the nabla derivative of F at ζ if, for any $\varepsilon > 0$, there is a neighborhood V of ζ, such that for all $s \in V$

$$\left|[F(\rho(\zeta)) - F(s)] - F^{\nabla}(\zeta)[\rho(\zeta) - s]\right| \leq \varepsilon|\rho(\zeta) - s|. \tag{9}$$

Furthermore, F is said to be nabla differentiable on $\mathbb{T}_{\kappa}$ if it is nabla differentiable at each $\zeta \in \mathbb{T}_{\kappa}$.

A function $F : \mathbb{T} \to \mathbb{R}$ is said to be right-dense continuous (rd-continuous) if F is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits exist at all left-dense points in $\mathbb{T}$.

In a similar manner, a function $F : \mathbb{T} \to \mathbb{R}$ is said to be left-dense continuous (ld-continuous) if F is continuous at all left-dense points in $\mathbb{T}$ and its right-sided limits exist at all right-dense points in $\mathbb{T}$.

The delta integration by parts on time scales is given by the following formula

$$\int_a^b g^{\Delta}(\zeta)F(\zeta)\Delta\zeta = g(b)F(b) - g(a)F(a) - \int_a^b g^{\sigma}(\zeta)F(\zeta)\Delta\zeta, \tag{10}$$

whereas the nabla integration by parts on time scales is given by

$$\int_a^b g^{\nabla}(\zeta)F(\zeta)\nabla\zeta = g(b)F(b) - g(a)F(a) - \int_a^b g^{\rho}(\zeta)F(\zeta)\nabla\zeta. \tag{11}$$

The following relations will be used.

(i) If $\mathbb{T} = \mathbb{R}$, then

$$\begin{gathered} \sigma(\zeta) = \rho(\zeta) = \zeta, \quad \mu(\zeta) = \nu(\zeta) = 0, \quad F^{\Delta}(\zeta) = F^{\nabla}(\zeta) = F'(\zeta), \\ \int_a^b F(\zeta)\Delta\zeta = \int_a^b F(\zeta)\nabla\zeta = \int_a^b F(\zeta)d\zeta. \end{gathered} \tag{12}$$

(ii) If $\mathbb{T} = \mathbb{Z}$, then

$$\begin{gathered} \sigma(\zeta) = \zeta + 1, \quad \rho(\zeta) = \zeta - 1, \quad \mu(\zeta) = \nu(\zeta) = 1, \\ F^{\Delta}(\zeta) = \Delta F(\zeta), \quad F^{\nabla}(\zeta) = \nabla F(\zeta), \\ \int_a^b F(\zeta)\Delta\zeta = \sum_{\zeta=a}^{b-1} F(\zeta), \quad \int_a^b F(\zeta)\nabla\zeta = \sum_{\zeta=a+1}^{b} F(\zeta), \end{gathered} \tag{13}$$

where Δ and ∇ are the forward and backward difference operators, respectively.

Now we will introduce the diamond-α calculus on time scales, and we refer the interested reader to [8,9] for further details on the definitions of nabla and delta integrals and derivatives.

If $\mathbb{T}$ is a time scale, and F is a function that is delta and nabla differentiable on $\mathbb{T}$, then, for any $\zeta \in \mathbb{T}$, the diamond-α dynamic derivative of F at ζ, denoted by $F^{\diamondsuit_\alpha}(\zeta)$, is defined by

$$F^{\diamondsuit_\alpha}(\zeta) = \alpha F^{\Delta}(\zeta) + (1-\alpha)F^{\nabla}(\zeta), \qquad 0 \le \alpha \le 1. \tag{14}$$

We conclude from the last relation that a function F is diamond-α differentiable if and only if it is both delta and nabla differentiable. For $\alpha = 1$, the diamond-α derivative boils down to a delta derivative, and for $\alpha = 0$ it boils down to a nabla derivative.

Assume F, $g : \mathbb{T} \to \mathbb{R}$ are diamond-α differentiable functions at $\zeta \in \mathbb{T}$, and let $k \in \mathbb{R}$. Then

(*i*) $(F+g)^{\diamondsuit_\alpha}(\zeta) = F^{\diamondsuit_\alpha}(\zeta) + g^{\diamondsuit_\alpha}(\zeta)$;
(*ii*) $(kf)^{\diamondsuit_\alpha}(\zeta) = kf^{\diamondsuit_\alpha}(\zeta)$;
(*iii*) $(fg)^{\diamondsuit_\alpha}(\zeta) = F^{\diamondsuit_\alpha}(\zeta)g(\zeta) + \alpha F^{\sigma}(\zeta)g^{\Delta}(\zeta) + (1-\alpha)F^{\rho}(\zeta)g^{\nabla}(\zeta)$.

Let $F : \mathbb{T} \to \mathbb{R}$ be a continuous function. Then the definite diamond-α integral of F is defined by

$$\int_a^b F(\zeta)\diamondsuit_\alpha\zeta = \alpha\int_a^b F(\zeta)\Delta\zeta + (1-\alpha)\int_a^b F(\zeta)\nabla\zeta, \qquad 0 \le \alpha \le 1, \quad a, b \in \mathbb{T}. \tag{15}$$

Let a, b, $c \in \mathbb{T}$, $k \in \mathbb{R}$. Then,

(*i*) $\int_a^b [F(\zeta) + g(\zeta)]\diamondsuit_\alpha\zeta = \int_a^b F(\zeta)\diamondsuit_\alpha\zeta + \int_a^b g(\zeta)\diamondsuit_\alpha\zeta$;
(*ii*) $\int_a^b kf(\zeta)\diamondsuit_\alpha\zeta = k\int_a^b F(\zeta)\diamondsuit_\alpha\zeta$;
(*iii*) $\int_a^b F(\zeta)\diamondsuit_\alpha\zeta = \int_a^c F(\zeta)\diamondsuit_\alpha\zeta + \int_c^b F(\zeta)\diamondsuit_\alpha\zeta$;
(*iv*) $\int_a^b F(\zeta)\diamondsuit_\alpha\zeta = -\int_b^a F(\zeta)\diamondsuit_\alpha\zeta$;
(*v*) $\int_a^a F(\zeta)\diamondsuit_\alpha\zeta = 0$;
(*vi*) if $F(\zeta) \ge 0$ on $[a,b]_{\mathbb{T}}$, then $\int_a^b F(\zeta)\diamondsuit_\alpha\zeta \ge 0$;
(*vii*) if $F(\zeta) \ge g(\zeta)$ on $[a,b]_{\mathbb{T}}$, then $\int_a^b F(\zeta)\diamondsuit_\alpha\zeta \ge \int_a^b g(\zeta)\diamondsuit_\alpha\zeta$;
(*viii*) $\left|\int_a^b F(\zeta)\diamondsuit_\alpha\zeta\right| \le \int_a^b |F(\zeta)|\diamondsuit_\alpha\zeta$.

Let F be a *diamond* $-$ α differentiable function on $[a,b]_{\mathbb{T}}$. Then F is increasing if $F^{\diamondsuit_\alpha}(\zeta) > 0$, non-decreasing if $F^{\diamondsuit_\alpha}(\zeta) \ge 0$, decreasing if $F^{\diamondsuit_\alpha}(\zeta) < 0$, and non-increasing if $F^{\diamondsuit_\alpha}(\zeta) \le 0$ on $[a,b]_{\mathbb{T}}$.

Next, we write Hölder's inequality and Jensen's inequality on time scales.

Lemma 1 (Dynamic Hölder's Inequality [3])**.** *Suppose u, $v \in \mathbb{T}$ with $u < v$. Assume F^*, $g^* \in CC^1([u,v]_{\mathbb{T}} \times [u,v]_{\mathbb{T}}, \mathbb{R})$ be integrable functions and $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$ then*

$$\begin{aligned}\int_u^v\int_u^v |F^*(r^*,\Im^*)g^*(r^*,\Im^*)|\diamondsuit_\alpha r^*\diamondsuit_\alpha\Im^* \quad \le \quad & \left[\int_u^v\int_u^v |F^*(r^*,\Im^*)|^p\diamondsuit_\alpha r^*\diamondsuit_\alpha\Im^*\right]^{\frac{1}{p}} \\ & \times\left[\int_u^v\int_u^v |g^*(r^*,\Im^*)|^q\diamondsuit_\alpha r^*\diamondsuit_\alpha\Im^*\right]^{\frac{1}{q}}.\end{aligned} \tag{16}$$

This inequality is reversed if $0 < p < 1$ and if $p < 0$ or $q < 0$.

Lemma 2 (Dynamic Jensen's inequality [3])**.** *Let r^*, $\Im^* \in R$ and $-\infty \leqslant m^*, n^* \leqslant \infty$. If $F^* \in CC^1(\mathbb{R}, (m^*, n^*))$ and $\Phi : (m^*, n^*) \longrightarrow \mathbb{R}$ is convex then*

$$\phi\left(\frac{\int_u^v\int_\omega^s F^*(r^*,\Im^*)\diamondsuit_{\alpha_1}r^*\diamondsuit_{\alpha_2}\Im^*}{\int_u^v\int_\omega^s \diamondsuit_{\alpha_1}r^*\diamondsuit_{\alpha_2}\Im^*}\right) \leqslant \frac{\int_u^v\int_\omega^s \phi(F^*(r^*,\Im^*))\diamondsuit_{\alpha_1}r^*\diamondsuit_{\alpha_2}\Im^*}{\int_u^v\int_\omega^s \diamondsuit_{\alpha_1}r^*\diamondsuit_{\alpha_2}\Im^*}. \tag{17}$$

This inequality is reversed if $\phi \in C\big((c,d), \mathbb{R}\big)$ is concave.

Definition 1. Φ *is called a supermultiplicative function on* $[0,\infty)$ *if*

$$\Phi(\vartheta\varsigma) \geqslant \Phi(\vartheta)\Phi(\varsigma), \ \textit{for all} \ \vartheta, \varsigma \geqslant 0. \tag{18}$$

In this paper, we extend some generalizations of the integral Hardy–Hilbert inequality to a general time scale using diamond alpha calculus. As special cases of our results, we will recover some dynamic integral and discrete inequalities known in the literature.

Now we are ready to state and prove our main results.

2. Main Results

First, we enlist the following assumptions for the proof of our main results:

(S_1) $\mathbb{T}$ be time scales with $\Im_0$, ϑ_ℓ, ς_ℓ, s_ℓ, $\Im_\ell \in \mathbb{T}$, $(\ell = 1, \ldots, n)$.
(S_2) $F_\ell(s_\ell, \Im_\ell)$ are non-negative, diamond-Alpha integrable functions defined on $[\Im_0, \vartheta_\ell)_\mathbb{T} \times [\Im_0, \varsigma_\ell)_\mathbb{T}$ $(\ell = 1, \ldots, n)$.
(S_3) $F_\ell(s_\ell, \Im_\ell)$ have partial $\diamondsuit_\alpha$-derivatives $F_\ell^{\diamondsuit_{\alpha_1}}(s_\ell, \Im_\ell)$ and $F_\ell^{\diamondsuit_{\alpha_2}}(s_\ell, \Im_\ell)$ with respect s_ℓ and $\Im_\ell$, respectively.
(S_4) All functions used in this section are integrable according to $\diamondsuit_\alpha$ sense.
(S_5) $F_\ell(s_\ell, \Im_\ell) \in C^2\big([\Im_0, \vartheta_\ell)_\mathbb{T} \times [\Im_0, \varsigma_\ell)_\mathbb{T}, [0,\infty)\big)$ $(\ell = 1, \ldots, n)$.
(S_6) $p_\ell(\xi_\ell, \tau_\ell)$ are n positive diamond-Alpha integrable functions defined for $\xi_\ell \in (\Im_0, s_\ell)_\mathbb{T}$, $\tau_\ell \in (\Im_0, \Im_\ell)_\mathbb{T}$.
(S_7) $p_\ell(\xi_\ell)$ and $q_\ell(\tau_\ell)$ are positive diamond-Alpha integrable functions defined for $\xi_\ell \in (\Im_0, s_\ell)_\mathbb{T}$, $\tau_\ell \in (\Im_0, \Im_\ell)_\mathbb{T}$.
(S_8) Φ_ℓ $(\ell = 1, \ldots, n)$ are n real-valued non-negative concave and supermultiplicative functions defined on $(0,\infty)$.
(S_9) ϑ_ℓ and ς_ℓ are positive real numbers.
(S_{10}) $s_\ell \in [\Im_0, \vartheta_\ell)_\mathbb{T}$ and $\Im_\ell \in [\Im_0, \varsigma_\ell)_\mathbb{T}$.
(S_{11}) $F_\ell(\Im_0, \Im_\ell) = F_\ell(s_\ell, \Im_0) = 0$, $(\ell = 1, \ldots, n)$.
(S_{12}) $F_\ell^{\diamondsuit_{\alpha_1}\diamondsuit_{\alpha_2}}(s_\ell, \Im_\ell) = F_\ell^{\diamondsuit_{\alpha_2}\diamondsuit_{\alpha_1}}(s_\ell, \Im_\ell)$.
(S_{13}) $P_\ell(s_\ell, \Im_\ell) = \int_{\Im_0}^{\Im_\ell}\int_{\Im_0}^{s_\ell} p_\ell(\xi_\ell) q_\ell(\tau_\ell) \diamondsuit_\alpha \xi_\ell \diamondsuit_\alpha \tau_\ell$.
(S_{14}) $\tilde{F}_\ell(s_\ell, \Im_\ell) = \int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell} F_\ell(\xi_\ell, \tau_\ell) \diamondsuit_\alpha \xi_\ell \diamondsuit_\alpha \tau_\ell$.
(S_{15}) $P_\ell(s_\ell, \Im_\ell) = \int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell} p_\ell(\xi_\ell, \tau_\ell) \diamondsuit_\alpha \xi_\ell \diamondsuit_\alpha \tau_\ell$.
(S_{16}) $\tilde{F}_\ell(s_\ell, \Im_\ell) = \frac{1}{P_\ell(\xi_\ell, \tau_\ell)} \int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell} p_\ell(\xi_\ell, \tau_\ell) F_\ell(\xi_\ell, \tau_\ell) \diamondsuit_\alpha \xi_\ell \diamondsuit_\alpha \tau_\ell$.
(S_{17}) $\gamma_\ell \in (1,\infty)$, $\gamma'_\ell = 1 - \gamma_\ell$, $\gamma = \sum_{\ell=1}^{n} \gamma_\ell$, and $\gamma' = \sum_{\ell=1}^{n} \gamma'_\ell = n - \gamma$, $(\ell = 1, \ldots, n)$.
(S_{18}) $0 < \beta_\ell < 1$.
(S_{19}) $h_\ell \geqslant 2$.
(S_{20}) $\sum_{\ell=1}^{n} \frac{1}{\gamma_\ell} = \frac{1}{\gamma}$.
(S_{21}) $h_\ell \geqslant 1$.
(S_{22}) $F_\ell(\xi_\ell) \in C^1[\Im_0, \vartheta_\ell]_\mathbb{T}$, $(\ell = 1, \ldots, n)$.
(S_{23}) ϑ_ℓ is positive real number.
(S_{24}) $\tilde{F}_\ell(s_\ell) = \int_{\Im_0}^{s_\ell} F_\ell(\xi_\ell) \diamondsuit_\alpha \xi_\ell$.
(S_{25}) $s_\ell \in [\Im_0, \vartheta_\ell)_\mathbb{T}$.
(S_{26}) $p_\ell(\xi_\ell)$ are n positive functions.
(S_{27}) $P_\ell(s_\ell) = \int_{\Im_0}^{s_\ell} p_\ell(\xi_\ell) \diamondsuit_\alpha \xi_\ell$.
(S_{28}) $\tilde{F}_\ell(s_\ell) = \frac{1}{P_\ell(s_\ell)} \int_{\Im_0}^{s_\ell} p(\xi_\ell) F(\xi_\ell) \diamondsuit_\alpha \xi_\ell$.
(S_{29}) $F_\ell(\Im_0) = 0$.

Now, we are ready to state and prove the main results that extend several results in the literature.

Theorem 1. *Let* S_1, S_4, S_5, S_{14}, S_6, S_{15}, *and* S_8 *be satisfied. Then for* S_{10}, S_{18} *and* S_{20} *we find that*

$$\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\vartheta_n}\int_{\Im_0}^{\varsigma_n}\frac{\prod_{\ell=1}^{n}\Phi_\ell(\tilde{F}_\ell(s_\ell,\Im_\ell))}{\left(\gamma\sum_{\ell=1}^{n}\frac{1}{\gamma_\ell}(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\frac{1}{\gamma}}}\diamondsuit_\alpha s_1\diamondsuit_\alpha\Im_1\ldots\diamondsuit_\alpha s_n\diamondsuit_\alpha\Im_n$$
$$\geqslant L(\vartheta_1\varsigma_1,\ldots,\vartheta_n\varsigma_n)$$
$$\times\prod_{\ell=1}^{n}\left(\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}(\rho(\vartheta_\ell)-s_\ell)(\rho(\varsigma_\ell)-\Im_\ell)\left(p_\ell(s_\ell,\Im_\ell)\Phi_\ell\left(\frac{F_\ell(s_\ell,\Im_\ell)}{p_\ell(s_\ell,\Im_\ell)}\right)\right)^{\beta_\ell}\diamondsuit_\alpha s_\ell\diamondsuit_\alpha\Im_\ell\right)^{\frac{1}{\beta_\ell}} \tag{19}$$

where

$$L(\vartheta_1\varsigma_1,\ldots,\vartheta_n\varsigma_n)=\prod_{\ell=1}^{n}\left(\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}\left(\frac{\Phi_\ell(P_\ell(s_\ell,\Im_\ell))}{P_\ell(s_\ell,\Im_\ell)}\right)^{\gamma_\ell}\diamondsuit_\alpha s_\ell\diamondsuit_\alpha\Im_\ell\right)^{\frac{1}{\gamma_\ell}}.$$

Proof. From the hypotheses of Theorem 1, S_{14}, S_{15}, and S_8, it is easy to observe that

$$\begin{aligned}\Phi_\ell(\tilde{F}_\ell(s_\ell,\Im_\ell)) &= \Phi_\ell\left(\frac{P_\ell(s_\ell,\Im_\ell)\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}p_\ell(\xi_\ell,\tau_\ell)\left(\frac{F_\ell(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell,\tau_\ell)}\right)\diamondsuit_\alpha\xi_\ell\diamondsuit_\alpha\tau_\ell}{\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}p_\ell(\xi_\ell,\tau_\ell)\diamondsuit_\alpha\xi_\ell\diamondsuit_\alpha\tau_\ell}\right)\\ &\geqslant \Phi_\ell(P_\ell(s_\ell,\Im_\ell))\Phi_\ell\left(\frac{\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}p_\ell(\xi_\ell,\tau_\ell)\left(\frac{F_\ell(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell,\tau_\ell)}\right)\diamondsuit_\alpha\xi_\ell\diamondsuit_\alpha\tau_\ell}{\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}p_\ell(\xi_\ell,\tau_\ell)\diamondsuit_\alpha\xi_\ell\diamondsuit_\alpha\tau_\ell}\right).\end{aligned} \tag{20}$$

By using inverse Jensen dynamic inequality, we obtain that

$$\Phi_\ell(\tilde{F}_\ell(s_\ell,\Im_\ell)) \geqslant \frac{\Phi_\ell(P_\ell(s_\ell,\Im_\ell))}{P_\ell(s_\ell,\Im_\ell)}\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}p_\ell(\xi_\ell,\tau_\ell)\Phi_\ell\left(\frac{F_\ell(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell,\tau_\ell)}\right)\diamondsuit_\alpha\xi_\ell\diamondsuit_\alpha\tau_\ell. \tag{21}$$

Applying inverse Hölder's inequality on the right hand side of (21) with indices γ_ℓ and β_ℓ, it is easy to observe that

$$\Phi_\ell(\tilde{F}_\ell(s_\ell,\Im_\ell))\geqslant\frac{\Phi_\ell(P_\ell(s_\ell,\Im_\ell))}{P_\ell(s_\ell,\Im_\ell)}\left[(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right]^{\frac{1}{\gamma_\ell}}\left(\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}\left(p_\ell(\xi_\ell,\tau_\ell)\Phi_\ell\left(\frac{F_\ell(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell,\tau_\ell)}\right)\right)^{\beta_\ell}\diamondsuit_\alpha\xi_\ell\diamondsuit_\alpha\tau_\ell\right)^{\frac{1}{\beta_\ell}}. \tag{22}$$

By using the following inequality on the term $\left[(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right]^{\frac{1}{\gamma_\ell}}$,

$$\prod_{\ell=1}^{n}m_\ell^{\frac{1}{\gamma_\ell}}\geqslant\left(\gamma\sum_{\ell=1}^{n}\frac{1}{\gamma_\ell}m_\ell\right)^{\frac{1}{\gamma}}, \tag{23}$$

we get that

$$\frac{\prod_{\ell=1}^{n}\Phi_\ell(\tilde{F}_\ell(s_\ell,\Im_\ell))}{\left(\gamma\sum_{\ell=1}^{n}\frac{1}{\gamma_\ell}(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\frac{1}{\gamma}}}$$
$$\geqslant\prod_{\ell=1}^{n}\frac{\Phi_\ell(P_\ell(s_\ell,\Im_\ell))}{P_\ell(s_\ell,\Im_\ell)}\left(\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}\left(p_\ell(\xi_\ell,\tau_\ell)\Phi_\ell\left(\frac{F_\ell(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell,\tau_\ell)}\right)\right)^{\frac{1}{\beta_\ell}}\diamondsuit_\alpha\xi_\ell\diamondsuit_\alpha\tau_\ell\right)^{\frac{1}{\beta_\ell}}. \tag{24}$$

Integrating both sides of (24) over s_ℓ, $\Im_\ell$ from $\Im_0$ to ϑ_ℓ, ς_ℓ $(\ell=1,\ldots,n)$, we obtain that

$$\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\vartheta_n}\int_{\Im_0}^{\varsigma_n}\frac{\prod_{\ell=1}^{n}\Phi_\ell(\tilde{F}_\ell(s_\ell,\Im_\ell))}{\left(\gamma\sum_{\ell=1}^{n}\frac{1}{\gamma_\ell}(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\frac{1}{\gamma}}}\diamondsuit_\alpha s_1\diamondsuit_\alpha\Im_1\ldots\diamondsuit_\alpha s_n\diamondsuit_\alpha\Im_n$$
$$\geqslant\prod_{\ell=1}^{n}\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}\frac{\Phi_\ell(P_\ell(s_\ell,\Im_\ell))}{P_\ell(s_\ell,\Im_\ell)}\left(\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}\left(p_\ell(\xi_\ell,\tau_\ell)\Phi_\ell\left(\frac{F_\ell(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell,\tau_\ell)}\right)\right)^{\beta_\ell}\diamondsuit_\alpha\xi_\ell\diamondsuit_\alpha\tau_\ell\right)^{\frac{1}{\beta_\ell}}\diamondsuit_\alpha s_\ell\diamondsuit_\alpha\Im_\ell. \tag{25}$$

Applying inverse Hölder's inequality on the right hand side of (25) with indices γ_ℓ and β_ℓ, it is easy to observe that

$$\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\vartheta_n}\int_{\Im_0}^{\varsigma_n}\frac{\prod_{\ell=1}^{n}\Phi_\ell(\tilde{F}_\ell(s_\ell,\Im_\ell))}{\left(\gamma\sum_{\ell=1}^{n}\frac{1}{\gamma_\ell}(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\frac{1}{\gamma}}}\diamondsuit_\alpha s_1\diamondsuit_\alpha\Im_1\ldots\diamondsuit_\alpha s_n\diamondsuit_\alpha\Im_n$$
$$\geqslant\prod_{\ell=1}^{n}\left(\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}\left(\frac{\Phi_\ell(P_\ell(s_\ell,\Im_\ell))}{P_\ell(s_\ell,\Im_\ell)}\right)^{\gamma_\ell}\diamondsuit_\alpha s_\ell\diamondsuit_\alpha\Im_\ell\right)^{\frac{1}{\gamma_\ell}} \tag{26}$$
$$\times\prod_{\ell=1}^{n}\left(\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}\left(\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}\left(p_\ell(\xi_\ell,\tau_\ell)\Phi_\ell\left(\frac{F_\ell(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell,\tau_\ell)}\right)\right)^{\beta_\ell}\diamondsuit_\alpha\xi_\ell\diamondsuit_\alpha\tau_\ell\right)\diamondsuit_\alpha s_\ell\diamondsuit_\alpha\Im_\ell\right)^{\frac{1}{\beta_\ell}}.$$

Using Fubini's theorem, we observe that

$$\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\vartheta_n}\int_{\Im_0}^{\varsigma_n}\frac{\prod_{\ell=1}^{n}\Phi_\ell(\tilde{F}_\ell(s_\ell,\Im_\ell))}{\left(\gamma\sum_{\ell=1}^{n}\frac{1}{\gamma_\ell}(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\frac{1}{\gamma}}}\diamondsuit_\alpha s_1\diamondsuit_\alpha\Im_1\ldots\diamondsuit_\alpha s_n\diamondsuit_\alpha\Im_n$$
$$\geqslant L(\vartheta_1\varsigma_1,\ldots,\vartheta_n\varsigma_n)$$
$$\times\prod_{\ell=1}^{n}\left(\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}(\vartheta_\ell-s_\ell)(\varsigma_\ell-\Im_\ell)\left(p_\ell(s_\ell,\Im_\ell)\Phi_\ell\left(\frac{F_\ell(s_\ell,\Im_\ell)}{p_\ell(s_\ell,\Im_\ell)}\right)\right)^{\beta_\ell}\diamondsuit_\alpha s_\ell\diamondsuit_\alpha\Im_\ell\right)^{\frac{1}{\beta_\ell}}.$$

By using the fact $\vartheta_\ell\geqslant\rho(\vartheta_\ell)$, and $\varsigma_\ell\geqslant\rho(\varsigma_\ell)$, we get that

$$\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\vartheta_n}\int_{\Im_0}^{\varsigma_n}\frac{\prod_{\ell=1}^{n}\Phi_\ell(\tilde{F}_\ell(s_\ell,\Im_\ell))}{\left(\gamma\sum_{\ell=1}^{n}\frac{1}{\gamma_\ell}(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\frac{1}{\gamma}}}\diamondsuit_\alpha s_1\diamondsuit_\alpha\Im_1\ldots\diamondsuit_\alpha s_n\diamondsuit_\alpha\Im_n$$
$$\geqslant L(\vartheta_1\varsigma_1,\ldots,\vartheta_n\varsigma_n)$$
$$\times\prod_{\ell=1}^{n}\left(\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}(\rho(\vartheta_\ell)-s_\ell)(\rho(\varsigma_\ell)-\Im_\ell)\left(p_\ell(s_\ell,\Im_\ell)\Phi_\ell\left(\frac{F_\ell(s_\ell,\Im_\ell)}{p_\ell(s_\ell,\Im_\ell)}\right)\right)^{\beta_\ell}\diamondsuit_\alpha s_\ell\diamondsuit_\alpha\Im_\ell\right)^{\frac{1}{\beta_\ell}}.$$

This completes the proof. □

Remark 1. *In Theorem 1, if $\mathbb{T}=\mathbb{R}$, $\alpha=1$ we get the result due to Zhao et al. ([10], Theorem 2).*

As a special case of Theorem 1, when $\mathbb{T}=\mathbb{Z}$, $\alpha=1$ we have $\rho(n)=n-1$, we get the following result.

Corollary 1. *Let $\{a_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}}\}$ and $\{p_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}}\}$, $(\ell = 1,\ldots,n)$ be n sequences of non-negative numbers defined for $m_{s_\ell} = 1,\ldots,k_{s_\ell}$, and $m_{\Im_\ell} = 1,\ldots,k_{\Im_\ell}$, and define*

$$\begin{aligned} A_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}} &= \sum_{m_{\xi_\ell}}^{m_{s_\ell}} \sum_{m_{\eta_\ell}}^{m_{\Im_\ell}} a_{s_\ell,\Im_\ell,m_{\xi_\ell},m_{\eta_\ell}} \\ P_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}} &= \sum_{m_{\xi_\ell}}^{m_{s_\ell}} \sum_{m_{\eta_\ell}}^{m_{\Im_\ell}} p_{s_\ell,\Im_\ell,m_{\xi_\ell},m_{\eta_\ell}}. \end{aligned} \tag{27}$$

Then

$$\begin{aligned} &\sum_{m_{s_1}}^{k_{s_1}} \sum_{m_{\Im_1}}^{k_{\Im_1}} \cdots \sum_{m_{s_n}}^{k_{s_n}} \sum_{m_{\Im_n}}^{k_{\Im_n}} \frac{\prod_{\ell=1}^{n} \Phi_\ell(A_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}})}{\left(\gamma \sum_{\ell=1}^{n} \frac{1}{\gamma_\ell}(m_{s_\ell} m_{\Im_\ell})\right)^{\frac{1}{\gamma}}} \\ &\geqslant C(k_{s_1}k_{\Im_1},\ldots,k_{s_n}k_{\Im_n}) \\ &\times \prod_{\ell=1}^{n} \left(\sum_{m_{s_\ell}}^{k_{s_\ell}} \sum_{m_{\Im_\ell}}^{k_{\Im_\ell}} (k_{s_\ell} - (m_{s_\ell}-1))(k_{\Im_\ell} - (m_{\Im_\ell}-1)) \left(P_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}} \Phi_\ell\left(\frac{a_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}}}{P_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}}} \right)\right)^{\beta_\ell} \right)^{\frac{1}{\beta_\ell}} \end{aligned}$$

where

$$C(k_{s_1}k_{\Im_1},\ldots,k_{s_n}k_{\Im_n}) = \prod_{\ell=1}^{n} \left(\sum_{m_{s_\ell}}^{k_{s_\ell}} \sum_{m_{\Im_\ell}}^{k_{\Im_\ell}} \left(\frac{\Phi_\ell(P_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}})}{P_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}}} \right) \right)^{\beta_\ell} \right)^{\frac{1}{\beta_\ell}}.$$

Remark 2. *Let $F_\ell(\xi_\ell,\tau_\ell)$, $p_\ell(\xi_\ell,\tau_\ell)$, $P_\ell(\xi_\ell,\tau_\ell)$, and $F_\ell(\xi_\ell,\tau_\ell)$ change to $F_\ell(\xi_\ell)$, $p_\ell(\xi_\ell)$, $P_\ell(s_\ell)$ and $F_\ell(s_\ell)$, respectively; with suitable changes, we have the following new corollary:*

Corollary 2. *Let S_{22}, S_{23}, S_{24}, S_{26}, S_{27} and S_8 be satisfied. Then for S_{18}, S_{20} and S_{25} we have that*

$$\begin{aligned} &\int_{\Im_0}^{\vartheta_1} \cdots \int_{\Im_0}^{\vartheta_n} \frac{\prod_{\ell=1}^{n} \Phi_\ell(\tilde{F}_\ell(s_\ell))}{\left(\gamma \sum_{\ell=1}^{n} \frac{1}{\gamma_\ell}(s_\ell - \Im_0)\right)^{\frac{1}{\gamma}}} \diamondsuit_\alpha s_1 \ldots \diamondsuit_\alpha s_n \\ &\geqslant L^*(\vartheta_1,\ldots,\vartheta_n) \prod_{\ell=1}^{n} \left(\int_{\Im_0}^{\vartheta_\ell} (\rho(\vartheta_\ell) - s_\ell) \left(p_\ell(s_\ell) \Phi_\ell\left(\frac{F_\ell(s_\ell)}{p_\ell(s_\ell)}\right)\right)^{\beta_\ell} \diamondsuit_\alpha s_\ell \right)^{\frac{1}{\beta_\ell}} \end{aligned} \tag{28}$$

where

$$L^*(\vartheta_1,\ldots,\vartheta_n) = \prod_{\ell=1}^{n} \left(\int_{\Im_0}^{\vartheta_\ell} \left(\frac{\Phi_\ell(P_\ell(s_\ell))}{P_\ell(s_\ell)} \right)^{\gamma_\ell} \diamondsuit_\alpha s_\ell \right)^{\frac{1}{\gamma_\ell}}.$$

Corollary 3. *In Corollary 2, if we take $n = 2$, $\beta_\ell = \frac{1}{2}$ then the inequality (28) changes to*

$$\begin{aligned} \int_{\Im_0}^{\vartheta_1} \int_{\Im_0}^{\vartheta_2} \frac{\Phi_1(\tilde{F}_1(s_1))\Phi_2(\tilde{F}_2(s_2))}{((s_1 - \Im_0) + (s_2 - \Im_0))^{-2}} \diamondsuit_\alpha s_1 \diamondsuit_\alpha s_2 &\geqslant L^{**}(\vartheta_1,\vartheta_2) \left(\int_{\Im_0}^{\vartheta_1} (\rho(\vartheta_1) - s_1) \left(p_1(s_1) \Phi\left(\frac{F_1(s_1)}{p_1(s_1)}\right)\right)^2 \diamondsuit_\alpha s_1 \right)^{\frac{1}{2}} \\ &\times \left(\int_{\Im_0}^{\vartheta_2} (\rho(\vartheta_2) - s_2) \left(p_2(s_2) \Psi\left(\frac{F_2(s_2)}{p_2(s_2)}\right)\right)^2 \diamondsuit_\alpha s_2 \right)^{\frac{1}{2}} \end{aligned} \tag{29}$$

where

$$L^{**}(\vartheta_1,\vartheta_2) = 4\left(\int_{\Im_0}^{\vartheta_2} \left(\frac{\Phi_1(P_1(s_1))}{P_1(s_1)} \right)^{-1} \diamondsuit_\alpha s_1 \right)^{-1} \left(\int_{\Im_0}^{\vartheta_2} \left(\frac{\Phi_2(P_2(s_2))}{P_2(s_2)} \right)^{-1} \diamondsuit_\alpha s_2 \right)^{-1}$$

Remark 3. *In Corollary 3, if we take $\mathbb{T} = \mathbb{R}$, then the inequality (29) changes to*

$$\int_0^{\vartheta_1}\int_0^{\tilde{\vartheta}_1} \frac{\Phi_1(F_1(s_1))\Phi_2(\tilde{F}_2(s_2))}{(s_1+s_2)^{-2}} ds_1 ds_2 \geqslant L^{**}(\vartheta_1,\vartheta_2)\left(\int_0^{\vartheta_1}(\vartheta_1 - s_1)\left(p_1(s_1)\Phi\left(\frac{F_1(s_1)}{p_1(s_1)}\right)\right)^2 ds_1\right)^{\frac{1}{2}} \times\left(\int_0^{\vartheta_2}(\vartheta_2 - s_2)\left(p_2(s_2)\Psi\left(\frac{F_2(s_2)}{p_2(s_2)}\right)\right)^2 ds_2\right)^{\frac{1}{2}} \tag{30}$$

where

$$L^{**}(\vartheta_1,\vartheta_2) = 4\left(\int_0^{\vartheta_1}\left(\frac{\Phi_1(P_1(s_1))}{P_1(s_1)}\right)^{-1} ds_1\right)^{-1}\left(\int_0^{\vartheta_2}\left(\frac{\Phi_2(P_2(s_2))}{P_2(s_2)}\right)^{-1} ds_2\right)^{-1}.$$

This is an inverse of the inequality (6) which was proved by Pachpatte [7].

Corollary 4. *In Corollary 2, if we take $\beta_\ell = \frac{n-1}{n}$ the inequality (28) becomes*

$$\int_{\Im_0}^{\vartheta_1}\cdots\int_{\Im_0}^{\vartheta_n} \frac{\prod_{\ell=1}^{n}\Phi_\ell(\tilde{F}_\ell(s_\ell))}{\left(\sum_{\ell=1}^{n}(s_\ell - \Im_0)\right)^{\frac{-n}{n-1}}}\diamondsuit_\alpha s_1\ldots\diamondsuit_\alpha s_n \geqslant L^*(\vartheta_1,\ldots,\vartheta_n)\prod_{\ell=1}^{n}\left(\int_{\Im_0}^{\vartheta_\ell}(\rho(\vartheta_\ell) - s_\ell)\left(p_\ell(s_\ell)\Phi_\ell\left(\frac{F_\ell(s_\ell)}{p_\ell(s_\ell)}\right)\right)^{\frac{n-1}{n}}\diamondsuit_\alpha s_\ell\right)^{\frac{n}{n-1}}$$

where

$$L^*(\vartheta_1,\ldots,\vartheta_n) = n^{\frac{n}{n-1}}\prod_{\ell=1}^{n}\left(\int_{\Im_0}^{\vartheta_\ell}\left(\frac{\Phi_\ell(P_\ell(s_\ell))}{P_\ell(s_\ell)}\right)^{-(n-1)}\diamondsuit_\alpha s_\ell\right)^{\frac{-1}{n-1}}.$$

Theorem 2. *Let $S_1, S_4, S_5, S_6, S_9, S_{15}$, and S_{16} be satisfied. Then for S_{10}, S_{18} and S_{20} we have that*

$$\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\vartheta_n}\int_{\Im_0}^{\varsigma_n} \frac{\prod_{\ell=1}^{n}P_\ell(s_\ell,\Im_\ell)\Phi_\ell(\tilde{F}_\ell(s_\ell,\Im_\ell))}{\left(\gamma\sum_{\ell=1}^{n}\frac{1}{\gamma_\ell}(s_\ell - \Im_0)(\Im_\ell - \Im_0)\right)^{\frac{1}{\gamma}}}\diamondsuit_\alpha s_1\diamondsuit_\alpha\Im_1\ldots\diamondsuit_\alpha s_n\diamondsuit_\alpha\Im_n \geqslant \prod_{\ell=1}^{n}\left[(\vartheta_\ell - \Im_0)(\varsigma_\ell - \Im_0)\right]^{\frac{1}{\gamma_\ell}}\left(\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}(\rho(\vartheta_\ell) - s_\ell)(\rho(\varsigma_\ell) - \Im_\ell)(p_\ell(s_\ell,\Im_\ell)\Phi_\ell(F_\ell(s_\ell,\Im_\ell)))^{\beta_\ell}\diamondsuit_\alpha s_\ell\diamondsuit_\alpha\Im_\ell\right)^{\frac{1}{\beta_\ell}} \tag{31}$$

Proof. From the hypotheses of Theorem 2, and by using inverse Jensen dynamic inequality, we have

$$\Phi_\ell(\tilde{F}_\ell(s_\ell,\Im_\ell)) = \Phi_\ell\left(\frac{1}{P_\ell(s_\ell,\Im_\ell)}\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell} p_\ell(\xi_\ell,\tau_\ell)F_\ell(\xi_\ell,\tau_\ell)\diamondsuit_\alpha\xi_\ell\diamondsuit_\alpha\tau_\ell\right) \geqslant \frac{1}{P_\ell(s_\ell,\Im_\ell)}\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell} p_\ell(\sigma_\ell,\tau_\ell)\Phi_\ell(F_\ell(\xi_\ell,\tau_\ell))\diamondsuit_\alpha\xi_\ell\diamondsuit_\alpha\tau_\ell. \tag{32}$$

Applying inverse Hölder's inequality on the right hand side of (32) with indices γ_ℓ and β_ℓ, it is easy to observe that

$$\Phi_\ell(\tilde{F}_\ell(s_\ell,\Im_\ell)) \geqslant \frac{1}{P_\ell(s_\ell,\Im_\ell)}\left[(s_\ell - \Im_0)(\Im_\ell - \Im_0)\right]^{\frac{1}{\gamma_\ell}}\left(\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}(p_\ell(\xi_\ell,\tau_\ell)\Phi_\ell(F_\ell(\xi_\ell,\tau_\ell)))^{\beta_\ell}\diamondsuit_\alpha\xi_\ell\diamondsuit_\alpha\tau_\ell\right)^{\frac{1}{\beta_\ell}}.$$

By using the inequality (23), on the term $\left[(s_\ell - \Im_0)(\Im_\ell - \Im_0)\right]^{\frac{1}{\gamma_\ell}}$ we get that

$$\frac{P_\ell(s_\ell,\Im_\ell)\Phi_\ell(\tilde{F}_\ell(s_\ell,\Im_\ell))}{\left(\gamma\sum\limits_{\ell=1}^{n}\frac{1}{\gamma_\ell}(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\frac{1}{\gamma}}} \geqslant \left(\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}\left(p_\ell(\xi_\ell,\tau_\ell)\Phi_\ell\big(F_\ell(\xi_\ell,\tau_\ell)\big)\right)^{\beta_\ell}\diamondsuit_\alpha\xi_\ell\diamondsuit_\alpha\tau_\ell\right)^{\frac{1}{\beta_\ell}}. \tag{33}$$

Integrating both sides of (33) over s_ℓ, $\Im_\ell$ from $\Im_0$ to ϑ_ℓ, ς_ℓ $(\ell=1,\dots,n)$, we get that

$$\begin{aligned}&\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\vartheta_n}\int_{\Im_0}^{\varsigma_n}\frac{\prod\limits_{\ell=1}^{n}P_\ell(s_\ell,\Im_\ell)\Phi_\ell(\tilde{F}_\ell(s_\ell,\Im_\ell))}{\left(\gamma\sum\limits_{\ell=1}^{n}\frac{1}{\gamma_\ell}(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\frac{1}{\gamma}}}\diamondsuit_\alpha s_1\diamondsuit_\alpha\Im_1\dots\diamondsuit_\alpha s_n\diamondsuit_\alpha\Im_n\\ &\geqslant\prod_{\ell=1}^{n}\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}\left(\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}\left(p_\ell(\xi_\ell,\tau_\ell)\Phi_\ell\big(F_\ell(\xi_\ell,\tau_\ell)\big)\right)^{\beta_\ell}\diamondsuit_\alpha\sigma_\ell\diamondsuit_\alpha\tau_\ell\right)^{\frac{1}{\beta_\ell}}.\end{aligned}$$

Applying inverse Hölder's inequality on the right hand side of (34) with indices γ_ℓ and β_ℓ, it is easy to observe that

$$\begin{aligned}&\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\vartheta_n}\int_{\Im_0}^{\varsigma_n}\frac{\prod\limits_{\ell=1}^{n}P_\ell(s_\ell,\Im_\ell)\Phi_\ell(\tilde{F}_\ell(s_\ell,\Im_\ell))}{\left(\gamma\sum\limits_{\ell=1}^{n}\frac{1}{\gamma_\ell}(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\frac{1}{\gamma}}}\diamondsuit_\alpha s_1\diamondsuit_\alpha\Im_1\dots\diamondsuit_\alpha s_n\diamondsuit_\alpha\Im_n\\ &\geqslant\prod_{\ell=1}^{n}\Big[(\vartheta_\ell-\Im_0)(\varsigma_\ell-\Im_0)\Big]^{\frac{1}{\gamma_\ell}}\left(\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}\left(p_\ell(\xi_\ell,\tau_\ell)\Phi_\ell\big(F_\ell(\xi_\ell,\tau_\ell)\big)\right)^{\beta_\ell}\diamondsuit_\alpha\xi_\ell\diamondsuit_\alpha\tau_\ell\diamondsuit_\alpha s_\ell\diamondsuit_\alpha\Im_\ell\right)^{\frac{1}{\beta_\ell}}.\end{aligned} \tag{34}$$

By using Fubini's theorem, we observe that

$$\begin{aligned}&\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\vartheta_n}\int_{\Im_0}^{\varsigma_n}\frac{\prod\limits_{\ell=1}^{n}P_\ell(s_\ell,\Im_\ell)\Phi_\ell(\tilde{F}_\ell(s_\ell,\Im_\ell))}{\left(\gamma\sum\limits_{\ell=1}^{n}\frac{1}{\gamma_\ell}(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\frac{1}{\gamma}}}\diamondsuit_\alpha s_1\diamondsuit_\alpha\Im_1\dots\diamondsuit_\alpha s_n\diamondsuit_\alpha\Im_n\\ &\geqslant\prod_{\ell=1}^{n}\Big[(\vartheta_\ell-\Im_0)(\varsigma_\ell-\Im_0)\Big]^{\frac{1}{\gamma_\ell}}\left(\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}(\vartheta_\ell-s_\ell)(\varsigma_\ell-\Im_\ell)\left(p_\ell(s_\ell,\Im_\ell)\Phi_\ell\big(F_\ell(s_\ell,\Im_\ell)\big)\right)^{\beta_\ell}\diamondsuit_\alpha s_\ell\diamondsuit_\alpha\Im_\ell\right)^{\frac{1}{\beta_\ell}}.\end{aligned}$$

By using the fact $\vartheta_\ell \geqslant \rho(\vartheta_\ell)$, and $\varsigma_\ell \geqslant \rho(\varsigma_\ell)$, we get that

$$\begin{aligned}&\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\vartheta_n}\int_{\Im_0}^{\varsigma_n}\frac{\prod\limits_{\ell=1}^{n}P_\ell(s_\ell,\Im_\ell)\Phi_\ell(\tilde{F}_\ell(s_\ell,\Im_\ell))}{\left(\gamma\sum\limits_{\ell=1}^{n}\frac{1}{\gamma_\ell}(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\frac{1}{\gamma}}}\diamondsuit_\alpha s_1\diamondsuit_\alpha\Im_1\dots\diamondsuit_\alpha s_n\diamondsuit_\alpha\Im_n\\ &\geqslant\prod_{\ell=1}^{n}\Big[(\vartheta_\ell-\Im_0)(\varsigma_\ell-\Im_0)\Big]^{\frac{1}{\gamma_\ell}}\left(\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}(\rho(\vartheta_\ell)-s_\ell)(\rho(\varsigma_\ell)-\Im_\ell)\left(p_\ell(s_\ell,\Im_\ell)\Phi_\ell\big(F_\ell(s_\ell,\Im_\ell)\big)\right)^{\beta_\ell}\diamondsuit_\alpha s_\ell\diamondsuit_\alpha\Im_\ell\right)^{\frac{1}{\beta_\ell}}.\end{aligned}$$

This completes the proof. □

Remark 4. *In Theorem 2, if $\mathbb{T}=\mathbb{R}$, $\alpha=1$ we get the result due to Zhao et al. ([10], Theorem 3).*

As a special case of Theorem 2, when $\mathbb{T}=\mathbb{Z}$, $\alpha=1$ we have $\rho(n)=n-1$, we get the following result.

Corollary 5. *Let* $\{a_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}}\}$ *and* $\{p_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}}\}$, $(\ell=1,\dots,n)$ *be* n *sequences of non-negative numbers defined for* $m_{s_\ell}=1,\dots,k_{s_\ell}$, *and* $m_{\Im_\ell}=1,\dots,k_{\Im_\ell}$, *and define*

$$\begin{aligned} A_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}} &= \frac{1}{P_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}}}\sum_{m_{\xi_\ell}}^{m_{s_\ell}}\sum_{m_{\eta_\ell}}^{m_{\Im_\ell}} a_{s_\ell,\Im_\ell,m_{\xi_\ell},m_{\eta_\ell}}p_{s_\ell,\Im_\ell,m_{\xi_\ell},m_{\eta_\ell}}, \\ P_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}} &= \sum_{m_{\xi_\ell}}^{m_{s_\ell}}\sum_{m_{\eta_\ell}}^{m_{\Im_\ell}} p_{s_\ell,\Im_\ell,m_{\xi_\ell},m_{\eta_\ell}}. \end{aligned} \tag{35}$$

Then

$$\sum_{m_{s_1}}^{k_{s_1}}\sum_{m_{\Im_1}}^{k_{\Im_1}}\cdots\sum_{m_{s_n}}^{k_{s_n}}\sum_{m_{\Im_n}}^{k_{\Im_n}}\frac{\prod_{\ell=1}^{n}P_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}}\Phi_\ell(A_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}})}{\left(\gamma\sum_{\ell=1}^{n}\frac{1}{\gamma_\ell}(m_{s_\ell}m_{\Im_\ell})\right)^{\frac{1}{\gamma}}}$$
$$\geqslant \prod_{\ell=1}^{n}(k_{s_\ell}k_{\Im_\ell})^{\frac{1}{\gamma_\ell}}\left(\sum_{m_{s_\ell}}^{k_{s_\ell}}\sum_{m_{\Im_\ell}}^{k_{\Im_\ell}}(k_{s_\ell}-(m_{s_\ell}-1))(k_{\Im_\ell}-(m_{\Im_\ell}-1))\left(p_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}}\Phi_\ell\left(a_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}}\right)\right)^{\beta_\ell}\right)^{\frac{1}{\beta_\ell}},$$

Remark 5. *Let* $F_\ell(\xi_\ell,\tau_\ell)$, $p_\ell(\xi_\ell,\tau_\ell)$, $P_\ell(\xi_\ell,\tau_\ell)$ *and*

$$\tilde{F}_\ell(s_\ell,\Im_\ell)=\frac{1}{P_\ell(s_\ell,\Im_\ell)}\int_{\Im_0}^{s_\ell}\int_0^{\Im_\ell}p_\ell(\xi_\ell,\tau_\ell)F_\ell(\xi_\ell,\tau_\ell)\diamondsuit_\alpha\xi_\ell\diamondsuit_\alpha\tau_\ell$$

changes to $\tilde{F}_\ell(\xi_\ell)$, $p_\ell(\xi_\ell)$, $P_\ell(s_\ell)$, *and*

$$\tilde{F}_\ell(s_\ell)=\frac{1}{P_\ell(s_\ell)}\int_{\Im_0}^{s_\ell}p_\ell(\xi_\ell)F_\ell(\xi_\ell)\diamondsuit_\alpha\xi_\ell,$$

respectively, and with suitable changes, we have the following new corollary:

Corollary 6. *Let* S_{22}, S_{23}, S_{26}, S_{27} *and* S_{28} *be satisfied. Then, for* S_{18}, S_{20} *and* S_{25}, *we have that*

$$\begin{aligned} &\int_{\Im_0}^{\vartheta_1}\cdots\int_{\Im_0}^{\vartheta_n}\frac{\prod_{\ell=1}^{n}P_\ell(s_\ell)\Phi_\ell(\tilde{F}_\ell(s_\ell)}{\left(\gamma\sum_{\ell=1}^{n}\frac{1}{\gamma_\ell}(s_\ell-\Im_0)\right)^{\frac{1}{\gamma}}}\diamondsuit_\alpha s_1\dots\diamondsuit_\alpha s_n \\ &\geqslant \prod_{\ell=1}^{n}(\vartheta_\ell-\Im_0)^{\frac{1}{\gamma_\ell}}\left(\int_{\Im_0}^{\vartheta_\ell}(\rho(\vartheta_\ell)-s_\ell)\Big(p_\ell(s_\ell)\Phi_\ell(F_\ell(s_\ell))\Big)^{\beta_\ell}\diamondsuit_\alpha s_\ell\right)^{\frac{1}{\beta_\ell}}. \end{aligned} \tag{36}$$

Corollary 7. *In Corollary 6, if we take* $n=2$, $\beta_\ell=\frac{1}{2}$ *then the inequality (28) changes to*

$$\begin{aligned} &\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\vartheta_2}\frac{P_1(s_1)P_2(s_2)\Phi_1(\tilde{F}_1(s_1))\Phi_2(\tilde{F}_2(s_2))}{((s_1-\Im_0)+(s_2-\Im_0))^{-2}}\diamondsuit_\alpha s_1\diamondsuit_\alpha s_2 \geqslant 4\big[(\vartheta_1-\Im_0)(\vartheta_1-\Im_0)\big]^{-1} \\ &\times\left(\int_{\Im_0}^{\vartheta_1}(\rho(\vartheta_1)-s_1)\Big(p_1(s_1)\Phi_1(F_1(s_1))\Big)^2\diamondsuit_\alpha s_1\right)^{\frac{1}{2}}\left(\int_{\Im_0}^{\vartheta_2}(\rho(\vartheta_2)-s_2)\Big(p_2(s_2)\Phi_2\Big(F_2(s_2)\Big)\Big)^2\diamondsuit_\alpha s_2\right)^{\frac{1}{2}}. \end{aligned} \tag{37}$$

Remark 6. *In Corollary 7, if we take* $\mathbb{T}=\mathbb{R}$, *then the inequality (37) changes to*

$$\int_0^{\vartheta_1}\int_0^{\vartheta_1}\frac{P_1(s_1)P_2(s_2)\Phi_1(\tilde{F}_1(s_1))\Phi_2(\tilde{F}_2(s_2))}{(s_1+s_2)^{-2}}ds_1ds_2 \geqslant 4\big[\vartheta_1\vartheta_1\big]^{-1}$$
$$\times\bigg(\int_0^{\vartheta_1}(\vartheta_1-s_1)\Big(p_1(s_1)\Phi_1\big(F_1(s_1)\big)\Big)^2 ds_1\bigg)^{\frac{1}{2}}\bigg(\int_0^{\vartheta_2}(\vartheta_2-s_2)\Big(p_2(s_2)\Phi_2\Big(F_2(s_2)\Big)\Big)^2 ds_2\bigg)^{\frac{1}{2}}. \quad (38)$$

This is an inverse of the inequality (7), which was proven by Pachpatte [7].

Corollary 8. *In Corollary 7, let $p_1(s_1) = p_2(s_2) = 1$, then $P_1(s_1) = s_1$, $P_2(s_2) = s_2$. Therefore, inequality (37) changes to*

$$\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\vartheta_2}\frac{\Phi_1(\tilde{F}_1(s_1))\Phi_2(\tilde{F}_2(s_2))}{(s_1s_2)^{-1}\big((s_1-\Im_0)+(s_2-\Im_0)\big)^{-2}}\Diamond_\alpha s_1\Diamond_\alpha s_2 \geqslant 4\big[(\vartheta_1-\Im_0)(\vartheta_1-\Im_0)\big]^{-1}$$
$$\times\bigg(\int_{\Im_0}^{\vartheta_1}(\rho(\vartheta_1)-s_1)\Big(\Phi_1\big(F_1(s_1)\big)\Big)^2\Diamond_\alpha s_1\bigg)^{\frac{1}{2}}\bigg(\int_{\Im_0}^{\vartheta_2}(\rho(\vartheta_2)-s_2)\Big(\Phi_2\Big(F_2(s_2)\Big)\Big)^2\Diamond_\alpha s_2\bigg)^{\frac{1}{2}}. \quad (39)$$

Remark 7. *In Corollary 8, if we take $\mathbb{T}=\mathbb{R}$, then inequality (39) changes to*

$$\int_0^{\vartheta_1}\int_0^{\vartheta_1}\frac{\Phi_1(\tilde{F}_1(s_1))\Phi_2(\tilde{F}_2(s_2))}{(s_1s_2)^{-1}(s_1+s_2)^{-2}}ds_1ds_2 \geqslant 4\big[\vartheta_1\vartheta_1\big]^{-1}$$
$$\times\bigg(\int_0^{\vartheta_1}(\vartheta_1-s_1)\Big(\Phi_1\big(F_1(s_1)\big)\Big)^2 ds_1\bigg)^{\frac{1}{2}}\bigg(\int_0^{\vartheta_2}(\vartheta_2-s_2)\Big(\Phi_2\Big(F_2(s_2)\Big)\Big)^2 ds_2\bigg)^{\frac{1}{2}}.$$

This is an inverse inequality of the following inequality which was proven by Pachpatte [10].

$$\int_0^{\vartheta}\int_0^{\varsigma}\frac{\Phi(F(s))\Psi(G(\Im))}{(s\Im)^{-1}(s+\Im)}dsd\Im \leqslant \frac{1}{2}\big[\vartheta\varsigma\big]^{\frac{1}{2}}$$
$$\times\bigg(\int_0^{\vartheta}(\vartheta-s_1)\Big(\Phi\big(F(s)\big)\Big)^2 ds\bigg)^{\frac{1}{2}}\bigg(\int_0^{\varsigma}(\varsigma-\Im)\Big(\Psi\Big(g(\Im)\Big)\Big)^2 d\Im\bigg)^{\frac{1}{2}}.$$

Corollary 9. *In Corollary 6, if we take $\beta_\ell = \frac{n-1}{n}$ $(\ell = 1,\dots,n)$ the inequality (36)*

$$\int_{\Im_0}^{\vartheta_1}\cdots\int_{\Im_0}^{\vartheta_n}\frac{\prod_{\ell=1}^{n}P_\ell(s_\ell)\Phi_\ell(\tilde{F}_\ell(s_\ell)}{\bigg(\sum_{\ell=1}^{n}(s_\ell-\Im_0)\bigg)^{\frac{-n}{n-1}}}\Diamond_\alpha s_1\dots\Diamond_\alpha s_n$$
$$\geqslant n^{\frac{n}{n-1}}\prod_{\ell=1}^{n}(\vartheta_\ell-\Im_0)^{\frac{-1}{n-1}}\bigg(\int_{\Im_0}^{\vartheta_\ell}(\rho(\vartheta_\ell)-s_\ell)\Big(p_\ell(s_\ell)\Phi_\ell(F_\ell(s_\ell))\Big)^{\frac{n-1}{n}}\Diamond_\alpha s_\ell\bigg)^{\frac{n}{n-1}}.$$

Theorem 3. *Let S_1, S_4, S_2, S_9, S_{11}, S_7, S_{13}, S_3, S_{12}, S_8 and S_{17} be satisfied. Then, for S_{10} we have that*

$$\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\vartheta_n}\int_{\Im_0}^{\varsigma_n}\prod_{\ell=1}^{n}\frac{\Phi_\ell(\tilde{F}_\ell(s_\ell,\Im_\ell))}{\bigg(\frac{1}{\gamma'}\sum_{\ell=1}^{n}\gamma'_\ell(s_\ell-\Im_0)(\Im_\ell-\Im_0)\bigg)^{\gamma'}}\Diamond_\alpha s_1\Diamond_\alpha\Im_1\dots\Diamond_\alpha s_n\Diamond_\alpha\Im_n \quad (40)$$
$$\geqslant G(\vartheta_1\varsigma_1,\dots,\vartheta_n\varsigma_n)$$
$$\times\prod_{\ell=1}^{n}\bigg(\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}(\rho(\vartheta_\ell)-s_\ell)(\rho(\varsigma_\ell)-\Im_\ell)\bigg(p_\ell(s_\ell)q_\ell(\Im_\ell)\Phi_\ell\bigg(\frac{F_\ell^{\Diamond_{\alpha_2}\Diamond_{\alpha_1}}(s_\ell,\Im_\ell)}{p_\ell(s_\ell)q_\ell(\Im_\ell)}\bigg)\bigg)^{\frac{1}{\gamma_\ell}}\Diamond_\alpha s_\ell\Diamond_\alpha\Im_\ell\bigg)^{\gamma_\ell}$$

where

$$G(\vartheta_1\varsigma_1,\ldots,\vartheta_n\varsigma_n)=\prod_{\ell=1}^{n}\left(\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}\left(\frac{\Phi_\ell(P_\ell(s_\ell,\Im_\ell))}{P_\ell(s_\ell,\Im_\ell)}\right)^{\frac{1}{\gamma'_\ell}}\diamondsuit_\alpha s_\ell\diamondsuit_\alpha\Im_\ell\right)^{\gamma'_\ell}.$$

Proof. From the hypotheses of Theorem 3, we obtain

$$F_\ell(s_\ell,\Im_\ell)=\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}F_\ell^{\diamondsuit_{\alpha_2}\diamondsuit_{\alpha_1}}(\xi_\ell,\tau_\ell)\diamondsuit_\alpha\xi_\ell\diamondsuit_\alpha\tau_\ell. \tag{41}$$

From (41) and S_8, it is easy to observe that

$$\begin{aligned}\Phi_\ell(F_\ell(s_\ell,\Im_\ell))&=\Phi_\ell\left(\frac{P_\ell(s_\ell,\Im_\ell)\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}p_\ell(\xi_\ell)q_\ell(\tau_\ell)\left(\frac{F_\ell^{\diamondsuit_{\alpha_2}\diamondsuit_{\alpha_1}}(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell)q_\ell(\tau_\ell)}\right)\diamondsuit_\alpha\xi_\ell\diamondsuit_\alpha\tau_\ell}{\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}p_\ell(\xi_\ell)q_\ell(\tau_\ell)\diamondsuit_\alpha\xi_\ell\diamondsuit_\alpha\tau_\ell}\right)\\&\geqslant\Phi_\ell(P_\ell(s_\ell,\Im_\ell))\Phi_\ell\left(\frac{\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}p_\ell(\xi_\ell)q_\ell(\tau_\ell)\left(\frac{F_\ell^{\diamondsuit_{\alpha_2}\diamondsuit_{\alpha_1}}(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell)q_\ell(\tau_\ell)}\right)\diamondsuit_\alpha\xi_\ell\diamondsuit_\alpha\tau_\ell}{\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}p_\ell(\xi_\ell)q_\ell(\tau_\ell)\diamondsuit_\alpha\xi_\ell\diamondsuit_\alpha\tau_\ell}\right).\end{aligned} \tag{42}$$

By using inverse Jensen's dynamic inequality, we get that

$$\Phi_\ell(F_\ell(s_\ell,\Im_\ell))\geqslant\frac{\Phi_\ell(P_\ell(s_\ell,\Im_\ell))}{P_\ell(s_\ell,\Im_\ell)}\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}p_\ell(\xi_\ell)q_\ell(\tau_\ell)\Phi_\ell\left(\frac{F_\ell^{\diamondsuit_{\alpha_2}\diamondsuit_{\alpha_1}}(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell)q_\ell(\tau_\ell)}\right)\diamondsuit_\alpha\xi_\ell\diamondsuit_\alpha\tau_\ell. \tag{43}$$

Applying inverse Hölder's inequality on the right hand side of (43) with indices $1/\gamma_\ell$ and $1/\gamma'_\ell$, we obtain

$$\begin{aligned}\Phi_\ell(F_\ell(s_\ell,\Im_\ell))&\geqslant\frac{\Phi_\ell(P_\ell(s_\ell,\Im_\ell))}{P_\ell(s_\ell,\Im_\ell)}[(s_\ell-\Im_0)(\Im_\ell-\Im_0)]^{\gamma'_\ell}\\&\times\left(\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}\left(p_\ell(\xi_\ell)q_\ell(\tau_\ell)\Phi_\ell\left(\frac{F_\ell^{\diamondsuit_{\alpha_2}\diamondsuit_{\alpha_1}}(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell)q_\ell(\tau_\ell)}\right)\right)^{\frac{1}{\gamma_\ell}}\diamondsuit_\alpha\xi_\ell\diamondsuit_\alpha\tau_\ell\right)^{\gamma_\ell}.\end{aligned} \tag{44}$$

Using the following inequality on the term $[(s_\ell-\Im_0)(\Im_\ell-\Im_0)]^{\gamma'_\ell}$ where $\gamma'_\ell<0$ and $\lambda_\ell>0$.

$$\prod_{\ell=1}^{n}\lambda_\ell^{\gamma'_\ell}\geqslant\left(\frac{1}{\gamma'}\left(\sum_{\ell=1}^{n}\gamma'_\ell\lambda_\ell\right)\right)^{\gamma'}. \tag{45}$$

We obtain that

$$\begin{aligned}\prod_{\ell=1}^{n}\Phi_\ell(F_\ell(s_\ell,\Im_\ell))&\geqslant\prod_{\ell=1}^{n}\frac{\Phi_\ell(P_\ell(s_\ell,\Im_\ell))}{P_\ell(s_\ell,\Im_\ell)}\left(\frac{1}{\gamma'}\sum_{\ell=1}^{n}\gamma'_\ell(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\gamma'}\\&\times\left(\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}\left(p_\ell(\xi_\ell)q_\ell(\tau_\ell)\Phi_\ell\left(\frac{F_\ell^{\diamondsuit_{\alpha_2}\diamondsuit_{\alpha_1}}(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell)q_\ell(\tau_\ell)}\right)\right)^{\frac{1}{\gamma_\ell}}\diamondsuit_\alpha\xi_\ell\diamondsuit_\alpha\tau_\ell\right)^{\gamma_\ell}.\end{aligned} \tag{46}$$

From (46), we have that

$$\begin{aligned}&\prod_{\ell=1}^{n}\frac{\Phi_\ell(F_\ell(s_\ell,\Im_\ell))}{\left(\frac{1}{\gamma'}\sum_{\ell=1}^{n}\gamma'_\ell(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\gamma'}}\\&\geqslant\prod_{\ell=1}^{n}\frac{\Phi_\ell(P_\ell(s_\ell,\Im_\ell))}{P_\ell(s_\ell,\Im_\ell)}\left(\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}\left(p_\ell(\xi_\ell)q_\ell(\tau_\ell)\Phi_\ell\left(\frac{F_\ell^{\diamondsuit_{\alpha_2}\diamondsuit_{\alpha_1}}(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell)q_\ell(\tau_\ell)}\right)\right)^{\frac{1}{\gamma_\ell}}\diamondsuit_\alpha\xi_\ell\diamondsuit_\alpha\tau_\ell\right)^{\gamma_\ell}.\end{aligned} \tag{47}$$

Integrating both sides of (47) over s_ℓ, $\Im_\ell$ from $\Im_0$ to ϑ_ℓ, ς_ℓ $(\ell=1,\ldots,n)$, we get that

$$\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\vartheta_n}\int_{\Im_0}^{\varsigma_n}\prod_{\ell=1}^{n}\frac{\Phi_\ell(F_\ell(s_\ell,\Im_\ell))}{\left(\frac{1}{\gamma'}\sum_{\ell=1}^{n}\gamma'_\ell(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\gamma'}}\diamondsuit_\alpha s_1\diamondsuit_\alpha\Im_1\ldots\diamondsuit_\alpha s_n\diamondsuit_\alpha\Im_n$$
$$\geqslant\prod_{\ell=1}^{n}\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}\frac{\Phi_\ell(P_\ell(s_\ell,\Im_\ell))}{P_\ell(s_\ell,\Im_\ell)}\left(\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}\left(p_\ell(\xi_\ell)q_\ell(\tau_\ell)\Phi_\ell\left(\frac{F_\ell^{\diamondsuit_{\alpha_2}\diamondsuit_{\alpha_1}}(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell)q_\ell(\tau_\ell)}\right)\right)^{\frac{1}{\gamma_\ell}}\diamondsuit_\alpha\xi_\ell\diamondsuit_\alpha\tau_\ell\right)^{\gamma_\ell}\diamondsuit_\alpha s_\ell\diamondsuit_\alpha\Im_\ell. \tag{48}$$

Applying inverse Hölder's inequality on the right hand side of (48) with indices $1/\gamma_\ell$ and $1/\gamma'_\ell$, we obtain

$$\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\vartheta_n}\int_{\Im_0}^{\varsigma_n}\prod_{\ell=1}^{n}\frac{\Phi_\ell(F_\ell(s_\ell,\Im_\ell))}{\left(\frac{1}{\gamma'}\sum_{\ell=1}^{n}\gamma'_\ell(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\gamma'}}\diamondsuit_\alpha s_1\diamondsuit_\alpha\Im_1\ldots\diamondsuit_\alpha s_n\diamondsuit_\alpha\Im_n$$
$$\geqslant\prod_{\ell=1}^{n}\left(\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}\left(\frac{\Phi_\ell(P_\ell(s_\ell,\Im_\ell))}{P_\ell(s_\ell,\Im_\ell)}\right)^{\frac{1}{\gamma'_\ell}}\diamondsuit_\alpha s_\ell\diamondsuit_\alpha\Im_\ell\right)^{\gamma'_\ell} \tag{49}$$
$$\times\prod_{\ell=1}^{n}\left(\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}\left(\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}\left(p_\ell(\xi_\ell)q_\ell(\tau_\ell)\Phi_\ell\left(\frac{F_\ell^{\diamondsuit_{\alpha_2}\diamondsuit_{\alpha_1}}(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell)q_\ell(\tau_\ell)}\right)\right)^{\frac{1}{\gamma_\ell}}\diamondsuit_\alpha\xi_\ell\diamondsuit_\alpha\tau_\ell\right)\diamondsuit_\alpha s_\ell\diamondsuit_\alpha\Im_\ell\right)^{\gamma_\ell}.$$

By using Fubini's theorem, we observe that

$$\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\vartheta_n}\int_{\Im_0}^{\varsigma_n}\prod_{\ell=1}^{n}\frac{\Phi_\ell(F_\ell(s_\ell,\Im_\ell))}{\left(\frac{1}{\gamma'}\sum_{\ell=1}^{n}\gamma'_\ell(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\gamma'}}\diamondsuit_\alpha s_1\diamondsuit_\alpha\Im_1\ldots\diamondsuit_\alpha s_n\diamondsuit_\alpha\Im_n$$
$$\geqslant G(\vartheta_1\varsigma_1,\ldots,\vartheta_n\varsigma_n) \tag{50}$$
$$\times\prod_{\ell=1}^{n}\left(\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}(\vartheta_\ell-s_\ell)(\varsigma_\ell-\Im_\ell)\left(p_\ell(s_\ell)q_\ell(\Im_\ell)\Phi_\ell\left(\frac{F_\ell^{\diamondsuit_{\alpha_2}\diamondsuit_{\alpha_1}}(s_\ell,\Im_\ell)}{p_\ell(s_\ell)q_\ell(\Im_\ell)}\right)\right)^{\frac{1}{\gamma_\ell}}\diamondsuit_\alpha s_\ell\diamondsuit_\alpha\Im_\ell\right)^{\gamma_\ell}.$$

By using the fact $\vartheta_\ell\geqslant\rho(\vartheta_\ell)$, and $\varsigma_\ell\geqslant\rho(\varsigma_\ell)$, we get that

$$\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\vartheta_n}\int_{\Im_0}^{\varsigma_n}\prod_{\ell=1}^{n}\frac{\Phi_\ell(F_\ell(s_\ell,\Im_\ell))}{\left(\frac{1}{\gamma'}\sum_{\ell=1}^{n}\gamma'_\ell(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\gamma'}}\diamondsuit_\alpha s_1\diamondsuit_\alpha\Im_1\ldots\diamondsuit_\alpha s_n\diamondsuit_\alpha\Im_n$$
$$\geqslant G(\vartheta_1\varsigma_1,\ldots,\vartheta_n\varsigma_n)$$
$$\times\prod_{\ell=1}^{n}\left(\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}(\rho(\vartheta_\ell)-s_\ell)(\rho(\varsigma_\ell)-\Im_\ell)\left(p_\ell(s_\ell)q_\ell(\Im_\ell)\Phi_\ell\left(\frac{F_\ell^{\diamondsuit_{\alpha_2}\diamondsuit_{\alpha_1}}(s_\ell,\Im_\ell)}{p_\ell(s_\ell)q_\ell(\Im_\ell)}\right)\right)^{\frac{1}{\gamma_\ell}}\diamondsuit_\alpha s_\ell\diamondsuit_\alpha\Im_\ell\right)^{\gamma_\ell}.$$

This completes the proof. □

Remark 8. *In Theorem 3, if $\mathbb{T}=\mathbb{Z}$, $\alpha=1$ we get the result due to Zhao et al. ([11], Theorem 1.5).*

Remark 9. *In Theorem 3, if we take $\mathbb{T}=\mathbb{R}$, $\alpha=1$ we get the result due to Zhao et al. ([11], Theorem 1.6).*

Remark 10. *Let S_1, S_2, S_9, S_{11}, S_7, S_{13}, S_3 and S_{12} be satisfied and let Φ_ℓ, γ_ℓ, γ'_ℓ, γ, and γ' be as Theorem 3. Similar to proof of Theorem 3, we have*

$$
\begin{aligned}
&\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\vartheta_n}\int_{\Im_0}^{\varsigma_n}\prod_{\ell=1}^{n}\frac{\Phi_\ell(F_\ell(s_\ell,\Im_\ell))}{\left(\frac{1}{\gamma'}\sum_{\ell=1}^{n}\gamma'_\ell(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\gamma'}}\diamondsuit_\alpha s_1\diamondsuit_\alpha\Im_1\ldots\diamondsuit_\alpha s_n\diamondsuit_\alpha\Im_n\\
&\leqslant G^*(\vartheta_1\varsigma_1,\ldots,\vartheta_n\varsigma_n)\\
&\times\prod_{\ell=1}^{n}\left(\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}(\sigma(\vartheta_\ell)-s_\ell)(\sigma(\varsigma_\ell)-\Im_\ell)\left(p_\ell(s_\ell)q_\ell(\Im_\ell)\Phi_\ell\left(\frac{F_\ell^{\diamondsuit_{\alpha_2}\diamondsuit_{\alpha_1}}(s_\ell,\Im_\ell)}{p_\ell(s_\ell)q_\ell(\Im_\ell)}\right)\right)^{\frac{1}{\gamma_\ell}}\diamondsuit_\alpha s_\ell\diamondsuit_\alpha\Im_\ell\right)^{\gamma_\ell}.
\end{aligned}
$$

where

$$
G^*(\vartheta_1\varsigma_1,\ldots,\vartheta_n\varsigma_n)=\frac{1}{(\gamma')^{\gamma'}}\prod_{\ell=1}^{n}\left(\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}\left(\frac{\Phi_\ell(P_\ell(s_\ell,\Im_\ell))}{P_\ell(s_\ell,\Im_\ell)}\right)^{\frac{1}{\gamma'_\ell}}\diamondsuit_\alpha s_\ell\diamondsuit_\alpha\Im_\ell\right)^{\gamma'_\ell}.
$$

This is an inverse form of the inequality (40).

Corollary 10. *Let S_{22}, S_{23}, S_{25}, S_{26}, S_{27}, S_{29}, S_{17} and S_8 be satisfied. Then we have that*

$$
\begin{aligned}
&\int_{\Im_0}^{\vartheta_1}\cdots\int_{\Im_0}^{\vartheta_n}\prod_{\ell=1}^{n}\frac{\Phi_\ell(F_\ell(s_\ell))}{\left(\frac{1}{\gamma'}\sum_{\ell=1}^{n}\gamma'_\ell(s_\ell-\Im_0)\right)^{\gamma'}}\diamondsuit_\alpha s_1\ldots\diamondsuit_\alpha s_n\\
&\geqslant G^{**}(\vartheta_1,\ldots,\vartheta_n)\\
&\times\prod_{\ell=1}^{n}\left(\int_{\Im_0}^{\vartheta_\ell}(\rho(\vartheta_\ell)-s_\ell)\left(p_\ell(s_\ell)\Phi_\ell\left(\frac{F_\ell^{\diamondsuit_\alpha}(s_\ell)}{p_\ell(s_\ell)}\right)\right)^{\frac{1}{\gamma_\ell}}\diamondsuit_\alpha s_\ell\right)^{\gamma_\ell}.
\end{aligned}
\tag{51}
$$

where

$$
G^{**}(\vartheta_1,\ldots,\vartheta_n)=\prod_{\ell=1}^{n}\left(\int_{\Im_0}^{\vartheta_\ell}\left(\frac{\Phi_\ell(P_\ell(s_\ell))}{P_\ell(s_\ell)}\right)^{\frac{1}{\gamma'_\ell}}\diamondsuit_\alpha s_\ell\right)^{\gamma'_\ell}.
$$

Remark 11. *In Corollary* 10, *if we take $\mathbb{T}=\mathbb{Z}$, $\alpha=1$ we get an inverse form of inequality* (3), *which was given by Handley et al.*

Remark 12. *In Corollary* 10, *if we take $\mathbb{T}=\mathbb{R}$, $\alpha=1$ we get an inverse form of inequality* (4), *which was given by Handley et al.*

Remark 13. *In inequality* (51) *taking $n=2$, $\gamma_1=\gamma_2=2$, then $\gamma'_1=\gamma'_2=-1$, we have*

$$
\begin{aligned}
&\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\vartheta_2}\prod_{\ell=1}^{n}\frac{\Phi_1(F_1(s_1))\Phi_1(F_2(s_2))}{\left((s_1-\Im_0)+(s_2-\Im_0)\right)^{-2}}\diamondsuit_\alpha s_1\diamondsuit_\alpha s_2\\
&\geqslant D(\vartheta_1,\vartheta_2)\left(\int_{\Im_0}^{\vartheta_1}(\rho(\vartheta_1)-s_1)\left(p_1(s_1)\Phi_1\left(\frac{F_1^{\diamondsuit_\alpha}(s_1)}{p_1(s_1)}\right)\right)^{\frac{1}{2}}\diamondsuit_\alpha s_1\right)^2\\
&\times\left(\int_{\Im_0}^{\vartheta_2}(\rho(\vartheta_2)-s_2)\left(p_2(s_2)\Phi_2\left(\frac{F_2^{\diamondsuit_\alpha}(s_2)}{p_2(s_2)}\right)\right)^{\frac{1}{2}}\diamondsuit_\alpha s_2\right)^2.
\end{aligned}
\tag{52}
$$

where

$$
D(\vartheta_1,\vartheta_2)=4\left(\int_{\Im_0}^{\vartheta_1}\left(\frac{\Phi_1(P_1(s_1))}{P_2(s_1)}\right)^{-1}\diamondsuit_\alpha s_1\right)^{-1}\left(\int_{\Im_0}^{\vartheta_2}\left(\frac{\Phi_2(P_2(s_2))}{P_2(s_2)}\right)^{-1}\diamondsuit_\alpha s_2\right)^{-1}.
$$

Remark 14. *If we take $\mathbb{T}=\mathbb{Z}$, $\alpha=1$ the inequality* (52) *is an inverse of inequality of* (1), *which was given by Pachpatte.*

Remark 15. *If we take* $\mathbb{T} = \mathbb{R}$, $\alpha = 1$ *the inequality* (52) *is an inverse of inequality of* (2), *which was given by Pachpatte.*

3. Conclusions

In this work, by applying $\Diamond_\alpha$ calculus, defined as a linear combination of the nabla and delta integrals, we introduced some novel results of Hardy–Hilbert-type inequalities on a general time-scale. Furthermore, we gave the multidimensional generalization for these inequalities to time scales. We also applied our inequalities to discrete and continuous calculus to obtain some new inequalities as special cases.

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