Article

# On the Lower and Upper Box Dimensions of the Sum of Two Fractal Functions 

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#### Abstract

Let $f$ and $g$ be two continuous functions. In the present paper, we put forward a method to calculate the lower and upper Box dimensions of the graph of $f+g$ by classifying all the subsequences tending to zero into different sets. Using this method, we explore the lower and upper Box dimensions of the graph of $f+g$ when the Box dimension of the graph of $g$ is between the lower and upper Box dimensions of the graph of $f$. In this case, we prove that the upper Box dimension of the graph of $f+g$ is just equal to the upper Box dimension of the graph of $f$. We also prove that the lower Box dimension of the graph of $f+g$ could be an arbitrary number belonging to a certain interval. In addition, some other cases when the Box dimension of the graph of $g$ is equal to the lower or upper Box dimensions of the graph of $f$ have also been studied.


Keywords: fractal dimension; lower Box dimension; upper Box dimension; sum of two continuous functions

## 1. Introduction

Let $I=[0,1]$ and $C_{I}$ be the set of all continuous functions on $I$. We know $C_{I}$ is a metric space consisting of differentiable functions and continuous functions that are not differentiable at certain points in $I$. It is well known that the Weierstrass function is an example of continuous functions differentiable nowhere on $I$ [1], which are usually called fractal functions, whose graphs have certain uncommon properties. Write

$$
\begin{equation*}
\Gamma(f, I)=\{(x, f(x)): x \in I\} \tag{1}
\end{equation*}
$$

as the graph of the function $f(x)$ on $I$. For a fractal function $f(x)$ on $I$, its most remarkable feature is that $\Gamma(f, I)$ has fractal dimensions larger than the topological dimension. Therefore, the studies of fractal dimensions of different types of fractal functions have drawn the attention of numerous researchers. In [2-4], self-affine curves and the corresponding fractal interpolation functions have been investigated. Barnsley and Ruan have made research on the linear fractal interpolation functions in [5,6], respectively. Moreover, there exist certain particular examples of one-dimensional fractal functions discussed in [7-14] and two-dimensional fractal functions constructed in [15,16]. For Hölder continuous functions, ref. $[17,18]$ estimated the Box dimension of their fractional integral.

As is commonly known, the Weierstrass function $[3,4,19,20$ ] and the Besicovitch function $[4,21,22]$ are two typical examples of fractal functions with different fractal dimensions. Here, we present their definitions as follows:

Example 1 ([3,4,19,20]). The Weierstrass function
Let $0<\alpha<1, \lambda>4$. The Weierstrass function is defined as

$$
W(x)=\sum_{j=1}^{\infty} \lambda^{-\alpha j} \sin \left(\lambda^{j} x\right)
$$

Example 2 ([4,21,22]). The Besicovitch function
Let $1<s<2, \lambda_{j} \nearrow \infty$. The Besicovitch function is defined as

$$
B(x)=\sum_{j \geq 1} \lambda_{j}^{s-2} \cos \left(\lambda_{j} x\right)
$$

Up to now, the Box dimension of the Weierstrass function has been calculated to be equal to $2-\alpha$ [3], although its Hausdorff dimension has not been investigated thoroughly [23-25]. In fact, Shen [26] proved that its Hausdorff dimension is equal to its Box dimension for integer $\lambda$ in Example 1, which can be regarded as a significant advance in estimating Hausdorff dimension of specific functions. In addition, ref. [4] says that the Box dimension of the Besicovitch function may not exist for suitably chosen $\left\{\lambda_{j}\right\}_{j=1}^{\infty}$ in Example 2, which can be an example of fractal functions that do not always have a Box dimension. However, we know a fractal function must have a lower Box dimension and an upper Box dimension, even if its Box dimension does not exist [3]. Now, we first give the definitions of lower Box dimension, upper Box dimension and Box dimension as below.

Definition 1 ([3]). Let $F(\neq \varnothing)$ be any bounded subset of $\mathbb{R}^{2}$ and $N_{\delta}(F)$ be the smallest number of sets of diameter at most $\delta$, which can cover $F$. Lower Box dimension and upper Box dimension of $F$ are defined as, respectively,

$$
\begin{equation*}
\underline{\operatorname{dim}}_{B}(F)=\varliminf_{\delta \rightarrow 0} \frac{\log N_{\delta}(F)}{-\log \delta} \tag{2}
\end{equation*}
$$

and

$$
\begin{equation*}
\overline{\operatorname{dim}}_{B}(F)=\varlimsup_{\delta \rightarrow 0} \frac{\log N_{\delta}(F)}{-\log \delta} . \tag{3}
\end{equation*}
$$

If (2) and (3) are equal, we refer to the common value as the Box dimension of $F$

$$
\operatorname{dim}_{B}(F)=\lim _{\delta \rightarrow 0} \frac{\log N_{\delta}(F)}{-\log \delta}
$$

From (1) and Definition 1, we can write the lower Box dimension, upper Box dimension and Box dimension of the graph of the function $f(x)$ on $I$ as

$$
\underline{\operatorname{dim}}_{B} \Gamma(f, I), \quad \overline{\operatorname{dim}}_{B} \Gamma(f, I) \quad \text { and } \quad \operatorname{dim}_{B} \Gamma(f, I)
$$

respectively. Then, it holds

$$
1<\operatorname{dim}_{B} \Gamma(W, I)=2-\alpha<2
$$

in Example 1 and

$$
\overline{\operatorname{dim}}_{B} \Gamma(B, I) \neq \underline{\operatorname{dim}}_{B} \Gamma(B, I)
$$

for suitably chosen $\left\{\lambda_{j}\right\}_{j=1}^{\infty}$ in Example 2. Hence, the Box dimension of $\Gamma(W, I)$ always exists, but the Box dimension of $\Gamma(B, I)$ does not always exist. Then, a question is naturally asked:

Question 1. If we choose $W(x)$ and $B(x)$ satisfying

$$
\underline{\operatorname{dim}}_{B} \Gamma(B, I)<\operatorname{dim}_{B} \Gamma(W, I)<\overline{\operatorname{dim}}_{B} \Gamma(B, I)
$$

does the Box dimension of $\Gamma(W+B, I)$ still exist? If the Box dimension of $\Gamma(W+B, I)$ does not exist, what can the lower and upper Box dimensions of $\Gamma(W+B, I)$ be, respectively?

It is essentially a problem of estimating fractal dimensions of the sum of two continuous functions. Actually, perhaps the first attempt to investigate fractal dimensions of the sum of two continuous functions was made by Wen [4]. On a fractal conference, Wen [4] said the possible value of the Box dimension of the sum of two continuous functions under
known Box dimensions of these two functions is an interesting and sophisticated problem. Until now, some research achievements of this problem in certain circumstances have been obtained. If these two functions have different Box dimensions, they can be found in [3]. Wang and Zhang [27] made research on the case when these two functions have the same Box dimension. Moreover, ref. [4] shows us the following conclusion when the lower Box dimension of one function is larger than the upper Box dimension of the other one:

Proposition 1 ([4]). Let $f(x), g(x) \in C_{I}$. We have

$$
\overline{\operatorname{dim}}_{B} \Gamma(f+g, I)=\overline{\operatorname{dim}}_{B} \Gamma(f, I)
$$

when

$$
\begin{equation*}
\overline{\operatorname{dim}}_{B} \Gamma(g, I)<\underline{\operatorname{dim}}_{B} \Gamma(f, I) . \tag{4}
\end{equation*}
$$

For $f(x), g(x) \in C_{I}$, Proposition 1 gives a result for calculating $\operatorname{dim}_{B} \Gamma(f+g, I)$. However, under the condition of (4), the estimation of $\operatorname{dim}_{B} \Gamma(f+g, I)$ has not been solved yet. Furthermore, if $\overline{\operatorname{dim}}_{B} \Gamma(g, I) \geq \underline{\operatorname{dim}}_{B} \Gamma(f, I)$, both $\overline{\operatorname{dim}}_{B} \Gamma(f+g, I)$ and $\underline{\operatorname{dim}}_{B} \Gamma(f+g, I)$ are unknown. All the above problems will be further explored in the present paper.

For the convenience of discussion, we first introduce the definitions of fractal function sets as follows:

Definition 2. Fractal functions sets.
(1) Let ${ }^{s} D_{I}$ be the set of all continuous functions whose Box dimensions exist and are equal to $s$ on I when $1 \leq s \leq 2$. That is, ${ }^{s} D_{I}$ is the set of $s$-dimensional continuous functions on I.
(2) Let ${ }_{s_{1}}^{s_{2}} D_{I}$ be the set of all continuous functions whose Box dimensions do not exist on I. Here, $s_{1}, s_{2}$ are, respectively, the lower and upper Box dimensions of the function on I as $1 \leq s_{1}<s_{2} \leq 2$.

Remark 1 (Remarks to Definition 2). Here, we give several examples belonging to fractal functions sets defined in Definition 2:
(1) The Weierstrass function $W(x) \in{ }^{2-\alpha} D_{I}$. The functions constructed in [15,16] belong to ${ }^{2} D_{I}$. In fact, for $\forall s \in[1,2],{ }^{s} D_{I}$ is non-empty.
(2) The Besicovitch function $B(x) \in{ }_{s_{1}}^{s_{2}} D_{I}$ if we choose a suitable sequence $\left\{\lambda_{j}\right\}_{j=1}^{\infty}$ [4]. In fact, for $\forall s_{1}, s_{2}$ satisfying $1 \leq s_{1}<s_{2} \leq 2,{ }_{s_{1}}^{s_{2}} D_{I}$ is non-empty, as well.

Suppose that $f(x) \in{ }_{s_{1}}^{s_{1}} D_{I}$ and $g(x) \in{ }^{s} D_{I}$. In this study, we mainly consider the problem of estimating the lower and upper Box dimensions of $\Gamma(f+g, I)$ when $s_{1}<s<s_{2}$. The rest of this paper is organized as follows:

In Section 2, we acquire a general method to calculate $\operatorname{dim}_{B} \Gamma(f+g, I)$ and $\overline{\operatorname{dim}}_{B} \Gamma(f+g, I)$ and give several basic results. For the above problem, we prove that $\overline{\operatorname{dim}}_{B} \Gamma(f+g, I)$ is equal to $s_{2}$. Additionally, an upper bound estimation of $\underline{\operatorname{dim}}_{B} \Gamma(f+g, I)$ has also been obtained which is $\underline{\operatorname{dim}}_{B} \Gamma(f+g, I) \leq s$. Then, we present some conclusions of fractal dimensions of the sum of two continuous functions when both of them have Box dimensions.

In Section 3, we investigate the calculation of $\underline{\operatorname{dim}}_{B} \Gamma(f+g, I)$ for the above problem by discussing whether $s$ is one of the accumulation points of $\Phi_{f}(\delta)$ (defined in Section 2.2) when $\delta \rightarrow 0$ or not. If $s$ is not one of the accumulation points of $\Phi_{f}(\delta)$ when $\delta \rightarrow 0$, we prove that $\operatorname{dim}_{B} \Gamma(f+g, I)$ is equal to $s$. If $s$ is one of the accumulation points of $\Phi_{f}(\delta)$ when $\delta \rightarrow 0$, we find that $\underline{\operatorname{dim}}_{B} \Gamma(f+g, I)$ could be any number belonging to $[1, s)$. Hence, we arrive at the conclusion that $\operatorname{dim}_{B} \Gamma(f+g, I)$ could be any number belonging to $[1, s]$, which means the above problem has been answered totally.

In Section 4, we make further research on two other cases when $s=s_{2}$ or $s=s_{1}$. Their results have been obtained by similar arguments to that in Section 3. In Section 5, as the end of the present paper, we give some conclusions and remarks.

## 2. Theoretical Basis

In Section 2.1, we give certain preliminary theories for the subsequent research. In Section 2.2, we put forward a method to calculate the lower and upper Box dimensions of the sum of two continuous functions and prove several basic results. Then, we present some conclusions of fractal dimensions of the sum of two continuous functions whose Box dimensions both exist in Section 2.3.

### 2.1. Preliminary

In the present paper, given a function $f(x)$ and an interval $[a, b]$, we write $R_{f}[a, b]$ for the maximum range of $f(x)$ over $[a, b]$ as

$$
R_{f}[a, b]=\sup _{a \leq x, y \leq b}|f(x)-f(y)|
$$

and denote $N_{\delta} \Gamma(f,[a, b])$ as the number of squares of the $\delta$-mesh that intersect $\Gamma(f,[a, b])$.
For $f(x), g(x) \in C_{I}$, our motivation is to seek the potential results for $\underline{\operatorname{dim}}_{B} \Gamma(f+g, I)$ and $\overline{\operatorname{dim}}_{B} \Gamma(f+g, I)$. From Definition 1, we can find that the calculation of $N_{\delta} \Gamma(f+g, I)$ is key to estimate $\underline{\operatorname{dim}}_{B} \Gamma(f+g, I)$ and $\operatorname{dim}_{B} \Gamma(f+g, I)$. So, in this subsection, we first show several conclusions about $N_{\delta} \Gamma(f+g, I)$.

Suppose that $0<\delta<\frac{1}{2}$ and $n$ is denoted as the largest integer less than or equal to $\delta^{-1}$. Now, we divide $I$ into $n$ subintervals written as $[i \delta,(i+1) \delta]$ with equal width $\delta(i=0,1,2, \cdots, n-1)$.

Since $f(x) \in C_{I}$, the estimation of $N_{\delta} \Gamma(f, I)$ can be transformed into the oscillation of $f(x)$ on the above subintervals. We note that the number of mesh squares of side $\delta$ in the column above the subinterval $[i \delta,(i+1) \delta]$ that intersect $\Gamma(f, I)$ is no less than

$$
\max \left\{\frac{R_{f}[i \delta,(i+1) \delta]}{\delta}, 1\right\}
$$

and no more than

$$
2+\frac{R_{f}[i \delta,(i+1) \delta]}{\delta}
$$

Summing over all the subintervals leads to the following estimation of $N_{\delta} \Gamma(f, I)$, which is adopted from ref. [3].

Lemma 1 ([3]). Let $f(x) \in C_{I}$. The range of $N_{\delta} \Gamma(f, I)$ can be estimated as

$$
\begin{equation*}
\sum_{i=0}^{n-1} \max \left\{\frac{R_{f}[i \delta,(i+1) \delta]}{\delta}, 1\right\} \leq N_{\delta} \Gamma(f, I) \leq \sum_{i=0}^{n-1}\left\{2+\frac{R_{f}[i \delta,(i+1) \delta]}{\delta}\right\} \tag{5}
\end{equation*}
$$

Now, we investigate $N_{\delta} \Gamma(f+g, I)$. From Lemma 1,

$$
N_{\delta} \Gamma(f+g, I) \leq \sum_{i=0}^{n-1}\left\{2+\frac{R_{f+g}[i \delta,(i+1) \delta]}{\delta}\right\} .
$$

In addition, we know

$$
R_{f+g}[i \delta,(i+1) \delta] \leq R_{f}[i \delta,(i+1) \delta]+R_{g}[i \delta,(i+1) \delta],
$$

which is a property for the maximum range of $f(x)+g(x)$ over $[i \delta,(i+1) \delta]$. Hence, the sum of the oscillation of $f(x)$ and $g(x)$ on subintervals can be used to estimate the upper bound of $N_{\delta} \Gamma(f+g, I)$, that is

$$
\begin{equation*}
N_{\delta} \Gamma(f+g, I) \leq 2 n+\sum_{i=0}^{n-1} \frac{R_{f}[i \delta,(i+1) \delta]}{\delta}+\sum_{i=0}^{n-1} \frac{R_{g}[i \delta,(i+1) \delta]}{\delta} . \tag{6}
\end{equation*}
$$

From (5) and (6), we find that $N_{\delta} \Gamma(f+g, I)$ seems to have a certain connection with $N_{\delta} \Gamma(f, I)$ and $N_{\delta} \Gamma(g, I)$. Here, we present an estimation of $N_{\delta} \Gamma(f+g, I)$ as the following theorem, which reveals the relationship among $N_{\delta} \Gamma(f, I), N_{\delta} \Gamma(g, I)$ and $N_{\delta} \Gamma(f+g, I)$.

Theorem 1. Let $f(x), g(x) \in C_{I}$. The range of $N_{\delta} \Gamma(f+g, I)$ can be estimated as

$$
\begin{equation*}
\frac{1}{3}\left|N_{\delta} \Gamma(f, I)-N_{\delta} \Gamma(g, I)\right| \leq N_{\delta} \Gamma(f+g, I) \leq 3 N_{\delta} \Gamma(f, I)+N_{\delta} \Gamma(g, I) \tag{7}
\end{equation*}
$$

Proof. On one hand, it follows from Lemma 1 that

$$
\sum_{i=0}^{n-1} \frac{R_{g}[i \delta,(i+1) \delta]}{\delta} \leq N_{\delta} \Gamma(g, I)
$$

and

$$
\begin{align*}
\sum_{i=0}^{n-1}\left\{2+\frac{R_{f}[i \delta,(i+1) \delta]}{\delta}\right\} & =\sum_{i=0}^{n-1}\left\{1+1+\frac{R_{f}[i \delta,(i+1) \delta]}{\delta}\right\} \\
& \leq 3 \sum_{i=0}^{n-1} \max \left\{\frac{R_{f}[i \delta,(i+1) \delta]}{\delta}, 1\right\}  \tag{8}\\
& \leq 3 N_{\delta} \Gamma(f, I) .
\end{align*}
$$

Combining (6), we obtain

$$
N_{\delta} \Gamma(f+g, I) \leq \sum_{i=0}^{n-1}\left\{2+\frac{R_{f}[i \delta,(i+1) \delta]}{\delta}\right\}+\sum_{i=0}^{n-1} \frac{R_{g}[i \delta,(i+1) \delta]}{\delta} \leq 3 N_{\delta} \Gamma(f, I)+N_{\delta} \Gamma(g, I) .
$$

On the other hand, similar with (8),

$$
\begin{aligned}
3 N_{\delta} \Gamma(f+g, I) & \geq \sum_{i=0}^{n-1}\left\{2+\frac{R_{f+g}[i \delta,(i+1) \delta]}{\delta}\right\} \\
& \geq 2 n+\left|\sum_{i=0}^{n-1} \frac{R_{f}[i \delta,(i+1) \delta]}{\delta}-\sum_{i=0}^{n-1} \frac{R_{g}[i \delta,(i+1) \delta]}{\delta}\right| \\
& \geq\left|N_{\delta} \Gamma(f, I)-N_{\delta} \Gamma(g, I)\right| .
\end{aligned}
$$

That is

$$
N_{\delta} \Gamma(f+g, I) \geq \frac{1}{3}\left|N_{\delta} \Gamma(f, I)-N_{\delta} \Gamma(g, I)\right| .
$$

This completes the proof of (7).
Corollary 1. Let $f(x), g(x) \in C_{I}$. Then,

$$
\overline{\operatorname{dim}}_{B} \Gamma(f+g, I) \leq \max \left\{\overline{\operatorname{dim}}_{B} \Gamma(f, I), \overline{\operatorname{dim}}_{B} \Gamma(g, I)\right\}
$$

Proof. It follows from Theorem 1 that

$$
\begin{aligned}
\overline{\operatorname{dim}}_{B} \Gamma(f+g, I) & =\varlimsup_{\delta \rightarrow 0} \frac{\log N_{\delta} \Gamma(f+g, I)}{-\log \delta} \\
& \leq \varlimsup_{\delta \rightarrow 0} \frac{\log \left(3 N_{\delta} \Gamma(f, I)+N_{\delta} \Gamma(g, I)\right)}{-\log \delta} \\
& \leq \varlimsup_{\delta \rightarrow 0} \frac{\log 4 \max \left\{N_{\delta} \Gamma(f, I), N_{\delta} \Gamma(g, I)\right\}}{-\log \delta} \\
& =\max \left\{\varlimsup_{\delta \rightarrow 0} \frac{\log N_{\delta} \Gamma(f, I)}{-\log \delta}, \varlimsup_{\delta \rightarrow 0} \frac{\log N_{\delta} \Gamma(g, I)}{-\log \delta}\right\} \\
& =\max \left\{\overline{\operatorname{dim}}_{B} \Gamma(f, I), \overline{\operatorname{dim}}_{B} \Gamma(g, I)\right\} .
\end{aligned}
$$

Thus, we get Corollary 1.
Theorem 1 implies that the value of $N_{\delta} \Gamma(f+g, I)$ can be controlled by certain linear combinations of $N_{\delta} \Gamma(f, I)$ and $N_{\delta} \Gamma(g, I)$. If we can figure out which of $N_{\delta} \Gamma(f, I)$ and $N_{\delta} \Gamma(g, I)$ is 'dominant' in a certain particular situation, the relationship between $N_{\delta} \Gamma(f+g, I)$ and the 'dominant' one of $N_{\delta} \Gamma(f, I)$ and $N_{\delta} \Gamma(g, I)$ may surface. In other words, we may discover some kind of link between fractal dimensions of $\Gamma(f+g, I)$ and fractal dimensions of $\Gamma(f, I)$ or $\Gamma(g, I)$, whose results will be obtained in Section 2.2.

### 2.2. Basic Results

For convenience of notation, let

$$
\Phi_{f}(\delta)=\frac{\log N_{\delta} \Gamma(f, I)}{-\log \delta}
$$

Here, $0<\delta<\frac{1}{2}$ and $f(x) \in C_{I}$. Then, the lower and upper Box dimensions of $\Gamma(f, I)$ can be written as

$$
\underline{\operatorname{dim}}_{B} \Gamma(f, I)=\underline{\lim }_{\delta \rightarrow 0} \Phi_{f}(\delta) \quad \text { and } \quad \overline{\operatorname{dim}}_{B} \Gamma(f, I)=\varlimsup_{\delta \rightarrow 0} \Phi_{f}(\delta)
$$

respectively. If Box dimension of $\Gamma(f, I)$ exists, $\operatorname{dim}_{B} \Gamma(f, I)=\lim _{\delta \rightarrow 0} \Phi_{f}(\delta)$ holds naturally.
It is universally acknowledged that $\lim _{\delta \rightarrow 0} \Phi_{f}(\delta)$ may exist or not. Actually, the number of the accumulation points of $\Phi_{f}(\delta)$ when $\delta \rightarrow 0$ is uncertain, which may be finite, countably infinite or uncountably infinite. For $f(x), g(x) \in C_{I}$, we first define some notations as follows:
(1) Let $\Omega_{f}=\left\{\mu_{j}\right\}_{j \in J_{1}}$ be the set of all the accumulation points of $\Phi_{f}(\delta)$ when $\delta \rightarrow 0$. Here, $J_{1}$ is the index set reflecting the number of the elements in $\Omega_{f}$. Then,

$$
\underline{\operatorname{dim}}_{B} \Gamma(f, I)=\inf _{j \in J_{1}}\left\{\mu_{j}\right\} \quad \text { and } \quad \overline{\operatorname{dim}}_{B} \Gamma(f, I)=\sup _{j \in J_{1}}\left\{\mu_{j}\right\} .
$$

(2) Let $\Omega_{g}=\left\{v_{j}\right\}_{j \in J_{2}}$ be the set of all the accumulation points of $\Phi_{g}(\delta)$ when $\delta \rightarrow 0$. Here, $J_{2}$ is the index set reflecting the number of the elements in $\Omega_{g}$. Then,

$$
\underline{\operatorname{dim}}_{B} \Gamma(g, I)=\inf _{j \in J_{2}}\left\{v_{j}\right\} \quad \text { and } \quad \overline{\operatorname{dim}}_{B} \Gamma(g, I)=\sup _{j \in J_{2}}\left\{v_{j}\right\} .
$$

(3) For $\forall j \in J_{1}$, we denote $\Delta_{j}$ as the set of a subsequence $\left\{\delta_{l_{k}}^{j}\right\}_{k=1}^{\infty}$ corresponding to $\mu_{j}$, which satisfies

$$
\lim _{k \rightarrow \infty} \Phi_{f}\left(\delta_{l_{k}}^{j}\right)=\mu_{j}, \quad \forall\left\{\delta_{l_{k}}^{j}\right\}_{k=1}^{\infty} \in \Delta_{j}
$$

Here $\lim _{k \rightarrow \infty} \delta_{l_{k}}^{j}=0$.
(4) For $\forall j \in J_{1}$, we denote $\alpha_{j}$ and $\beta_{j}$ as the minimum and the maximum value in the following set:

$$
S_{j}=\left\{\lim _{k \rightarrow \infty} \Phi_{f+g}\left(\delta_{l_{k}}^{j}\right):\left\{\delta_{l_{k}}^{j}\right\}_{k=1}^{\infty} \in \Delta_{j}\right\}
$$

respectively. Here, $\lim _{k \rightarrow \infty} \delta_{l_{k}}^{j}=0$.
Now, we present the following proposition, which provides a calculation of $\operatorname{dim}_{B} \Gamma(f+g, I)$ and $\overline{\operatorname{dim}}_{B} \Gamma(f+g, I)$ :

Proposition 2. Let $f(x), g(x) \in C_{I}$. It holds

$$
\begin{equation*}
\underline{\operatorname{dim}}_{B} \Gamma(f+g, I)=\inf _{j \in J_{1}}\left\{\alpha_{j}\right\} \tag{9}
\end{equation*}
$$

and

$$
\begin{equation*}
\overline{\operatorname{dim}}_{B} \Gamma(f+g, I)=\sup _{j \in J_{1}}\left\{\beta_{j}\right\} \tag{10}
\end{equation*}
$$

Proof. From the definition of $\Delta_{j}$, we know $\bigcup_{j \in J_{1}} \Delta_{j}$ covers all the possible subsequences verging to zero. Namely, $\bigcup_{j \in J_{1}} S_{j}$ contains all the accumulation points of $\Phi_{f+g}(\delta)$ when $\delta \rightarrow 0$. This means $\underline{\operatorname{dim}}_{B} \Gamma(f+g, I)$ and $\overline{\operatorname{dim}}_{B} \Gamma(f+g, I)$ are just the minimum and the maximum value in $\bigcup_{j \in J_{1}} S_{j}$, respectively, which leads to the conclusion of Proposition 2.

From Proposition 2, we observe that the key work to calculate $\operatorname{dim}_{B} \Gamma(f+g, I)$ and $\overline{\operatorname{dim}}_{B} \Gamma(f+g, I)$ is to figure out the values of $\alpha_{j}$ and $\beta_{j}$. In preparation for the subsequent work, we first prove a conclusion about sequences given in the following lemma.

Lemma 2. Let $f(x), g(x) \in C_{I}$. For any non-negative sequence $\left\{\delta_{l_{k}}\right\}_{k=1}^{\infty}$ satisfying $\lim _{k \rightarrow \infty} \delta_{l_{k}}=0$, it holds

$$
\varliminf_{k \rightarrow \infty} \Phi_{f+g}\left(\delta_{l_{k}}\right)=\varliminf_{k \rightarrow \infty} \Phi_{g}\left(\delta_{l_{k}}\right) \quad \text { and } \quad \varlimsup_{k \rightarrow \infty} \Phi_{f+g}\left(\delta_{l_{k}}\right)=\varlimsup_{k \rightarrow \infty} \Phi_{g}\left(\delta_{l_{k}}\right)
$$

when

$$
\varlimsup_{k \rightarrow \infty} \Phi_{f}\left(\delta_{l_{k}}\right)<\varliminf_{k \rightarrow \infty} \Phi_{g}\left(\delta_{l_{k}}\right) .
$$

Proof. Suppose that

$$
\varlimsup_{\delta_{l_{k}} \rightarrow 0} \frac{\log N_{\delta_{l_{k}}} \Gamma(f, I)}{-\log \delta_{l_{k}}}=s_{1} \quad \text { and } \quad \varliminf_{\delta_{l_{k} \rightarrow 0} \rightarrow 0} \frac{\log N_{\delta_{l_{k}}} \Gamma(g, I)}{-\log \delta_{l_{k}}}=s_{2}
$$

Given $0<\varepsilon \leq \frac{s_{2}-s_{1}}{4}$, there must exist a certain number $\delta_{0}>0$ such that

$$
\frac{\log N_{\delta_{l_{k}}} \Gamma(f, I)}{-\log \delta_{l_{k}}} \leq s_{1}+\varepsilon
$$

and

$$
\frac{\log N_{\delta_{l_{k}}} \Gamma(g, I)}{-\log \delta_{l_{k}}} \geq s_{2}-\varepsilon
$$

when $\delta_{l_{k}} \leq \delta_{0}$. Now, we have

$$
N_{\delta_{l_{k}}} \Gamma(f, I) \leq\left(\frac{1}{\delta_{l_{k}}}\right)^{s_{1}+\varepsilon}<\left(\frac{1}{\delta_{l_{k}}}\right)^{s_{2}-\varepsilon} \leq N_{\delta_{l_{k}}} \Gamma(g, I) .
$$

Thus,

$$
\frac{N_{\delta_{l_{k}}} \Gamma(f, I)}{N_{\delta_{l_{k}}} \Gamma(g, I)} \leq\left(\frac{1}{\delta_{l_{k}}}\right)^{s_{1}-s_{2}+2 \varepsilon} \leq\left(\frac{1}{\delta_{0}}\right)^{\frac{s_{1}-s_{2}}{2}}=\delta_{0}^{\frac{s_{2}-s_{1}}{2}}
$$

Let $C=\delta_{0}^{\frac{s_{2}-s_{1}}{2}}$. That is

$$
N_{\delta_{l_{k}}} \Gamma(f, I) \leq C \cdot N_{\delta_{l_{k}}} \Gamma(g, I) .
$$

Then, by Theorem 1,

$$
N_{\delta_{l_{k}}} \Gamma(f+g, I) \leq 3 N_{\delta_{l_{k}}} \Gamma(f, I)+N_{\delta_{l_{k}}} \Gamma(g, I) \leq(3 C+1) \cdot N_{\delta_{l_{k}}} \Gamma(g, I)
$$

and

$$
N_{\delta_{l_{k}}} \Gamma(f+g, I) \geq \frac{1}{3}\left(N_{\delta_{l_{k}}} \Gamma(g, I)-N_{\delta_{l_{k}}} \Gamma(f, I)\right) \geq \frac{1-C}{3} \cdot N_{\delta_{l_{k}}} \Gamma(g, I) .
$$

Thus,

$$
\varliminf_{\delta_{l_{k}} \rightarrow 0} \frac{\log N_{\delta_{l_{k}}} \Gamma(f+g, I)}{-\log \delta_{l_{k}}} \leq \varliminf_{\delta_{l_{k} \rightarrow 0} \rightarrow 0} \frac{\log (3 C+1) \cdot N_{\delta_{l_{k}}} \Gamma(g, I)}{-\log \delta_{l_{k}}}=\varliminf_{\delta_{l_{k}} \rightarrow 0} \frac{\log N_{\delta_{l_{k}}} \Gamma(g, I)}{-\log \delta_{l_{k}}}
$$

and

$$
\varliminf_{\delta_{l_{k} \rightarrow 0}} \frac{\log N_{\delta_{l_{k}}} \Gamma(f+g, I)}{-\log \delta_{l_{k}}} \geq \varliminf_{\delta_{l_{k} \rightarrow 0} \rightarrow 0} \frac{\log \frac{1-C}{3} \cdot N_{\delta_{l_{k}}} \Gamma(g, I)}{-\log \delta_{l_{k}}}=\varliminf_{\delta_{l_{k} \rightarrow 0}} \frac{\log N_{\delta_{l_{k}}} \Gamma(g, I)}{-\log \delta_{l_{k}}}
$$

This means

$$
\varliminf_{\delta_{l_{k} \rightarrow 0}} \frac{\log N_{\delta_{l_{k}}} \Gamma(f+g, I)}{-\log \delta_{l_{k}}}=\varliminf_{\delta_{l_{k} \rightarrow 0}} \frac{\log N_{\delta_{l_{k}}} \Gamma(g, I)}{-\log \delta_{l_{k}}} .
$$

That is

$$
\varliminf_{k \rightarrow \infty} \Phi_{f+g}\left(\delta_{l_{k}}\right)=\varliminf_{k \rightarrow \infty} \Phi_{g}\left(\delta_{l_{k}}\right)
$$

Similarly, we can also obtain

$$
\varlimsup_{k \rightarrow \infty} \Phi_{f+g}\left(\delta_{l_{k}}\right)=\varlimsup_{k \rightarrow \infty} \Phi_{g}\left(\delta_{l_{k}}\right) .
$$

Hence, Lemma 2 holds.
Now we can acquire several basic results of the lower and upper Box dimensions of the sum of two continuous functions. We begin by presenting the calculation of $\overline{\operatorname{dim}}_{B} \Gamma(f+g, I)$ in the following theorem.

Theorem 2. Let $f(x), g(x) \in C_{I}$. It holds

$$
\overline{\operatorname{dim}}_{B} \Gamma(f+g, I)=\max \left\{\overline{\operatorname{dim}}_{B} \Gamma(f, I), \overline{\operatorname{dim}}_{B} \Gamma(g, I)\right\}
$$

when

$$
\overline{\operatorname{dim}}_{B} \Gamma(f, I) \neq \overline{\operatorname{dim}}_{B} \Gamma(g, I) .
$$

Proof. On one hand, from Corollary 1,

$$
\begin{equation*}
\overline{\operatorname{dim}}_{B} \Gamma(f+g, I) \leq \max \left\{\overline{\operatorname{dim}}_{B} \Gamma(f, I), \overline{\operatorname{dim}}_{B} \Gamma(g, I)\right\} \tag{11}
\end{equation*}
$$

On the other hand, if

$$
\overline{\operatorname{dim}}_{B} \Gamma(f, I)>\overline{\operatorname{dim}}_{B} \Gamma(g, I),
$$

there must exist an index set $J_{1}^{\prime} \subset J_{1}$ such that

$$
\sup _{i \in J_{2}}\left\{v_{i}\right\}<\inf _{j \in J_{1}^{\prime}}\left\{\mu_{j}\right\} \leq \sup _{j \in J_{1}^{\prime}}\left\{\mu_{j}\right\}=\overline{\operatorname{dim}}_{B} \Gamma(f, I) .
$$

Thus, for $\forall j \in J_{1}^{\prime}$,

$$
\varlimsup_{k \rightarrow \infty} \Phi_{g}\left(\delta_{l_{k}}^{j}\right) \leq \sup _{i \in J_{2}}\left\{v_{i}\right\}<\mu_{j}=\varliminf_{k \rightarrow \infty} \Phi_{f}\left(\delta_{l_{k}}^{j}\right), \quad \forall\left\{\delta_{l_{k}}^{j}\right\}_{k=1}^{\infty} \in \Delta_{j}
$$

Then, it follows from Lemma 2 that for $\forall j \in J_{1}^{\prime}$,

$$
\varlimsup_{k \rightarrow \infty} \Phi_{f+g}\left(\delta_{l_{k}}^{j}\right)=\varlimsup_{k \rightarrow \infty} \Phi_{f}\left(\delta_{l_{k}}^{j}\right)=\mu_{j}, \quad \forall\left\{\delta_{l_{k}}^{j}\right\}_{k=1}^{\infty} \in \Delta_{j}
$$

This means $\beta_{j}=\mu_{j}$ for $\forall j \in J_{1}^{\prime}$. From (10), we can get

$$
\overline{\operatorname{dim}}_{B} \Gamma(f+g, I)=\sup _{j \in J_{1}}\left\{\beta_{j}\right\} \geq \sup _{j \in J_{1}^{\prime}}\left\{\beta_{j}\right\}=\sup _{j \in J_{1}^{\prime}}\left\{\mu_{j}\right\}=\overline{\operatorname{dim}}_{B} \Gamma(f, I) .
$$

If

$$
\overline{\operatorname{dim}}_{B} \Gamma(g, I)>\overline{\operatorname{dim}}_{B} \Gamma(f, I),
$$

similarly we can get

$$
\overline{\operatorname{dim}}_{B} \Gamma(f+g, I) \geq \overline{\operatorname{dim}}_{B} \Gamma(g, I) .
$$

Thus,

$$
\begin{equation*}
\overline{\operatorname{dim}}_{B} \Gamma(f+g, I) \geq \max \left\{\overline{\operatorname{dim}}_{B} \Gamma(f, I), \overline{\operatorname{dim}}_{B} \Gamma(g, I)\right\} \tag{12}
\end{equation*}
$$

when

$$
\overline{\operatorname{dim}}_{B} \Gamma(f, I) \neq \overline{\operatorname{dim}}_{B} \Gamma(g, I) .
$$

Hence, we can get the conclusion of Theorem 2 by (11) and (12).
Theorem 2 shows the conclusion of upper Box dimension of the sum of two continuous functions. If upper Box dimensions of two continuous functions are not equal, upper Box dimension of the sum of these two functions must be the maximum one. This means a continuous function with smaller upper Box dimension can be absorbed by another continuous function with bigger upper Box dimension.

From Theorem 2, we can immediately get Corollary 2, shown below.
Corollary 2. Let $f(x) \in{ }_{s_{1}}^{s_{2}} D_{I}$ and $g(x) \in{ }^{s} D_{I}$. If $s_{1}<s<s_{2}$, it holds

$$
\overline{\operatorname{dim}}_{B} \Gamma(f+g, I)=s_{2} .
$$

Next, we study the calculation of $\underline{\operatorname{dim}}_{B} \Gamma(f+g, I)$ under the condition of (4). Theorem 3 tells us its conclusion.

Theorem 3. Let $f(x), g(x) \in C_{I}$. It holds

$$
\underline{\operatorname{dim}}_{B} \Gamma(f+g, I)=\underline{\operatorname{dim}}_{B} \Gamma(f, I)
$$

when

$$
\overline{\operatorname{dim}}_{B} \Gamma(g, I)<\underline{\operatorname{dim}}_{B} \Gamma(f, I) .
$$

Proof. For $\forall j \in J_{1}$,

$$
\varlimsup_{k \rightarrow \infty} \Phi_{g}\left(\delta_{l_{k}}^{j}\right) \leq \sup _{i \in J_{2}}\left\{v_{i}\right\}<\inf _{i \in J_{1}}\left\{\mu_{i}\right\} \leq \varliminf_{k \rightarrow \infty} \Phi_{f}\left(\delta_{l_{k}}^{j}\right), \quad \forall\left\{\delta_{l_{k}}^{j}\right\}_{k=1}^{\infty} \in \Delta_{j}
$$

Then, it follows from Lemma 2 that for $\forall j \in J_{1}$,

$$
\varliminf_{k \rightarrow \infty} \Phi_{f+g}\left(\delta_{l_{k}}^{j}\right)=\varliminf_{k \rightarrow \infty} \Phi_{f}\left(\delta_{l_{k}}^{j}\right)=\mu_{j}, \quad \forall\left\{\delta_{l_{k}}^{j}\right\}_{k=1}^{\infty} \in \Delta_{j} .
$$

This means $\alpha_{j}=\mu_{j}$ for $\forall j \in J_{1}$. From (9), we can get

$$
\underline{\operatorname{dim}}_{B} \Gamma(f+g, I)=\inf _{j \in J_{1}}\left\{\alpha_{j}\right\}=\inf _{j \in J_{1}}\left\{\mu_{j}\right\}=\underline{\operatorname{dim}}_{B} \Gamma(f, I) .
$$

So, Theorem 3 holds.
Now, for $f(x), g(x) \in C_{I}$ satisfying

$$
\underline{\operatorname{dim}}_{B} \Gamma(f, I)<\operatorname{dim}_{B} \Gamma(g, I)<\operatorname{\operatorname {dim}}_{B} \Gamma(f, I)
$$

we have an upper bound estimation of $\underline{\operatorname{dim}}_{B} \Gamma(f+g, I)$ as follows.
Corollary 3. Let $f(x) \in{ }_{s_{1}}^{s_{2}} D_{I}, g(x) \in{ }^{s} D_{I}$. If $s_{1}<s<s_{2}$, it holds

$$
\underline{\operatorname{dim}}_{B} \Gamma(f+g, I) \leq s
$$

Proof. Let $H(x)=f(x)+g(x)$. If we suppose

$$
\underline{\operatorname{dim}}_{B} \Gamma(H, I)>s,
$$

it means

$$
\underline{\operatorname{dim}}_{B} \Gamma(H, I)>\operatorname{\operatorname {dim}}_{B} \Gamma(g, I)
$$

From Theorem 3,

$$
\underline{\operatorname{dim}}_{B} \Gamma(f, I)=\underline{\operatorname{dim}}_{B} \Gamma(-g+H, I)=\underline{\operatorname{dim}}_{B} \Gamma(H, I)>s .
$$

This is in contradiction with $s_{1}<s$. Thus,

$$
\operatorname{dim}_{B} \Gamma(f+g, I) \leq s
$$

So far, we have resolved a portion of the problem proposed in Section 1. If $f(x) \in$ ${ }_{s_{1}}^{s_{2}} D_{I}$ and $g(x) \in{ }^{s} D_{I}$ satisfying $s_{1}<s<s_{2}$, we can obtain the results that the upper Box dimension of $\Gamma(f+g, I)$ is equal to $s_{2}$ from Corollary 2 and the lower Box dimension of $\Gamma(f+g, I)$ is no more than $s$ from Corollary 3. Therefore, the lower Box dimension of $\Gamma(f+g, I)$ has not yet been studied thoroughly. In Section 2.3 , to prepare for the further research, we first present several conclusions of sum of two continuous functions if both of them have Box dimensions.

### 2.3. Sum of Two Continuous Functions Having Box Dimension

Firstly, we consider the sum of two continuous functions with different Box dimensions. The following assertion is adopted from [3].

Proposition 3 ([3]). Let $f(x), g(x) \in C_{I}$ with different Box dimensions. Then,

$$
\operatorname{dim}_{B} \Gamma(f+g, I)=\max \left\{\operatorname{dim}_{B} \Gamma(f, I), \operatorname{dim}_{B} \Gamma(g, I)\right\}
$$

Secondly, Theorems 4 and 5 show the conclusions of the sum of two continuous functions with the same Box dimension that is not equal to one.

Theorem 4. Let $f(x), g(x) \in C_{I}$ with the same Box dimension $s(1<s \leq 2)$. If the Box dimension of $\Gamma(f+g, I)$ exists, it could be any number belonging to $[1, s)$.

Proof. Firstly, let

$$
f(x)=-g(x)+W(x)
$$

Here, $W(x)$ is the Weierstrass function given in Example 1, and $\operatorname{dim}_{B} \Gamma(W, I)$ could be any number belonging to $(1, s)$. Then, by Proposition 3,

$$
\begin{aligned}
\operatorname{dim}_{B} \Gamma(f, I) & =\operatorname{dim}_{B} \Gamma(-g+W, I) \\
& =\max \left\{\operatorname{dim}_{B} \Gamma(g, I), \operatorname{dim}_{B} \Gamma(W, I)\right\} \\
& =\max \{s, 2-\beta\} \\
& =s
\end{aligned}
$$

Secondly, let

$$
f(x)=-g(x)+H(x)
$$

Here, $H(x) \in{ }^{1} D_{I}$. In the same way,

$$
\operatorname{dim}_{B} \Gamma(f, I)=\max \left\{\operatorname{dim}_{B} \Gamma(g, I), \operatorname{dim}_{B} \Gamma(H, I)\right\}=\max \{s, 1\}=s .
$$

From discussion above, we find that Box dimension of $\Gamma(f+g, I)$ exists and could be any number belonging to $[1, s)$.

Theorem 5. Let $f(x), g(x) \in C_{I}$ with the same Box dimension $s(1<s \leq 2)$. If the Box dimension of $\Gamma(f+g, I)$ does not exist,

$$
\begin{equation*}
1 \leq \underline{\operatorname{dim}}_{B} \Gamma(f+g, I)<\overline{\operatorname{dim}}_{B} \Gamma(f+g, I)<s \tag{13}
\end{equation*}
$$

Here, $\underline{\operatorname{dim}}_{B} \Gamma(f+g, I)$ and $\overline{\operatorname{dim}}_{B} \Gamma(f+g, I)$ could be any numbers satisfying (13).
Proof. Let

$$
f(x)=-g(x)+B(x)
$$

Here, $B(x)$ is the Besicovitch function given in Example 2. For suitably chosen $\left\{\lambda_{j}\right\}_{j=1}^{\infty}$, we have

$$
\begin{equation*}
1 \leq \underline{\operatorname{dim}}_{B} \Gamma(B, I)<\overline{\operatorname{dim}}_{B} \Gamma(B, I)<s . \tag{14}
\end{equation*}
$$

Here, $\underline{\operatorname{dim}}_{B} \Gamma(B, I)$ and $\overline{\operatorname{dim}}_{B} \Gamma(B, I)$ could be any numbers satisfying (14). From Theorems 2 and 3,

$$
\overline{\operatorname{dim}}_{B} \Gamma(f, I)=\overline{\operatorname{dim}}_{B} \Gamma(-g+B, I)=\max \left\{\overline{\operatorname{dim}}_{B} \Gamma(g, I), \overline{\operatorname{dim}}_{B} \Gamma(B, I)\right\}=s
$$

and

$$
\underline{\operatorname{dim}}_{B} \Gamma(f, I)=\underline{\operatorname{dim}}_{B} \Gamma(-g+B, I)=\underline{\operatorname{dim}}_{B} \Gamma(g, I)=s .
$$

This means

$$
\operatorname{dim}_{B} \Gamma(f, I)=s
$$

From (14), we know $\underline{\operatorname{dim}}_{B} \Gamma(f+g, I)$ and $\overline{\operatorname{dim}}_{B} \Gamma(f+g, I)$ could be any numbers satisfying (13).

Thirdly, in Theorems 4 and 5 if $s=1$, the result given below holds trivially.
Theorem 6. Let $f(x), g(x) \in C_{I}$ with the same Box dimension one. Then,

$$
\operatorname{dim}_{B} \Gamma(f+g, I)=1
$$

Proof. On one hand, it follows from (11) that

$$
\overline{\operatorname{dim}}_{B} \Gamma(f+g, I) \leq 1
$$

On the other hand, we know the lower Box dimension of any continuous functions is no less than one. That is

$$
\underline{\operatorname{dim}}_{B} \Gamma(f+g, I) \geq 1 .
$$

Thus,

$$
\operatorname{dim}_{B} \Gamma(f+g, I)=1
$$

Theorem 6 says that sum of two one-dimensional continuous functions on $I$ can keep the Box dimension closed, which implies that ${ }^{1} D_{I}$ is a linear space. However, we note from Theorems 4 and 5 that ${ }^{s} D_{I}$ is not linear when $s \neq 1$.

## 3. Further Research on $\underline{\operatorname{dim}}_{B} \Gamma(f+g, I)$

For $f(x) \in{ }_{s_{1}}^{s_{2}} D_{I}$ and $g(x) \in{ }^{s} D_{I}$ satisfying $s_{1}<s<s_{2}, \overline{\operatorname{dim}}_{B} \Gamma(f+g, I)$ and an upper bound estimation of $\operatorname{dim}_{B} \Gamma(f+g, I)$ have been obtained in Section 2. In this section, we make further research on the calculation of $\underline{\operatorname{dim}}_{B} \Gamma(f+g, I)$.

When $s_{1}<s<s_{2}$, there must exist two index sets denoted as $J_{1}^{(1)}$ and $J_{1}^{(2)}$, which satisfy

$$
J_{1}^{(1)} \bigcup J_{1}^{(2)}=J_{1}
$$

and

$$
\underline{\operatorname{dim}}_{B} \Gamma(f, I)=\inf _{j \in J_{1}^{(1)}}\left\{\mu_{j}\right\} \leq \sup _{j \in J_{1}^{(1)}}\left\{\mu_{j}\right\}<s \leq \inf _{j \in J_{1}^{(2)}}\left\{\mu_{j}\right\} \leq \sup _{j \in J_{1}^{(2)}}\left\{\mu_{j}\right\}=\overline{\operatorname{dim}}_{B} \Gamma(f, I) .
$$

For the convenience of discussion, write $\inf _{j \in J_{1}^{(2)}}\left\{\mu_{j}\right\}=\mu_{j_{*}}$. Here, $j_{*} \in J_{1}^{(2)}$. Since $s_{1}<s<s_{2}$, the element $s$ may belong to $\Omega_{f}$ or not. In other words, $\mu_{j_{*}}$ may be equal to $s$ or not. So, we should discuss two cases as follows.
3.1. $\mu_{j_{*}} \neq s$

From Lemma 2, we check every element $\mu_{j}$ in the set $\Omega_{f}$ and then obtain the following results.
(I) For $j \in J_{1}^{(1)}$, we know $\mu_{j}<s$. That is

$$
\varlimsup_{k \rightarrow \infty} \Phi_{f}\left(\delta_{l_{k}}^{j}\right)=\mu_{j}<s=\varliminf_{k \rightarrow \infty} \Phi_{g}\left(\delta_{l_{k}}^{j}\right), \quad \forall\left\{\delta_{l_{k}}^{j}\right\}_{k=1}^{\infty} \in \Delta_{j}
$$

Thus,

$$
\varliminf_{k \rightarrow \infty} \Phi_{f+g}\left(\delta_{l_{k}}^{j}\right)=\varliminf_{k \rightarrow \infty} \Phi_{g}\left(\delta_{l_{k}}^{j}\right)=s, \quad \forall\left\{\delta_{l_{k}}^{j}\right\}_{k=1}^{\infty} \in \Delta_{j} .
$$

This means $\alpha_{j}=s$.
(II) For $j \in J_{1}^{(2)}$, we know $\mu_{j}>s$. That is

$$
\varliminf_{k \rightarrow \infty} \Phi_{f}\left(\delta_{l_{k}}^{j}\right)=\mu_{j}>s=\varlimsup_{k \rightarrow \infty} \Phi_{g}\left(\delta_{l_{k}}^{j}\right), \quad \forall\left\{\delta_{l_{k}}^{j}\right\}_{k=1}^{\infty} \in \Delta_{j} .
$$

Thus,

$$
\varliminf_{k \rightarrow \infty} \Phi_{f+g}\left(\delta_{l_{k}}^{j}\right)=\varliminf_{k \rightarrow \infty} \Phi_{f}\left(\delta_{l_{k}}^{j}\right)=\mu_{j}, \quad \forall\left\{\delta_{l_{k}}^{j}\right\}_{k=1}^{\infty} \in \Delta_{j}
$$

This means $\alpha_{j}=\mu_{j}>s$.

So, in this case, we can assert that

$$
\underline{\operatorname{dim}}_{B} \Gamma(f+g, I)=\inf _{j \in J_{1}}\left\{\alpha_{j}\right\}=\min \left\{\inf _{j \in J_{1}^{(1)}}\left\{\alpha_{j}\right\}, \inf _{j \in J_{1}^{(2)}}\left\{\alpha_{j}\right\}\right\}=\min \left\{s, \inf _{j \in J_{1}^{(2)}}\left\{\mu_{j}\right\}\right\}=s
$$

3.2. $\mu_{j_{*}}=s$

In this case, we first introduce an auxiliary lemma as follows.
Lemma 3. Let $f(x) \in{ }_{s_{1}}^{s_{2}} D_{I}$ and $g(x) \in{ }^{s} D_{I}$. For any non-negative sequence $\left\{\delta_{l_{k}}\right\}_{k=1}^{\infty}$ satisfying $\lim _{k \rightarrow \infty} \delta_{l_{k}}=0$, if

$$
\lim _{k \rightarrow \infty} \Phi_{f}\left(\delta_{l_{k}}\right)=s \in(1,2],
$$

then $\varliminf_{k \rightarrow \infty} \Phi_{f+g}\left(\delta_{l_{k}}\right)$ could be any number belonging to $[1, s)$.
Proof. In the present paper, we know $\Delta_{j_{*}}$ is just the set of sequences satisfying the condition of this lemma. For $s \in(1,2]$, we choose any two possible functions $F(x)$ and $G(x)$ with the same Box dimension $s$, which means

$$
\lim _{k \rightarrow \infty} \Phi_{F}\left(\delta_{l_{k}}^{j_{*}}\right)=\lim _{k \rightarrow \infty} \Phi_{G}\left(\delta_{l_{k}}^{j_{*}}\right)=s, \quad \forall\left\{\delta_{l_{k}}^{j_{*}}\right\}_{k=1}^{\infty} \in \Delta_{j_{*}} .
$$

From Theorem 4, if the Box dimension of $\Gamma(F+G, I)$ exists, its value could be any number belonging to $[1, s)$. In other words, $\varliminf_{k \rightarrow \infty} \Phi_{F+G}\left(\delta_{l_{k}}^{j_{*}}\right)$ could be any number belonging to $[1, s)$. From Theorem 5, if Box dimension of $\Gamma(F+G, I)$ does not exist,

$$
\begin{equation*}
1 \leq \underline{\operatorname{dim}}_{B} \Gamma(F+G, I)<\overline{\operatorname{dim}}_{B} \Gamma(F+G, I)<s . \tag{15}
\end{equation*}
$$

Here, $\underline{\operatorname{dim}}_{B} \Gamma(F+G, I)$ and $\operatorname{dim}_{B} \Gamma(F+G, I)$ could be any numbers satisfying (15). Besides, we know

$$
1 \leq \underline{\operatorname{dim}}_{B} \Gamma(F+G, I) \leq \varliminf_{k \rightarrow \infty} \Phi_{F+G}\left(\delta_{l_{k}}^{j_{*}^{*}}\right) \leq \overline{\operatorname{dim}}_{B} \Gamma(F+G, I)<s .
$$

From arbitrariness of $\underline{\operatorname{dim}}_{B} \Gamma(F+G, I)$ and $\overline{\operatorname{dim}}_{B} \Gamma(F+G, I)$ satisfying (15),

$$
\begin{equation*}
1 \leq \varliminf_{k \rightarrow \infty} \Phi_{F+G}\left(\delta_{l_{k}}^{j_{*}^{*}}\right)<s, \quad \forall\left\{\delta_{l_{k}}^{j_{*}}\right\}_{k=1}^{\infty} \in \Delta_{j_{*}} \tag{16}
\end{equation*}
$$

Here, $\varliminf_{k \rightarrow \infty}^{\lim } \Phi_{F+G}\left(\delta_{l_{k}}^{j_{*}}\right)$ could be any number satisfying (16). Let $g(x)=G(x)$. Then, we investigate the connection between $f(x)$ and $F(x)$. It is obvious that for $\forall i \in J_{1}$,

$$
\lim _{k \rightarrow \infty} \Phi_{F}\left(\delta_{l_{k}}^{i}\right)=s, \quad \forall\left\{\delta_{l_{k}}^{i}\right\}_{k=1}^{\infty} \in \Delta_{i}
$$

For $j \in J_{1}$, now we define $\Psi_{j}$ as the set of $F_{j}$ satisfying

$$
\lim _{k \rightarrow \infty} \Phi_{F_{j}}\left(\delta_{l_{k}}^{i}\right)=\left\{\begin{array}{ll}
\mu_{j}, & i=j \\
s, & i \in J_{1} \backslash\{j\}^{\prime}
\end{array} \quad \forall F_{j} \in \Psi_{j}, \quad \forall\left\{\delta_{l_{k}}^{i}\right\}_{k=1}^{\infty} \in \Delta_{i}\right.
$$

For $\forall F_{j} \in \Psi_{j}$, we note that we only change the limitation of $\Phi_{F}\left(\delta_{l_{k}}^{j}\right)$ from $s$ to $\mu_{j}$ when $\delta_{l_{k}}^{j} \rightarrow 0$ for $\forall\left\{\delta_{l_{k}}^{j}\right\}_{k=1}^{\infty} \in \Delta_{j}$. For the convenience of notation, we denote this transformation
as

$$
T_{j}: \quad F \underset{s \rightarrow \mu_{j}}{\Delta_{j}} F_{j} .
$$

Write $T_{j} \odot F=F_{j}$. Then, we can acquire a series of transformations $\left\{T_{j}\right\}_{j \in J_{1}}$. We find that $\left\{T_{j}\right\}_{j \in J_{1}}$ can be divided into three different categories in terms of different effects on $F$, which have been discussed as follows.
(a) For $j \in J_{1}^{(1)}$, since $\mu_{j}<s$, we observe that the only different result for $F_{j}$ from $F$ is that

$$
\lim _{k \rightarrow \infty} \Phi_{F_{j}+g}\left(\delta_{l_{k}}^{j}\right)=\lim _{k \rightarrow \infty} \Phi_{g}\left(\delta_{l_{k}}^{j}\right)=s, \quad \forall F_{j} \in \Psi_{j}, \quad \forall\left\{\delta_{l_{k}}^{j}\right\}_{k=1}^{\infty} \in \Delta_{j}
$$

by Lemma 2. However, for other sets $\Delta_{i}\left(i \in J_{1} \backslash\{j\}\right)$, the results for $F_{j}$ are the same as $F$. Specially for $\Delta_{j^{\prime}}$

$$
\begin{equation*}
1 \leq \underline{\lim _{k \rightarrow \infty}} \Phi_{F_{j}+g}\left(\delta_{l_{k}}^{j_{*}}\right)<s, \quad \forall F_{j} \in \Psi_{j}, \quad \forall\left\{\delta_{l_{k}}^{j_{*}}\right\}_{k=1}^{\infty} \in \Delta_{j_{*}} . \tag{17}
\end{equation*}
$$

Here, $\varliminf_{k \rightarrow \infty} \Phi_{F_{j}+g}\left(\delta_{l_{k}}^{j_{*}}\right)$ could be any number satisfying (17).
(b) For $j=j_{*}$, since $\mu_{j_{*}}=s$, the results for $F_{j_{*}}$ are the same as $F$. Specially for $\Delta_{j_{*}}$,

$$
\begin{equation*}
1 \leq \varliminf_{k \rightarrow \infty}^{\lim } \Phi_{F_{j_{*}}+g}\left(\delta_{l_{k}}^{j_{*}}\right)<s, \quad \forall F_{j_{*}} \in \Psi_{j_{*},} \quad \forall\left\{\delta_{l_{k}}^{j_{*}}\right\}_{k=1}^{\infty} \in \Delta_{j_{*}} \tag{18}
\end{equation*}
$$

Here, $\varliminf_{k \rightarrow \infty}^{\varliminf_{1 m}} \Phi_{F_{j_{*}}+g}\left(\delta_{l_{k}}^{j_{j}^{*}}\right)$ could be any number satisfying (18).
(c) For $j \in J_{1}^{(2)} \backslash\left\{j_{*}\right\}$, since $\mu_{j}>s$, we observe that the only different result for $F_{j}$ from $F$ is that

$$
\lim _{k \rightarrow \infty} \Phi_{F_{j}+g}\left(\delta_{l_{k}}^{j}\right)=\lim _{k \rightarrow \infty} \Phi_{F_{j}}\left(\delta_{l_{k}}^{j}\right)=\mu_{j}, \quad \forall F_{j} \in \Psi_{j}, \quad \forall\left\{\delta_{l_{k}}^{j}\right\}_{k=1}^{\infty} \in \Delta_{j}
$$

by Lemma 2. However, for other sets $\Delta_{i}\left(i \in J_{1} \backslash\{j\}\right)$, the results for $F_{j}$ are the same as $F$. Specially for $\Delta_{j_{*}}$,

$$
\begin{equation*}
1 \leq \varliminf_{k \rightarrow \infty} \Phi_{F_{j}+g}\left(\delta_{l_{k}}^{j_{*}}\right)<s, \quad \forall F_{j} \in \Psi_{j}, \quad \forall\left\{\delta_{l_{k}}^{j_{*}}\right\}_{k=1}^{\infty} \in \Delta_{j_{*}} \tag{19}
\end{equation*}
$$

Here, $\varliminf_{k \rightarrow \infty} \Phi_{F_{j}+g}\left(\delta_{l_{k}}^{j_{*}}\right)$ could be any number satisfying (19).
Now, we do all the transformations $\left\{T_{j}\right\}_{j \in J_{1}}$ on $F$ denoted as $T_{j} \bigodot_{j \in J_{1}} F=F_{J_{1}}$. Define $\Psi_{J_{1}}$ as the set of $F_{J_{1}}$. From the discussion above, we know for $\forall i \in J_{1}$,

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \Phi_{F_{J_{1}}}\left(\delta_{l_{k}}^{i}\right)=\mu_{i}, \quad \forall F_{J_{1}} \in \Psi_{J_{1}}, \quad \forall\left\{\delta_{l_{k}}^{i}\right\}_{k=1}^{\infty} \in \Delta_{i} \tag{20}
\end{equation*}
$$

and for $\Delta_{j_{*}}$,

$$
\begin{equation*}
1 \leq \varliminf_{k \rightarrow \infty} \Phi_{F_{J_{1}}+g}\left(\delta_{l_{k}}^{j_{*}^{*}}\right)<s, \quad \forall F_{J_{1}} \in \Psi_{J_{1}}, \quad \forall\left\{\delta_{l_{k}}^{j_{*}}\right\}_{k=1}^{\infty} \in \Delta_{j_{*}} . \tag{21}
\end{equation*}
$$

Here, $\varliminf_{k \rightarrow \infty} \Phi_{F_{I_{1}}+g}\left(\delta_{l_{k}}^{j_{*}}\right)$ could be any number satisfying (21). From (20), we note that $f \in \Psi_{J_{1}}$. Let $F_{J_{1}}=f$. Thus, for $\Delta_{j_{*}}$,

$$
\begin{equation*}
1 \leq \varliminf_{k \rightarrow \infty} \Phi_{f+g}\left(\delta_{l_{k}}^{j_{*}^{*}}\right)<s, \quad \forall\left\{\delta_{l_{k}}^{j_{*}}\right\}_{k=1}^{\infty} \in \Delta_{j_{*}} . \tag{22}
\end{equation*}
$$

Here, $\varliminf_{k \rightarrow \infty} \Phi_{f+g}\left(\delta_{l_{k}}^{j_{*}}\right)$ could be any number satisfying (22).
This completes the proof of Lemma 3.

Similarly, we check every element $\mu_{j}$ in the set $\Omega_{f}$.
(I) For $j \in J_{1} \backslash\left\{j_{*}\right\}$, the result is the same with Section 3.1, that is

$$
\alpha_{j}=s, \quad j \in J_{1}^{(1)}
$$

and

$$
\alpha_{j}=\mu_{j}>s, \quad j \in J_{1}^{(2)} \backslash\left\{j_{*}\right\} .
$$

(II) For $j=j_{*}$, we know

$$
\lim _{k \rightarrow \infty} \Phi_{f}\left(\delta_{l_{k}}^{j_{*}^{*}}\right)=\lim _{k \rightarrow \infty} \Phi_{g}\left(\delta_{l_{k}}^{j_{*}}\right)=s, \quad \forall\left\{\delta_{l_{k}}^{j_{*}}\right\}_{k=1}^{\infty} \in \Delta_{j_{*}}
$$

Here, $s \in(1,2)$. Then, it follows from Lemma 3 that $\varliminf_{k \rightarrow \infty} \Phi_{f+g}\left(\delta_{l_{k}}^{j_{*}}\right)$ could be any number belonging to $[1, s)$, which implies that $\alpha_{j_{*}}$ could be any number belonging to $[1, s)$.
So, in this case, we can assert that

$$
\underline{\operatorname{dim}}_{B} \Gamma(f+g, I)=\inf _{j \in J_{1}}\left\{\alpha_{j}\right\}=\min \left\{s, \alpha_{j_{*} \prime} \inf _{j \in J_{1}^{(2)} \backslash\left\{j_{*}\right\}}\left\{\mu_{j}\right\}\right\}=\alpha_{j_{*} \prime}
$$

which means $\underline{\operatorname{dim}}_{B} \Gamma(f+g, I)$ could be any number belonging to $[1, s)$.

### 3.3. Conclusions of This Section

From discussion of Sections 3.1 and 3.2, we can obtain the result that $\underline{\operatorname{dim}}_{B} \Gamma(f+g, I)$ could be any number belonging to $[1, s]$. Hence, we have the following conclusion:

Theorem 7. Let $f(x) \in{ }_{s_{1}}^{s_{2}} D_{I}$ and $g(x) \in{ }^{s} D_{I}$. If $s_{1}<s<s_{2}$,

$$
f(x)+g(x) \in{ }_{v}^{s_{2}} D_{I}
$$

Here, $v$ could be any number belonging to $[1, s]$.
So far, the problems in Section 1 have been investigated totally. We find that the value of $\underline{\operatorname{dim}}_{B} \Gamma(f+g, I)$ depends on different situations of the accumulation points of $\Phi_{f}(\delta)$ when $\delta \rightarrow 0$. If $s$ is one of the elements in $\Omega_{f}$, the value of $\underline{\operatorname{dim}}_{B} \Gamma(f+g, I)$ can definitely not be equal to $s$. However, it may be equal to an arbitrary number belonging to $[1, s)$. If $s$ is not one of the elements in $\Omega_{f}$, the value of $\underline{\operatorname{dim}}_{B} \Gamma(f+g, I)$ can only be equal to $s$. In particular, if $s_{1}$ and $s_{2}$ are the only two elements in $\Omega_{f}$, we can directly obtain $\underline{\operatorname{dim}}_{B} \Gamma(f+g, I)=s$.

Furthermore, Lemma 3 shows us a method to seek the relationship between two fractal continuous functions. We find that the same accumulation point of $\Phi_{f}(\delta)$ and $\Phi_{g}(\delta)$ by the same subsequence is the "bridge" to connect $f(x)$ and $g(x)$. If we denote this same accumulation point as $s$, we also prove that the accumulation points of $\Phi_{f+g}(\delta)$ could be equal to any numbers belonging to $[1, s)$ by this subsequence.

## 4. Other Cases

In Section 3, we have figured out the fractal dimensions of the sum of two continuous functions $f(x)$ and $g(x)$ when $\operatorname{dim}_{B} \Gamma(g, I)$ is between $\underline{\operatorname{dim}}_{B} \Gamma(f, I)$ and $\overline{\operatorname{dim}}_{B} \Gamma(f, I)$. Now, we can further consider the following question:

Question 2. If $\operatorname{dim}_{B} \Gamma(g, I)$ is equal to $\operatorname{dim}_{B} \Gamma(f, I)$ or $\overline{\operatorname{dim}}_{B} \Gamma(f, I)$, can we acquire a similar result?
The purpose of this section is to make research on the above problem by the similar discussion with that in Section 3.
4.1. $\operatorname{dim}_{B} \Gamma(g, I)=\overline{\operatorname{dim}}_{B} \Gamma(f, I)$

In this case, we can get the following conclusion.
Theorem 8. Let $f(x) \in{ }_{s_{1}}^{s_{2}} D_{I}$ and $g(x) \in{ }^{s} D_{I}$. If $s_{1}<s=s_{2}$,

$$
f(x)+g(x) \in{ }_{v}^{s} D_{I} .
$$

Here, $v$ could be any number belonging to $[1, s)$.
Proof. From (11),

$$
\overline{\operatorname{dim}}_{B} \Gamma(f+g, I) \leq \max \left\{\overline{\operatorname{dim}}_{B} \Gamma(f, I), \overline{\operatorname{dim}}_{B} \Gamma(g, I)\right\}=s
$$

Let $H(x)=f(x)+g(x)$. If we suppose

$$
\overline{\operatorname{dim}}_{B} \Gamma(f+g, I)=\overline{\operatorname{dim}}_{B} \Gamma(H, I)<s,
$$

we can get

$$
\underline{\operatorname{dim}}_{B} \Gamma(f, I)=\underline{\operatorname{dim}}_{B} \Gamma(-g+H, I)=\underline{\operatorname{dim}}_{B} \Gamma(g, I)=s
$$

by Proposition 3. This is in contradiction with $s_{1}<s$. Thus,

$$
\overline{\operatorname{dim}}_{B} \Gamma(f+g, I)=s .
$$

For $\underline{\operatorname{dim}}_{B} \Gamma(f+g, I)$, we know

$$
\inf _{j \in J_{1}}\left\{\mu_{j}\right\}<\sup _{j \in J_{1}}\left\{\mu_{j}\right\}=s .
$$

Write $\sup _{j \in J_{1}}\left\{\mu_{j}\right\}=\mu_{j_{*}}$. Similar argument with that in Section 3, we can obtain the following results.
(I) For $j \in J_{1} \backslash\left\{j_{*}\right\}$, we have $\alpha_{j}=s$.
(II) For $j=j_{*}$,

$$
\lim _{k \rightarrow \infty} \Phi_{f}\left(\delta_{l_{k}}^{j_{*}}\right)=\lim _{k \rightarrow \infty} \Phi_{g}\left(\delta_{l_{k}}^{j_{*}}\right)=s, \quad \forall\left\{\delta_{l_{k}}^{j_{*}}\right\}_{k=1}^{\infty} \in \Delta_{j_{*}} .
$$

Here $s \in(1,2]$. From Lemma 3, we know $\alpha_{j_{*}}$ could be any number belonging to $[1, s)$.
Hence, we can assert that

$$
\underline{\operatorname{dim}}_{B} \Gamma(f+g, I)=\inf _{j \in J_{1}}\left\{\alpha_{j}\right\}=\min \left\{s, \alpha_{j_{*}}\right\}=\alpha_{j_{*}}
$$

which implies that $\operatorname{dim}_{B} \Gamma(f+g, I)$ could be any number belonging to $[1, s)$.
This completes the proof of Theorem 8.
4.2. $\operatorname{dim}_{B} \Gamma(g, I)=\operatorname{dim}_{B} \Gamma(f, I)$

In this case, we should discuss two situations according to whether $\operatorname{dim}_{B} \Gamma(g, I)$ is equal to one or not. Theorems 9 and 10 present their results, respectively.

Theorem 9. Let $f(x) \in{ }_{s_{1}}^{s_{2}} D_{I}$ and $g(x) \in{ }^{s} D_{I}$. If $1<s_{1}=s<s_{2}$,

$$
f(x)+g(x) \in{ }_{v}^{s_{2}} D_{I} .
$$

Here, $v$ could be any number belonging to $[1, s)$.
Proof. From Theorem 2,

$$
\overline{\operatorname{dim}}_{B} \Gamma(f+g, I)=s_{2}
$$

For $\underline{\operatorname{dim}}_{B} \Gamma(f+g, I)$, in this case we know

$$
1<s=\inf _{j \in J_{1}}\left\{\mu_{j}\right\}<\sup _{j \in J_{1}}\left\{\mu_{j}\right\} .
$$

Write $\inf _{j \in J_{1}}\left\{\mu_{j}\right\}=\mu_{j_{*}}$. Similar argument with that in Section 3, we can obtain the following results.
(I) For $j \in J_{1} \backslash\left\{j_{*}\right\}$, we have $\alpha_{j}=\mu_{j}>s$.
(II) $\operatorname{For} j=j_{*}$,

$$
\lim _{k \rightarrow \infty} \Phi_{f}\left(\delta_{l_{k}}^{j_{*}}\right)=\lim _{k \rightarrow \infty} \Phi_{g}\left(\delta_{l_{k}}^{j_{*}}\right)=s, \quad \forall\left\{\delta_{l_{k}}^{j_{*}}\right\}_{k=1}^{\infty} \in \Delta_{j_{*}}
$$

Here, $s \in(1,2)$. From Lemma 3, we know $\alpha_{j_{*}}$ could be any number belonging to $[1, s)$.
So, we can assert that

$$
\underline{\operatorname{dim}}_{B} \Gamma(f+g, I)=\inf _{j \in J_{1}}\left\{\alpha_{j}\right\}=\min \left\{\alpha_{j_{*} \prime} \inf _{j \in J_{1} \backslash\left\{j_{*}\right\}}\left\{\mu_{j}\right\}\right\}=\alpha_{j_{*}}
$$

which means $\operatorname{dim}_{B} \Gamma(f+g, I)$ could be any number belonging to $[1, s)$.
This completes the proof of Theorem 9.
Theorem 10. Let $f(x) \in{ }_{1}^{s_{2}} D_{I}$ and $g(x) \in{ }^{1} D_{I}$. If $s_{2}>1$,

$$
f(x)+g(x) \in{ }_{1}^{s_{2}} D_{I} .
$$

Proof. From Theorem 2,

$$
\overline{\operatorname{dim}}_{B} \Gamma(f+g, I)=s_{2} .
$$

For $\underline{\operatorname{dim}}_{B} \Gamma(f+g, I)$, in this case we know

$$
1=\inf _{j \in J_{1}}\left\{\mu_{j}\right\}<\sup _{j \in J_{1}}\left\{\mu_{j}\right\}
$$

Write $\inf _{j \in J_{1}}\left\{\mu_{j}\right\}=\mu_{j_{*}}$. Similar to the argument in Section 3, we can obtain the following results.
(I) For $j \in J_{1} \backslash\left\{j_{*}\right\}$, we have $\alpha_{j}=\mu_{j}>1$.
(II) $\operatorname{For} j=j_{*}$,

$$
\lim _{k \rightarrow \infty} \Phi_{f}\left(\delta_{l_{k}}^{j_{*}^{*}}\right)=\lim _{k \rightarrow \infty} \Phi_{g}\left(\delta_{l_{k}}^{j_{*}}\right)=1, \quad \forall\left\{\delta_{l_{k}}^{j_{*}}\right\}_{k=1}^{\infty} \in \Delta_{j_{*}}
$$

From Lemma 3, we can deduce that $\alpha_{j_{*}}=1$.
So, we can assert that

$$
\underline{\operatorname{dim}}_{B} \Gamma\left(f+g^{\prime} I\right)=\inf _{j \in J_{1}}\left\{\alpha_{j}\right\}=\min \left\{1, \inf _{j \in J_{1} \backslash\left\{j_{*}\right\}}\left\{\mu_{j}\right\}\right\}=1 .
$$

This completes the proof of Theorem 10.

## 5. Conclusions

In this last section, we give some remarks on our paper.

### 5.1. Main Results

Throughout the present paper, we mainly investigated fractal dimensions of the sum of two continuous functions $f$ and $g$ on $I$ with certain lower and upper Box dimensions. The main results we have obtained can be summarized as the following three aspects:
(1) If

$$
\underline{\operatorname{dim}}_{B} \Gamma(f, I)<\operatorname{dim}_{B} \Gamma(g, I)<{\operatorname{dim}_{B}} \Gamma(f, I)
$$

we prove that

$$
\overline{\operatorname{dim}}_{B} \Gamma(f+g, I)=\overline{\operatorname{dim}}_{B} \Gamma(f, I)
$$

Then, we study $\underline{\operatorname{dim}}_{B} \Gamma(f+g, I)$ by whether $\operatorname{dim}_{B} \Gamma(g, I)$ is one of the accumulation points of $\Phi_{f}(\delta)$ when $\delta \rightarrow 0$ or not.
(i) If $\operatorname{dim}_{B} \Gamma(g, I)$ is not one of the accumulation points of $\Phi_{f}(\delta)$ when $\delta \rightarrow 0$, we prove that

$$
\underline{\operatorname{dim}}_{B} \Gamma(f+g, I)=\operatorname{dim}_{B} \Gamma(g, I) ;
$$

(ii) If $\operatorname{dim}_{B} \Gamma(g, I)$ is one of the accumulation points of $\Phi_{f}(\delta)$ when $\delta \rightarrow 0$, we prove that $\underline{\operatorname{dim}}_{B} \Gamma(f+g, I)$ could be any number belonging to $\left[1, \operatorname{dim}_{B} \Gamma(g, I)\right)$.
In conclusion, $\underline{\operatorname{dim}}_{B} \Gamma(f+g, I)$ could be any number belonging to $\left[1, \operatorname{dim}_{B} \Gamma(g, I)\right]$, which answers the question proposed in Section 1.
(2) If

$$
\underline{\operatorname{dim}}_{B} \Gamma(f, I)<\operatorname{dim}_{B} \Gamma(g, I)=\overline{\operatorname{dim}}_{B} \Gamma(f, I)
$$

we prove that

$$
\overline{\operatorname{dim}}_{B} \Gamma(f+g, I)=\overline{\operatorname{dim}}_{B} \Gamma(f, I)
$$

and $\underline{\operatorname{dim}}_{B} \Gamma(f+g, I)$ could be any number belonging to $\left[1, \operatorname{dim}_{B} \Gamma(g, I)\right)$.
(3) If

$$
{\underset{\operatorname{dim}}{B}} \Gamma(f, I)=\operatorname{dim}_{B} \Gamma(g, I)<\overline{\operatorname{dim}}_{B} \Gamma(f, I)
$$

we prove that

$$
\overline{\operatorname{dim}}_{B} \Gamma(f+g, I)=\overline{\operatorname{dim}}_{B} \Gamma(f, I)
$$

Then we study $\operatorname{dim}_{B} \Gamma(f+g, I)$ by whether $\operatorname{dim}_{B} \Gamma(g, I)$ is equal to one or not.
(i) If $\operatorname{dim}_{B} \Gamma(g, I)>1$, we prove that $\operatorname{dim}_{B} \Gamma(f+g, I)$ could be any number belonging to $\left[1, \operatorname{dim}_{B} \Gamma(g, I)\right)$;
(ii) If $\operatorname{dim}_{B} \Gamma(g, I)=1$, we prove that $\operatorname{dim}_{B} \Gamma(f+g, I)$ is equal to one.

Meanwhile, we should point out that the presented results can be generalized to any closed interval $[a, b]$. This means all the results obtained in the present paper still hold for two continuous functions $f$ and $g$ defined on $[a, b]$.

### 5.2. Main Methods

We emphasize that the key work in the present paper is to propose the following two main methods:
(1) We put forward a general method to calculate $\underline{\operatorname{dim}}_{B} \Gamma(f+g, I)$ and $\overline{\operatorname{dim}}_{B} \Gamma(f+g, I)$. We classify all the subsequences into different sets by the accumulation points of $\Phi_{f}(\delta)$ when $\delta \rightarrow 0$. Then, we just have to explore the minimum and maximum accumulation point of $\Phi_{f+g}(\delta)$ by the subsequences in every set, respectively, so that the values of $\underline{\operatorname{dim}}_{B} \Gamma(f+g, I)$ and $\overline{\operatorname{dim}}_{B} \Gamma(f+g, I)$ can be directly acquired. Hence, we find the values of $\underline{\operatorname{dim}}_{B} \Gamma(f+g, I)$ and $\overline{\operatorname{dim}}_{B} \Gamma(f+g, I)$ depend on different situations of the accumulation points distribution of $\Phi_{f}(\delta)$ and $\Phi_{g}(\delta)$ when $\delta \rightarrow 0$. In other words, studying the relationship between the accumulation points of $\Phi_{f}(\delta)$ and $\Phi_{g}(\delta)$ when $\delta \rightarrow 0$ is the fundamental approach to figuring out $\operatorname{dim}_{B} \Gamma(f+g, I)$ and $\check{\operatorname{dim}}_{B} \Gamma(f+g, I)$.
(2) We also obtain a way to seek the relationship between two fractal continuous functions. We find that the same accumulation point of $\Phi_{f}(\delta)$ and $\Phi_{g}(\delta)$ by the same subsequence is the "bridge" to connect $f(x)$ and $g(x)$. If we denote this same accumulation point as $s$, accumulation points of $\Phi_{f+g}(\delta)$ could be equal to any numbers belonging to $[1, s)$ by this subsequence.

For the calculation of $\operatorname{dim}_{B} \Gamma(f+g, I)$, the equivalent reformulation is that

$$
\operatorname{dim}_{B} \Gamma(f+g, I) \leq \max \left\{\operatorname{dim}_{B} \Gamma(f, I), \operatorname{dim}_{B} \Gamma(g, I)\right\}
$$

and $\operatorname{dim}_{B} \Gamma(f, I)$ is just the infimum of $p$ such that the unit interval $I$ can be split into $C \delta^{-p}$ subintervals $I_{j}$. Here $C$ is a certain constant number and $j=1,2, \cdots, C \delta^{-p}$. We note that $\left|I_{j}\right| \leq \delta$ and $R_{f} I_{j} \leq \delta$ for $\forall \delta>0$. Making the common splitting for $f(x)$ and $g(x)$ merely doubles the number of such intervals, which can be another way to obtain $\operatorname{dim}_{B} \Gamma(f+g, I)$.

### 5.3. Applications in Specific Examples

The calculation of the fractal dimension has been widely applied in a variety of fields such as metal materials. The fracture surface topography regarding the fatigue of metals can be investigated by fractal features, which can be found in [28,29]. Moreover, ref. [30] shows that the fractal dimension is closely related to the parameters of areal surface of metals. It is well known that there are a number of ways to calculate the fractal dimension, and the results of different methods and resolutions are slightly different. The present paper mainly studies how to calculate fractal dimension by counting boxes and how to calculate the fractal dimension of the superposition of two fractal curves. People could further explore the calculation of fractal dimension of the superposition of two fractal surfaces and apply it to the study of fracture surface topography regarding to the fatigue of metals. In the future, we will continue to do this work by visualizing specific examples, which shows the utility of our study well.

### 5.4. Improvement and Further Research

There still exist several points worthy of improvement and further discussion in the present work. We should point out that our results for fractal dimensions estimation are only based on theoretical analysis. However, examples of fractal continuous functions should be given to support these theoretical results. People could make further research on this problem by numerical simulation of fractal dimensions estimation for specific examples of fractal continuous functions.

At the end of our paper, we put forward an open question below:
Question 3. Suppose that $f(x) \in{ }_{s_{1}}^{s_{2}} D_{I}$ and $g(x) \in{ }_{s_{3}}^{s_{4}} D_{I}$. What can the lower and upper Box dimensions of $\Gamma(f+g, I)$ be, respectively?

People could try to explore this question by discussing the relationship among $s_{1}, s_{2}$, $s_{3}$ and $s_{4}$ in the future. The method used to deal with it may be similar with the present paper. Here, we give our conjecture for three cases as follows:

Conjecture 1. Let $f(x) \in{ }_{s_{1}}^{s_{2}} D_{I}$ and $g(x) \in{ }_{s_{3}}^{s_{4}} D_{I}$.
(1) If $s_{1}<s_{2}<s_{3}<s_{4}$,

$$
f+g \in{ }_{s_{3}}^{s_{4}} D_{I}
$$

(2) If $s_{1}<s_{3}<s_{4}<s_{2}$,

$$
f+g \in{ }_{v}^{s_{2}} D_{I}
$$

Here, $v$ could be any number belonging to $\left[1, s_{3}\right]$.
(3) If $s_{1}<s_{3}<s_{2}<s_{4}$,

$$
f+g \in{ }_{v}^{s_{4}} D_{I}
$$

Here, $v$ could be any number belonging to $\left[1, s_{2}\right]$.

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