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Second Derivative Block Hybrid Methods for the Numerical Integration of Differential Systems

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Abstract: The second derivative block hybrid method for the continuous integration of differential systems within the interval of integration was derived. The second derivative block hybrid method maintained the stability properties of the Runge–Kutta methods suitable for solving stiff differential systems. The lack of such stability properties makes the continuous solution not reliable, especially in solving large stiff differential systems. We derive these methods by using one intermediate off-grid point in between the familiar grid points for continuous solution within the interval of integration. The new family had a high accuracy, non-overlapping piecewise continuous solution with very low error constants and converged under the suitable conditions of stability and consistency. The results of computational experiments are presented to demonstrate the efficiency and usefulness of the methods, which also indicate that the block hybrid methods are competitive with some strong stability stiff integrators.

Keywords: block hybrid method; continuous scheme; differential system; multistep collocation

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1. Introduction

The system whose numerical approximation is sought is written in the form

$$\begin{cases} \frac{dy}{dx} = f(x,y), \ \ (a \le x \le b), \\ y(x_0) = y_0 \ . \end{cases} \tag{1}$$

In Equation (1), $y : [a,b] \to R^m$ and $f : [a,b] \times R^m \to R^m$ are differentiable. To obtain accurate integration methods, which combine, to some extent, the advantages of the Runge–Kutta methods (RKMs) and linear multistep methods (LMMs), the use of the multistep collocation technique has been proposed by many authors, for example, Onumanyi et al. [1], Chollom and Jackiewicz [2], Chollom and Onumanyi [3], and Jator [4]. In this work, methods were designed for finding a continuous approximate solution of the system in Equation (1), where $y(x)$ belongs to $C^1([a, b], R^m)$ and the set of points defined as

$$\Omega : a = x_0 < x_1 < \cdots < x_{n+1} = b,$$

so that

$$x_n : x_n = x_0 + nh, \ n = 0,1,\cdots,N-1, h = x_{n+1} - x_n \text{ and } N = (b-a)/h.$$

The h in this paper, for simplicity, is a constant and N is a positive integer. Some of the methods derived for Equation (1) were, in fact, to evaluate the solution only at the first derivative of Equation (1). Long before our consideration of introducing the off-step points, Gragg and Stetter [5], Butcher [6,7], and Gear [8] had already considered introducing some off-step points and referred to them as generalized multistep predictor–corrector methods, a modified multistep method, and hybrid methods, respectively. Similarly, the introduction of the second derivative terms had already been considered by many authors. For example, earlier, Urabe [9] worked on a second derivative method with $y''(x) = g(x, y)$ to obtain a starting method for the single-step integrator in the paper. In [10], Mitsui changed slightly Urabe's PC pair to improve the performance of the method. To unify and extend this result after some years, Cash [11] generalized the PC pair of Urabe's type of method. Gupta, in [12], derived and implemented second-derivative methods. Shaintani [13,14] suggested some integration algorithms very similar to RKMs, with $y'(x) = f(x, y)$ and $y''(x) = g(x, y)$. In [15], the author constructed (p, q)-stage RKMs, which exhibit $y''(x) = g(x, y)$ evaluation. Chan and Tsai [16] considered explicit two-derivative RKMs, which are cheaper to calculate with fewer function evaluations than the standard RKMs. Recently, many authors have worked on methods to obtain better approximate solutions to differential equations or on stability properties to improve the accuracy and efficiency of solution of differential equations (see, for example, [17–24]).

In this article, we extend the work of Yakubu et al. [25] to derive block-hybrid methods that show a high order of accuracy with very low error constants and large regions of absolute stability and converge rapidly to the required solution. We should also point out that the effectiveness of this class of methods for the treatment of stiff systems is shown on the basis of their attractive properties and the efficient technique to deal with a large system of a stiff initial value problem of ordinary differential equations.

Definition 1 ([26])**.** *Let Y_m and F_m be vectors given by*

$$Y_m = (y_n, y_{n+1}, \ldots, y_{n+r-1})^T,$$

$$F_m = (f_n, f_{n+1}, \ldots, f_{n+r-1})^T.$$

Then the k-block method is of the form

$$Y_m = \sum_{i=1}^{k} A_i Y_{m-i} + \sum_{i=0}^{k} B_i F_{m-i}. \tag{2}$$

If r = 1, then the above equation in Equation (2) is just the classical k-step method. When $B_0 = 0$, Equation (2) is explicit; otherwise, it is implicit.

The below diagram depicts the idea of the new methods.

In Figure 1, [a, b] is divided into a series of equal lengths of a block of six points with size or length h. The approximate solutions $\{y_{n+u}, y_{n+1}, y_{n+v}, y_{n+2}, y_{n+w}, y_{n+3}\}$ are computed simultaneously in the block at the points $\{x_{n+u}, x_{n+1}, x_{n+v}, x_{n+2}, x_{n+w}, x_{n+3}\}$ in the kth block. Since the methods are self-starting, we do not need predictors to start the block methods.

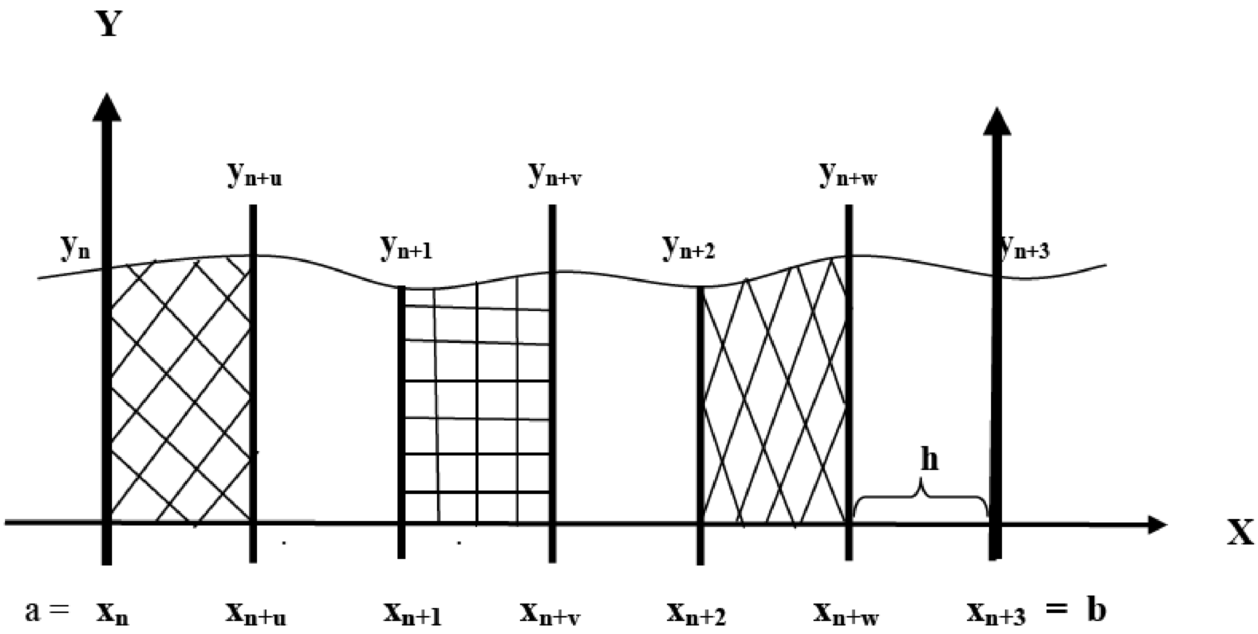


Figure 1. Schematic representation of the block hybrid methods for stiff system of initial value problems.

2. The Block Hybrid Methods

The block hybrid methods in this segment are based on the polynomial of the form

$$y(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_{p-1} x^{p-1} = \sum_{i=0}^{p-1} \alpha_i x^i \tag{3}$$

and are referred to as interpolation polynomials, which is twice continuously differentiable. The $y(x)$ is interpolated at $\{x_{n+j}\}$, and $y'(x)$ and $y''(x)$ are collocated at $\{c_{n+j}\}$ to obtain the system,

$$y(x_{n+j}) = y(x_n + jh), \qquad j \in \{0, 1, 2, \ldots, r-1\}, \tag{4}$$

$$y'(c_{n+j}) = f_{n+j} = f(x_n + jh, y(x_n + jh)), \quad j = 0, 1, 2, \ldots, s-1, \tag{5}$$

$$y''(c_{n+j}) = g_{n+j} = f_x + f_y y' = f_x + f f_y, \; j = 0, 1, 2, \ldots, t-1. \tag{6}$$

Following Yakubu et al. [25], we put Equations (4)–(6) as:

$$V\alpha = y \tag{7}$$

to have

$$V = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & \cdots & x_n^{p-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n+r-1} & x_{n+r-1}^2 & x_{n+r-1}^3 & x_{n+r-1}^4 & \cdots & x_{n+r-1}^{p-1} \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & \cdots & D' x_n^{p-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 2x_{n+s-1} & 3x_{n+s-1}^2 & 4x_{n+s-1}^3 & \cdots & D' x_{n+s-1}^{p-2} \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & \cdots & D'' x_n^{p-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 2\cdots & 6x_{n+t-1} & 12x_{n+t-1}^2 & \cdots & D'' x_{n+t-1}^{p-3} \end{bmatrix} \tag{8}$$

$$\alpha = (\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_{p-1})^T,\ y = (y_n, \cdots, y_{n+r-1}, f_n, \cdots, f_{n+s-1}, g_n, \cdots, g_{n+t-1})^T.$$

The $D' = (p-1)$ and $D'' = (p-1)(p-2)$ are first and second derivatives. From Equation (7), we have

$$\alpha = Uy,\ \text{where } U = V^{-1} \tag{9}$$

which is rearranged to have

$$y(x) = \sum_{j=0}^{r-1} \alpha_j(x) y_{n+j} + h\sum_{j=0}^{s-1} \beta_j(x) f_{n+j} + h^2 \sum_{j=0}^{t-1} \gamma_j(x) g_{n+j}, \tag{10}$$

where

$$\alpha_j(x) = \sum_{i=0}^{p-1} \alpha_{j,\,i+1} x^i,\ \ j = 0,\ 1,\ 2,\ \ldots,\ r-1, \tag{11a}$$

$$h\beta_j(x) = h\sum_{i=0}^{p-1} \beta_{j,\,i+1} x^i,\ \ j = 0,\ 1,\ 2,\ \ldots,\ s-1, \tag{11b}$$

$$h^2\gamma_j(x) = h^2\sum_{i=0}^{p-1} \gamma_{j,\,i+1} x^i,\ j = 0,\ 1,\ 2,\ \ldots,\ t-1. \tag{11c}$$

In fact, the coefficients in Equation (11) can be calculated from the inverse of the matrix U, as in Equation (9), or written as

$$U = V^{-1}. \tag{12}$$

Insert Equation (11) into Equation (10) (see Yakubu et al. [25,27]) to have

$$y(x) = \sum_{j=0}^{r-1} \sum_{i=0}^{r+s+t-1} \alpha_{i+1,j} y_{n+j} P_i(x) + h\sum_{j=0}^{s-1} \sum_{i=0}^{r+s+t-1} \beta_{i+1,j} f_{n+j} P_i(x) + h^2 \sum_{j=0}^{t-1} \sum_{i=0}^{r+s+t-1} \gamma_{i+1,j} g_{n+j} P_i(x)$$

which is factorized to obtain

$$\begin{aligned} y(x) &= \sum_{i=0}^{r+s+t-1} \left\{ \sum_{j=0}^{r-1} \alpha_{i+1,j} y_{n+j} + h\sum_{j=0}^{s-1} \beta_{i+1,j} f_{n+j} + h^2 \sum_{j=0}^{t-1} \gamma_{i+1,j} g_{n+j} \right\} P_i(x) \\ &= \sum_{i=0}^{r+s+t-1} \varphi_i P_i(x) \end{aligned} \tag{13}$$

where

$$\phi_i = \sum_{j=0}^{r-1} \alpha_{i+1,j} y_{n+j} + h\sum_{j=0}^{s-1} \beta_{i+1,j} f_{n+j} + h^2 \sum_{j=0}^{t-1} \gamma_{i+1,j} g_{n+j}.$$

Then Equation (13) becomes

$$\begin{aligned} y(x) = \Bigg\{ & \sum_{j=0}^{r-1} \alpha_{j,1} y_{n+j} + h\sum_{j=0}^{s-1} \beta_{j,1} f_{n+j} + h^2 \sum_{j=0}^{t-1} \gamma_{j,1} g_{n+j}, \\ & \sum_{j=0}^{r-1} \alpha_{j,2} y_{n+j} + h\sum_{j=0}^{s-1} \beta_{j,2} f_{n+j} + h^2 \sum_{j=0}^{t-1} \gamma_{j,2} g_{n+j}, \\ & \quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ & \sum_{j=0}^{r-1} \alpha_{j,r+s+t-1} y_{n+j} + h\sum_{j=0}^{s-1} \beta_{j,r+s+t-1} f_{n+j} + h^2 \sum_{j=0}^{t-1} \gamma_{j,r+s+t-1} g_{n+j} \Bigg\} \left(1, x, x^2, \cdots, x^{r+s+t-1}\right)^T. \end{aligned} \tag{14}$$

Expanding Equation (14) fully, we obtain

$$y(x) = (y_n, \cdots, y_{n+r-1}, f_n, \cdots, f_{n+s-1}, g_n, \cdots, g_{n+t-1})^T U^T (1, x, \cdots, x^{r+s+t-1})^T. \tag{15}$$

The T in Equation (15) denotes the transpose of.

3. Specification of the Multistep Block Hybrid Methods

3.1. Block Hybrid Method of Seventh Order

In this segment, we use the multistep approach for the construction of the new block hybrid method with symmetric points of order seven. We introduce three off-step points, $u = \frac{1}{2}$, $v = \frac{3}{2}$, and $w = \frac{5}{2}$, and $\eta = (x - x_n)$ for the construction of the continuous scheme. These points are carefully chosen to guarantee the convergence of the method, as pointed out by [28–30]. From Equation (10), putting $r = 1$ and $s = 7$ gives the block hybrid method of the form

$$y(x) = \alpha_0(x)y_n + h\sum_{j=0}^{6}\beta_j(x)f_{n+j}. \tag{16}$$

Simplifying Equation (16), the interpolation and collocation polynomial in Equation (10) reduces to the proposed continuous scheme of the form in Equation (15), as follows:

$$y(x) = \alpha_0(x)y_n + h[\beta_0(x)f_n + \beta_1(x)f_{n+u} + \beta_2(x)f_{n+1} + \beta_3(x)f_{n+v} + \beta_4(x)f_{n+2} + \beta_5(x)f_{n+w} + \beta_6(x)f_{n+3}] \tag{17}$$

where

$$\begin{aligned}
\alpha_0(x) &= 1,\\
\beta_0(x) &= \left[\frac{96\eta^7 - 1176h\eta^6 + 5880h^2\eta^5 - 15,435h^3\eta^4 + 22,736h^4\eta^3 - 18,522h^5\eta^2 + 7560h^6\eta}{7560h^6}\right],\\
\beta_1(x) &= \left[\frac{-24\eta^7 + 280h\eta^6 - 1302h^2\eta^5 + 3045h^3\eta^4 - 3654h^4\eta^3 + 1890h^5\eta^2}{315h^6}\right],\\
\beta_2(x) &= \left[\frac{480\eta^7 - 5320h\eta^6 + 23,016h^2\eta^5 - 48,405h^3\eta^4 + 49,140h^4\eta^3 - 18,900h^5\eta^2}{2520h^6}\right],\\
\beta_3(x) &= \left[\frac{-240\eta^7 + 2520h\eta^6 - 10,164h^2\eta^5 + 19,530h^3\eta^4 - 17,780h^4\eta^3 + 6300h^5\eta^2}{945h^6}\right],\\
\beta_4(x) &= \left[\frac{480\eta^7 - 4760h\eta^6 + 17,976h^2\eta^5 - 32,235h^3\eta^4 + 27,720h^4\eta^3 - 9450h^5\eta^2}{2520h^6}\right],\\
\beta_5(x) &= \left[\frac{-24\eta^7 + 224h\eta^6 - 798h^2\eta^5 + 1365h^3\eta^4 - 1134h^4\eta^3 + 378h^5\eta^2}{315h^6}\right],\\
\beta_6(x) &= \left[\frac{96\eta^7 - 840h\eta^6 + 2856h^2\eta^5 - 4725h^3\eta^4 + 3836h^4\eta^3 - 1260h^5\eta^2}{7560h^6}\right],
\end{aligned}$$

Evaluate Equation (17) at x_{n+u}, $x_{n+1}x_{n+v}$, x_{n+2}, x_{n+w}, and x_{n+3} to obtain the method:

$$y_{n+u} = y_n + \frac{h}{120,960}[19,087f_n + 65,112f_{n+u} - 46,461f_{n+1} + 37,504f_{n+v} - 20,211f_{n+2} + 6312f_{n+w} - 863f_{n+3}] \tag{18}$$

$$y_{n+1} = y_n + \frac{h}{7560}[1139f_n + 5640f_{n+u} + 33f_{n+1} + 1328f_{n+v} - 807f_{n+2} + 264f_{n+w} - 37f_{n+3}]$$

$$y_{n+v} = y_n + \frac{h}{4480}[685f_n + 3240f_{n+u} + 1161f_{n+1} + 2176f_{n+v} - 729f_{n+2} + 216f_{n+w} - 29f_{n+3}]$$

$$y_{n+2} = y_n + \frac{h}{945}[143f_n + 696f_{n+u} + 192f_{n+1} + 752f_{n+v} + 87f_{n+2} + 24f_{n+w} - 4f_{n+3}]$$

$$y_{n+w} = y_n + \frac{h}{24,192}[3715f_n + 17,400f_{n+u} + 6375f_{n+1} + 16,000f_{n+v} + 11,625f_{n+2} + 5640f_{n+w} - 275f_{n+3}]$$

$$y_{n+3} = y_n + \frac{h}{280}[41f_n + 216f_{n+u} + 27f_{n+1} + 272f_{n+v} + 27f_{n+2} + 216f_{n+w} + 41f_{n+3}]$$

3.2. Second-Derivative Block Hybrid Method of Order 14

Here, we introduce the second-derivative term to have the block hybrid method of order 14 whereby we have the following interpolation and collocation polynomial of the form

$$y(x) = \alpha_0(x)y_\eta + h\sum_{j=0}^{6}\beta_j(x)f_{n+j} + h^2\sum_{j=0}^{6}\gamma_j(x)g_{n+j} \tag{19}$$

Simplify Equation (19) to obtain the proposed continuous scheme of the form in Equation (15) as:

$$\begin{aligned} y(x) = \alpha_0(x)y_n + h[\beta_0(x)f_n + \beta_1(x)f_{n+u} + \beta_2(x)f_{n+1} + \beta_3(x)f_{n+v} + \beta_4(x)f_{n+2} + \beta_5(x)f_{n+w} + \beta_6(x)f_{n+3}] \\ +h^2[\gamma_0(x)g_n + \gamma_1(x)g_{n+u} + \gamma_2(x)g_{n+1} + \gamma_3(x)g_{n+v} + \gamma_4(x)g_{n+2} + \gamma_5(x)g_{n+w} + \gamma_6(x)g_{n+3}] \end{aligned} \tag{20}$$

where

$$\alpha_0(x) = 1,$$

$$\beta_0(x) = \begin{bmatrix} \frac{56\eta^{14}}{10{,}125h^{13}} - \frac{16{,}384\eta^{13}}{131{,}625h^{12}} + \frac{38{,}339\eta^{12}}{30{,}375h^{11}} - \frac{15{,}428\eta^{11}}{2025h^{10}} + \frac{1{,}028{,}069\eta^{10}}{33{,}750h^{9}} - \frac{2{,}578{,}814\eta^{9}}{30{,}375h^{8}} + \frac{54{,}733{,}637\eta^{8}}{324{,}000h^{7}} \\ -\frac{2{,}737{,}391\eta^{7}}{11{,}340h^{6}} + \frac{29{,}786{,}393\eta^{6}}{121{,}500h^{5}} - \frac{173{,}613{,}232\eta^{5}}{101{,}250h^{4}} + \frac{1{,}383{,}221\eta^{4}}{18{,}000h^{3}} - \frac{48{,}587\eta^{3}}{2700h^{2}} + \eta \end{bmatrix},$$

$$\beta_1(x) = \begin{bmatrix} \frac{352\eta^{14}}{3375h^{13}} - \frac{100{,}064\eta^{13}}{43{,}875h^{12}} + \frac{3016\eta^{12}}{135h^{11}} - \frac{320{,}048\eta^{11}}{2475h^{10}} + \frac{2{,}762{,}014\eta^{10}}{5625h^{9}} - \frac{12{,}992{,}474\eta^{9}}{10{,}125h^{8}} \\ +\frac{1{,}588{,}042\eta^{8}}{675h^{7}} - \frac{14{,}281{,}196\eta^{7}}{4725h^{6}} + \frac{2{,}995{,}066\eta^{6}}{1125h^{5}} - \frac{2{,}864{,}446\eta^{5}}{1875h^{4}} + \frac{12{,}798\eta^{4}}{25h^{3}} - \frac{375\eta^{3}}{5h^{2}} \end{bmatrix},$$

$$\beta_2(x) = \begin{bmatrix} \frac{8\eta^{14}}{27h^{13}} - \frac{2192\eta^{13}}{351h^{12}} + \frac{1583\eta^{12}}{27h^{11}} - \frac{31984\eta^{11}}{99h^{10}} + \frac{104{,}287\eta^{10}}{90h^{9}} - \frac{229{,}583\eta^{9}}{81h^{8}} + \\ \frac{4{,}159{,}523\eta^{8}}{864h^{7}} - \frac{2{,}142{,}395\eta^{7}}{378h^{6}} + \frac{36{,}235\eta^{6}}{8h^{5}} - \frac{35{,}194\eta^{5}}{15h^{4}} + \frac{2865\eta^{4}}{4h^{3}} - \frac{100\eta^{3}}{h^{2}} \end{bmatrix},$$

$$\beta_3(x) = \begin{bmatrix} \frac{256\eta^{13}}{1053h^{12}} - \frac{128\eta^{12}}{27h^{11}} + \frac{36{,}224\eta^{11}}{891h^{10}} - \frac{5440\eta^{10}}{27h^{9}} + \frac{154{,}928\eta^{9}}{243h^{8}} - \frac{12{,}088\eta^{8}}{9h^{7}} + \\ \frac{1{,}079{,}840\eta^{7}}{567h^{6}} - \frac{48{,}448\eta^{6}}{27h^{5}} + \frac{436{,}624\eta^{5}}{405h^{4}} - \frac{10{,}160\eta^{4}}{27h^{3}} + \frac{1600\eta^{3}}{27h^{2}} \end{bmatrix},$$

$$\beta_4(x) = \begin{bmatrix} -\frac{8\eta^{14}}{27h^{13}} + \frac{2176\eta^{13}}{351h^{12}} - \frac{1559\eta^{12}}{27h^{11}} + \frac{31{,}244\eta^{11}}{99h^{10}} - \frac{101{,}107\eta^{10}}{90h^{9}} + \frac{221{,}290\eta^{9}}{81h^{8}} \\ -\frac{3{,}999{,}539\eta^{8}}{864h^{7}} + \frac{4{,}132{,}745\eta^{7}}{756h^{6}} - \frac{158{,}999\eta^{6}}{36h^{5}} + \frac{70{,}051\eta^{5}}{30h^{4}} - \frac{11{,}745\eta^{4}}{16h^{3}} + \frac{425\eta^{3}}{4h^{2}} \end{bmatrix},$$

$$\beta_5(x) = \begin{bmatrix} -\frac{352\eta^{14}}{3375h^{13}} + \frac{92{,}128\eta^{13}}{43{,}875h^{12}} - \frac{63{,}496\eta^{12}}{3375h^{11}} + \frac{244{,}912\eta^{11}}{2475h^{10}} - \frac{1{,}908{,}574\eta^{10}}{5625h^{9}} + \frac{8{,}060{,}938\eta^{9}}{10{,}125h^{8}} \\ -\frac{4{,}402{,}922\eta^{8}}{3375h^{7}} + \frac{1{,}411{,}868\eta^{7}}{945h^{6}} - \frac{146{,}794\eta^{6}}{125h^{5}} + \frac{1{,}136{,}462\eta^{5}}{1875h^{4}} - \frac{23{,}334\eta^{4}}{125h^{3}} + \frac{664\eta^{3}}{25h^{2}} \end{bmatrix},$$

$$\beta_6(x) = \begin{bmatrix} -\frac{56\eta^{14}}{10{,}125h^{13}} + \frac{14{,}192\eta^{13}}{131{,}625h^{12}} - \frac{1139\eta^{12}}{1215h^{11}} + \frac{9712\eta^{11}}{2025h^{10}} - \frac{180{,}403\eta^{10}}{11{,}250h^{9}} + \frac{1{,}117{,}297\eta^{9}}{30{,}375h^{8}} \\ -\frac{3{,}827{,}857\eta^{8}}{64{,}800h^{7}} + \frac{1{,}884{,}301\eta^{7}}{28{,}350h^{6}} - \frac{12{,}502{,}381\eta^{6}}{243{,}000h^{5}} + \frac{1{,}326{,}917\eta^{5}}{50{,}625h^{4}} - \frac{21{,}559\eta^{4}}{2700h^{3}} + \frac{152\eta^{3}}{135h^{2}} \end{bmatrix},$$

$$\gamma_0(x) = \begin{bmatrix} \frac{8\eta^{14}}{14{,}175h^{12}} - \frac{112\eta^{13}}{8775h^{11}} + \frac{791\eta^{12}}{6075h^{10}} - \frac{392\eta^{11}}{495h^{9}} + \frac{21{,}581\eta^{10}}{6750h^{8}} - \frac{18{,}277\eta^{9}}{2025h^{7}} + \\ \frac{1{,}184{,}153\eta^{8}}{64{,}800h^{6}} - \frac{3613\eta^{7}}{135h^{5}} + \frac{685{,}307\eta^{6}}{24{,}300h^{4}} - \frac{23{,}569\eta^{5}}{1125h^{3}} + \frac{37{,}849\eta^{4}}{3600h^{2}} - \frac{49\eta^{3}}{15h} + \frac{\eta^{2}}{2} \end{bmatrix},$$

$$\gamma_1(x) = \begin{bmatrix} \frac{32\eta^{14}}{1575h^{12}} - \frac{1312\eta^{13}}{2925h^{11}} + \frac{40\eta^{12}}{95h^{10}} - \frac{4304\eta^{11}}{165h^{9}} + \frac{12{,}594\eta^{10}}{125h^{8}} - \frac{60{,}514\eta^{9}}{225h^{7}} + \\ \frac{22{,}796\eta^{8}}{45h^{6}} - \frac{212{,}308\eta^{7}}{315h^{5}} + \frac{46{,}658\eta^{6}}{75h^{4}} - \frac{47618\eta^{5}}{125h^{3}} + \frac{702\eta^{4}}{5h^{2}} - \frac{24\eta^{3}}{h} \end{bmatrix},$$

$$\gamma_2(x) = \left[\begin{array}{c} \frac{8\eta^{14}}{63h^{12}} - \frac{320\eta^{13}}{117h^{11}} + \frac{79\eta^{12}}{3h^{10}} - \frac{1644\eta^{11}}{11h^9} + \frac{16{,}649\eta^{10}}{30h^8} - \frac{38{,}182\eta^9}{27h^7} + \\ \frac{725{,}969\eta^8}{288h^6} - \frac{791{,}491\eta^7}{252h^5} + \frac{21{,}449\eta^6}{8h^4} - \frac{29{,}929\eta^5}{20h^3} + \frac{495\eta^4}{h^2} - \frac{75\eta^3}{h} \end{array} \right],$$

$$\gamma_3(x) = \left[\begin{array}{c} \frac{128\eta^{14}}{567h^{12}} - \frac{128\eta^{13}}{27h^{11}} + \frac{10{,}784\eta^{12}}{243h^{10}} - \frac{6592\eta^{11}}{27h^9} + \frac{118{,}264\eta^{10}}{135h^8} - \frac{19{,}352\eta^9}{9h^7} + \\ \frac{298{,}168\eta^8}{81h^6} - \frac{830{,}608\eta^7}{189h^5} + \frac{872{,}360\eta^6}{243h^4} - \frac{258{,}952\eta^5}{135h^3} + \frac{5480\eta^4}{9h^2} - \frac{800\eta^3}{9h} \end{array} \right],$$

$$\gamma_4(x) = \left[\begin{array}{c} \frac{8\eta^{14}}{63h^{12}} - \frac{304\eta^{13}}{117h^{11}} + \frac{71\eta^{12}}{3h^{10}} - \frac{1392\eta^{11}}{11h^9} + \frac{13{,}229\eta^{10}}{30h^8} - \frac{28{,}373\eta^9}{27h^7} + \\ \frac{503{,}201\eta^8}{288h^6} - \frac{255{,}529\eta^7}{126h^5} + \frac{19{,}361\eta^6}{12h^4} - \frac{4208\eta^5}{5h^3} + \frac{4185\eta^4}{16h^2} - \frac{75\eta^3}{2h} \end{array} \right],$$

$$\gamma_5(x) = \left[\begin{array}{c} \frac{32\eta^{14}}{1575h^{12}} - \frac{1184\eta^{13}}{2925h^{11}} + \frac{808\eta^{12}}{225h^{10}} - \frac{3088\eta^{11}}{165h^9} + \frac{7954\eta^{10}}{125h^8} - \frac{33{,}338\eta^9}{225h^7} + \\ \frac{54{,}256\eta^8}{225h^6} - \frac{86{,}468\eta^7}{315h^5} + \frac{5366\eta^6}{25h^4} - \frac{13{,}786\eta^5}{125h^3} + \frac{846\eta^4}{25h^2} - \frac{24\eta^3}{5h} \end{array} \right],$$

$$\gamma_6(x) = \left[\begin{array}{c} \frac{8\eta^{14}}{14{,}175h^{12}} - \frac{32\eta^{13}}{2925h^{11}} + \frac{23\eta^{12}}{243h^{10}} - \frac{716\eta^{11}}{1485h^9} + \frac{10{,}841\eta^{10}}{6750h^8} - \frac{7436\eta^9}{2025h^7} + \\ \frac{76{,}213\eta^8}{12{,}960h^6} - \frac{8317\eta^7}{1260h^5} + \frac{247{,}819\eta^6}{48{,}600h^4} - \frac{35{,}009\eta^5}{13{,}500h^3} + \frac{71\eta^4}{90h^2} - \frac{\eta^3}{9h} \end{array} \right],$$

Evaluating the continuous scheme in Equation (20) as usual at x_{n+u}, $x_{n+1}x_{n+v}$, x_{n+2}, x_{n+w}, and x_{n+3}, we obtain the method

$$\begin{array}{l} y_{n+u} = y_n + \frac{h}{1{,}245{,}404{,}160{,}000}\left[\begin{array}{c} 199,368,819,177 f_n - 68,951,829,552 f_{n+u} - 380,416,470,375 f_{n+1} + 300,642,304,000 f_{n+v} \\ +457,138,998,375 f_{n+2} + 110,327,270,448 f_{n+w} + 4,592,987,927 f_{n+3} \end{array} \right] \\ \frac{h^2}{249{,}080{,}832{,}000}\left[\begin{array}{c} 1,784,098,013 g_n - 33,488,665,488 g_{n+u} - 71,514,207,675 g_{n+1} - 77,935,000,000 g_{n+v} \\ -3,164,886,075 g_{n+2} - 3,963,034,512 g_{n+w} - 90,441,763 g_{n+3} \end{array} \right] \end{array} \quad (21)$$

$$\begin{array}{l} y_{n+1} = y_n + \frac{h}{4{,}864{,}860{,}000}\left[\begin{array}{c} 783,720,817 f_n + 706,775,424 f_{n+u} - 457,058,625 f_{n+1} + 1,387,808,000 f_{n+v} \\ +1,957,353,375 f_{n+2} + 466,919,808 f_{n+w} + 19,341,201 f_{n+3} \end{array} \right] \\ \frac{h^2}{972{,}972{,}000}\left[\begin{array}{c} 7,057,013 g_n - 117,681,984 g_{n+u} - 337,970,925 g_{n+1} - 336,793,600 g_{n+v} \\ -134,615,475 g_{n+2} - 16,742,592 g_{n+w} - 380,629 g_{n+3} \end{array} \right] \end{array}$$

$$\begin{array}{l} y_{n+v} = y_n + \frac{h}{5{,}125{,}120{,}000}\left[\begin{array}{c} 826,473,395 f_n + 775,497,456 f_{n+u} + 688,759,875 f_{n+1} + 2,699,264,000 f_{n+v} \\ +2,168,488,125 f_{n+2} + 508,254,480 f_{n+w} + 20,942,669 f_{n+3} \end{array} \right] \\ \frac{h^2}{1{,}025{,}024{,}000}\left[\begin{array}{c} 7,450,095 g_n - 123,030,576 g_{n+u} - 333,689,625 g_{n+1} - 390,561,600 g_{n+v} \\ -147,584,025 g_{n+2} - 18,189,360 g_{n+w} - 411,921 g_{n+3} \end{array} \right] \end{array}$$

$$\begin{array}{l} y_{n+2} = y_n + \frac{h}{152{,}026{,}875}\left[\begin{array}{c} 24,532,563 f_n + 23,488,800 f_{n+u} + 23,587,500 f_{n+1} + 116,768,000 f_{n+v} \\ +99,037,875 f_{n+2} + 15,993,312 f_{n+w} + 645,700 f_{n+3} \end{array} \right] \\ \frac{h^2}{30{,}405{,}375}\left[\begin{array}{c} 221,317 g_n - 3,633,120 g_{n+u} - 9,727,200 g_{n+1} - 10,524,800 g_{n+v} \\ -5,041,125 g_{n+2} - 567,648 g_{n+w} - 12,680 g_{n+3} \end{array} \right] \end{array}$$

$$\begin{array}{l} y_{n+w} = y_n + \frac{h}{1{,}992{,}646{,}656}\left[\begin{array}{c} 322,126,585 f_n + 322,599,120 f_{n+u} + 379,475,625 f_{n+1} + 1,617,920,000 f_{n+v} \\ +1,719,564,375 f_{n+2} + 609,445,680 f_{n+w} + 10,485,255 f_{n+3} \end{array} \right] \\ \frac{h^2}{1{,}992{,}646{,}656}\left[\begin{array}{c} 14,560,225 g_n - 235,515,600 g_{n+u} - 614,964,375 g_{n+1} - 623,480,000 g_{n+v} \\ -210,324,375 g_{n+2} - 64,098,000 g_{n+w} - 1,010,975 g_{n+3} \end{array} \right] \end{array}$$

$$\begin{array}{l} y_{n+3} = y_n + \frac{h}{20{,}020{,}000}\left[\begin{array}{c} 3,310,219 f_n + 5,014,656 f_{n+u} + 11,161,125 f_{n+1} + 21,088,000 f_{n+v} \\ +11,161,125 f_{n+2} + 5,014,656 f_{n+w} + 3,310,219 f_{n+3} \end{array} \right] \\ \frac{h^2}{4{,}004{,}000}\left[\begin{array}{c} 30,711 g_n - 409,536 g_{n+u} - 726,975 g_{n+1} + 726,975 g_{n+2} \\ +409,536 g_{n+w} - 30,711 g_{n+3} \end{array} \right] \end{array}$$

The order and error constants for the constructed block hybrid methods are presented in Table 1. It is clear from the table that the members of the block hybrid method without a second-derivative evaluation are all of order seven except the last member in the block, which has an order higher than the remaining members in the block (order eight). The members of the block hybrid method with a second derivative are of uniform accuracy of order 14 with smaller error constants and, hence, are more accurate than those without a second derivative.

Table 1. Order and error constants for the block hybrid methods.

Method	Order	Error Constants
Block method (18)	(i) $y_n + u$, $P = 7$	$C8 = 4.4403 \times 10^{-5}$
	(ii) $y_n + 1$, $P = 7$	$C8 = 3.3068 \times 10^{-5}$
	(iii) $y_n + v$, $P = 7$	$C8 = 3.9236 \times 10^{-5}$
	(iv) $y_n + 2$, $P = 7$	$C8 = 3.3068 \times 10^{-5}$
	(v) $y_n + w$, $P = 7$	$C8 = 4.4403 \times 10^{-5}$
	(vi) $y_n + 3$, $P = 8$	$C9 = 1.2555 \times 10^{-5}$
Uniform order block method (21)	(i) $y_n + u$, $P = 14$	$C15 = 1.4789 \times 10^{-12}$
	(ii) $y_n + 1$, $P = 14$	$C15 = 1.5718 \times 10^{-12}$
	(iii) $y_n + v$, $P = 14$	$C15 = 1.5989 \times 10^{-12}$
	(iv) $y_n + 2$, $P = 14$	$C15 = 1.6261 \times 10^{-12}$
	(v) $y_n + w$, $P = 14$	$C15 = 1.7190 \times 10^{-12}$
	(vi) $y_n + 3$, $P = 14$	$C15 = 3.1979 \times 10^{-12}$

4. Regions of Absolute Stability (RAS) of the Block Hybrid Methods

Generally, in designing a new numerical method, it is very important to consider the stability properties of the method. Therefore, in this paper, we reformulate the block hybrid methods, as in [31,32], by the partitioning $(s+r) \times (s+r)$ of the form

$$\begin{bmatrix} Y^{[n]} \\ y^{[n-1]} \end{bmatrix} = \begin{bmatrix} \frac{A|\,U}{B\,|\,V} \end{bmatrix} \begin{bmatrix} hf\left(Y^{[n]}\right) \\ y^{[n]} \end{bmatrix}, \; n = 1,\, 2,\, \ldots,\, \mathrm{N}, \tag{22a}$$

where

$$Y^{[n]} = \begin{bmatrix} Y_1^{[n]} \\ Y_2^{[n]} \\ \vdots \\ Y_s^{[n]} \end{bmatrix},\; y^{[n-1]} = \begin{bmatrix} y_1^{[n-1]} \\ y_2^{[n-1]} \\ \vdots \\ y_r^{[n-1]} \end{bmatrix},\; f\left(Y^{[n]}\right) = \begin{bmatrix} f(Y_1^{[n]}) \\ f(Y_2^{[n]}) \\ \vdots \\ f(Y_s^{[n]}) \end{bmatrix},\; y^{[n]} = \begin{bmatrix} y_1^{[n]} \\ y_2^{[n]} \\ \vdots \\ y_r^{[n]} \end{bmatrix},$$

$$A = \begin{bmatrix} 0 & 0 \\ A & B \end{bmatrix},\; U = \begin{bmatrix} & I & 0 & 0 \\ 0 & \mu & e-\mu \end{bmatrix},\; B = \begin{bmatrix} A & B \\ 0 & 0 \\ \nu^T & \omega^T \end{bmatrix},\; V = \begin{bmatrix} I & \mu & e-\mu \\ 0 & 0 & I \\ 0 & 0 & I-\theta \end{bmatrix},$$

and $e = [1, \cdots, 1]^T \in R^m$.

Thus, Equation (22a) is

$$\begin{bmatrix} Y_1^{[n]} \\ Y_2^{[n]} \\ \vdots \\ Y_s^{[n]} \\ - \\ y_1^{[n]} \\ \vdots \\ y_r^{[n]} \end{bmatrix} = \begin{bmatrix} \frac{A \mid U}{B \mid V} \end{bmatrix} \begin{bmatrix} hf(Y_1^{[n]}) \\ hf(Y_2^{[n]}) \\ \vdots \\ hf(Y_s^{[n]}) \\ - \\ y_1^{[n-1]} \\ \vdots \\ y_r^{[n-1]} \end{bmatrix}. \tag{22b}$$

The values r and s denote output and stage values, respectively. Applying Equation (22) to the linear test equation $y' = \lambda y, x \geq 0$ and $\lambda \in C$, we have $M(z)$ as

$$M(z) = V + zB(1 - zA)^{-1}U \tag{23}$$

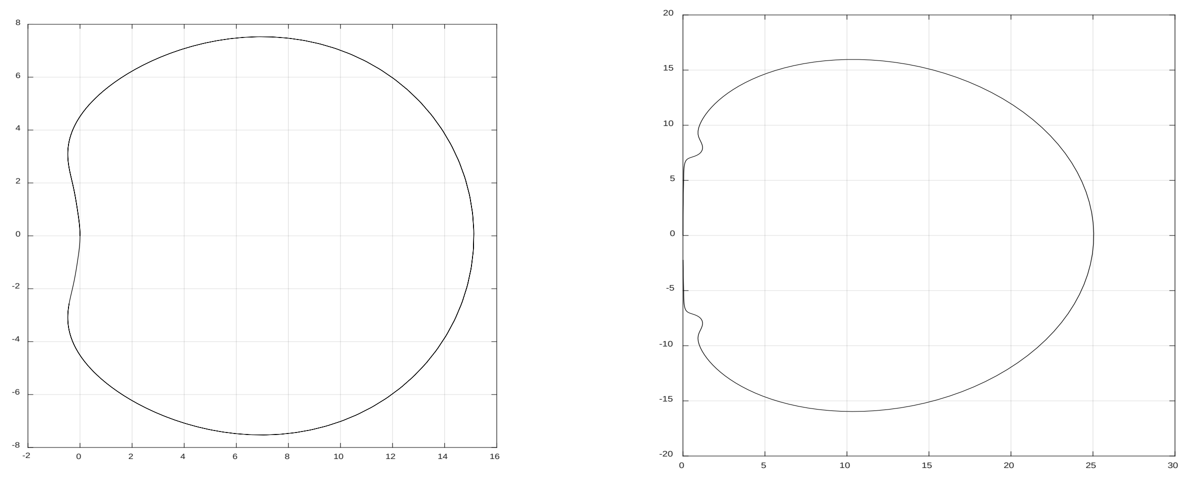
and the stability polynomial $\rho(\eta, z)$ of the method can easily be obtained as

$$\rho(\eta, z) = det(\eta I - M(z)). \tag{24}$$

The region of absolute stability $\Re$ of the method is defined as

$$\Re = x \in C : \rho(\eta, z) = 1 \Rightarrow |\eta| \leq 1.$$

Computing the stability function gives the stability polynomial of the method, which is plotted to produce the required graph of the region of absolute stability of the method, as shown in Figure 2.



(a) **(b)**

Figure 2. Regions of absolute stability of the *block hybrid methods.* (**a**) Method (18) is *A(α)-stable*. (**b**) Method (21) is *A*-stable.

Remark 1. *In the stable block hybrid second-derivative implicit method, we added the matrix D1 obtained from the coefficients of h^2 to the matrices A, C, B, and D, which enabled us to plot the region of absolute stability of the new method. The region of absolute stability of method (18) is A(α)-stable while the region of absolute stability of the second-derivative implicit method (21) is A-stable since the region contains the complex plane outside the enclosed figure.*

5. Numerical Illustrations

For the illustration of the performance of the derived methods, we consider both linear and nonlinear challenging systems. To provide a direct comparison, Matlab software codes were written for the preliminary test experiments using a fixed step length. We present the calculated results in tables and depict the curves in figures. Here, *nfe* and *Ext* are the function evaluations and exact values, respectively.

Example 1. *The Kaps problem [30].*

We consider the nonlinear ***Kaps*** *stiff system,*

$$\begin{bmatrix} y'_1(x) \\ y'_2(x) \end{bmatrix} = \begin{bmatrix} -1002\, y_1(x) + 1000\, y_2^2(x) \\ y_1(x) - y_2(x)(1 + y_2(x)) \end{bmatrix}, \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

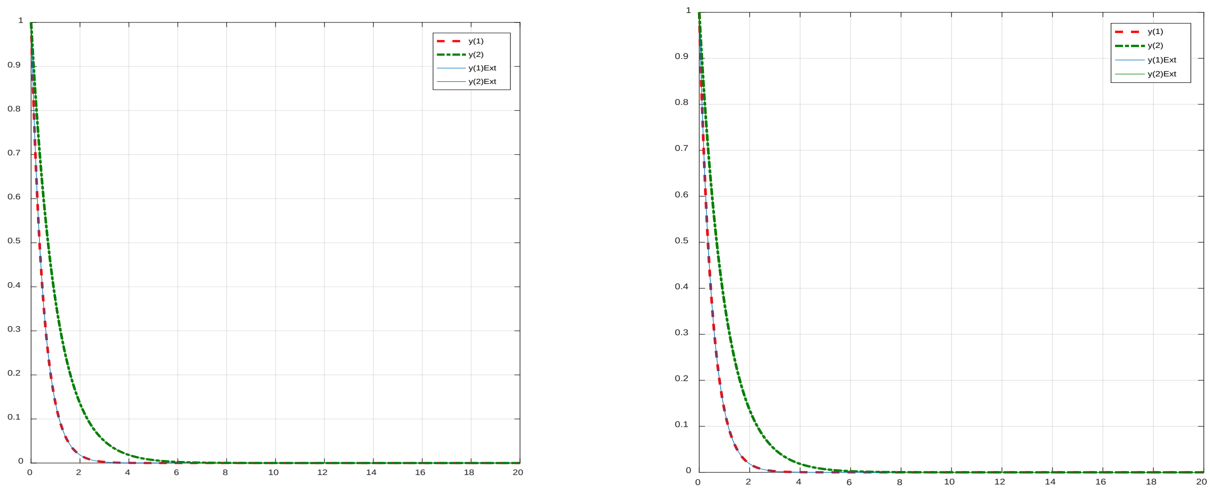
The exact value of the system is

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} \exp(-2x) \\ \exp(-x) \end{bmatrix}.$$

The solutions of this example are shown in Table 2 and the solution curves are depicted in Figure 3.

Table 2. Absolute errors in the numerical integration of Example 1.

x	y_i	Method (18)	Method (21)
5	y_1	$1.223052805026881 \times 10^{-3}$	$1.228938367083599 \times 10^{-3}$
	y_2	$1.290570363021715 \times 10^{-6}$	$1.800318343625484 \times 10^{-6}$
50	y_1	$3.320709446422848 \times 10^{-5}$	$3.325679258575631 \times 10^{-5}$
	y_2	$9.887815172193726 \times 10^{-8}$	$5.804723043345561 \times 10^{-7}$
250	y_1	$3.619658989642897 \times 10^{-12}$	$3.622719245691676 \times 10^{-12}$
	y_2	$2.523305960607913 \times 10^{-11}$	$2.101212666995355 \times 10^{-10}$
500	y_1	$7.167561881971770 \times 10^{-21}$	$7.173620185942641 \times 10^{-21}$
	y_2	$1.122741130992365 \times 10^{-15}$	$9.350493168888896 \times 10^{-15}$



(**a**) (**b**)

Figure 3. Graphical plots of Example 1 using block hybrid methods with *nfe* = 500. (**a**) Solution curve of Example 1 using (18); (**b**) solution curve of Example 1 using (21).

Example 2. *Consider the linear stiff system.*

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \\ y_3'(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 25 & 1 & -25 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}, \begin{bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

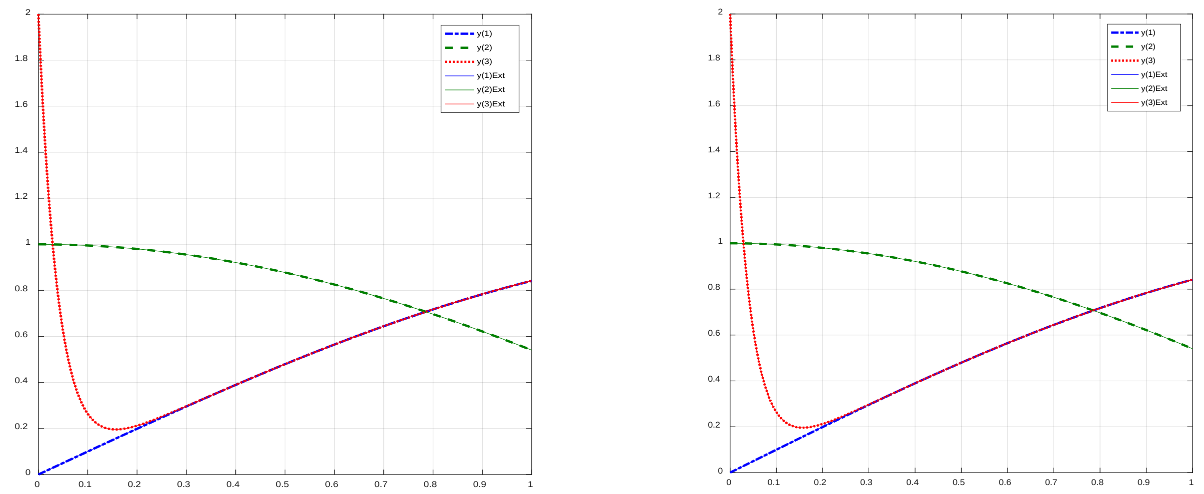
The exact value is

$$\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix} = \begin{bmatrix} sin(x) \\ cos(x) \\ sin(x) + 2\, exp(Lx) \end{bmatrix}.$$

The results of using the newly constructed methods are shown in Table 3 and the solution curves are in Figure 4.

Table 3. Absolute errors in the numerical integration of Example 2.

x	y_i	Method (18)	Method (21)
5	y_1	0	0
	y_2	$1.110223024625157 \times 10^{-16}$	$1.110223024625157 \times 10^{-16}$
	y_3	$7.993605777301127 \times 10^{-15}$	0
50	y_1	$5.551115123125783 \times 10^{-17}$	$4.163336342344337 \times 10^{-17}$
	y_2	$3.330669073875470 \times 10^{-16}$	$3.330669073875470 \times 10^{-16}$
	y_3	$1.038058528024521 \times 10^{-14}$	0
250	y_1	$2.220446049250313 \times 10^{-16}$	$1.110223024625157 \times 10^{-16}$
	y_2	$1.110223024625157 \times 10^{-16}$	$1.110223024625157 \times 10^{-16}$
	y_3	$3.330669073875470 \times 10^{-16}$	$1.665334536937735 \times 10^{-16}$
500	y_1	$4.440892098500626 \times 10^{-16}$	$3.330669073875470 \times 10^{-16}$
	y_2	$2.220446049250313 \times 10^{-16}$	$1.110223024625157 \times 10^{-16}$
	y_3	$3.330669073875470 \times 10^{-16}$	$2.220446049250313 \times 10^{-16}$



(a) **(b)**

Figure 4. Graphical plots of Example 2 using block hybrid methods with *nfe* = 500. (**a**) Solution curve of Example 2 using (18); (**b**) solution curve of Example 2 using (21).

Example 3. *The linear problem by Enright [33] is given by:*

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \\ y_3'(x) \\ y_4'(x) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -10 & 0 & 0 \\ -1 & 0 & -100 & 0 \\ -1 & 0 & 0 & -1000 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}, \begin{bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \\ y_4(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

The results of the integration are largely self-explanatory. If we examine the accuracy obtained, however, we see that the newly constructed methods are considerably accurate (see Table 4). The plotted curves are displayed in Figure 5.

Table 4. Absolute errors in the numerical integration of Example 3.

x	y_i	Method (18)	Method (21)
5	y_1	0	0
	y_2	0	0
	y_3	0	0
	y_4	$1.110223024625157 \times 10^{-16}$	$1.110223024625157 \times 10^{-16}$
50	y_1	$2.220446049250313 \times 10^{-16}$	$2.220446049250313 \times 10^{-16}$
	y_2	$4.440892098500626 \times 10^{-16}$	$4.440892098500626 \times 10^{-16}$
	y_3	$1.110223024625157 \times 10^{-16}$	$1.110223024625157 \times 10^{-16}$
	y_4	$2.220446049250313 \times 10^{-16}$	$1.110223024625157 \times 10^{-16}$
250	y_1	$5.551115123125783 \times 10^{-16}$	$5.551115123125783 \times 10^{-16}$
	y_2	$7.771561172376096 \times 10^{-16}$	$7.771561172376096 \times 10^{-16}$
	y_3	$7.771561172376096 \times 10^{-16}$	$1.110223024625157 \times 10^{-16}$
	y_4	$5.204170427930421 \times 10^{-18}$	$5.204170427930421 \times 10^{-18}$
500	y_1	$2.220446049250313 \times 10^{-16}$	$2.220446049250313 \times 10^{-16}$
	y_2	$5.551115123125783 \times 10^{-16}$	$5.551115123125783 \times 10^{-16}$
	y_3	$3.885780586188048 \times 10^{-16}$	$2.775557561562891 \times 10^{-16}$
	y_4	$1.626303258728257 \times 10^{-19}$	$3.388131789017201 \times 10^{-20}$

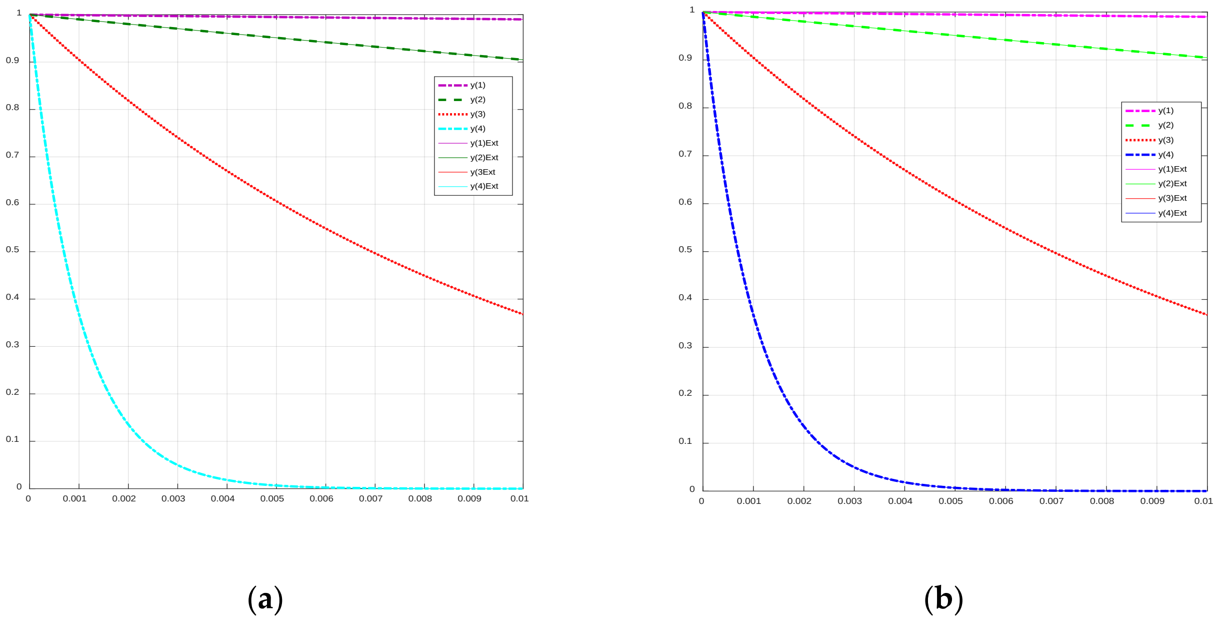
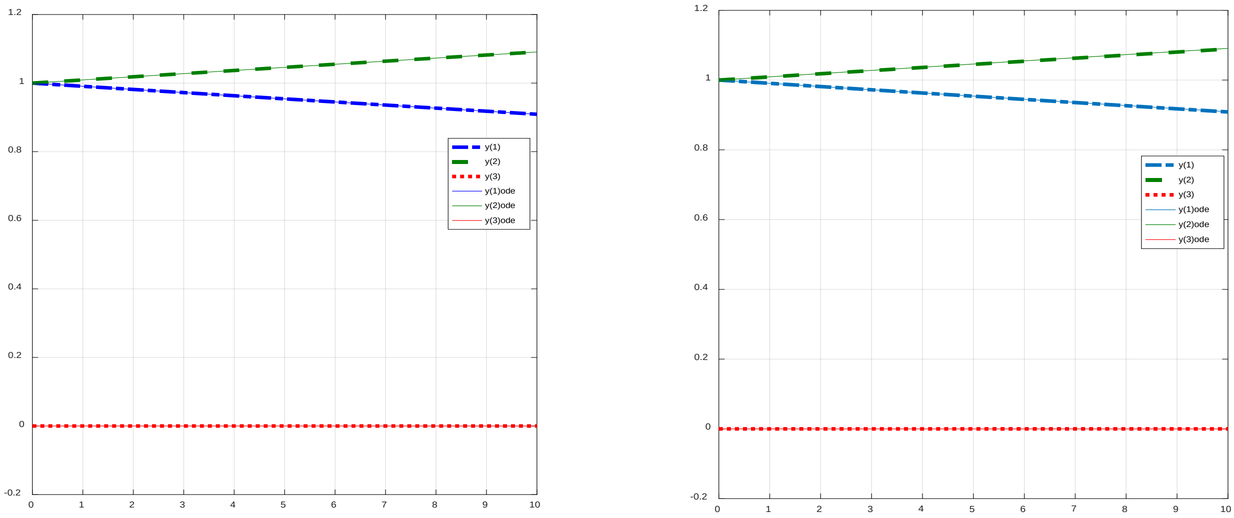


Figure 5. Graphical plots of Example 3 using block hybrid methods with *nfe* = 500. (**a**) Solution curve of Example 3 using (18); (**b**) solution curve of Example 3 using (21).

Example 4. *This is given by Gear [34]:*

$$
\begin{aligned}
&y'_1 == -0.013\, y_1 - 1000\, y_1 y_3, && y_1(0) = 1,\\
&y'_2 = -2500\, y_2 y_3, && y_2(0) = 1,\\
&y'_3 = -0.013\, y_1 - 1000\, y_1 y_3 - 2500\, y_2 y_3, && y_3(0) = 0\,.
\end{aligned}
$$

We solve this problem, and the solution curves are presented in Figure 6.



(a) **(b)**

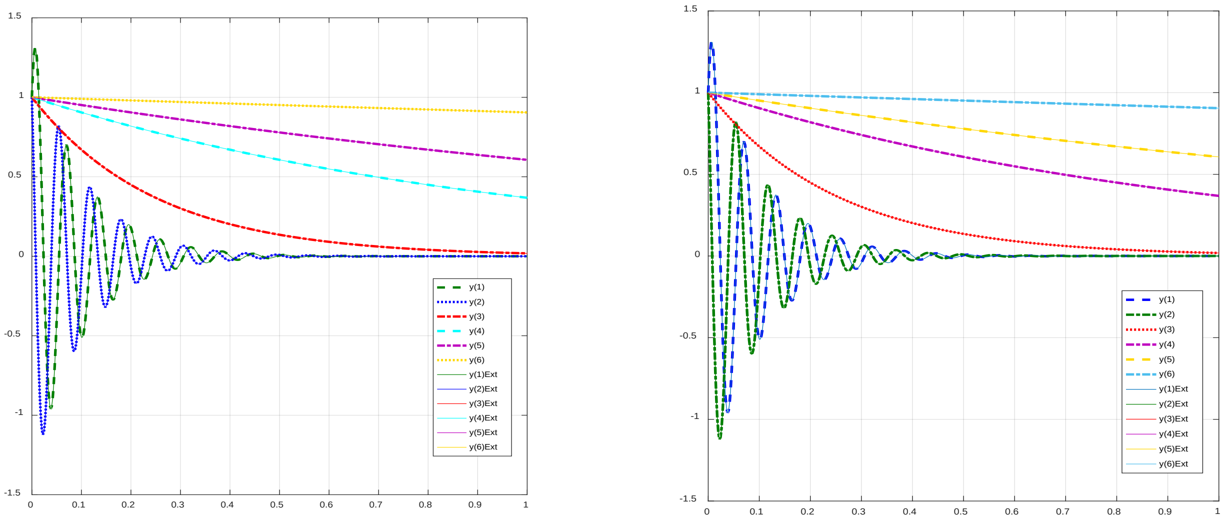
Figure 6. Graphical plots of Example 4 using the block hybrid methods with *nfe* = 500. (**a**) Solution curve of Example 4 using (18); (**b**) solution curve of Example 4 using (21).

Example 5. *Here, the present problem was solved by [35]. Therefore, for comparison, we present the graphical plots of this example in Figure 7, comparing with the exact solution curves. The application of the newly derived methods to this problem is to demonstrate their performance. However, we considered only the first four components* $\{y_1, y_2, y_3, y_4\}$*, as shown in Table 5.*

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \\ y_3'(x) \\ y_4'(x) \\ y_5'(x) \\ y_6'(x) \end{bmatrix} = \begin{bmatrix} -10 & 100 & 0 & 0 & 0 & 0 \\ -100 & -10 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.1 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \\ y_5(x) \\ y_6(x) \end{bmatrix}, \begin{bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \\ y_4(0) \\ y_5(0) \\ y_6(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Table 5. Absolute errors of numerical integration of Example 5.

x	$\mathbf{y_i}$	Method (18)	Method (21)
5	y_1	$2.024105327791403 \times 10^{-10}$	$2.220446049250313 \times 10^{-16}$
	y_2	$4.056337835067758 \times 10^{-10}$	$1.318389841742373 \times 10^{-16}$
	y_3	0	0
	y_4	0	0
50	y_1	$1.721994824510631 \times 10^{-9}$	$3.330669073875470 \times 10^{-16}$
	y_2	$1.453979242560521 \times 10^{-9}$	$7.771561172376096 \times 10^{-16}$
	y_3	$4.440892098500626 \times 10^{-16}$	$4.440892098500626 \times 10^{-16}$
	y_4	0	$1.110223024625157 \times 10^{-16}$
250	y_1	$2.077217382476237 \times 10^{-10}$	$6.591949208711867 \times 10^{-17}$
	y_2	$1.233960850166582 \times 10^{-11}$	$1.734723475976807 \times 10^{-18}$
	y_3	$2.775557561562891 \times 10^{-17}$	$8.326672684688674 \times 10^{-17}$
	y_4	$6.661338147750939 \times 10^{-16}$	$6.661338147750939 \times 10^{-16}$
500	y_1	$2.711908290569135 \times 10^{-12}$	$6.810144895924575 \times 10^{-19}$
	y_2	$6.195955749334348 \times 10^{-13}$	$2.710505431213761 \times 10^{-20}$
	y_3	$6.938893903907228 \times 10^{-18}$	$4.857225732735060 \times 10^{-17}$
	y_4	$3.885780586188048 \times 10^{-16}$	$3.330669073875470 \times 10^{-16}$



(**a**) (**b**)

Figure 7. Graphical plots of Example 5 using block hybrid methods with *nfe* = 500. (**a**) Solution curve of Example 5 using method (18); (**b**) solution curve of Example 5 using method (21).

6. Concluding Remarks

The presented second-derivative block hybrid method for a stiff system of ordinary differential equations is suitable for large systems. The second-derivative block hybrid time integrator provides good performance. Numerical results for the new second-derivative block hybrid method are promising and are demonstrably comparable to those obtained from popular high-order stiff time integrators found in the literature. Their stability properties, based on Remark 1, indicate that they are good candidates for large stiff systems. The next step of our research is to further apply some new methods to modeled differential equations that arise in other areas of scientific fields, such as chemical reaction, enzyme kinetics, cardiac electrophysiology, models of drug magnetic nanoparticle transport, and a model of tumor immune interaction, to mention just a few.

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