



Article On the Approximate Solution of the Cauchy Problem in a Multidimensional Unbounded Domain

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Abstract: In this paper, the Carleman matrix is constructed, and based on it we found explicitly a regularized solution of the Cauchy problem for the matrix factorization of the Helmholtz equation in a multidimensional unbounded domain in \mathbb{R}^m , ($m = 2k, k \ge 2$). The corresponding theorems on the stability of the solution of problems are proved.

Keywords: integral formula; regularization of the Cauchy problem; approximate solution; Carleman matrix; family of vector functions; Bessel and Hankel functions

MSC: 35J46; 35J56

1. Introduction

The most actively developing modern area of scientific knowledge is the theory of correctly and incorrectly posed problems, most of which have practical value and require decision-making in uncertain or contradictory conditions. The development and justification of methods for solving such a complex class of problems as ill-posed ones is an intensely studied problem at the present time. The theory of ill-posed problems is an apparatus of scientific research for many scientific areas, such as the differentiation of approximately given functions, solving inverse boundary value problems, solving problems of linear programming and control systems, solving degenerate or ill-conditioned systems of linear equations, etc.

J. Hadamard [1] introduced the concept of a well-posed problem. It is due to their view that any mathematical problem corresponding to a physical or technological problem must be well-posed. J. Hadamard was concerned about what physical interpretation a solution might have if a small arbitrary change in the data could lead to major changes in the solution. He specified that it was difficult to apply approximation methods to such issues. Therefore, the opportunity arose to study ill-posed problems. For ill-posed problems, two questions were asked. The first one was: what is meant by approximate solution? It should be defined in such a way that it is stable in the event of minor changes to the original information. A second question was: what algorithms can we use to build such solutions? Tikhonov [2] gave the basic answers to these questions.

Formulas that allow one to find a solution to an elliptic equation in the case when the Cauchy data are known only on a part of the boundary of the domain are called Carlemantype formulas. In [3], Carleman established a formula giving a solution to the Cauchy– Riemann equations in a domain of a special form. Developing their idea, G.M. Goluzin and V.I. Krylov [4] derived a formula for determining the values of analytic functions from



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data known only on a portion of the boundary, already for arbitrary domains. A formula of the Carleman type, in which the fundamental solution of a differential operator with special properties (the Carleman function) is used, was obtained by M.M. Lavrent'ev (see, for instance [5–7]). Using this method, Sh.Ya. Yarmukhamedov (see, for instance [8–10]) constructed the Carleman functions for the Laplace and Helmholtz operators. Carlemantype formulas for various elliptic equations and systems were also obtained in [5–26]. A multidimensional analogue of Carleman's formula for analytic functions of several variables was constructed in [11]. In [15], an integral formula was proved for systems of equations of elliptic type of the first order, with constant coefficients in a bounded domain. Using the methodology of [8,9], Ikehata [22] considered the probe method and Carleman functions for the Helmholtz equation in the three-dimensional domain. Using exponentially growing solutions, Ikehata [23] obtained a formula for solving the Helmholtz equation with a variable coefficient for regions in space where the unknown data were located on a section of the hypersurface. In [20], the Cauchy problem was considered for the Helmholtz equation in an arbitrary bounded plane domain with Cauchy data, known only on the region boundary. The solvability criterion for the Cauchy problem for the Laplace equation in the space \mathbb{R}^m was considered by Shlapunov in [16]. In [27], the continuation of the problem for the Helmholtz equation was investigated and the results of numerical experiments were presented.

The concept of conditional correctness first appeared in the work of Tikhonov [2], and then in the studies of Lavrent'ev [5-7]. In a theoretical study of the conditional correctness (correctness according to Tikhonov) of an ill-posed problem of the existence of a solution and its belonging to the correctness set, it was postulated in the very formulation of the problem. The study of uniqueness issues in a conditionally well-posed formulation does not essentially differ from the study in a classically well-posed formulation, and the stability of the solution from the data of the problem is required only from those variations of the data that do not deduce solutions from the well-posedness set. After establishing the uniqueness and stability theorems in the study of the conditional correctness of illposed problems, the question arises of constructing effective solution methods, i.e., the construction of regularizing operators. The paper studied the construction of exact and approximate solutions to the ill-posed Cauchy problem for the matrix factorization of the Helmholtz equation. Such problems naturally arise in mathematical physics and in various fields of natural science (for example, in electrogeological exploration, in cardiology, in electrodynamics, etc.). In general, the theory of ill-posed problems for elliptic systems of equations has been sufficiently developed thanks to the works of A.N. Tikhonov, V.K. Ivanov, M.M. Lavrent'ev, N.N. Tarkhanov and others famous mathematicians. Among them, the most important for applications are the so-called conditionally well-posed problems, characterized by stability in the presence of additional information about the nature of the problem data. One of the most effective ways to study such problems is to construct regularizing operators. For example, this can be the Carleman-type formulas (as in a complex analysis) or iterative processes (the Kozlov-Maz'ya-Fomin algorithm, etc.) [15].

Based on the works from [8–10,24], in this paper, we construct the Carleman matrix and based on it the approximate solution of the Cauchy problem for the matrix factorization of the Helmholtz equation, in a multidimensional unbounded domain of \mathbb{R}^m . Based on the results of the previous work, we similarly obtain better results with approximate estimates for a multidimensional unbounded domain. When solving correct problems, sometimes it is not possible to find the value of the vector function on the entire boundary. Finding the value of a vector function on the entire boundary for systems of elliptic type with constant coefficients (see, for example, [26]) is an important problem in differential equations theory.

In many well-posed problems it is not possible to compute the values of the function on the whole boundary. Thus, one of the important problems in the theory of differential equations is the reconstruction of the solution of systems of equations of the first-order elliptic type, factorizing the Helmholtz operator. We recall that ill-posed problems of mathematical physics have been studied in recent years. Some papers in these directions, regarding the Laplace equation are [5,6,8–10]. Other results have been established in [9,13–20,22–26]. Boundary value problems and numerical solutions of some problems can be consulted in [28–33].

We consider $k \in \mathbb{N}$, $k \ge 1$, m = 2k, and the Euclidean space \mathbb{R}^m . Let

$$x = (x_1, \ldots, x_m) \in \mathbb{R}^m, \quad y = (y_1, \ldots, y_m) \in \mathbb{R}^m,$$

and

$$x' = (x_1, \ldots, x_{m-1}) \in \mathbb{R}^{m-1}, \quad y' = (y_1, \ldots, y_{m-1}) \in \mathbb{R}^{m-1}.$$

We also consider an unbounded simply connected domain $\Omega \subset \mathbb{R}^m$, having a piecewise smooth boundary $\partial \Omega = \Sigma \bigcup D$, where Σ is a smooth surface lying in the half-space $y_m > 0$ and D is the plane $y_m = 0$.

The following notations are used below:

$$r = |y - x|, \quad \alpha = |y' - x'|, \quad z = i\sqrt{a^2 + \alpha^2} + y_m, \quad a \ge 0.$$

$$\partial_x = (\partial_{x_1}, \dots, \partial_{x_m})^T, \quad \partial_x \to \xi^T, \quad \xi^T = \begin{pmatrix} \xi_1 \\ \dots \\ \xi_m \end{pmatrix} \text{-transposed vector } \xi,$$

$$V(x) = (V_1(x), \dots, V_n(x))^T, \quad v^0 = (1, \dots, 1) \in \mathbb{R}^n, \quad n = 2^m, \quad m \ge 2,$$

$$E(u) = \begin{vmatrix} u_1 & 0 & \cdots & 0 \\ 0 & u_2 & \cdots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & 0 & u_n \end{vmatrix} -\text{diagonal matrix}, \quad u = (u_1, \dots, u_n) \in \mathbb{R}^n.$$

 $P(\xi^T)$ is an $(n \times n)$ -dimensional matrix satisfying:

$$P^{*}(\xi^{T})P(\xi^{T}) = E((|\xi|^{2} + \lambda^{2})v^{0}),$$

where $P^*(\xi^T)$ is the Hermitian conjugate matrix of $P(\xi^T)$, $\lambda \in \mathbb{R}$, and the elements of the matrix $P(\xi^T)$ are linear functions with constant coefficients from \mathbb{C} .

Definition 1. Let us consider the following first-order system

$$P(\partial_x)V(x) = 0, \ x \in \Omega,\tag{1}$$

where $P(\partial_x)$ is the matrix of first-order differential operators. Furthermore, consider the set

$$S(\Omega) = \{ V : \overline{\Omega} \longrightarrow \mathbb{R}^n \},\$$

V being continuous on $\overline{\Omega}$, satisfying the system (1).

2. Statement of the Cauchy Problem

The Cauchy problem for system (1) is formulated as follows:

Definition 2. Let $f : \Sigma \longrightarrow \mathbb{R}^n$ be a continuous given function on Σ . Suppose $V(y) \in S(\Omega)$ and

$$V(y)|_{\Sigma} = f(y), \quad y \in \Sigma.$$
⁽²⁾

Our purpose is to determine the function V(y) *in the domain* Ω *when its values are known* Σ *.*

If $V(y) \in S(\Omega)$, then

$$V(x) = \int_{\partial \Omega} L(y, x; \lambda) V(y) ds_y, \quad x \in \Omega,$$
(3)

where

$$L(y, x; \lambda) = \left(E\left(\varphi_m(\lambda r)v^0\right) P^*(\partial_x) \right) P(t^T),$$

 $t = (t_1, ..., t_m)$ is the unit exterior normal at a point $y \in \partial \Omega$, $\varphi_m(\lambda r)$ is the fundamental solution of the Helmholtz equation in \mathbb{R}^m , and

$$\varphi_m(\lambda r) = N_m \lambda^{(m-2)/2} \frac{H_{(m-2)/2}^{(1)}(\lambda r)}{r^{(m-2)/2}},$$

$$N_m = \frac{1}{2i(2\pi)^{(m-2)/2}}, \quad m = 2k, \quad k \ge 2.$$
(4)

Here, $H_{(m-2)/2}^{(1)}(\lambda r)$ is the Hankel function of the first kind of order (m-2)/2 (see [34,35]). Let K(z) be an entire function taking real values for real z, $(z = a + ib, a, b \in \mathbb{R})$. We suppose

$$K(a) \neq 0, \quad \sup_{b \ge 1} \left| b^p K^{(p)}(z) \right| = B(a, p) < \infty,$$

$$-\infty < a < \infty, \quad p = 0, 1, \dots, m.$$
(5)

Define, for $y \neq x$,

$$\Psi(y, x; \lambda) = \frac{1}{c_m K(x_m)} \frac{\partial^{k-1}}{\partial s^{k-1}} \int_0^\infty \operatorname{Im}\left[\frac{K(z)}{z - x_m}\right] \frac{a I_0(\lambda a)}{\sqrt{a^2 + \alpha^2}} da,$$

$$z = i\sqrt{a^2 + \alpha^2} + y_m,$$
(6)

where $c_m = (-1)^{k-1}(k-1)!(m-2)\omega_m$; $I_0(\lambda a) = J_0(i\lambda a)$ is the Bessel function of the first kind of order zero [34,35], and ω_m is the area of a unit sphere in \mathbb{R}^m .

Remark that (3) holds if we replace $\varphi_m(\lambda r)$ by

$$\Psi(y, x; \lambda) = \varphi_m(\lambda r) + g(y, x; \lambda), \tag{7}$$

where g(y, x) is the regular solution of the Helmholtz equation with respect to y, including the case y = x.

Hence, (3) becomes

$$V(x) = \int_{\partial\Omega} L(y, x; \lambda) V(y) ds_y, \quad x \in \Omega,$$

$$L(y, x; \lambda) = \left(E\left(\Psi(y, x; \lambda) v^0\right) P^*(\partial_x) \right) P(t^T).$$
(8)

Formula (8) can be generalized for an unbounded domain
$$\Omega$$
.

We therefore consider an unbounded domain $\Omega \subset \mathbb{R}^m$, finitely connected, with a piecewise smooth boundary $\partial \Omega$ extending to infinity.

Let Ω_R be the part of Ω situated inside the circle of radius *R*, centered at zero:

$$\Omega_R = \{y: y\in \Omega, \quad |y| < R\}, \quad \Omega_R^\infty = \Omega ackslash \Omega_R, \quad R > 0$$

Assume that Ω is situated inside the portion

$$0 < y_m < h, \quad h = \frac{\pi}{\rho}, \quad \rho > 0,$$

of smallest width, and $\partial \Omega$ extends to infinity.

Theorem 1. Let $V(y) \in S(\Omega)$. If $\forall x \in \Omega$, x fixed, we have

$$\lim_{R \to \infty} \int_{\Omega_R^{\infty}} L(y, x; \lambda) V(y) ds_y = 0,$$
(9)

then the formula (8) is true.

Proof. Let us fix $x \in \Omega$ (|x| < R). Using (8), we get

$$\int_{\partial\Omega} L(y, x; \lambda) V(y) ds_y = \int_{\partial\Omega_R} L(y, x; \lambda) V(y) ds_y$$
$$+ \int_{\partial\Omega_R^{\infty}} L(y, x; \lambda) V(y) ds_y = V(x) + \int_{\partial\Omega_R^{\infty}} L(y, x; \lambda) V(y) ds_y, \quad x \in \Omega_R.$$

Taking into account condition (9), for $R \to \infty$, we obtain (8). Furthermore, suppose, for $d_0 > 0$,

$$\int_{\partial\Omega} \exp\left[-d_0\rho_0 |y'|\right] ds_y < \infty, \quad 0 < \rho_0 < \rho.$$
⁽¹⁰⁾

Suppose $V(y) \in S(\Omega)$ so that it satisfies the boundary growth condition

$$|V(y)| \le \exp\left[\exp\rho_2|y'|\right], \quad \rho_2 < \rho, \quad y \in \Omega.$$
(11)

In (6), we put

$$K(z) = \exp\left[-d i\rho_1\left(z - \frac{h}{2}\right) - d_1 i\rho_0\left(d - \frac{h}{2}\right)\right],$$

$$K(x_m) = \exp\left[d \cos\rho_1\left(x_m - \frac{h}{2}\right) + d_1 \cos i\rho_0\left(x_m - \frac{h}{2}\right)\right],$$

$$0 < \rho_1 < \rho, \quad 0 < x_m < h,$$
(12)

where

$$d = 2c \exp(
ho_1 |x'|), \quad d_1 > rac{d_0}{\cos\left(
ho_0 rac{h}{2}
ight)}, \quad c \ge 0, \quad d > 0.$$

Then, the integral representation (8) is true.

For $x \in \Omega$, x fixed and $y \to \infty$, we estimate $\Psi(y, x; \lambda)$ and its derivatives $\frac{\partial \Psi(y, x; \lambda)}{\partial y_j}$, (j = 1, ..., m - 1), $\frac{\partial \Psi(y, x; \lambda)}{\partial y_m}$. For the estimation of $\frac{\partial \Psi(y, x; \lambda)}{\partial y_j}$ we use

$$\frac{\partial \Psi(y, x; \lambda)}{\partial y_j} = \frac{\partial \Psi(y, x; \lambda)}{\partial s} \frac{\partial s}{\partial y_j} = 2(y_j - x_j) \frac{\partial \Psi(y, x; \lambda)}{\partial s},$$

$$s = \alpha^2, \quad j = 1, \dots, m - 1.$$
(13)

Really,

$$\left| \exp\left[-d\,i\rho_1\left(z - \frac{h}{2}\right) - d_1i\rho_0\left(z - \frac{h}{2}\right) \right] \right|$$
$$= \exp\operatorname{Re}\left[-d\,i\rho_1\left(z - \frac{h}{2}\right) - d_1i\rho_0\left(z - \frac{h}{2}\right) \right]$$
$$= \exp\left[-d\,\rho_1\sqrt{a^2 + \alpha^2}\cos\rho_1\left(y_m - \frac{h}{2}\right) - d_1\rho_0\sqrt{a^2 + \alpha^2}\cos\rho_0\left(y_m - \frac{h}{2}\right) \right].$$
As
$$-\frac{\pi}{2} \le -\frac{\rho_1}{\rho} \cdot \frac{\pi}{2} \le \frac{\rho_1}{\rho} \cdot \frac{\pi}{2} < \frac{\pi}{2},$$

$$-\frac{\pi}{2} \leq -\frac{\rho_1}{\rho} \cdot \frac{\pi}{2} \leq \rho_0 \left(y_m - \frac{h}{2} \right) \leq \frac{\rho_1}{\rho} \cdot \frac{\pi}{2} < \frac{\pi}{2}.$$

Consequently,

$$\cos
hoig(y_m-rac{h}{2}ig)>0, \quad \cos
ho_0ig(y_m-rac{h}{2}ig)\geq\cosrac{h
ho_0}{2}>\delta_0>0,$$

it does not vanish in the region Ω and

$$\begin{split} |\Psi(y,x;\lambda)| &= O[\exp(-\varepsilon\rho_1|y'|)], \quad \varepsilon > 0, \quad y \to \infty, \quad y \in \Omega \bigcup \partial \Omega, \\ \left| \frac{\partial \Psi(y,x;\lambda)}{\partial y_j} \right| &= O[\exp(-\varepsilon\rho_1|y'|)], \quad \varepsilon > 0, \quad y \to \infty, \quad y \in \Omega \bigcup \partial \Omega, \ j = 1, \dots, m-1. \\ \left| \frac{\partial \Psi(y,x;\lambda)}{\partial y_m} \right| &= O[\exp(-\varepsilon\rho_1|y'|)], \quad \varepsilon > 0, \quad y \to \infty, \quad y \in \Omega \bigcup \partial \Omega. \end{split}$$

We now choose ρ_1 with the condition $\rho_2 < \rho_1 < \rho$. Then, condition (10) is fulfilled and the integral formula (8) is true. \Box

Condition (12) can be weakened. Consider now the class $S_{\rho}(\Omega)$ of vector-valued functions from $S(\Omega)$, satisfying the following growth condition:

$$S_{\rho}(\Omega) = \{ V(y) \in S(\Omega), \quad |V(y)| \le \exp\left[o\left(\exp\rho\left|y'\right|\right)\right], \quad y \to \infty, \quad y \in \Omega \}.$$
(14)

We obtain the following theorem:

Theorem 2. Suppose $V(y) \in S_{\rho}(\Omega)$ so that it satisfies the growth condition

$$|V(y)| \le C \exp\left[c \cos \rho_1 \left(y_m - \frac{h}{2}\right) \exp\left(\rho_1 |y'|\right)\right],$$

$$c \ge 0, \quad 0 < \rho_1 < \rho, \quad y \in \partial\Omega,$$
(15)

where C is some constant. Then, formula (8) is valid.

Proof. Divide the area Ω by a line $y_m = \frac{h}{2}$ into two areas

$$\Omega_1 = \left\{ y : 0 < y_m < \frac{h}{2} \right\} \text{ and } \Omega_2 = \left\{ y : \frac{h}{2} < y_m < h \right\}.$$

Consider the domain Ω_1 . In formula (6) we put together K(z) and $K_1(z)$

$$K_{1}(z) = K(z) \exp\left[-\delta i\tau \left(z - \frac{h}{2}\right) - \delta_{1}i\rho \left(z - \frac{h}{2}\right)\right],$$

$$\rho < \tau < 2\rho, \quad \delta > 0, \quad \delta_{1} > 0,$$
(16)

K(z) being given by (12). With this notation, (10) is true. Really,

$$\left| \exp\left[-i\tau\left(z-\frac{h}{4}\right)-\delta_{1}i\rho\left(z-\frac{h}{4}\right)\right] \right|$$
$$= \exp\left[-\delta\tau\sqrt{a^{2}+\alpha^{2}}\cos\tau\left(y_{m}-\frac{h}{4}\right)\right]$$
$$= \exp\left[-\delta\tau\sqrt{a^{2}+\alpha^{2}}\right] \le \exp\left[-\delta\exp\tau|y'|\right],$$

as

$$-\frac{\pi}{2} \le -\tau\frac{\pi}{4} \le \tau\left(y_m - \frac{h}{4}\right) \le \tau\frac{\pi}{2} < \frac{h}{2} \text{ and } \cos\tau\left(y_m - \frac{h}{4}\right) \ge \cos\tau\frac{h}{4} \ge \delta_0 > 0$$

Denote the corresponding $\Psi(y, x; \lambda)$ by $\Psi^+(y, x; \lambda)$. As

$$\cos au \left(y_m-rac{h}{4}
ight)\geq \delta_0, \quad y\in \Omega_1igcup \partial\Omega_1,$$

then for a fixed $x \in \Omega_1$, $y \in \Omega_1 \bigcup \partial \Omega_1$, for $\Psi^+(y, x; \lambda)$, and its derivatives are true asymptotic estimates

$$\begin{split} \Psi^{+}(y,x;\lambda) &|= O[\exp(-\delta_{0}\,\exp(\tau|y'|)], \quad y \to \infty, \quad \rho < \tau < 2\rho, \\ \left| \frac{\partial \Psi^{+}(y,x;\lambda)}{\partial y_{j}} \right| &= O[\exp(-\delta_{0}\,\exp(\tau|y'|)], \quad y \to \infty, \quad \rho < \tau < 2\rho, \quad j = 1, \dots, m-1. \\ \left| \frac{\partial \Psi^{+}(y,x;\lambda)}{\partial y_{m}} \right| &= O[\exp(-\delta_{0}\,\exp(\tau|y'|)], \quad y \to \infty, \quad \rho < \tau < 2\rho. \end{split}$$

Suppose $V(y) \in S_{\rho}(\Omega_1)$ so that in a domain Ω_1 satisfies

$$|V(y)| \le C \exp\left[\exp(2\rho - \varepsilon)|y'|\right], \quad \varepsilon > 0.$$
(17)

Consider τ in (16) such that

$$2\rho - \varepsilon < \tau < 2\rho. \tag{18}$$

Then, (16) is satisfied in Ω_1 , hence:

$$V(x) = \int_{\partial \Omega_1} L(y, x; \lambda) V(y) ds_y, \quad x \in \Omega_1,$$
(19)

where

$$L(y,x;\lambda) = \left(E\left(\Psi^+(y,x;\lambda)v^0\right) P^*(\partial_x) \right) P(t^T).$$

If $V(y) \in S_{\rho}(\Omega_2)$ satisfies the growth condition (15) in Ω_2 , then for τ satisfying (18), similarly, we obtain

$$V(x) = \int_{\partial \Omega_2} L(y, x; \lambda) V(y) ds_y, \quad x \in \Omega_2,$$

$$L(y, x; \lambda) = \left(E\left(\Psi^{-}(y, x; \lambda) v^0\right) P^*(\partial_x) \right) P(t^T).$$
(20)

Here, $\Psi^{-}(y, x; \lambda)$ is defined by the formula (6), K(z) being replaced by

$$K_2(z) = K(z) \exp\left[-\delta i\tau(z-h_1) - \delta_1 i\rho\left(z-\frac{h}{2}\right)\right],\tag{21}$$

where

$$h_1 = \frac{h}{2} + \frac{h}{4}, \quad \frac{h}{2} < y_m < h, \quad \frac{h}{2} < x_m < h_1, \quad \delta > 0, \quad \delta_1 > 0.$$

In the formulas obtained with this formula, the integrals (according to (11)) converge uniformly for $\delta \ge 0$, when $V(y) \in S_{\rho}(\Omega)$. In these formulas, we put $\delta = 0$, hence

$$V(x) = \int_{\partial \Omega} L(y, x; \lambda) V(y) ds_y, \quad x \in \Omega, \quad x_m \neq \frac{h}{2},$$
(22)

where

$$L(y, x; \lambda) = \left(E\left(\tilde{\Psi}(y, x; \lambda)v^0\right) P^*(\partial_x) \right) P(t^T)$$

(the integrals over the cross section $y_m = \frac{h}{2}$ are mutually destroyed)

$$\tilde{\Psi}(y,x;\lambda) = (\Psi^+(y,x;\lambda))_{\delta=0} = (\Psi^-(y,x;\lambda))_{\delta=0}.$$

 $\tilde{\Psi}(y, x; \lambda)$ is given by (6), in which K(z) is determined by (16), for $\delta = 0$. Using the continuation principle, (22) holds, $\forall x \in \Omega$. If (17) is satisfied, formula (22) is valid, $\forall \delta_1 \ge 0$. Supposing $\delta_1 = 0$, Theorem 2 is proved. \Box

In formula (6), choosing

$$K(z) = \frac{1}{(z - x_m + 2h)^k} \exp(\sigma z), \quad k \ge 2,$$

$$K(x_m) = \frac{1}{(2h)^k} \exp(\sigma x_m), \quad 0 < x_m < h, \quad h = \frac{\pi}{\rho},$$
(23)

we get

$$\Psi_{\sigma}(y,x) = -\frac{e^{-\sigma x_m}}{c_m(2h)^{-k}} \frac{\partial^{k-1}}{\partial s^{k-1}} \int_0^\infty \operatorname{Im} \frac{\exp(\sigma z)}{(z-x_m+2h)^k(z-x_m)} \frac{a I_0(\lambda a)}{\sqrt{a^2+\alpha^2}} da.$$
(24)

Hence, (8) becomes:

$$V(x) = \int_{\partial \Omega} L_{\sigma}(y, x; \lambda) V(y) ds_y, \quad x \in \Omega,$$
(25)

where

$$L_{\sigma}(y, x; \lambda) = \left(E\left(\Psi_{\sigma}(y, x; \lambda) v^0 \right) P^*(\partial_x) \right) P(t^T).$$

3. Regularization of the Cauchy Problem and Estimation of Conditional Stability

Theorem 3. Let $V(y) \in S_{\rho}(\Omega)$ satisfying in the following inequality

$$|V(y)| \le M, \quad y \in D. \tag{26}$$

If

$$V_{\sigma}(x) = \int_{\Sigma} L_{\sigma}(y, x; \lambda) V(y) ds_{y}, \quad x \in \Omega,$$
(27)

then

$$|V(x) - V_{\sigma}(x)| \le K_{\rho}(\lambda, x)\sigma^{k}Me^{-\sigma x_{m}}, \quad x \in \Omega,$$
(28)

where $K_{\rho}(\lambda, x)$ are bounded functions on compact subsets of Ω .

Proof. Using (25) and (27), we have

$$egin{aligned} V(x) &= \int\limits_{\Sigma} L_{\sigma}(y,x;\lambda) U(y) ds_y + \int\limits_{D} L_{\sigma}(y,x;\lambda) V(y) ds_y \ &= L_{\sigma}(x) + \int\limits_{D} L_{\sigma}(y,x;\lambda) V(y) ds_y, \quad x \in \Omega. \end{aligned}$$

According to (26), we get

$$|V(x) - V_{\sigma}(x)| \leq \left| \int_{D} L_{\sigma}(y, x; \lambda) V(y) ds_{y} \right|$$

$$\leq \int_{D} |L_{\sigma}(y, x; \lambda)| |V(y)| ds_{y} \leq M \int_{D} |L_{\sigma}(y, x; \lambda)| ds_{y}, \quad x \in \Omega.$$
(29)

We estimate now the following integrals $\int_{D} |\Psi_{\sigma}(y, x; \lambda)| ds_y$, $\int_{D} \left| \frac{\partial \Psi_{\sigma}(y, x; \lambda)}{\partial y_j} \right| ds_y$, (j = 1, ..., m-1) and $\int_{D} \left| \frac{\partial \Psi_{\sigma}(y, x; \lambda)}{\partial y_m} \right| ds_y$ on $D: y_m = 0$.

We separate the imaginary part of (24), hence we get

$$\Psi_{\sigma}(y, x) = \frac{e^{\sigma(y_m - x_m)}}{c_m(2h)^{-k}} \frac{\partial^{k-1}}{\partial s^{k-1}} \left[\int_0^\infty \left(\frac{(\beta + \beta_1) \cos \sigma \alpha_1}{(\alpha_1^2 + \beta_1^2) (\alpha_1^2 + \beta^2)} + \frac{(-\alpha_1^2 + \beta_1 \beta)}{(\alpha_1^2 + \beta_1^2) (\alpha_1^2 + \beta^2)} \frac{\sin \sigma \alpha_1}{\alpha_1} \right) a I_0(\lambda a) da \right],$$
(30)

where

$$\alpha_1^2 = a^2 + \alpha^2, \quad \beta = y_m - x_m, \quad \beta_1 = y_m - x_m + 2h.$$

Given (30) and the inequality

$$I_0(\lambda a) \le \sqrt{\frac{2}{\lambda \pi a}},\tag{31}$$

we have

$$\int_{D} |\Psi_{\sigma}(y, x; \lambda)| ds_{y} \leq K_{\rho}(\lambda, x) \sigma^{k} e^{-\sigma x_{m}}, \quad \sigma > 1, \quad x \in \Omega.$$
(32)

Next, we use:

$$\frac{\partial \Psi_{\sigma}(y, x; \lambda)}{\partial y_{j}} = \frac{\partial \Phi_{\sigma}(y, x; \lambda)}{\partial s} \frac{\partial s}{\partial y_{j}} = 2(y_{j} - x_{j}) \frac{\partial \Psi_{\sigma}(y, x; \lambda)}{\partial s},$$

$$s = \alpha^{2}, \quad j = 1, \dots, m - 1.$$
(33)

According to (30), (31) and (33), we have

$$\int_{D} \left| \frac{\partial \Psi_{\sigma}(y, x; \lambda)}{\partial y_{j}} \right| ds_{y} \leq K_{\rho}(\lambda, x) \sigma^{k} e^{-\sigma x_{m}}, \quad \sigma > 1, \quad x \in \Omega,$$
(34)

Now, we estimate the integral $\int_{D} \left| \frac{\partial \Psi_{\sigma}(y, x; \lambda)}{\partial y_m} \right| ds_y$, and we obtain

$$\int_{D} \left| \frac{\partial \Psi_{\sigma}(y, x; \lambda)}{\partial y_{m}} \right| ds_{y} \leq K_{\rho}(\lambda, x) \sigma^{k} e^{-\sigma x_{m}}, \quad \sigma > 1, \quad x \in \Omega,$$
(35)

Using (32), (34), (35) and (29), (28) is proved. □

Corollary 1.

$$\lim_{\sigma\to\infty}V_{\sigma}(x)=V(x),$$

holds uniformly on every compact set from Ω .

Suppose that the boundary of the domain Ω consists of a hyper plane $y_m = 0$ and a smooth surface Σ extending to infinity and lying in the layer

$$0 < y_m < h, \quad h = \frac{\pi}{\rho}, \quad \rho > 0.$$

We assume that Σ is given by the equation

$$y_m = \psi(y'), \quad y' \in \mathbb{R}^{m-1},$$

and $\psi(y')$ satisfies the condition

$$|\psi'(y')| \le M < \infty, \quad M = const.$$

Theorem 4. If $V(y) \in S_{\rho}(\Omega)$ satisfies the condition in (26) and

$$|V(y)| \le \delta, \quad 0 < \delta < 1, \ y \in \Sigma, \tag{36}$$

then

$$|V(x)| \le K_{\rho}(\lambda, x)\sigma^{k}M^{1-\frac{x_{m}}{h}}\delta^{\frac{x_{m}}{h}}, \quad \sigma > 1, \quad x \in \Omega.$$
(37)

Proof. From (25), we get

$$V(x) = \int_{\Sigma} L_{\sigma}(y, x; \lambda) V(y) ds_y + \int_{D} L_{\sigma}(y, x; \lambda)) V(y) ds_y, \quad x \in \Omega.$$
(38)

We estimate the following

$$|V(x)| \leq \left| \int_{\Sigma} L_{\sigma}(y, x; \lambda) V(y) ds_{y} \right| + \left| \int_{D} L_{\sigma}(y, x; \lambda) V(y) ds_{y} \right|, \quad x \in \Omega.$$
(39)

We have

$$\left| \int_{\Sigma} L_{\sigma}(y, x; \lambda) V(y) ds_{y} \right| \leq \int_{\Sigma} |L_{\sigma}(y, x; \lambda)| |V(y)| ds_{y}
\leq \delta \int_{\Sigma} |L_{\sigma}(y, x; \lambda)| ds_{y}, \quad x \in \Omega.$$
(40)

Using (30) and (31), we obtain

$$\int_{\Sigma} |\Psi_{\sigma}(y, x; \lambda)| ds_{y} \leq K_{\rho}(\lambda, x) \sigma^{k} e^{\sigma(h - x_{m})}, \quad \sigma > 1, \quad x \in \Omega.$$
(41)

Using (30)–(32), we get

$$\int_{\Sigma} \left| \frac{\partial \Psi_{\sigma}(y, x; \lambda)}{\partial y_{j}} \right| ds_{y} \leq K_{\rho}(\lambda, x) \sigma^{k} e^{\sigma(h - x_{m})}, \quad \sigma > 1, \quad x \in \Omega.$$
(42)

Furthermore, from (30) and (31), we have

$$\int_{\Sigma} \left| \frac{\partial \Psi_{\sigma}(y, x; \lambda)}{\partial y_m} \right| ds_y \le K_{\rho}(\lambda, x) \sigma^k e^{\sigma(h - x_m)}, \quad \sigma > 1, \quad x \in \Omega.$$
(43)

From (41)–(43) and applying (40), we get

$$\left| \int_{\Sigma} L_{\sigma}(y, x; \lambda) V(y) ds_y \right| \le K_{\rho}(\lambda, x) \sigma^k \delta e^{\sigma(h - x_m)}, \quad \sigma > 1, \quad x \in \Omega.$$
(44)

We know that

$$\left| \int_{D} L_{\sigma}(y, x; \lambda) V(y) ds_{y} \right| \leq K_{\rho}(\lambda, x) \sigma^{k} M e^{-\sigma x_{m}}, \quad \sigma > 1, \quad x \in \Omega.$$
(45)

Now, taking into account (44) and (45) and using (39) and (40), we have

.

$$|V(x)| \le \frac{K_{\rho}(\lambda, x)\sigma^{k}}{2} (\delta e^{\sigma h} + M)e^{-\sigma x_{m}}, \quad \sigma > 1, \quad x \in \Omega.$$
(46)

Considering

$$\sigma = \frac{1}{h} \ln \frac{M}{\delta},\tag{47}$$

(37) is proved. \Box

Consider now $V(y) \in S_{\rho}(\Omega)$ and instead of V(y) on Σ , its continuous approximations $f_{\delta}(y)$ are given, with error $0 < \delta < 1$. Then,

$$\max_{\Sigma} |V(y) - f_{\delta}(y)| \le \delta.$$
(48)

We put

$$V_{\sigma(\delta)}(x) = \int_{\Sigma} L_{\sigma}(y, x; \lambda) f_{\delta}(y) ds_y, \quad x \in \Omega.$$
(49)

Theorem 5. If $V(y) \in S_{\rho}(\Omega)$ satisfies (26) on the plane $y_m = 0$, then

$$\left| V(x) - V_{\sigma(\delta)}(x) \right| \le K_{\rho}(\lambda, x) \sigma^k M^{1 - \frac{x_m}{h}} \delta^{\frac{x_m}{h}}, \quad \sigma > 1, \quad x \in \Omega.$$
(50)

Proof. From the integral formulas (25) and (49), we have

$$V(x) - V_{\sigma(\delta)}(x) = \int_{\partial\Omega} L_{\sigma}(y, x; \lambda) L(y) ds_{y}$$
$$- \int_{\Sigma} L_{\sigma}(y, x; \lambda) f_{\delta}(y) ds_{y} = \int_{\Sigma} L_{\sigma}(y, x; \lambda) V(y) ds_{y}$$
$$+ \int_{D} L_{\sigma}(y, x; \lambda) V(y) ds_{y} - \int_{\Sigma} L_{\sigma}(y, x; \lambda) f_{\delta}(y) ds_{y}$$
$$= \int_{\Sigma} L_{\sigma}(y, x; \lambda) \{V(y) - f_{\delta}(y)\} ds_{y} + \int_{D} L_{\sigma}(y, x; \lambda) L(y) ds_{y}.$$

Using conditions (26) and (48), we have:

$$\begin{split} \left| V(x) - V_{\sigma(\delta)}(x) \right| &= \left| \int_{\Sigma} L_{\sigma}(y, x; \lambda) \{ V(y) - f_{\delta}(y) \} ds_{y} \right| \\ &+ \left| \int_{D} L_{\sigma}(y, x; \lambda) V(y) ds_{y} \right| \leq \int_{\Sigma} |L_{\sigma}(y, x; \lambda)| |\{ V(y) - f_{\delta}(y) \} |ds_{y} \\ &+ \int_{D} |L_{\sigma}(y, x; \lambda)| |V(y)| ds_{y} \leq \delta \int_{\Sigma} |L_{\sigma}(y, x; \lambda)| ds_{y} \\ &+ M \int_{D} |L_{\sigma}(y, x; \lambda)| |ds_{y}. \end{split}$$

As in the proof of Theorems 3 and 4, we get

$$\left|V(x)-V_{\sigma(\delta)}(x)\right|\leq rac{K_{
ho}(\lambda,x)\sigma^{k}}{2}(\delta e^{\sigma h}+M)e^{-\sigma x_{m}},$$

From here, choosing σ from equality (47), we obtain an estimate (50). \Box

Corollary 2.

$$\lim_{\delta\to 0} V_{\sigma(\delta)}(x) = V(x)$$

holds uniformly on each compact set of Ω .

Example 1. Consider the following system of partial differential equations of first order:

$$\begin{aligned} \frac{\partial V_1}{\partial x_1} - \frac{\partial V_2}{\partial x_2} + iV_4 &= 0, \\ \frac{\partial V_1}{\partial x_2} + \frac{\partial V_2}{\partial x_1} + iV_3 &= 0, \\ -\frac{\partial V_3}{\partial x_1} + \frac{\partial V_4}{\partial x_2} - iV_2 &= 0, \\ \frac{\partial V_3}{\partial x_2} + \frac{\partial V_4}{\partial x_1} + iV_1 &= 0. \end{aligned}$$

Check that the following relations hold:

$$P^*(\xi^T)P(\xi^T) = E((|\xi|^2 + \lambda^2)v^0), \quad v^0 = (1, \dots, 1) \in \mathbb{R}^n.$$
(51)

Assuming
$$\frac{\partial}{\partial x_1} \to \xi_1 \text{ and } \frac{\partial}{\partial x_2} \to \xi_2, \text{ compose the following matrices}$$

$$P(\xi^T) = \begin{pmatrix} \xi_1 & \xi_2 & 0 & i \\ -\xi_2 & \xi_1 & -i & 0 \\ 0 & i & -\xi_1 & \xi_2 \\ i & 0 & \xi_2 & \xi_1 \end{pmatrix}, \quad P^*(\xi^T) = \begin{pmatrix} \xi_1 - \xi_2 & 0 & -i \\ \xi_2 & \xi_1 & -i & 0 \\ 0 & i & -\xi_1 & \xi_2 \\ -i & 0 & \xi_2 & \xi_1 \end{pmatrix}.$$

The relation (51) *is easily checked.*

4. Conclusions

We built in this paper a family of vector-functions $V_{\sigma(\delta)}(x) = V(x, f_{\delta})$ (called a regularized solution of the problem for matrix factorizations of the Helmholtz equation), in a multidimensional unbounded domain in \mathbb{R}^m , m = 2k, $k \ge 2$, depending on a parameter σ , and we proved that the family $V_{\sigma(\delta)}(x)$ converged in the usual sense to a solution V(x) at a point $x \in \Omega$, if certain conditions were set on the parameter $\sigma = \sigma(\delta)$, at $\delta \to 0$. A regularized solution determines a stable method of finding an approximate solution to the problems (1) and (2).

Hence, functional $V_{\sigma(\delta)}(x)$ determines the regularization of the solution of the problem for matrix factorizations of the Helmholtz equation.

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