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Article

A Numerical Strategy for the Approximate Solution of the Nonlinear Time-Fractional Foam Drainage Equation

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Abstract: This study develops a numerical strategy for finding the approximate solution of the nonlinear foam drainage (NFD) equation with a time-fractional derivative. In this paper, we formulate the idea of the Laplace homotopy perturbation transform method (LHPTM) using Laplace transform and the homotopy perturbation method. This approach is free from the heavy calculation of integration and the convolution theorem for the recurrence relation and obtains the solution in the form of a series. Two-dimensional and three-dimensional graphical models are described at various fractional orders. This paper puts forward a practical application to indicate the performance of the proposed method and reveals that all the outputs are in excellent agreement with the exact solutions.

Keywords: Laplace transform; nonlinear foam drainage equation; homotopy perturbation method; approximate solution

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1. Introduction

Modern calculus includes the study of fractional calculus (FC), where the fractional order derivative of a function can be utilized to determine various long-term dynamics and other helpful data about the intended phenomenon. Numerous fields including signal processing, electrical community optics, and manipulating ideas of dynamical systems, have successfully used fractional differential equations to simulate these uses of FC. The topic of FC tackles the study of integrals and derivatives of fractional order. In the past century, it has gained more scientific attention generally [1]. The concept of non-integral order of integration can be traced back to some insights of G.W. Leibniz and L. Euler. These definitions provide a new aspect of study for scientists to conceive and characterize the fractional differential equations. In the discipline of FC, numerous models have been developed with the passage of time. Different physical phenomena in science and technology, such as fluid dynamics, physics, thermodynamics, biology, and dynamics of compounds have been modeled by nonlinear partial differential equations (PDEs). These nonlinear problems may not be convenient for finding exact solutions [2]. A broad collection of analytical and numerical strategies was employed to deal with these issues [3–5].

The foam drainage equation is a practical nonlinear differential equation that plays a significant role in both natural and industrial activities. Consider the NFD problem with the time-fractional derivative

$$D_t^{\alpha}\Psi + \frac{\partial}{\partial x}\left(\Psi^2 - \frac{\sqrt{\Psi}}{2}\frac{\partial \Psi}{\partial x}\right) = 0, \quad (1)$$

with condition

$$\Psi(x,0) = f(x), \quad (2)$$

where Ψ represents the cross section of a channel, x indicates the scaled level, and t represents the time, respectively. α is the fractional order and $D_t^\alpha = \frac{\partial^\alpha}{\partial t^\alpha}$ is the Caputo fractional derivative of Ψ.

The recent study of foam has focused on three main topics; see Figure 1. If $\alpha = 1$, Equation (1) changes to the traditional equation that occurs in daily life with our personal use items such as food, lotions, creams, cleaning of clothes, and scrubbing. The study of foam has a feature that when the liquid ripples out of foam due to gravitational force, it is called free drainage. Many researchers have applied different schemes to solve the NFD problem such as the (G′/G)-expansion method [6], the variational iteration method [7], the exp-function approach [8], the power series method [9,10], the homotopy perturbation method (HPM) [11], the reduced differential transform method, the homotopy analysis method [12], and the Haar wavelets method [13]. New developments of HPM can be seen in [14,15]. The performance of LHPTM shows its high validity for the propagation of weakly nonlinear acoustic waves and other nonlinear problems [16,17].

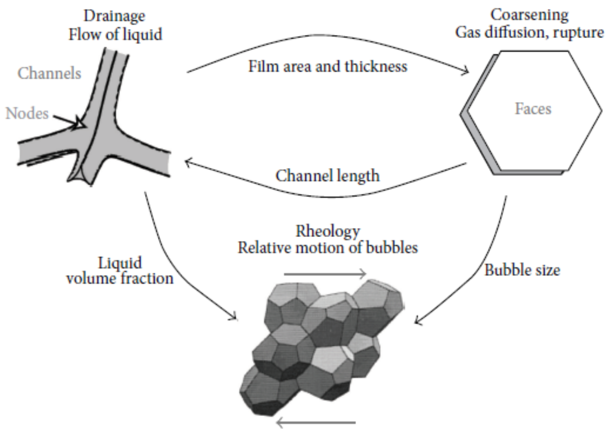

Figure 1. Demonstration of drainage, coarsening, and rheology of the foam.

In this paper, LHPTM is applied to solve the NFD equation with a time-fractional derivative in such a way that it produces a recurrence relation without any assumptions, where the calculation is simple and HPM is implemented to obtain the series solution. This research is summarized as follows: In Section 2, we introduce certain definitions and facts of FC. In Section 3, we construct the idea of LHPTM mathematically and present the convergence analysis in Section 4. We include numerical examples and provide the results and discussion in Sections 5 and 6 to show the reliability and suitability of the presented strategy, while the conclusion is in Section 7.

2. Preliminaries

In this section, we provide some definitions and facts of the Riemann–Liouville and Caputo fractional derivatives which can be stated as follows:

Definition 1. *The Riemann–Liouville fractional integral of a function $\vartheta(x,t)$ is given as [18]*

$$D^\alpha \vartheta(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\eta)^{\alpha-1} \vartheta(x,\eta)\, d\eta. \quad \alpha,\ t > 0,$$

where $D^0 \vartheta(x,t) = \vartheta(x,t)$, and Γ represents the gamma function, i.e.,

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-t}\, dt.$$

where the real part of the complex number z is strictly positive, i.e., $\Re(z) > 0$.

Definition 2. *Suppose that $\alpha > 0,\ t > a$, and $a,\ \alpha, t\ \in \mathbb{N}$. Then,*

$$D^{\alpha}\vartheta(x,t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t \frac{\vartheta(x,\eta)}{(t-\eta)^{\ \alpha-n+1}}d\eta, \qquad n-1<\alpha<n\in\mathbb{N}$$

is called the Riemann–Liouville fractional derivative [18].

Definition 3. *Let $\alpha > 0,\ t > a$, and $a,\ \alpha, t\ \in \mathbb{N}$. Then, the Caputo fractional derivative is [19]*

$$D^{\alpha}\vartheta(x,t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{\vartheta^{\ (n)}(x,\eta)}{(t-\eta)^{\ \alpha-n+1}}d\eta. \qquad n-1<\alpha<n\in\mathbb{N}$$

Proposition 1. *If $\vartheta(x,t) = t^{\alpha}$, then its Laplace transform (LT) is [20]*

$$\mathscr{L}[t^{\alpha}] = \int_0^{\infty} e^{-st}t^{\alpha}dt = \frac{\Gamma(\alpha+1)}{s^{(\alpha+1)}}.$$

Proposition 2. *The LT of a fractional derivative in the Caputo sense is [20]*

$$\mathscr{L}\Big[D^{\alpha}\vartheta(x,t)\Big] = s^{\alpha}\,\mathscr{L}\vartheta(x,t) - \sum_{m=0}^{n-1} s^{\alpha-m-1}\vartheta^{m}(x,0), \quad n-1<\alpha<n.$$

3. Basic Idea of LHPTM

The basic idea of LHPTM for the solution of nonlinear physical phenomena provides the direct results in the form of power series, which converge to the exact solution very quickly. To illustrate the simple concept of LHPTM, we assume a nonlinear general time-fractional equation:

$$D_t^{\alpha}\vartheta(x,t) = L_1[\vartheta(x,t)] + L_2[\vartheta(x,t)] + g(x,t), \quad n-1<\alpha\leq n,\ t>0,\ x\in\mathbb{R} \tag{3}$$

with the initial condition

$$\vartheta(x,0) = h(x), \tag{4}$$

where $D_t^{\alpha} = \dfrac{\partial^{\alpha}}{\partial t^{\alpha}}$ is the Caputo fractional derivative, L_1 and L_2 represent the linear and nonlinear differential operators, respectively, and $g(x,t)$ is the given source term.

Taking the LT of Equation (3)

$$\mathscr{L}\Big[D_t^{\alpha}\vartheta(x,t)\Big] = \mathscr{L}\Big[L_1(\vartheta(x,t) + L_2(\vartheta(x,t) + g(x,t)\Big].$$

Applying Proposition 2

$$s^{\alpha}\,\mathscr{L}\Big[\vartheta(x,t)\Big] - s^{\alpha-1}\vartheta(x,0) = \mathscr{L}\Big[L_1(\vartheta(x,t)) + L_2(\vartheta(x,t)) + g(x,t)\Big].$$

Using the condition (4), we obtain

$$\mathscr{L}\Big[\vartheta(x,t)\Big] = \frac{h(x)}{s} + \frac{1}{s^{\alpha}}\,\mathscr{L}\Big[g(x,t)\Big] + \frac{1}{s^{\alpha}}\,\mathscr{L}\Big[L_1(\vartheta(x,t) + L_2(\vartheta(x,t)\Big].$$

Thus, the inverse LT yields

$$\vartheta(x,t) = G(x,t) + \mathscr{L}^{-1}\left[\frac{1}{s^{\alpha}}\,\mathscr{L}\left\{L_1(\vartheta(x,t) + L_2(\vartheta(x,t)\right\}\right], \tag{5}$$

which is said to be the recurrence relation of $\vartheta(x,t)$ and

$$G(x,t) = \mathscr{L}^{-1}\left[\frac{h(x)}{s} + \frac{1}{s^{\alpha}}\mathscr{L}\left\{g(x,t)\right\}\right].$$

The solution of Equation (3) can be written as

$$\vartheta(x,t) = \sum_{n=0}^{\infty} p^{n}\vartheta_{n}(x,t), \tag{6}$$

where

$$L_{2}\vartheta(x,t) = \sum_{n=0}^{\infty} p^{n} H_{n}\vartheta(x,t), \tag{7}$$

where $H_n(\vartheta)$ is He's polynomial and is defined as:

$$H(\vartheta_0, \vartheta_1, \vartheta_2, \dots, \vartheta_n) = \frac{1}{n!}\frac{\partial^{n}}{\partial p^{n}} L_2\left(\sum_{i=0}^{\infty}\vartheta_i p^{i}\right)_{p=0}. \qquad n = 0, 1, 2, \dots \tag{8}$$

Puting the Equations (6) and (7) into Equation (5), we obtain

$$\sum_{n=0}^{\infty} p^{n}u_{n}(x,t) = G(x,t) + p\,\mathscr{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathscr{L}\left\{L_1\left(\sum_{n=0}^{\infty} p^{n}u_{n}(x,t)\right) + \sum_{n=0}^{\infty} p^{n}H_{n}(\vartheta(x,t)\right\}\right]. \tag{9}$$

Equating the highest powers of p, we obtain the successive estimates:

$$\begin{aligned}
p^{0} &: \vartheta_0(x,t) = G(x,t),\\
p^{1} &: \vartheta_1(x,t) = \mathscr{L}^{-1}\left[\frac{1}{s^{\alpha}}\left\{\mathscr{L}\Big(L_1\vartheta_0(x,t) + H_0\Big)\right\}\right],\\
p^{2} &: \vartheta_2(x,t) = \mathscr{L}^{-1}\left[\frac{1}{s^{\alpha}}\left\{\mathscr{L}\Big(L_1\vartheta_1(x,t) + H_1\Big)\right\}\right],\\
p^{3} &: \vartheta_3(x,t) = \mathscr{L}^{-1}\left[\frac{1}{s^{\alpha}}\left\{\mathscr{L}\Big(L_1\vartheta_2(x,t) + H_2\Big)\right\}\right],\\
p^{4} &: \vartheta_4(x,t) = \mathscr{L}^{-1}\left[\frac{1}{s^{\alpha}}\left\{\mathscr{L}\Big(L_1\vartheta_3(x,t) + H_3\Big)\right\}\right],\\
&\ \vdots
\end{aligned}$$

continuing the comparable process, the series result is accordingly combined as

$$\vartheta(x,t) = p^{0}\vartheta_0(x,t) + p^{1}\vartheta_1(x,t) + p^{2}\vartheta_2(x,t) + \cdots = \lim_{p\to 1}\sum_{n=0}^{\infty} p^{n}u_{n}(x,t).$$

Most of the time, the above series converges and its rate of convergence is dependent on the nonlinear operator L_2. Eslami and Mirzazadeh [21] investigated the convergence of the finite approximate series for two-dimensional linear volterra integral equations of the first kind. The decisions below were built by [22,23],

(i) The second order derivative of $L_2(\vartheta)$ according to ϑ should be smaller as the parameter p becomes larger, i.e., $p \to 1$

(ii) $\left\| L_1^{-1}\left(\frac{\partial L_2}{\partial \vartheta}\right) \right\| < 1$, so that the series converges, whereas L_1^{-1} represents the inverse of the linear operator L_1.

4. Convergence Analysis

Let P and Q be Banach spaces where $X : P \to Q$ is a nonlinear mapping. If the series produced by HPM is

$$\vartheta_n(P,x) = X(\vartheta_{n-1}(P,x)) = \sum_{i=0}^{n-1} \vartheta_i(P,x), \quad n = 1,2,3\ldots$$

the following conditions must be true:

(1) $\|\vartheta_n(P,x) - \vartheta(P,x)\| \leq \varphi^n \|\vartheta(P,x) - \vartheta(P,x)\|$;

(2) $\vartheta_n(P,x)$ is forever in the neighborhood of $\vartheta(P,x)$ meaning $\vartheta_n(P,x) \in B(\vartheta(P,x),r) = \{\vartheta^*(P,x)/\|\vartheta^*(P,x) - \vartheta(P,x)\|\}$;

(3) $\lim_{n\to\infty} \vartheta_n(P,x) = \vartheta(P,x)$.

Proof. (1) We demonstrate condition (1) by recognition on n, such as $\|\vartheta_1 - \vartheta\| = \|G(\vartheta_0) - \vartheta\|$, and the Banach fixed point theorem states that X has a fixed point ϑ, i.e., $X(\vartheta) = \vartheta$; therefore,

$$\|\vartheta_1 - \vartheta\| = \|G(\vartheta_0) - \vartheta\| = \|G(\vartheta_0) - G(\vartheta)\| \leq \varphi\|\vartheta_0 - \vartheta\| = \varphi\|\vartheta(P,x) - \vartheta\|$$

since X is a nonlinear mapping. If we consider that $\|\vartheta_{n-1} - \vartheta\| \leq \varphi^{n-1}\|\vartheta(P,0) - \vartheta(P,x)\|$ is an induction hypothesis, then

$$\|\vartheta_n - \vartheta\| = \|G(\vartheta_{n-1}) - G(\vartheta)\| \leq \varphi\|\vartheta_{n-1} - \vartheta\| \leq \varphi\varphi^{n-1}\|\vartheta(P,x) - \vartheta\|$$

(2) Our initial challenge is to demonstrate the $\vartheta(P,x) \in B(\vartheta(P,x),r)$, which is attained by replacing on m. Thus, for $m = 1$, $\|\vartheta(P,x) - \vartheta(P,x)\| = \|\vartheta(P,0) - \vartheta(P,x)\| \leq r$ with $\vartheta(P,0)$ as an initial condition. If we consider that $\|\vartheta(P,x) - \vartheta(P,x)\| \leq r$ for $m-2$ is an induction theory, then

$$\begin{aligned}
\|\vartheta(P,x) - \vartheta(P,x)\| &= \vartheta_{m-2}(P,x) - \frac{f_m(P)}{\Gamma(\delta - m + 1)}x^{\delta - m}\| \\
&\leq \|\vartheta_{m-1}(P,x) - \vartheta(P,x)\| + \left\|\frac{f_m(P)}{\Gamma(\delta - m + 1)}x^{\delta - m}\right\| \\
&= r.
\end{aligned}$$

Now, $\forall\, n \geq 1$, using (1) we obtain

$$\|\vartheta_n - \vartheta\| \leq \varphi^n\|\vartheta(P,x) - \vartheta\| \leq \varphi^n r \leq r.$$

(3) Using condition (2) and $\lim_{n\to\infty} \varphi^n = 0$, it provides $\lim_{n\to\infty}\|\vartheta_n - \vartheta\| = 0$; hence,

$$\lim_{n\to\infty} \vartheta_n = \vartheta;$$

thus, ϑ converges. □

Theorem 1. *Let H be the Hilbert space defined as $H = L^2((a,b) \times [0,T])$ the set of applications*

$$\vartheta : (\alpha,\beta)\times[0,T] \to \quad with \int_{(\alpha,\beta)\times[0,T]} \vartheta^2(x,s)dsd\theta \quad < +\infty$$

Now, we consider the time-fractional foam drainage equation in the above assumptions, and let us denote

$$L(\vartheta) = \frac{\partial^{\alpha}\vartheta}{\partial t^{\alpha}};$$

then, the time-fractional foam drainage equation can be written in an operator form as

$$L(\vartheta) = \frac{1}{2}\vartheta\vartheta_{xx} - 2\vartheta^2\vartheta_x + \vartheta_x^2$$

The LHPTM is convergent if the following two assumptions are fulfilled:

- $(L(\vartheta) - L(w), \vartheta - w) \geq k \parallel \vartheta - w \parallel^2; k > 0, \forall \vartheta, w \in H;$
- whatever may be $M > 0$, there exists a constant $C(M) > 0$ such that for $\vartheta, w \epsilon H$ with $\parallel \vartheta \parallel \leq M, \parallel w \parallel \leq M$, we have $(L(\vartheta) - L(w), \vartheta - w) \leq C(M) \parallel \vartheta - w \parallel \parallel v \parallel$ for every $v \in H$.

5. Numerical Applications

In this segment, an NFD problem of time-fractional order with an initial condition is considered to validate the accuracy and applicability of the proposed algorithm.

5.1. Example 1

Let us apply the following transformation in Equation (1) such that

$$\Psi(x,t) = \vartheta^2(x,t). \tag{10}$$

Thus, Equation (1) becomes

$$\begin{aligned}
D_t^{\alpha}\vartheta^2 + \frac{\partial}{\partial x}\left(\vartheta^4 - \frac{\vartheta}{2}.\frac{\partial\vartheta^2}{\partial x}\right) &= 0,\\
D_t^{\alpha}\vartheta^2 + \frac{\partial}{\partial x}\left(\vartheta^4 - \vartheta^2\frac{\partial\vartheta}{\partial x}\right) &= 0,\\
2uD_t^{\alpha}\vartheta + 4\vartheta^3\frac{\partial\vartheta}{\partial x} - 2\vartheta\frac{\partial\vartheta}{\partial x}.\frac{\partial\vartheta}{\partial x} - \vartheta^2\frac{\partial^2\vartheta}{\partial x^2} &= 0,
\end{aligned}$$

which can be written as

$$D_t^{\alpha}\vartheta(x,t) = \frac{1}{2}\vartheta(x,t)\vartheta_{xx}(x,t) - 2\vartheta^2(x,t)\vartheta_x(x,t) + \vartheta_x^2(x,t), \tag{11}$$

with condition

$$\vartheta(x,0) = -\sqrt{c}\tanh(\sqrt{c}x), \tag{12}$$

here, c represents the speed of the wave front [7,8].

Applying LT on (11), we obtain

$$\begin{aligned}
\mathcal{L}\left[D_t^{\alpha}\vartheta(x,t)\right] &= \mathcal{L}\left[\frac{1}{2}\vartheta(x,t)\vartheta_{xx}(x,t) - 2\vartheta^2(x,t)\vartheta_x(x,t) + \vartheta_x^2(x,t)\right],\\
s^{\alpha}\,\mathcal{L}\left[\vartheta(x,t)\right] - s^{\alpha-1}\vartheta(x,0) &= \mathcal{L}\left[\frac{1}{2}\vartheta(x,t)\vartheta_{xx}(x,t) - 2\vartheta^2(x,t)\vartheta_x(x,t) + \vartheta_x^2(x,t)\right],\\
\mathcal{L}\left[\vartheta(x,t)\right] &= \frac{\vartheta(x,0)}{s} + \frac{1}{s^{\alpha}}\,\mathcal{L}\left[\frac{1}{2}\vartheta(x,t)\vartheta_{xx}(x,t) - 2\vartheta^2(x,t)\vartheta_x(x,t) + \vartheta_x^2(x,t)\right].
\end{aligned}$$

The inverse LT of the above equation tends to:

$$\vartheta(x,t) = \vartheta(x,0) + \mathscr{L}^{-1}\left[\frac{1}{s^{\alpha}}\,\mathscr{L}\left\{\frac{1}{2}\vartheta(x,t)\vartheta_{xx}(x,t) - 2\vartheta^{2}(x,t)\vartheta_{x}(x,t) + \vartheta_{x}^{2}(x,t)\right\}\right].$$

Now, applying He's HPM properly

$$\sum_{n=0}^{\infty} p^{n}\vartheta_{n}(x,t) = \vartheta(x,0) + p\,\mathscr{L}^{-1}\left[\frac{1}{s^{\alpha}}\,\mathscr{L}\left\{\frac{1}{2}\left(\sum_{n=0}^{\infty} p^{n}\vartheta_{n}(x,t)\right)\left(\sum_{n=0}^{\infty} p^{n}\vartheta_{n}(x,t)\right)_{xx} - 2\left(\sum_{n=0}^{\infty} p^{n}\vartheta_{n}(x,t)\right)^{2}\left(\sum_{n=0}^{\infty} p^{n}\vartheta_{n}(x,t)\right)_{x} + \left(\sum_{n=0}^{\infty} p^{n}\vartheta_{n}(x,t)\right)_{x}^{2}\right\}\right].$$

Equating the similar identity of p, we obtain

$$\begin{aligned}
p^{0} &: \vartheta_{0}(x,t) = \vartheta(x,0),\\
p^{1} &: \vartheta_{1}(x,t) = \mathscr{L}^{-1}\left[\frac{1}{s^{\alpha}}\left\{\mathscr{L}\left(\frac{1}{2}\vartheta_{0}\vartheta_{0,xx} - 2\vartheta_{0}^{2}\vartheta_{0,x} + \vartheta_{0,x}^{2}\right)\right\}\right],\\
p^{2} &: \vartheta_{2}(x,t) = \mathscr{L}^{-1}\left[\frac{1}{s^{\alpha}}\left\{\mathscr{L}\left(\frac{1}{2}\left(\vartheta_{1}\vartheta_{0,xx} + \vartheta_{0}\vartheta_{1,xx}\right) - 2\left(\vartheta_{0}^{2}\vartheta_{1,x} + 2\vartheta_{0}\vartheta_{1}\vartheta_{0,x}\right) + 2\vartheta_{0,x}\vartheta_{1,x}\right)\right\}\right],\\
p^{3} &: \vartheta_{3}(x,t) = \mathscr{L}^{-1}\left[\frac{1}{s^{\alpha}}\left\{\mathscr{L}\left(\frac{1}{2}\left(\vartheta_{0}\vartheta_{2,xx} + \vartheta_{1}\vartheta_{1,xx} + \vartheta_{2}\vartheta_{0,xx}\right) - 2\left(\vartheta_{0}^{2}\vartheta_{2,x} + 2\vartheta_{0}\vartheta_{1}\vartheta_{1,x} + \vartheta_{1}^{2}\vartheta_{0,x}r\right.\right.\right.\right.\\
&\quad\left.\left.\left.\left. + 2\vartheta_{0}\vartheta_{2}\vartheta_{0,x}\right) + \left(\vartheta_{1,x}^{2} + 2\vartheta_{0,x}\vartheta_{2,x}\right)\right)\right\}\right],
\end{aligned} \tag{13}$$

Solving the above system of equations, we obtain

$$\begin{aligned}
\vartheta_{0}(x,t) &= -\sqrt{c}\tanh(\sqrt{c}x),\\
\vartheta_{1}(x,t) &= \left(\operatorname{sech}^{4}(\sqrt{c}x) + \tanh^{2}(\sqrt{c}x)\operatorname{sech}^{2}(\sqrt{c}x)\right)\frac{c^{2}t^{\alpha}}{\Gamma(\alpha+1)},\\
\vartheta_{2}(x,t) &= \frac{2c^{7/2}t^{2\alpha}\tanh(\sqrt{c}x)\operatorname{sech}^{2}(\sqrt{c}x)}{\Gamma(2\alpha+1)},\\
\vartheta_{3}(x,t) &= \frac{c^{5}t^{3\alpha}(\cosh(2\sqrt{c}x) - 2)\operatorname{sech}^{6}(\sqrt{c}x)\left(\Gamma(\alpha+1)^{2}(\cosh(2\sqrt{c}x) + 3) - \Gamma(2\alpha+1)\right)}{\Gamma(\alpha+1)^{2}\Gamma(3\alpha+1)},\\
&\vdots
\end{aligned}$$

We may write these iterations such as

$$\begin{aligned}
\vartheta(x,t) &= \vartheta_{0}(x,t) + \vartheta_{1}(x,t) + \vartheta_{2}(x,t) + \vartheta_{3}(x,t) + \dots,\\
&= -\sqrt{c}\tanh(\sqrt{c}x) + \frac{c^{2}t^{\alpha}\left(\operatorname{sech}^{4}(\sqrt{c}x) + \tanh^{2}(\sqrt{c}x)\operatorname{sech}^{2}(\sqrt{c}x)\right)}{\Gamma(\alpha+1)} + \frac{2c^{7/2}t^{2\alpha}\tanh(\sqrt{c}x)\operatorname{sech}^{2}(\sqrt{c}x)}{\Gamma(2\alpha+1)}\\
&+ \frac{c^{5}t^{3\alpha}(\cosh(2\sqrt{c}x) - 2)\operatorname{sech}^{6}(\sqrt{c}x)\left(\Gamma(\alpha+1)^{2}(\cosh(2\sqrt{c}x) + 3) - \Gamma(2\alpha+1)\right)}{\Gamma(\alpha+1)^{2}\Gamma(3\alpha+1)} + \cdots,
\end{aligned} \tag{14}$$

Thus, this series converges to the approximate solution for $\alpha = 1$,

$$\vartheta(x,t) = -\sqrt{c}\tanh\left(\sqrt{c}(x - ct)\right). \tag{15}$$

5.2. Example 2

Again, we consider Equation (12) with the following condition

$$\vartheta(x,0) = -\frac{1}{2} + \frac{1}{1+e^x}. \tag{16}$$

Proceeding with the strategy of LHPTM, the system of Equation (13) provides the following iterations,

$$\begin{aligned}
\vartheta_0(x,t) &= -\frac{1}{2} + \frac{1}{1+e^x},\\
\vartheta_1(x,t) &= \left(\frac{2e^x\left(\frac{1}{e^x+1}-\frac{1}{2}\right)^2}{(e^x+1)^2} + \frac{1}{2}\left(\frac{2e^{2x}}{(e^x+1)^3} - \frac{e^x}{(e^x+1)^2}\right)\left(\frac{1}{e^x+1}-\frac{1}{2}\right) + \frac{e^{2x}}{(e^x+1)^4}\right)\frac{t^\alpha}{\Gamma(\alpha+1)},\\
\vartheta_2(x,t) &= \frac{e^x(e^x-1)t^{2\alpha}}{16(e^x+1)^3\Gamma(2\alpha+1)},\\
\vartheta_3(x,t) &= \frac{e^x\left(-4e^x+e^{2x}+1\right)\S^{3\alpha}\left(\left(6e^x+e^{2x}+1\right)\Gamma(\alpha+1)^2 - 2e^x\Gamma(2\alpha+1)\right)}{64(e^x+1)^6\Gamma(\alpha+1)^2\Gamma(3\alpha+1)},\\
&\vdots
\end{aligned}$$

we may write these iterations such as

$$\begin{aligned}
\vartheta(x,t) &= \vartheta_0(x,t) + \vartheta_1(x,t) + \vartheta_2(x,t) + \vartheta_3(x,t) + \dots\\
&= -\frac{1}{2} + \frac{1}{1+e^x} + \left(\frac{2e^x\left(\frac{1}{e^x+1}-\frac{1}{2}\right)^2}{(e^x+1)^2} + \frac{1}{2}\left(\frac{2e^{2x}}{(e^x+1)^3} - \frac{e^x}{(e^x+1)^2}\right)\left(\frac{1}{e^x+1}-\frac{1}{2}\right) + \frac{e^{2x}}{(e^x+1)^4}\right)\frac{t^\alpha}{\Gamma(\alpha+1)}\\
&+ \frac{e^x(e^x-1)t^{2\alpha}}{16(e^x+1)^3\Gamma(2\alpha+1)} + \frac{e^x\left(-4e^x+e^{2x}+1\right)t^{3\alpha}\left(\left(6e^x+e^{2x}+1\right)\Gamma(\alpha+1)^2 - 2e^x\Gamma(2\alpha+1)\right)}{64(e^x+1)^6\Gamma(\alpha+1)^2\Gamma(3\alpha+1)} + \cdots,
\end{aligned} \tag{17}$$

which can be approaches to the precise solution at $\alpha = 1$

$$\vartheta(x,t) = -\frac{1}{2} + \frac{1}{1+e^{x-\frac{1}{4}t}} \tag{18}$$

6. Results and Discussion

Figure 2a,b describe the approximate solutions of Equations (14) and (15) for different values of α at $0 \le x \le 2\pi$ with fixed point $t = 0.2$. It is observed in Figure 2c–f for $\alpha = 1$, the approximate solution approaches to the precise solution with the increase of the values of α. Similarly, Figure 3a,b describe the approximate solutions of Equations (17) and (18) for different values of α at $0 \le x \le 2\pi$ with fixed point $t = 0.2$. It is observed in Figure 3c–f for $\alpha = 1$, the approximate solution approaches to the precise solution with the increase of the values of α. We demonstrate the solution graphs in both 2-D and 3-D to analyze the accuracy of our obtained results. Table 1 reveals the correlation of the estimated and the precise solutions of Equations (14) and (15), whereas Table 2 provides these for Equations (17) and (18) respectively, showing significant results at $\alpha = 1$. These figures confirm the accuracy of this scheme for evaluating the estimated solution of the time-fractional NFD model. The LHPTM revealed the results in the order of series that were very near to the precise solution.

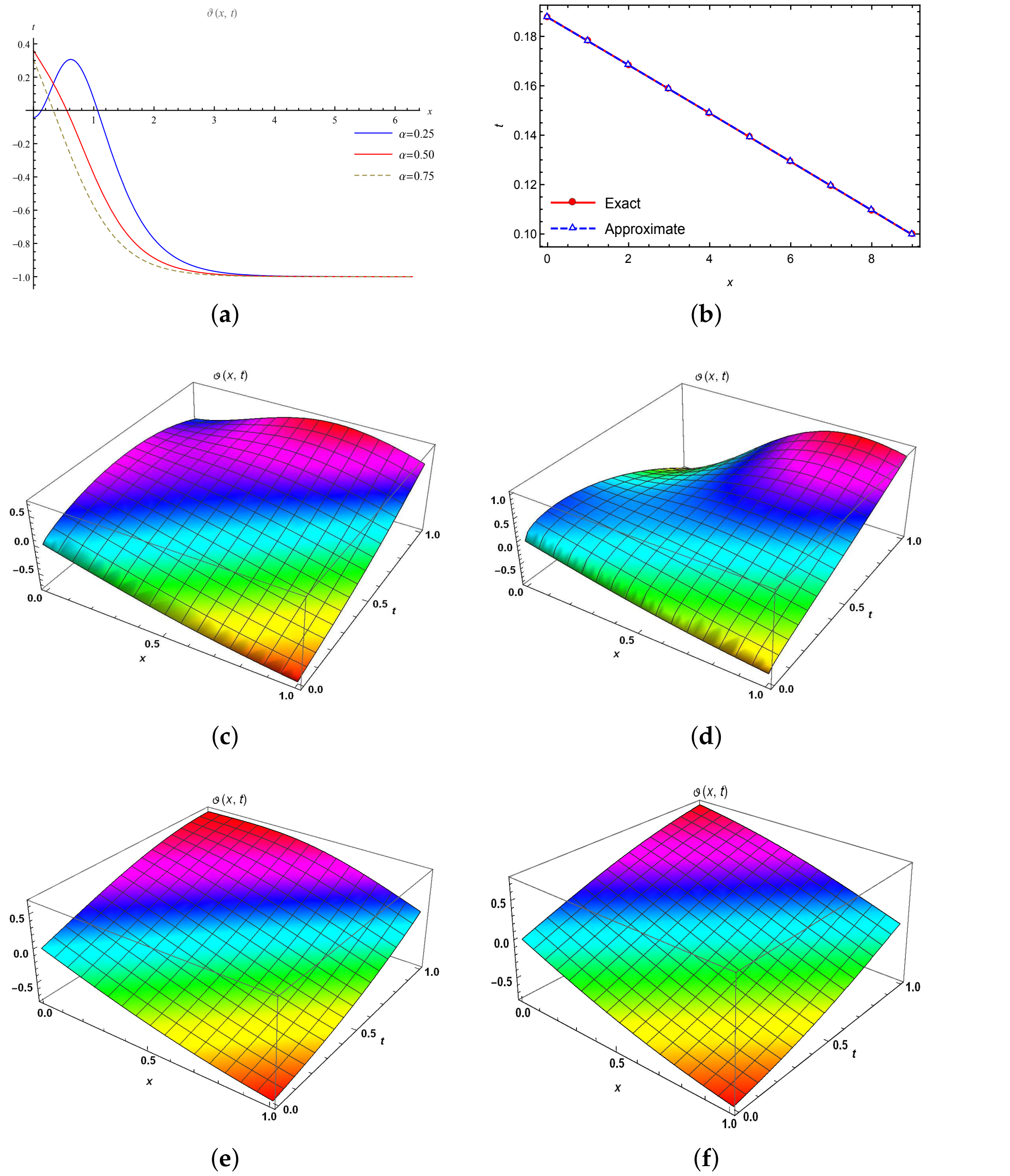

Figure 2. The different surfaces of approximate solutions and the exact solution. (**a**) Approximate solution plot of Equation (14) for different values of α. (**b**) Approximate solution of Equation (14) and exact solution of Equation (15) at $\alpha = 1$. (**c**) The surface solution of Equation (14) at $\alpha = 0.50$. (**d**) The surface solution of Equation (14) at $\alpha = 0.75$. (**e**) The surface solution of Equation (14) at $\alpha = 1$. (**f**) The surface solution of Equation (15) at $\alpha = 1$.

Table 1. Comparison between approximate solutions of Equations (14) and (15) when $t = 0.2$.

x	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 1$	Exact
0.01	0.315188	0.291273	0.187715	0.187746
0.02	0.309270	0.282554	0.178066	0.178081
0.03	0.303405	0.273793	0.168371	0.168381
0.04	0.297590	0.264991	0.158649	0.158649
0.05	0.294341	0.256149	0.148896	0.148885

Table 2. Comparison between approximate solutions of Equations (17) and (18) when $t = 0.5$.

x	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 1$	Exact
0.2	0.0196119	0.0177632	0.0125034	0.012974
0.4	−0.02963	−0.0318867	−0.0374168	−0.0374298
0.6	−0.0782875	−0.0809143	−0.0865996	−0.0866176
0.8	−0.125486	−0.128406	−0.134115	−0.134136
1.0	−0.170451	−0.173558	−0.179158	−0.1791179

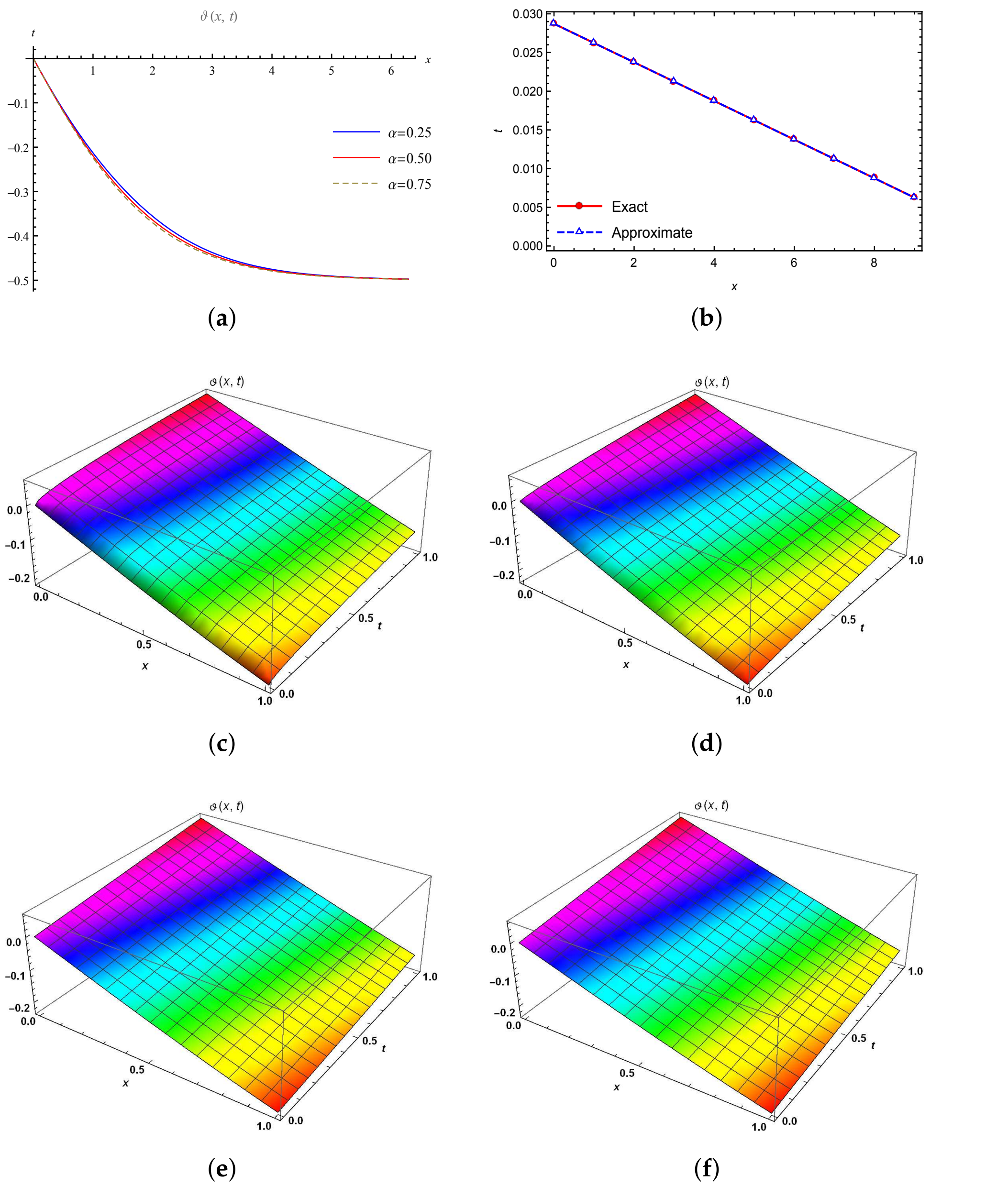

Figure 3. The surface solutions with different fractional orders. (**a**) Approximate solution plot of Equation (17) for different values of α. (**b**) Approximate solution of Equation (17) and exact solution of Equation (18) at $\alpha = 1$. (**c**) The surface solution of Equation (17) at $\alpha = 0.50$. (**d**) The surface solution of Equation (17) at $\alpha = 0.75$. (**e**) The surface solution of Equation (17) at $\alpha = 1$. (**f**) The surface solution of Equation (18) at $\alpha = 1$.

7. Conclusions

We have favorably applied the LHPTM to obtain the estimated results of the time-fractional NFD problem. During the investigation, we revealed that the performance of the Laplace transform with HPM does not require any assumptions and hypotheses to construct a recurrence relation. We depicted the obtained solutions in terms of the plots with various values of the fractional order, which indicate the nature of the considered complex problem. The outcomes showed that the LHPTM operated the series solution to the exact solution very rapidly only after a few iterations. Thus, we can declare that the suggested approach is highly powerful for a broad group of nonlinear fractional-order problems in various disciplines of science and technology.

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