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Convergence Rate of the High-Order Finite Difference Method for Option Pricing in a Markov Regime-Switching Jump-Diffusion Model

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Abstract: The high-order finite difference method for option pricing is one of the most popular numerical algorithms. Therefore, it is of great significance to study its convergence rate. Based on the relationship between this algorithm and the trinomial tree method, as well as the definition of local remainder estimation, a strict mathematical proof is derived for the convergence rate of the high-order finite difference method for option pricing in a Markov regime-switching jump-diffusion model. The theoretical result shows that the convergence rate of this algorithm is $O(\Delta\tau)$. Moreover, the results also hold in the case of Brownian motion and jump-diffusion models that are specialized forms of the given model.

Keywords: convergence rate; high-order finite difference method; Markov regime-switching jump-diffusion model; partial integro-differential equations

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1. Introduction

1.1. Background

Partial integro-differential equations (PIDEs) in a Markov regime-switching jump-diffusion model are popular in financial engineering ([1–14]). The advantages of this model lie in two aspects: on the one hand, the Markov chain reflects the information of market environments; on the other hand, it accurately describes the behavior of the underlying asset. However, it is difficult to solve the PIDEs due to the close relation to the Markov chain.

Some numerical methods, such as the high-order finite difference scheme, have been widely used to solve the PIDEs. The principle of the high-order difference method is to obtain finite difference approximations for high-order derivatives in the truncation error by operating on the differential equations as an auxiliary relation. The high-order schemes in a central difference approximation increase the order of accuracy. During and Fournie ([15–17]) derived a high-order difference scheme under the Heston model in 2012 and extended this method to non-uniform grids in 2014 and to multiple space dimensions in 2015. In 2019, During and Pitkin [18] applied this approach to stochastic volatility jump models. Additionally, some other scholars have put forward an improved algorithm based on higher-order finite difference in their papers ([19–25]). Rambeerich and Pantelous [4] developed a high-order finite element scheme to approximate the spatial terms of PIDE using linear and quadratic basis polynomial approximations and solved the resulting initial value problem using exponential time integration. Patel [6] proposed a fourth-order compact finite difference scheme for the solution of PIDE under regime-switching jump-diffusion models. Tour et al. [7] developed a high-order radial basis function finite difference (RBF-FD) approximation on a five-point stencil under the regime-switching stochastic volatility models with log-normal and contemporaneous jumps. Ma et al. [26] presented the high-order equivalence between the finite difference method and trinomial trees method for regime-switching models and proved the convergence rates of trinomial trees for pricing options with state-dependent switching rates using the theory of the FDMs.

It is of great importance to investigate the convergence rate of algorithms based on the Markov chain with finite difference schemes. In 2010, Alfonsi [27] presented weak second and third-order schemes for the CIR process and gave a general recursive construction method for obtaining weak second-order schemes. In 2017, Altmayer and Neuenkirch [28] established a weak convergence rate of order one under mild assumptions regarding the smoothness of the payoff. Zheng [29] derived that the weak convergence rate of a time-discrete scheme for the Heston stochastic volatility model was 2 for all parameter regimes. In 2018, Bossy and Olivero [30] studied the rate of convergence of a symmetrized version of the Milstein scheme applied to the solution of the one-dimensional SDE. Briani et al. studied the rate of weak convergence of Markov chains to diffusion processes under suitable, but quite general, assumptions in [31] and developed stability properties of a hybrid approximation of the functional of the Bates jump model with the stochastic interest rate in [32]. Lesmana and Wang [33] presented the consistency, stability, convergence, and numerical simulations of American options with transaction cost under a jump-diffusion process.

However, these papers all show the efficiency of this algorithm via numerical examples. It is important to give strict mathematical proof to guarantee the correctness of the high-order difference method. The objective of this article is to investigate the convergence rate of the high-order difference scheme (5)–(15) for option pricing assuming a Markov regime-switching jump-diffusion model (1) followed by the underlying asset.

1.2. The PIDEs in a Markov Regime-Switching Jump-Diffusion Model

Under the risk-neutral measure, the underlying $x_t = \log S_t$ will be modelled by a Markov regime-switching jump-diffusion model.

$$dx_t = [r(\alpha_t) - \beta(\alpha_t)\lambda(\alpha_t)]dt + \sigma(\alpha_t)dW_t + [\eta(\alpha_t) - 1]dQ_t \tag{1}$$

where $\{W_t\}_{t\geq 0}$ is a standard Brownian motion, $\{\alpha_t\}$ is a continuous-time Markov chain with finite states $\{1, 2, \cdots, n\}$, $r(\alpha_t) = r_i$ is the risk-free rate, $\sigma(\alpha_t) = \sigma_i$ denotes the constant volatility, $\{Q_t\}$ represents the compound Poisson process with intensity $\lambda(\alpha_t) = \lambda_i$ at state i, $[\eta(\alpha_t) - 1] = \eta_i - 1$ denotes the function which jump from S_t to $S_t\eta_i$. The expectation of this function is then given by $\beta(\alpha_t) = \beta_i$ where $\beta_i = E(\eta_i - 1)$. We assume that the stochastic processes $\{W_t\}_{t\geq 0}$ and $\{Q_t\}_{t\geq 0}$ in (1) are mutually independent in this paper.

Let $A = (\rho_{il}), i, l = 1, 2, \cdots, n$, be the generator matrix of the Markov chain process whose elements are constants satisfying $\rho_{il} \geq 0$ for $i \neq l$ and $\sum_{l=1}^{n} \rho_{il} = 0$ for $i = 1, 2, \cdots, n$.

Let the underlying x_t satisfy (1). Then, the value of a European option $V^i(x, \tau)$ satisfies the following PIDE:

$$\begin{cases} \frac{\partial V^i(x,\tau)}{\partial \tau} = \mathcal{L}V^i(x,\tau) + \mathcal{I}V^i(x,\tau) + \sum\limits_{l=1}^{n} \rho_{il} V^l(x,\tau),\ i = 1, 2, \cdots, n,\ (x,\tau) \in R \times [0, T], \\ V^i(x,\tau) = \begin{cases} 0,\ x \to -\infty, \\ Ke^x - Ke^{-r_i\tau},\ x \to +\infty. \end{cases} \end{cases} \tag{2}$$

where

$$\mathcal{L}V^i(x,\tau) = \left(r_i - \lambda_i\beta_i - \frac{1}{2}\sigma_i^2\right)\frac{\partial V^i(x,\tau)}{\partial x} + \frac{1}{2}\sigma_i^2\frac{\partial^2 V^i(x,\tau)}{\partial x^2} - (r_i + \lambda_i)\, V^i(x,\tau) \tag{3}$$

$$\mathcal{I}V^i(x,\tau) = \lambda_i \int_{-\infty}^{+\infty} V^i(x,\tau) f^i(z - x)dz \tag{4}$$

and the density function $f^i(z-x)$ is given by [34]

$$f^i(z-x) = \frac{1}{\sqrt{2\pi}\gamma_i}\exp\left[-\frac{(z-x-\alpha_i)^2}{2\gamma_i^2}\right].$$

1.3. High-Order Finite Difference Method

The high-order finite difference method has been developed for option pricing [15–24]. The idea of this method is to obtain finite difference approximations for high-order derivatives in the truncation error. The high-order schemes in a central difference approximation increase the order of accuracy.

We divide the domain $(-\infty, +\infty)$ into three parts: $(-\infty, x_{\min})$, $[x_{\min}, x_{\max})$, and $[x_{\max}, +\infty)$ and introduce uniform grids with $\Delta x = (x_{\max} - x_{\min})/M$ and $\Delta\tau = (T-t)/N$ where M and N denote the number of space and time intervals, respectively. T is the maturity date of the option. Furthermore, let the mesh points be $x_m = x_{\min} + m\Delta x$ for $m = 0, 1, \cdots, M$ and $\tau_j = j\,\Delta\tau$ for $j = 0, 1, 2, \cdots, N$.

For the integral term in Equation (2), by choosing the appropriate interval$[x_{\min}, x_{\max}]$, we can assure that the integral value beyond this range can be ignored, that is,

$$\lambda_i \int_{-\infty}^{x_{\min}} V^i(x,\tau) f^i(z-x)dz \approx \lambda_i \int_{-\infty}^{x_{\min}} \max(1-\exp(\xi),0) f^i(\xi)d\xi = 0 \tag{5}$$

$$\lambda_i \int_{x_{\max}}^{+\infty} V^i(x,\tau) f^i(z-x)dz \approx \lambda_i \int_{x_{\max}}^{+\infty} \max(1-\exp(\xi),0) f^i(\xi)d\xi = 0. \tag{6}$$

By using the composite Simpson's rule and Equations (5) and (6), we obtain

$$\begin{aligned} \mathcal{I}V^i(x,\tau) &= \lambda_i \textstyle\int_{-\infty}^{+\infty} V^i(x,\tau) f^i(z-x)dz \approx \lambda_i \int_{x_{\min}}^{x_{\max}} V^i(x,\tau) f^i(z-x)dz \\ &\approx \tfrac{\lambda_i \Delta x}{3} \sum_{m=1}^{m_x/2} [V^i(x_{2m-2},\tau) f^i(x_{2m-2}-x) + 4V^i(x_{2m-1},\tau) f^i(x_{2m-1}-x) \\ &\qquad + V^i(x_{2m},\tau) f^i(x_{2m}-x)] \end{aligned} \tag{7}$$

For the differential term in Equation (2), we define $V^i_{m,j} \equiv V^i(x_m, \tau_j), j = 1, 2, \cdots, N$. Then, the standard central difference approximation to Equation (3) at point (x_m, τ_j) for regime i is

$$\mathcal{L}V^i_{m,j} = (r_i - \lambda_i\beta_i - \frac{1}{2}\sigma_i^2)\delta_x V^i_{m,j} + \frac{1}{2}\sigma_i^2\delta_x^2 V^i_{m,j} - (r_i + \lambda_i)V^i_{m,j} + \varepsilon_m^{(i)} \tag{8}$$

where δ_x and δ_x^2 are the first- and second-order central difference approximations with respect to x, respectively. The truncation error is given by

$$\varepsilon_m^{(i)} = \frac{(\Delta x)^2}{12}\left(2r_i - 2\lambda_i\beta_i - \sigma_i^2\right)\frac{\partial^3 V^i_{m,j}}{\partial x^3} + \frac{1}{24}(\Delta x)^2\sigma_i^2\frac{\partial^4 V^i_{m,j}}{\partial x^4} + \mathcal{O}\left((\Delta x)^4\right) \tag{9}$$

Differentiating Equation (3) with respect to x, we have

$$\frac{\partial^3 V^i_{m,j}}{\partial x^3} = \frac{2}{\sigma_i^2}\frac{\partial \mathcal{L}V^i_{m,j}}{\partial x} - \frac{2r_i - 2\lambda_i\beta_i - \sigma_i^2}{\sigma_i^2}\frac{\partial^2 V^i_{m,j}}{\partial x^2} + \frac{2(r_i+\lambda_i)}{\sigma_i^2}\frac{\partial V^i_{m,j}}{\partial x} \tag{10}$$

$$\frac{\partial^4 V^i_{m,j}}{\partial x^4} = \frac{2}{\sigma_i^2}\frac{\partial^2 \mathcal{L}V^i_{m,j}}{\partial x^2} - \frac{2(2r_i - 2\lambda_i\beta_i - \sigma_i^2)}{\sigma_i^4}\frac{\partial \mathcal{L}V^i_{m,j}}{\partial x} + \frac{(2r_i - 2\lambda_i\beta_i - \sigma_i^2)^2}{\sigma_i^4}\frac{\partial^2 V^i_{m,j}}{\partial x^2} - \frac{4(r_i+\lambda_i)(r_i - \lambda_i\beta_i - \sigma_i^2)}{\sigma_i^4}\frac{\partial V^i_{m,j}}{\partial x} \tag{11}$$

We substitute Equations (10) and (11) into (9) to obtain a new expression of the error term $\varepsilon_l^{(i)}$ that only includes terms which are either $\mathcal{O}\left((\Delta x)^4\right)$ or $\mathcal{O}\left((\Delta x)^2\right)$ multiplied by derivatives of V, which can be approximated to $\mathcal{O}\left((\Delta x)^2\right)$ within the compact stencil. Inserting this new expression for the error term in (8), we obtain

$$\begin{aligned}&\left[r_i-\lambda_i\beta_i-\tfrac{\sigma_i^2}{2}+\tfrac{(r_i+\lambda_i)(r_i-\lambda_i\beta_i)(\Delta x)^2}{6\sigma_i^2}\right]\delta_x V_{m,j}^i+\left[\tfrac{\sigma_i^2}{2}-\tfrac{\left(2r_i-2\lambda_i\beta_i-\sigma_i^2\right)^2(\Delta x)^2}{24\sigma_i^2}\right]\delta_x^2 V_{m,j}^i-(r_i+\lambda_i)V_{m,j}^i\\&=\mathcal{L}V_{m,j}^i-\tfrac{\left(2r_i-2\lambda_i\beta_i-\sigma_i^2\right)(\Delta x)^2}{12\sigma_i^2}\delta_x\mathcal{L}V_{m,j}^i-\tfrac{(\Delta x)^2}{12}\delta_x^2\mathcal{L}V_{m,j}^i\end{aligned} \tag{12}$$

According to Equations (7) and (12), we obtain the discretization of PIDE (2) at point (x_m,τ_j) for regime i

$$V_{m,j}^i=a_iV_{m+1,j+1}^i+b_iV_{m,j+1}^i+c_iV_{m-1,j+1}^i$$

where

$$a_i=\frac{1}{1+(r_i+\lambda_i)\Delta\tau}\frac{(\Delta x)^2[(r_i+\lambda_i-\rho_{ii})\Delta\tau-1]-(\Delta\tau-1)\left[\sigma_i^2-\frac{\left(2r_i-2\lambda_i\beta_i-\sigma_i^2\right)^2(\Delta x)^2}{12\sigma_i^2}\right]}{(r_i+\lambda_i-\rho_{ii})\Delta\tau(\Delta x)^2-(\Delta x)^2+\left[\sigma_i^2-\frac{\left(2r_i-2\lambda_i\beta_i-\sigma_i^2\right)^2(\Delta x)^2}{12\sigma_i^2}\right]} \tag{13}$$

$$b_i=\frac{1}{1+(r_i+\lambda_i)\Delta\tau}\frac{\left[\frac{\sigma_i^2}{2}-\frac{\left(2r_i-2\lambda_i\beta_i-\sigma_i^2\right)^2(\Delta x)^2}{24\sigma_i^2}\right]\Delta\tau+\frac{1}{2}\left[r_i-\lambda_i\beta_i-\frac{\sigma_i^2}{2}+\frac{(r_i+\lambda_i)(r_i-\lambda_i\beta_i)(\Delta x)^2}{6\sigma_i^2}\right]\Delta\tau\Delta x}{(r_i+\lambda_i-\rho_{ii})\Delta\tau(\Delta x)^2-(\Delta x)^2+\left[\sigma_i^2-\frac{\left(2r_i-2\lambda_i\beta_i-\sigma_i^2\right)^2(\Delta x)^2}{12\sigma_i^2}\right]} \tag{14}$$

$$c_i=\frac{1}{1+(r_i+\lambda_i)\Delta\tau}\frac{\left[\frac{\sigma_i^2}{2}-\frac{\left(2n_i-2\lambda_i\beta_i-\sigma_i^2\right)^2(\Delta x)^2}{24\sigma_i^2}\right]\Delta\tau-\frac{1}{2}\left[n_i-\lambda_i\beta_i-\frac{\sigma_i^2}{2}+\frac{(r_i+\lambda_i)(r_i-\lambda_i\beta_i)(\Delta x)^2}{6\sigma_i^2}\right]\Delta\tau\Delta x}{(r_i+\lambda_i-\rho_{ii})\Delta\tau(\Delta x)^2-(\Delta x)^2+\left[\sigma_i^2-\frac{\left(2r_i-2\lambda_i\beta_i-\sigma_i^2\right)^2(\Delta x)^2}{12\sigma_i^2}\right]} \tag{15}$$

1.4. Outline of This Paper

The rest of this paper is organized as follows. In Section 2, the relationship between the high-order difference scheme and the trinomial tree algorithm is investigated, and then the convergence rate of the high-order difference algorithm for option pricing in a Markov regime-switching model is obtained. Section 3 summarizes the main conclusions.

2. Main Results

In this section, we investigate the relationship between the high-order difference method and the trinomial tree approach and propose the estimation of the local remainder of this algorithm. After this, we can obtain the convergence rate.

2.1. The Two Lemmas

Lemma 1. *If* $\Delta\tau\leq\frac{1+\left(2r_i-2\lambda_i\beta_i-\sigma_i^2\right)^2}{r_i+\lambda_i-\rho_{ii}}$ *and* $\Delta x\leq\sigma\sqrt{\frac{\Delta\tau}{1-(r_i+\lambda_i-\rho_{ii})\Delta\tau+\left(2r_i-2\lambda_i\beta_i-\sigma_i^2\right)^2}}$ *, the high-order finite difference method is equivalent to a trinomial tree approach, that is, for the defined high-order finite difference (13)–(15), the following result holds for regime* $i=1,2,\cdots,n$.

$$a_i+b_i+c_i=\frac{1}{1+(r_i+\lambda_i)\Delta\tau}\text{ and }a_i\geq 0,b_i\geq 0,c_i\geq 0$$

Proof. Equations (13)–(15) imply that

$$\begin{aligned}
a_i+b_i+c_i = {} & \frac{1}{1+(r_i+\lambda_i)\,\Delta\tau}\frac{(\Delta x)^2[(r_i+\lambda_i-\rho_{ii})\,\Delta\tau-1]-(\Delta\tau-1)\left[\sigma_i^2-\frac{\left(2r_i-2\lambda_i\beta_i-\sigma_i^2\right)^2(\Delta x)^2}{12\sigma_i^2}\right]}{(r_i+\lambda_i-\rho_{ii})\,\Delta\tau(\Delta x)^2-(\Delta x)^2+\left[\sigma_i^2-\frac{\left(2r_i-2\lambda_i\beta_i-\sigma_i^2\right)^2(\Delta x)^2]}{12\sigma_i^2}\right]} \\
& +\frac{1}{1+(r_i+\lambda_i)\,\Delta\tau}\frac{\left[\frac{\sigma_i^2}{2}-\frac{\left(2r_i-2\lambda_i\beta_i-\sigma_i^2\right)^2(\Delta x)^2}{24\sigma_i^2}\right]\Delta\tau+\frac{1}{2}\left[r_i-\lambda_i\beta_i-\frac{\sigma_i^2}{2}+\frac{(r_i+\lambda_i)\,(r_i-\lambda_i\beta_i)\,(\Delta x)^2}{6\sigma_i^2}\right]\Delta\tau\Delta x}{(r_i+\lambda_i-\rho_{ii})\,\Delta\tau(\Delta x)^2-(\Delta x)^2+\left[\sigma_i^2-\frac{\left(2r_i-2\lambda_i\beta_i-\sigma_i^2\right)^2(\Delta x)^2}{12\sigma_i^2}\right]} \\
& +\frac{1}{1+(r_i+\lambda_i)\,\Delta\tau}\frac{\left[\frac{\sigma_i^2}{2}-\frac{\left(2r_i-2\lambda_i\beta_i-\sigma_i^2\right)^2(\Delta x)^2}{24\sigma_i^2}\right]\Delta\tau-\frac{1}{2}\left[r_i-\lambda_i\beta_i-\frac{\sigma_i^2}{2}+\frac{(r_i+\lambda_i)\,(r_i-\lambda_i\beta_i)\,(\Delta x)^2}{6\sigma_i^2}\right]\Delta\tau\Delta x}{(r_i+\lambda_i-\rho_{ii})\,\Delta\tau(\Delta x)^2-(\Delta x)^2+\left[\sigma_i^2-\frac{\left(2r_i-2\lambda_i\beta_i-\sigma_i^2\right)^2(\Delta x)^2}{12\sigma_i^2}\right]} \\
= {} & \frac{1}{1+(r_i+\lambda_i)\,\Delta\tau}
\end{aligned}$$

Under the condition in Lemma 1, it is easy to show $a_i \geq 0, b_i \geq 0, c_i \geq 0$. Therefore, the expressions $a_i[1+(r_i+\lambda_i)\Delta\tau]$, $b_i[1+(r_i+\lambda_i)\Delta\tau]$, and $c_i[1+(r_i+\lambda_i)\Delta\tau]$ can be interpreted as the probabilities of moving from x_m to x_m, x_{m+1} and x_{m-1}, respectively. □

Let $V\left(x_m,\tau_j,i\right)$ denote a high-order finite difference approximation value at the node $\left(x_m,\tau_j\right)$ for regime i. Then, from Lemma 1, $V\left(x_m,\tau_j,i\right)$ can be calculated by

$$V\left(x_m,\tau_j,i\right)=e^{-r_i\Delta\tau}\sum_{l=1}^{n}\left[P_{il}\left(a_iV\left(x_{m+1},\tau_{j+1},l\right)+b_iV\left(x_m,\tau_{j+1},l\right)+c_iV\left(x_{m-1},\tau_{j+1},l\right)\right)\right] \tag{16}$$

where P_{il} is the transition probability from regime i to l, satisfying the following equation

$$(P_{il})_{n\times n}=I+\sum_{l=1}^{\infty}\frac{(\Delta\tau)^lA^l}{l!} \tag{17}$$

in which I denotes the unit matrix and A is the generation matrix of the Markov chain.

Define the local remainder of $V^i(x,\tau)$ for regime i at $\left(x_m,\tau_j\right)$ by

$$\begin{aligned}
R_j^i = V^i\left(x_m,\tau_j\right)-e^{-r_i\Delta\tau}\sum_{l=1}^{n}[P_{il}(a_iV\left(x_{m+1},\tau_{j+1},l\right)+b_iV\left(x_m,\tau_{j+1},l\right) \\
+c_iV(x_{m-1},\tau_{j+1},l))]
\end{aligned} \tag{18}$$

where $V^i\left(x_m,\tau_j\right)$ denotes the exact European option value for regime i at $\left(x_m,\tau_j\right)$.

Lemma 2. *Let $V(x,\tau)$ be a function for which the partial derivatives $\frac{\partial V}{\partial x}$, $\frac{\partial^2 V}{\partial x^2}$ and $\frac{\partial^3 V}{\partial x^3}$ are defined and continuous. The estimation of the local remainder R_j^i in (18) is given by $R_j^i=\mathcal{O}\left((\Delta\tau)^2\right)$ for regime $i=1,2,\cdots,n$.*

Proof. By applying Taylor expansion to $V^i\left(x_m,\tau_j\right)$, $V^i\left(x_{m+1},\tau_{j+1}\right)$ and $V^i\left(x_{m-1},\tau_{j+1}\right)$, $i=1,2,\ \cdots,\ n$ at τ_{j+1}, we have

$$V^i\left(x_m,\tau_j\right)=V^i\left(x_m,\tau_{j+1}\right)-\frac{\partial V^i\left(x_m,\tau_{j+1}\right)}{\partial\tau}\Delta\tau+\mathcal{O}\left((\Delta\tau)^2\right) \tag{19}$$

$$\begin{aligned}
V^i\left(x_{m+1},\tau_{j+1}\right)= {} & V^i\left(x_m,\tau_{j+1}\right)+\frac{\partial V^i\left(x_m,\tau_{j+1}\right)}{\partial x}\Delta x_m \\
& +\frac{1}{2}\frac{\partial^2 V^i\left(x_m,\tau_{j+1}\right)}{\partial x^2}(\Delta x_m)^2+\frac{1}{6}\frac{\partial^3 V^i\left(x_m,\tau_{j+1}\right)}{\partial x^3}(\Delta x_m)^3+\mathcal{O}((\Delta x_m)^4)
\end{aligned} \tag{20}$$

and

$$\begin{aligned}V^i(x_{m-1},\tau_{j+1}) = {} & V^i(x_m,\tau_{j+1}) + \frac{\partial V^i(x_m,\tau_{j+1})}{\partial x}(-\Delta x_m) \\ & + \frac{1}{2}\frac{\partial^2 V^i(x_m,\tau_{j+1})}{\partial x^2}(-\Delta x_m)^2 + \frac{1}{6}\frac{\partial^3 V^i(x_m,\tau_{j+1})}{\partial x^3}(-\Delta x_m)^3 + \mathcal{O}((-\Delta x_m)^4)\end{aligned} \tag{21}$$

Substituting (19)–(21) into (18), we have

$$\begin{aligned}R_j^i &= V^i(x_m,\tau_j) - e^{-r_i\Delta\tau}\sum_{l=1}^{n} P_{il}\left(a_i V^l(x_{m+1},\tau_{j+1}) + b_i V^l(x_m,\tau_{j+1}) + c_i V^l(x_{m-1},\tau_{j+1})\right) \\ &= V^i(x_m,\tau_{j+1}) - \frac{\partial V^i(x_m,\tau_{j+1})}{\partial \tau}\Delta\tau + \mathcal{O}((\Delta\tau)^2) \\ &\quad - e^{-r_i\Delta\tau}\sum_{l=1}^{n} P_{il}\left\{\begin{array}{c} a_i\left[\begin{array}{c} V^l(x_m,\tau_{j+1}) + \frac{\partial V^l(x_m,\tau_{j+1})}{\partial x}\Delta x_m + \frac{1}{2}\frac{\partial^2 V^l(x_m,\tau_{j+1})}{\partial x^2}(\Delta x_m)^2 \\ + \frac{1}{6}\frac{\partial^3 V^l(x_m,\tau_{j+1})}{\partial x^3}(\Delta x_m)^3 + \mathcal{O}((\Delta x_m)^4)\end{array}\right] + b_i V^l(x_m,\tau_{j+1}) \\ + c_i\left[\begin{array}{c} V^l(x_m,\tau_{j+1}) + \frac{\partial V^l(x_m,\tau_{j+1})}{\partial x}(-\Delta x_m) + \frac{1}{2}\frac{\partial^2 V^l(x_m,\tau_{j+1})}{\partial x^2}(-\Delta x_m)^2 \\ + \frac{1}{6}\frac{\partial^3 V^l(x_m,\tau_{j+1})}{\partial x^3}(-\Delta x_m)^3 + \mathcal{O}((-\Delta x_m)^4)\end{array}\right]\end{array}\right\} \\ &= V^i(x_m,\tau_{j+1}) - \frac{\partial V^i(x_m,\tau_{j+1})}{\partial \tau}\Delta\tau + \mathcal{O}((\Delta\tau)^2) \\ &\quad - e^{-r_i\Delta\tau}\sum_{l=1}^{n} P_{il}\left\{\begin{array}{c} (a_i+b_i+c_i)V^l(x_m,\tau_{j+1}) + (a_i-c_i)\frac{\partial V^l(x_m,\tau_{j+1})}{\partial x}\Delta x_m \\ + \frac{1}{2}(a_i+c_i)\frac{\partial^2 V^l(x_m,\tau_{j+1})}{\partial x^2}(\Delta x_m)^2 + \frac{1}{6}(a_i-c_i)\frac{\partial^3 V^l(x_m,\tau_{j+1})}{\partial x^3}(\Delta x_m)^3 + (a_i+c_i)\mathcal{O}((\Delta x_m)^4)\end{array}\right\}\end{aligned} \tag{22}$$

Using Lemma 1 and Equation (17), we obtain

$$\begin{aligned}R_j^i &= \mathcal{O}((\Delta\tau)^2)\left[\begin{array}{c} \frac{\partial V^i(x_m,\tau_{j+1})}{\partial \tau} - \left(r_i - \beta_i - \frac{1}{2}\sigma_i^2\right)\frac{\partial V^i(x_m,\tau_{j+1})}{\partial x} - \frac{1}{2}\sigma_i^2\frac{\partial^2 V^i(x_m,\tau_{j+1})}{\partial x^2} \\ -(r_i+\lambda_i)V^i(x_m,\tau_{j+1}) - \sum_{j=1}^{n}\rho_{ij}V^i(x_m,\tau_{j+1}) - \lambda_i\int_{-\infty}^{+\infty} V(z,\tau,i)f(z-x,i)dz\end{array}\right]\Delta\tau \\ &= \mathcal{O}((\Delta\tau)^2)\end{aligned}$$

□

Based on Lemmas 1 and 2, the convergence rate of the high-order finite difference algorithm is investigated as follows.

2.2. The Main Theorem

Theorem 1. (*Convergence rate of the high-order finite difference method*). *We define the error of high-order finite difference at the node (x_m,τ_j) by*

$$\varepsilon_i^j(x_m) = V^i(x_m,\tau_j) - V(x_m,\tau_j,i),\ i = 1,2,\cdots,n \tag{23}$$

and the infinity norm by

$$\|\varepsilon_i^j\|_\infty = \max\left|\varepsilon_i^j(x_m)\right|, i = 1,2,\cdots,n \tag{24}$$

Then, the convergence rate of the high-order finite difference is estimated by

$$\|\varepsilon_i^j\|_\infty = |\mathcal{O}(\Delta\tau)|, i = 1,2,\cdots,n \tag{25}$$

Proof. According to Equation (18),

$$V^i(x_m,\tau_j) = R_j^i + e^{-r_i\Delta\tau}\sum_{l=1}^{n} P_{il}\left(a_i V^l(x_{m+1},\tau_{j+1}) + b_i V^l(x_m,\tau_{j+1}) + c_i V^l(x_{m-1},\tau_{j+1})\right) \tag{26}$$

Then, we note from Equations (16) and (26) that

$$\varepsilon_i^j(x_m) = e^{-r_i\Delta\tau}\sum_{l=1}^{n} P_{il}\left(a_i\varepsilon_l^{j+1}(x_{m+1}) + b_i\varepsilon_l^{j+1}(x_m) + c_i\varepsilon_l^{j+1}(x_{m-1})\right) + R_j^i$$

Therefore, the following inequality holds:

$$\begin{aligned}
\left|\varepsilon_i^j(x_m)\right| &\leq \left|R_j^i\right| + e^{-r_i\Delta\tau}\sum_{l=1}^{n} P_{il}(a_i|\varepsilon_i^{j+1}(x_{m+1})| + b_i|\varepsilon_i^{j+1}(x_m)| + c_i|\varepsilon_i^{j+1}(x_{m-1})|) \\
&\leq \left|R_j^i\right| + e^{-r_i\Delta\tau}\sum_{l=1}^{n} P_{il}\|\varepsilon_l^{j+1}\|_\infty \\
&= e^{-r_i\Delta\tau}\sum_{l=1}^{n} P_{il}\|\varepsilon_l^{k+1}\|_\infty + |\mathcal{O}((\Delta\tau)^2)|
\end{aligned} \tag{27}$$

The last line of (27) is obtained from Lemma 2. Therefore, by using Equation (17), we have

$$\begin{aligned}
\sum_{i=1}^{n}\|\varepsilon_i^j\|_\infty &\leq \sum_{i=1}^{n} e^{-r_i\Delta\tau}\sum_{l=1}^{2} P_{il}\|\varepsilon_l^{j+1}\|_\infty + |\mathcal{O}((\Delta\tau)^2)| \\
&\leq \sum_{i=1}^{n}\left[1 + \sum_{i=1}^{n} a_{il}\Delta\tau + |\mathcal{O}\left((\Delta\tau)^2\right)|\right]\|\varepsilon_i^{j+1}\|_\infty + |\mathcal{O}((\Delta\tau)^2)|
\end{aligned} \tag{28}$$

The term $\sum_{i=1}^{n} a_{il}\Delta\tau + |\mathcal{O}((\Delta\tau)^2)| \geq 0$ in Equation (28) implies

$$\sum_{i=1}^{n}\|\varepsilon_i^j\|_\infty \leq \sum_{i=1}^{n}\|\varepsilon_i^{j+1}\|_\infty + \left\{\sum_{j=1}^{n}\left[\sum_{i=1}^{n} a_{ij}\Delta\tau + |\mathcal{O}((\Delta\tau)^2)|\right]\right\}\sum_{i=1}^{n}\|\varepsilon_i^{j+1}\|_\infty + |\mathcal{O}((\Delta\tau)^2)|$$

Since $\sum_{j=1}^{n} a_{ij} = 0, i = 1, 2, \cdots, n$, the following inequality can be obtained:

$$\sum_{i=1}^{n}\|\varepsilon_i^j\|_\infty \leq [1 + |\mathcal{O}((\Delta\tau)^2)|]\sum_{i=1}^{n}\|\varepsilon_i^{j+1}\|_\infty + |\mathcal{O}((\Delta\tau)^2)| \tag{29}$$

By iterating (29), we have

$$\sum_{i=1}^{n}\|\varepsilon_i^j\|_\infty \leq [1 + |\mathcal{O}((\Delta\tau)^2|]^{n-j}\left(\sum_{i=1}^{n}\|\varepsilon_i^n\|_\infty\right) - 1 + [1 + |\mathcal{O}((\Delta\tau)^2)|]^{n-j} \tag{30}$$

At the final step $\tau_n = 0$, the following expression holds:

$$\varepsilon_i^n = V^i(x_m, \tau_n) - V(x_m, \tau_n, i) = 0, i = 1, 2, \cdots, n$$

According to (30), we have

$$\begin{aligned}
\sum_{i=1}^{n}\|\varepsilon_i^j\|_\infty &\leq -1 + [1 + |\mathcal{O}((\Delta\tau)^2)|]^{n-j} \\
&= -1 + \sum_{l=0}^{n-j} C_{n-j}^l|\mathcal{O}((\Delta\tau)^2)|^l \\
&= C_{n-j}^1\mathcal{O}((\Delta\tau)^2) + \sum_{l=2}^{n-j} C_{n-j}^l|\mathcal{O}((\Delta\tau)^2)|^l \\
&\leq n\mathcal{O}((\Delta\tau)^2) = \tfrac{T}{\Delta\tau}\mathcal{O}((\Delta\tau)^2) = \mathcal{O}(\Delta\tau)
\end{aligned}$$

□

3. Conclusions

In this paper, we have investigated the convergence rate of the high-order finite difference method for option pricing in a Markov regime-switching jump-diffusion model by employing the relationship between this algorithm and the trinomial tree approach. The result shows that the convergence rate of this algorithm is $\mathcal{O}(\Delta\tau)$. This theoretical proof ensures the validation of the high-order finite difference method for option pricing.

For future research, it is worth investigating the convergence rate of the high-order finite difference method for options with stochastic volatility jump models in the case of infinite states for the Markov chain.

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