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Article

New Outcomes Regarding the Existence of Hilfer Fractional Stochastic Differential Systems via Almost Sectorial Operators

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Abstract: In this paper, we focus on the existence of Hilfer fractional stochastic differential systems via almost sectorial operators. The main results are obtained by using the concepts and ideas from fractional calculus, multivalued maps, semigroup theory, sectorial operators, and the fixed-point technique. We start by confirming the existence of the mild solution by using Dhage's fixed-point theorem. Finally, an example is provided to demonstrate the considered Hilferr fractional stochastic differential systems theory.

Keywords: existence; mild solution; Hilfer fractional (HF) system; stochastic differential system; almost sectorial operators; fixed-point technique

MSC: 26A33; 34A08; 34K30; 47D09

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1. Introduction

Fractional calculus, which is a logical extension of conventional calculus, allows for the definition of integrals and derivatives for any real order. Since the invention of fractional calculus in the 17th century, a number of novel derivatives have been developed, including R-L, Hadamard, Grunwald–Lenikov, and Caputo, to name just a few [1–4]. Many authors described the fractional order models with the most common definitions of fractional derivatives defined by Caputo and Riemann–Liouville sense. The Caputo derivative is of use to modeling phenomena which takes account of interactions within the past and also problems with nonlocal properties. As extensions of the classical integer order partial differential equations [5], fractional order partial differential equations are increasingly employed to describe issues in finance, fluid flow, and other areas of application. In a diffusion or dispersion model, increased diffusion results when a fractional derivative takes the place of the second derivative (also called super-diffusion). Laplace–Fourier transform techniques can be used to acquire analytical solutions for fractional partial differential equations with constant coefficients. However, a model with variable coefficients is necessary for many real-world issues [6]. Meerschaert and Tadjeran [7] used the finite difference method to solve two-point boundary differential equations with fractional order derivatives. The numerical solution for the initial boundary value time fractional partial differential equations was provided by Podlubny [8]. Langlands and Henry [9] covered the numerical approaches for the time-fractional diffusion equations $\frac{\partial^{\gamma} u(x,t)}{\partial t^{\gamma}} = \frac{\partial^2 u(x,t)}{\partial x^2}$. The suitable fractional derivative depends on the system under consideration, which is why there are so many publications denoting different fractional operators. Scientific models address a wide range of real-world behaviours, such as anomalous diffusion, ecological impacts, blood circulation issues, disease propagation, control mechanisms, etc., by using fractional order differential and integral operators, which are nonlocal in nature. Fractional calculus offers a wide range of applications, which has prompted numerous scholars to work on the theoretical elements of this branch of modern analysis. In fractional calculus, there has been a great deal of progress [1–4,8,10–12]. In particular, the articles "Design, Analysis and Comparison of a Nonstandard Computational Method for the Solution of a General Stochastic Fractional Epidemic Model" [13], "Structure Preserving Algorithm for Fractional Order Mathematical Model of COVID-19," and "Analysis of the fractional diarrhea model with Mittag–Leffler kernel" [14,15], and "A Fractal Fractional Model for Cervical Cancer due to Human Papillomavirus Infection," and "Optimal existence of fractional order computer virus epidemic model and numerical simulations" [16,17] are important.

Over the past ten years, fractional calculus has emerged as one of the most helpful methods for comprehending lengthy operations. These models appeal to pure mathematicians, scientists, and designers alike. The most effective fractional equations in these models are those with fractional order derivatives. Academics are also concentrating on the qualitative characteristics of fractional dynamical systems, such as their stability, existence, and controllability. Recently, stochastic partial differential equations have attracted a great deal of attention because they arise naturally in mathematical modeling of various phenomena in the social and natural sciences [18]. Numerous writers [19,20] have looked at the qualitative characteristics of stochastic partial differential equations in infinite dimensional spaces, including existence, stability, controllability, and invariant measures, among others. Because stochastic fluctuation cannot be avoided in real-world applications, computational issues for stochastic differential equations must be researched [21]. The applicability of stochastic differential equations in several scientific and engineering fields has attracted attention. It should be remembered that noise or stochastic discomfort cannot be avoided in nature and not even in artificial systems. Due to its widespread application in the modeling of a variety of complex dynamical systems in the biological, physical, and medical areas, stochastic differential systems have attracted attention; one can see [22–25] for examples. Differential inclusions tools make it easier to investigate dynamical systems with velocities that are not only governed by the system's state.

In order to examine the almost sectorial operators, the authors in [26] created new spaces and developed a functional calculus concept. As $\wp$ approached zero, they explored the properties of both the mild and classic semigroups, as well as possible explanations for their existence. In [27], the functional calculus was used to construct two sets of operators, and the Caputo derivative was used to solve various fractional Cauchy problems. The existence of mild solutions for fractional differential evolution systems with impulse employing sectorial operators was investigated by the authors of [28,29]. In [21,23], the existence of almost periodic fractional differential equation solutions was investigated. In the context of finite dimensional Banach spaces, the authors of [30] are concerned with a novel generic class of nonlocal fractional differential systems with impulse via the Lipschitz multivalued function and a linear sectorial operator. Researchers have made significant progress in the field of sectorial operators in [26,31–34].

Hilfer introduced additional fractional order derivatives, including the R-L and Caputo fractional derivatives [35–38]. Furthermore, the importance and applicability of the *HF* derivative have been discovered in conceptual simulations of dielectric relaxation in crystal materials, polymer science, rheological constitutive modeling, engineering, and other fields. Gu and Trujillo [39] in particular used a *MNC* method and a fixed-point approach to justify the existence of an integral solution to the evolution problem with the *HF* derivative. He created the recent parameter $\rho \in [0, 1]$ and a fractional parameter ν, in which $\rho = 0$ generates the R-L derivative and $\nu = 1$ generates the Caputo derivative, to define the order of the derivative. Numerous publications have been written about *HF* calculus [23,39–42]. In [43–45], investigators demonstrate the existence of the mild solution for Hilfer fractional differential systems via almost sectorial operators by using the fixed-point method. By using a fixed point method, refs. [46–48] investigated the solvability and controllability of differential systems.

The existence of HF stochastic differential systems with almost sectorial operators is discussed in this article, which is prompted by the theory presented in the preceding publications. In this manuscript, we will focus on the following subject: Hilfer fractional stochastic differential inclusions contain almost sectorial operators:

$$D_{0^+}^{\nu,\rho}z(\wp) \in Az(\wp) + \mathfrak{G}(\wp, z(\wp))\frac{dW(\wp)}{d\wp}, \qquad \wp \in \mathcal{J}' = (0, b], \tag{1}$$

$$I_{0^+}^{(1-\nu)(1-\rho)}z(0) = z_0. \tag{2}$$

In the above, $z(\cdot)$ accepts the value in a Hilbert space $\mathcal{Z}$ with $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$; $D_{0^+}^{\nu,\rho}$ represents the HF derivative of sequence $\nu \in (0,1)$ and type $\rho \in [0,1]$. An almost sectorial operator A is of the analytic semigroup $\{T(\wp), \wp \geq 0\}$ on $\mathcal{Z}$. Assume that U is a separable Hilbert space with $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$. For a U-valued Wiener process with a finite trace nuclear covariance operator $Q \geq 0$, we have $\{W(\wp)\}_{\wp\geq 0}$. The function $\mathfrak{G} : \mathcal{J} \times \mathcal{Z} \to L_2^0(\mathcal{J}, \mathcal{Z})$ is a nonempty, bounded, closed, and convex multivalued map with $\mathcal{J} = [0, b]$.

Our article is organised as follows: Section 2 covers fractional calculus, multivalued maps, semigroup theory, and sectorial operators, which includes several important concepts and well-known results. We present the existence of the mild solution in Section 3 and give an example in Section 4 to explain our primary claims. In the end, some conclusions are provided.

2. Preliminaries

We provide the required theorems and results in this section, which will be used throughout the essay to get the new results.

The symbols $(\mathcal{Z}, \|\cdot\|)$ and $(U, \|\cdot\|)$ signify two real Hilbert spaces. Consider that $(\Omega, \mathscr{E}, P)$ is a complete probability space associated to the whole family of right continuous increasing sub σ-algebras $\{\mathscr{E}_\wp : \wp \in \mathcal{J}\}$ fulfilling $\mathscr{E}_\wp \subset \mathscr{E}$. Let $W = (W_\wp)_{\wp\geq 0}$ be a Q-Wiener process with the covariance operator Q such that $Tr(Q) < \infty$ defined on $(\Omega, \mathscr{E}, P)$. We assume that U contains an orthonormal system $\{e_n\}_{n\geq 1}$, a bounded sequence of nonnegative real numbers β_n such that $Qe_n = \beta_n e_n$, $n = 1, 2, \cdots$ and $\{\mu_n\}$ of independent Brownian motions such that

$$(W(\wp), e)_U = \sum_{k=1}^{\infty} \sqrt{\beta_n}(e_n, e)\mu_n(\wp), \quad e \in U, \ \wp \geq 0.$$

Assume that $L_2^0 = L_2(Q^{\frac{1}{2}}U, \mathcal{Z})$ denotes for the space of all Q–Hilbert–Schmidt operators $\Psi : Q^{\frac{1}{2}}U \to \mathcal{Z}$ with the inner product $\|\Psi\|_Q^2 = \langle\Psi, \Psi\rangle = Tr(\Psi Q \Psi)$ is a Hilbert space. Assume $0 \in \varrho(A)$, is the resolvent set of A, and $\mathbf{S}(\cdot)$ is uniformly bounded, i.e., $\|\mathbf{S}(\wp)\| \leq M$, $M \leq 1$, and $\wp \geq 0$. For $\eta \in (0,1]$, the fractional power operator A^η on its domain $D(A^\eta)$ can thus be determined. Furthermore, $D(A^\eta)$ is dense in $\mathcal{Z}$.

The essential characteristics of A^η are as follows.

Theorem 1.

1. *Suppose* $0 < \eta \leq 1$, *accompanying* $\mathcal{Z}_\eta := D(A^\eta)$ *is a Banach space with* $\|z\|_\eta = \|A^\eta z\|$, $z \in \mathcal{Z}_\eta$.
2. *Suppose* $0 < \gamma < \eta \leq 1$, *corresponding* $D(A^\eta) \to D(A^\gamma)$ *and the implementation are compact whenever* A *is compact.*
3. *For all* $0 < \eta \leq 1$, *there exists* $C_\eta > 0$ *such that*

$$\|A^\eta \mathbf{S}(\wp)\| \leq \frac{C_\eta}{\wp^\eta}, \ 0 < \wp \leq b.$$

The collection of all strongly measurable, square-integrable, $\mathcal{Z}$-valued random variables, denoted by $L_2(\Omega, \mathcal{Z})$, is a Hilbert space associated with $\|z(\cdot)\|_{L_2(\Omega,\mathcal{Z})} = (E\|z(.,W)\|^2)^{\frac{1}{2}}$, where E is defined as $E(z) = \int_\Omega z(W)dP$. An important subspace of $L_2(\Omega, \mathcal{Z})$ is given by

$$L_2^0(\Omega, \mathcal{Z}) = \{z \in L_2(\Omega, \mathcal{Z}),\ z \text{ is } \mathscr{E}_0 - \text{measurable}\}.$$

For $b > 0$, let $\mathcal{J} = [0, b]$ and $\mathcal{J}' = (0, b]$. Denote $C(\mathcal{J}', L_2(\Omega, \mathcal{Z})) = \mathcal{C}$ as the Banach space of all continuous functions from $\mathcal{J}'$ to $L_2(\Omega, \mathcal{Z})$ that satisfy the condition $\sup_{\wp\in\mathcal{J}'} E\|z(\wp)\|^2 < \infty$. Consider, $\Delta = \{z \in C(\mathcal{J}', L_2(\Omega, \mathcal{Z})) : \lim_{\wp\to 0} \wp^{1-\rho+\rho\nu-\nu\vartheta} z(\wp)$ exists and finite$\}$, is a Hilbert space with $\|\cdot\|_\Delta$ and $\|z\|_\Delta = (\sup_{\wp\in\mathcal{J}'} E\|\wp^{1-\rho+\rho\nu-\nu\vartheta} z(\wp)\|^2)^{\frac{1}{2}}$.

We write

1. *For $\rho = 1$, $\Delta = \mathcal{C}$ and $\|z\|_\Delta = \sup_{\wp\in\mathcal{J}'} \|z(\wp)\|$.*
2. *For $\rho = 0$, $\|z\|_\Delta = \sup_{\wp\in\mathcal{J}'} \|\wp^{(1+\nu\vartheta)} z(\wp)\|$.*
3. *Let $z(\wp) = \wp^{1-\rho+\rho\nu-\nu\vartheta} u(\wp)$, $\wp \in \mathcal{J}'$, then $u \in \Delta \iff u \in \mathcal{C}$ and $\|z\|_\Delta = \|u\|$.*

Fix $\mathscr{B}_r(\mathcal{J}) = \{u \in \mathcal{C}$ such that $\|u\| \le r\}$ and $B_r^\Delta(\mathcal{J}) = \{z \in \Delta$ such that $\|z\|_\Delta \le r\}$.

Definition 1. *(See [4]). Thefractional integral of order ν for a function $\mathscr{G}(\wp)$ is given by*

$$I_{0+}^\nu \mathscr{G}(\wp) = \frac{1}{\Gamma(\nu)} \int_0^\wp \mathscr{G}(\ell)(\wp - \ell)^{\nu-1} d\ell, \quad \wp > 0, \quad \nu > 0.$$

The gamma function is represented by the aforementioned formula, $\Gamma(\cdot)$, and right side is defined point-wise on $[0, \infty)$.

Definition 2. *(See [4]). The R-L fractional derivative of a function $\mathscr{G}(\wp) : [0, \infty) \to \mathbb{R}$ with order ν may be expressed as*

$$^L D^\nu \mathscr{G}(\wp) = \frac{1}{\Gamma(1-\nu)} \frac{d}{d\wp} \int_0^\wp \frac{\mathscr{G}(\ell)}{(\wp - \ell)^\nu} d\ell, \quad \wp > 0, \quad 0 < \nu < 1.$$

Definition 3. *(See [4]). For $0 < \nu < 1$, the Caputo fractional derivative of order ν for a function $\mathscr{G}(\wp) : [0, \infty) \to \mathbb{R}$ is denoted by*

$$^C D^\nu \mathscr{G}(\wp) = \frac{1}{\Gamma(1-\nu)} \int_0^\wp \frac{\mathscr{G}'(\ell)}{(\wp - \ell)^\nu} d\ell.$$

Definition 4. *(See [39]). Let $0 < \nu < 1$ and $0 \le \rho \le 1$. The HF derivative of order ν and type ρ is denoted as*

$$D_{0+}^{\nu,\rho} \mathscr{G}(\wp) = I_{0+}^{\nu(1-\rho)} \frac{d}{d\wp} I_{0+}^{(1-\nu)(1-\rho)} \mathscr{G}(\wp).$$

Definition 5. *(See [26,49]). Let $0 < \delta < \Psi$ and $0 < \vartheta < 1$. We define $\mathcal{S}_\delta^0 = \{v \in \mathbb{C}\backslash\{0\}$ such that $|arg\ v| < \delta\}$ and its closure by S_δ,*

$$\text{i.e., } \mathcal{S}_\delta = \{v \in \mathbb{C}\backslash\{0\} \text{ such that } |arg\ v| \le \delta\} \cup \{0\}.$$

Consider $0 < \vartheta < 1$, $0 < \varpi < \frac{\pi}{2}$, we determine $\Theta_\varpi^{-\vartheta}$ is the set of all closed and linear operators $A : D(A) \subset \mathcal{Z} \to \mathcal{Z}$ that fulfills

(a) *$\sigma(A) \subseteq \mathcal{S}_\varpi$;*
(b) *$R(v, A)_{L(\mathcal{Z})} \le M_\delta |v|^{-\vartheta}$, given by for all $\varpi < \delta < \pi$ and $\exists\ M_\delta$ be a constant ,*

then $A \in \Theta_\varpi^{-\vartheta}$ is identified as almost sectorial operator on $\mathcal{Z}$.

Proposition 1. *(See [49]). Let $T(\wp)$ be the compact semigroup defined in [26] and $A \in \Theta_\varpi^{-\vartheta}$ for $0 < \vartheta < 1$ and $0 < \varpi < \frac{\pi}{2}$. Then the proceeding outcomes are fulfilled:*

1. $T(\wp)$ is analytic and $\frac{d^n}{d\wp^n}T(\wp) = (-A)^n T(\wp)$, $\wp \in \mathcal{S}_{\frac{\pi}{2}-\varpi}$;
2. $T(\wp+\ell) = T(\wp)T(\ell)$, for all $\ell, \wp \in \mathcal{S}_{\frac{\pi}{2}-\varpi}$;
3. $\|T(\wp)\|_{L(\mathcal{Z})} \leq \beta_0 \wp^{\vartheta-1}$, $\wp > 0$; where $\beta_0 > 0$ be the constant;
4. If $\Sigma_T = \{z \in \mathcal{Z} : \lim_{t\to 0^+} T(\wp)z = z\}$. Then $D(A^\theta) \subset \Sigma_T$, if $\theta > 1+\vartheta$;
5. $(v-A)^{-1} = \int_0^\infty e^{-v\ell} T(\ell) d\ell$, $v \in \mathbb{C}$ and $Re(v) > 0$.

Definition 6. *(See [50]). The Wright function $M_\nu(\beta)$ is defined by*

$$M_\nu(\beta) = \sum_{n\in\mathbb{N}} \frac{(-\beta)^{n-1}}{\Gamma(1-\nu n)(n-1)!}, \qquad \beta \in \mathbb{C}, \tag{3}$$

with the following property

$$\int_0^\infty \theta^\iota M_\nu(\theta) d\theta = \frac{\Gamma(1+\iota)}{\Gamma(1+\nu\iota)}.$$

Theorem 2. *(See [4]). In the uniform operator topology, $\mathbf{S}_{\nu,\rho}(\wp)$ and $\mathbf{Q}_\nu(\wp)$ are continuous for $\wp > 0$, for all $b > 0$, and the continuity is uniform on $[b,\infty)$.*

Definition 7. *An $\mathscr{E}_\wp$-adapted stochastic process $z(\wp) \in C(\mathcal{J}', \mathcal{Z})$ is called a mild solution of the Cauchy problem (1) and (2), given $I_0^{(1-\nu)(1-\rho)} z(0) = z_0$; $z_0 \in L_2^0(\Omega, \mathcal{Z})$ and there exists $\mathscr{G} \in L^2(\Omega, \mathcal{Z})$ such that $\mathscr{G}(\wp) \in \mathfrak{G}(\wp, z(\wp))$ on $\wp \in \mathcal{J}'$ and that satisfied*

$$z(\wp) = \mathbf{S}_{\nu,\rho}(\wp) z_0 + \int_0^\wp \mathbf{K}_\nu(\wp)(\wp-\ell)\mathfrak{G}(\ell, z(\ell)) dW(\ell), \qquad \wp \in \mathcal{J}', \tag{4}$$

where $\mathbf{S}_{\nu,\rho}(\wp) = I_0^{\rho(1-\nu)} \mathbf{K}_\nu(\wp)$, $\mathbf{K}_\nu(\wp) = \wp^{\nu-1}\mathbf{Q}_\nu(\wp)$ and $\mathbf{Q}_\nu(\wp) = \int_0^\infty \nu\xi M_\nu(\xi) T(\wp^\nu \xi) d\xi$.

Lemma 1. *(See [44]).*

1. $\mathbf{K}_\nu(\wp)$ *and* $\mathbf{S}_{\nu,\rho}(\wp)$ *are strongly continuous, for* $\wp > 0$.
2. *The bounded linear operators on $\mathcal{Z}$ are $\mathbf{K}_\nu(\wp)$ and $\mathbf{S}_{\nu,\rho}(\wp)$, for all fixed $\wp \in \mathcal{S}_{\frac{\pi}{2}-\varpi}$, and we have*

$$\|\mathbf{K}_\nu(\wp)z\| \leq \mathcal{L}_1 \wp^{-1+\nu\vartheta}\|z\|, \qquad \|\mathbf{Q}_\nu(\wp)z\| \leq \mathcal{L}_1 \wp^{-\nu+\nu\vartheta}\|z\|,$$
$$\|\mathbf{S}_{\nu,\rho}(\wp)z\| \leq \mathcal{L}_2 \wp^{-1+\rho-\rho\nu+\nu\vartheta}\|z\|,$$

where $\mathcal{L}_1 = \frac{\beta_p \Gamma(\vartheta)}{\Gamma(\nu\vartheta)}$, $\quad \mathcal{L}_2 = \frac{\beta_p \Gamma(\vartheta)}{\Gamma(\rho(1-\nu)+\nu\vartheta)}$.

Lemma 2. *(See [51]). Let $\mathcal{J}$ be a compact real interval, and $BCC(\mathcal{Z})$ is the family of all bounded, closed, convex, and nonempty subset of $\mathcal{Z}$. Let $\mathfrak{G}$ be the L^2-Caratheodory multivalued map that fulfills*

$$\mathcal{S}_{\mathfrak{G},z} = \{\mathscr{G} \in L^2(\mathcal{J}, L(U,\mathcal{Z})) : \mathscr{G}(\wp) \in \mathfrak{G}(\wp, z(\wp)), \quad \wp \in \mathcal{J}\}, \tag{5}$$

which is nonempty. Let $\Upsilon : L^2(\mathcal{J}, \mathcal{Z}) \to \mathcal{C}$ be the linear continuous function, then

$$\Upsilon \circ \mathcal{S}_{\mathfrak{G}} : \mathcal{C} \to BCC(\mathcal{C}), \quad z \to (\Upsilon \circ \mathcal{S}_{\mathfrak{G}})(z) = \Upsilon(\mathcal{S}_{\mathfrak{G},z}), \tag{6}$$

is closed graph operator in $\mathcal{C} \times \mathcal{C}$.

Lemma 3. *(Dhage fixed-point theorem) (See [52,53]). Let $\mathcal{Z}$ be a Hilbert space, and $\sigma_1 : \mathcal{Z} \to BCC(\mathcal{Z})$ and $\sigma_2 : \mathcal{Z} \to BCC(\mathcal{Z})$ are any two multivalued operators that satisfy:*

1. σ_1 *is contraction and*
2. σ_2 *is completely continuous. Then either*
   (a) *the operator inclusions $\mu z \in \sigma_1 z + \sigma_2 z$ has a solution for $\mu = 1$, or*
   (b) *the set $\Lambda = \{z \in \mathcal{Z} : \mu z \in \sigma_1 z + \sigma_2 z, \ \mu > 1\}$ is unbounded.*

3. Existence of Mild Solution

We are mostly interested in the existence of (1) and (2). Before we begin looking at the key outcomes, we make the proceeding assumptions:

(H_1) The operator $\{T(\wp),\ \wp \geq 0\}$ is compact.

(H_2) The multivalued map $\mathfrak{G} : \mathcal{J} \times \mathcal{Z} \to BCC(L(U, \mathcal{Z}))$ is measurable to $\wp$ for all fixed $z \in \mathcal{Z}$, u.s.c. to z for every $\wp \in \mathcal{J}$ and for all $z \in \mathcal{C}$ the set

$$\mathcal{S}_{\mathfrak{G},z} = \{\mathscr{G} \in L^2(\mathcal{J}, L(U, \mathcal{Z})) : \mathscr{G}(\wp) \in \mathfrak{G}(\wp, z(\wp)), \quad \wp \in \mathcal{J}\}$$

is nonempty.

(H_3) There exists a constant $q_1 \in (0, q)$ and $L_{\mathfrak{G},r}(\wp) \in L^{\frac{1}{q_1}}(\mathcal{J}', \mathbb{R}^+)$ satisfying

$$\lim_{\wp \to 0^+} \wp^{1-\rho+\rho\nu-\nu\vartheta} I_{0^+}^{\nu\vartheta} L_{\mathfrak{G},r}(\wp) = 0 \text{ and } \sup\{E\|\mathscr{G}\|^2 : \mathscr{G}(\wp) \in \mathfrak{G}(\wp, z(\wp))\} \leq L_{\mathfrak{G},r}(\wp),$$

for all $z(\wp) \in B_r^{\Delta}$ and a.e. $\wp \in \mathcal{J}$.

(H_4) For the function $\ell \to (\wp - \ell)^{2(\nu\vartheta-1)} L_{\mathfrak{G},r}(\ell) \in L^1(\mathcal{J}, \mathbb{R}^+)$, there exists $M_r > 0$ such that

$$\liminf_{r \to \infty} \frac{\wp^{2(1-\rho+\rho\nu-\nu\vartheta)} \int_0^{\wp} (\wp - \ell)^{2(\nu\vartheta-1)} L_{\mathfrak{G},r}(\ell) d\ell}{r} = M_r < \infty. \tag{7}$$

Theorem 3. *Assume the hypotheses (H_1)–(H_4) are fulfilled. Then the Hilfer fractional stochastic differential systems (1) and (2) have a mild solution on $\mathcal{J}$, provided $2Tr(Q)\beta_p^2 M_r < 1$, and $z_0 \in D(A^{\theta})$ with $\theta > 1 + \vartheta$.*

Proof. We approach the operator $\Gamma : \mathcal{C} \to 2^{\mathcal{C}}$ is denoted by Γz the set of $\Psi \in \mathcal{C}$ such that

$$\Psi(z(\wp)) = \begin{cases} 0, & \wp = 0, \\ \wp^{1-\rho+\rho\nu-\nu\vartheta}\left[\mathbf{S}_{\nu,\rho}(\wp) z_0 + \int_0^{\wp} (\wp - \ell)^{\nu-1} \mathbf{Q}_{\nu}(\wp - \ell) \mathscr{G}(\ell) dW(\ell)\right], & \wp \in (0, b], \end{cases} \tag{8}$$

where $\mathscr{G} \in S_{\mathfrak{G},z}$. To prove that Ψ seems to have a fixed point. Now we'll look at an operator $\Psi = \Psi_1 + \Psi_2$, where $\Psi_k : \mathscr{B}_r(\mathcal{J}) \to \mathscr{B}_r(\mathcal{J}), \quad k = 1, 2.$

$$\Psi_1 z(\wp) = \wp^{1-\rho+\rho\nu-\nu\vartheta} \mathbf{S}_{\nu,\rho}(\wp) z_0,$$
$$\Psi_2 z(\wp) = \wp^{1-\rho+\rho\nu-\nu\vartheta} \int_0^{\wp} (\wp - \ell)^{\nu-1} \mathbf{Q}_{\nu}(\wp - \ell) \mathscr{G}(\ell) dW(\ell).$$

Step 1: To show that Ψ_1 is a closed, convex subset of $\mathcal{C}$ for all $z \in \mathscr{B}_r(\mathcal{J})$ we will now prove Ψ_1 has a bounded value in $\mathscr{B}_r(\mathcal{J})$.

$$\begin{aligned} E\|\Psi_1 z(\wp)\|^2 &\leq \sup_{\wp \in [0,b]} \left\{ \wp^{2(1-\rho+\rho\nu-\nu\vartheta)} E\|\mathbf{S}_{\nu,\rho}(\wp) z_0\|^2 \right\} \\ &\leq \sup_{\wp \in [0,b]} \left\{ \wp^{2(1-\rho+\rho\nu-\nu\vartheta)} \left(\frac{\Gamma(\vartheta)}{\Gamma(\rho(1-\nu)+\nu\vartheta)} \right)^2 \beta_p^2 \wp^{2(-1+\rho-\rho\nu+\nu\vartheta)} \|z_0\|^2 \right\} \\ &\leq b^{2(1-\rho+\rho\nu-\nu\vartheta)} \left(\frac{\Gamma(\vartheta)}{\Gamma(\rho(1-\nu)+\nu\vartheta)} \right)^2 \beta_p^2 b^{2(-1+\rho-\rho\nu+\nu\vartheta)} \|z_0\|^2 \\ &\leq b^{2(1-\rho+\rho\nu-\nu\vartheta)} \mathcal{L}^*, \end{aligned}$$

where $\mathcal{L}^* = \left(\frac{\Gamma(\vartheta)}{\Gamma(\rho(1-\nu)+\nu\vartheta)} \right)^2 \beta_p^2 b^{2(-1+\rho-\rho\nu+\nu\vartheta)} \|z_0\|^2.$

Hence Ψ_1 is bounded.

Step 2: To prove, Ψ_1 is contraction on $\mathscr{B}_r(\mathcal{J})$.
Consider $z_1, z_2 \in \mathscr{B}_r(\mathcal{J})$,

$$E\left\|\Psi_1 z_1(\wp) - \Psi_1 z_2(\wp)\right\|^2 \leq \wp^{2(1-\rho+\rho\nu-\nu\vartheta)} E\|\mathbf{S}_{\nu,\rho}(\wp)z_0 - \mathbf{S}_{\nu,\rho}(\wp)z_0\|^2.$$

We do not need to show anything because Ψ_1 is a contraction.
Step 3: For the Ψ_2 is completely continuous of Ψ_2.
Claim 1: For all $z \in \mathscr{B}_r(\mathcal{J})$, $\Psi_2 z$ is convex.
Let $\Psi_1, \Psi_2 \in \{\Psi_2 z(\wp)\}$ and $\mathscr{G}_1, \mathscr{G}_2 \in \mathcal{S}_{\mathfrak{G},z}$ such that $\wp \in \mathcal{J}$, and we know

$$\Psi_k = \wp^{1-\rho+\rho\nu-\nu\vartheta} \int_0^{\wp} (\wp - \ell)^{\nu-1} \mathbf{Q}_\nu(\wp - \ell) \mathscr{G}_k(\ell) dW(\ell), \quad k = 1, 2.$$

Consider $\chi \in [0,1]$ then each of $\wp \in \mathcal{J}$, we obtain

$$\chi\Psi_1 + (1-\chi)\Psi_2(\wp) = \wp^{1-\rho+\rho\nu-\nu\vartheta} \int_0^{\wp} (\wp - \ell)^{\nu-1} \mathbf{Q}_\nu(\wp - \ell) \left[\chi\mathscr{G}_1(\ell) + (1-\chi)\mathscr{G}_2(\ell)\right] dW(\ell).$$

We know that $\mathcal{S}_{\mathfrak{G},z}$ is convex. So, $\chi\mathscr{G}_1 + (1-\chi)\mathscr{G}_2 \in \mathcal{S}_{\mathfrak{G},z}$.
Therefore,

$$\chi\Psi_1 + (1-\chi)\Psi_2 \in \Psi_2 z(\wp),$$

and hence Ψ_2 is convex.
Claim 2: In $\mathscr{B}_r(\mathcal{J})$, Ψ_2 mapping bounded sets into bounded sets. It is sufficient to show that there exists $\Lambda > 0$ such that for all $z \in \mathscr{B}_r(\mathcal{J})$, and that $\|\Psi_2 z\|^2 \leq \Lambda$ exists. Consider that for all $z \in \mathscr{B}_r(\mathcal{J})$, we have

$$\begin{aligned}
E\left\|\Psi_2 z(\wp)\right\|^2 &\leq \sup_{\wp\in[0,b]} \wp^{2(1-\rho+\rho\nu-\nu\vartheta)} E\left\| \int_0^{\wp} (\wp - \ell)^{\nu-1} \mathbf{Q}_\nu(\wp - \ell) \mathscr{G}(\ell) dW(\ell)\right\|^2 \\
&\leq Tr(Q) \sup_{\wp\in[0,b]} \wp^{2(1-\rho+\rho\nu-\nu\vartheta)} \int_0^{\wp} \beta_p^2 (\wp - \ell)^{2(\nu\vartheta-1)} E\|\mathscr{G}(\ell)\|^2 d\ell \\
&\leq Tr(Q) \sup_{\wp\in[0,b]} \wp^{2(1-\rho+\rho\nu-\nu\vartheta)} \beta_p^2 \frac{L_{\mathfrak{G},r}(\wp)}{(2\nu\vartheta - 1)} \wp^{2\nu\vartheta-1} \\
&\leq Tr(Q) b^{1-2\rho(1-\nu)} \beta_p^2 \frac{L_{\mathfrak{G},r}(b)}{(2\nu\vartheta - 1)} \\
&\leq \Lambda.
\end{aligned}$$

As a result, it is bounded.

Claim 3: To show that Ψ_2 mapping bounded set to equicontinuous set of $\mathscr{B}_r(\mathcal{J})$. Consider $0 < \wp_1 < \wp_2 \leq b$ and there exists $\mathscr{G} \in \mathcal{S}_{\mathfrak{G},z}$, we get

$$
\begin{aligned}
E\left\|\Psi_2 z(\wp_2) - \Psi_2 z(\wp_1)\right\|^2 &\leq E\left\|\wp_2^{1-\rho+\rho\nu-\nu\vartheta}\int_0^{\wp_2}(\wp_2-\ell)^{\nu-1}\mathbf{Q}_\nu(\wp_2-\ell)\mathscr{G}(\ell)dW(\ell)\right.\\
&\quad\left.-\wp_1^{1-\rho+\rho\nu-\nu\vartheta}\int_0^{\wp_1}(\wp_1-\ell)^{\nu-1}\mathbf{Q}_\nu(\wp_1-\ell)\mathscr{G}(\ell)dW(\ell)\right\|^2\\
&\leq E\left\|\wp_2^{1-\rho+\rho\nu-\nu\vartheta}\int_0^{\wp_1}(\wp_2-\ell)^{\nu-1}\mathbf{Q}_\nu(\wp_2-\ell)\mathscr{G}(\ell)dW(\ell)\right.\\
&\quad+\wp_2^{1-\rho+\rho\nu-\nu\vartheta}\int_{\wp_1}^{\wp_2}(\wp_2-\ell)^{\nu-1}\mathbf{Q}_\nu(\wp_2-\ell)\mathscr{G}(\ell)dW(\ell)\\
&\quad\left.-\wp_1^{1-\rho+\rho\nu-\nu\vartheta}\int_0^{\wp_1}(\wp_1-\ell)^{\nu-1}\mathbf{Q}_\nu(\wp_1-\ell)\mathscr{G}(\ell)dW(\ell)\right\|^2\\
&\leq 3E\left\|\wp_2^{1-\rho+\rho\nu-\nu\vartheta}\int_{\wp_1}^{\wp_2}(\wp_2-\ell)^{\nu-1}\mathbf{Q}_\nu(\wp_2-\ell)\mathscr{G}(\ell)dW(\ell)\right\|^2\\
&\quad+3E\left\|\wp_2^{1-\rho+\rho\nu-\nu\vartheta}\int_0^{\wp_1}(\wp_2-\ell)^{\nu-1}\mathbf{Q}_\nu(\wp_2-\ell)\mathscr{G}(\ell)dW(\ell)\right.\\
&\quad\left.-\wp_1^{1-\rho+\rho\nu-\nu\vartheta}\int_0^{\wp_1}(\wp_1-\ell)^{\nu-1}\mathbf{Q}_\nu(\wp_2-\ell)\mathscr{G}(\ell)dW(\ell)\right\|^2\\
&\quad+3E\left\|\wp_1^{1-\rho+\rho\nu-\nu\vartheta}\int_0^{\wp_1}(\wp_1-\ell)^{\nu-1}\mathbf{Q}_\nu(\wp_2-\ell)\mathscr{G}(\ell)dW(\ell)\right.\\
&\quad\left.-\wp_1^{1-\rho+\rho\nu-\nu\vartheta}\int_0^{\wp_1}(\wp_1-\ell)^{\nu-1}\mathbf{Q}_\nu(\wp_1-\ell)\mathscr{G}(\ell)dW(\ell)\right\|^2\\
&= I_1 + I_2 + I_3,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= 3E\left\|\wp_2^{1-\rho+\rho\nu-\nu\vartheta}\int_{\wp_1}^{\wp_2}(\wp_2-\ell)^{\nu-1}\mathbf{Q}_\nu(\wp_2-\ell)\mathscr{G}(\ell)dW(\ell)\right\|^2\\
&\leq 3Tr(Q)\wp_2^{2(1-\rho+\rho\nu-\nu\vartheta)}\int_{\wp_1}^{\wp_2}(\wp_2-\ell)^{2(\nu-1)}\|\mathbf{Q}_\nu(\wp_2-\ell)\|^2 E\|\mathscr{G}(\ell)\|^2 d\ell\\
&\leq 3Tr(Q)\beta_p^2\wp_2^{2(1-\rho+\rho\nu-\nu\vartheta)}\int_{\wp_1}^{\wp_2}(\wp_2-\ell)^{2(\nu\vartheta-1)}L_{\mathfrak{G},r}(\ell)d\ell\\
&\leq 3Tr(Q)\beta_p^2\left\{\left[\wp_2^{2(1-\rho+\rho\nu-\nu\vartheta)}\int_0^{\wp_2}(\wp_2-\ell)^{2(\nu\vartheta-1)}L_{\mathfrak{G},r}(\ell)d\ell\right.\right.\\
&\quad\left.-\wp_1^{2(1-\rho+\rho\nu-\nu\vartheta)}\int_0^{\wp_1}(\wp_1-\ell)^{2(\nu\vartheta-1)}L_{\mathfrak{G},r}(\ell)d\ell\right]\\
&\quad\left.+\int_0^{\wp_1}\left[\wp_1^{2(1-\rho+\rho\nu-\nu\vartheta)}(\wp_1-\ell)^{2(\nu\vartheta-1)}-\wp_2^{2(1-\rho+\rho\nu-\nu\vartheta)}(\wp_2-\ell)^{2(\nu\vartheta-1)}\right]L_{\mathfrak{G},r}(\ell)d\ell\right\}.
\end{aligned}
$$

Then $I_1 \to 0$ as $\wp_2 \to \wp_1$ by using (H_2) and the Lebesgue dominated convergent theorem:

$$
\begin{aligned}
I_2 &= 3E\bigg\|\wp_2^{(1-\rho+\rho\nu-\nu\vartheta)}\int_0^{\wp_1}(\wp_2-\ell)^{\nu-1}\mathbf{Q}_\nu(\wp_2-\ell)\mathscr{G}(\ell)dW(\ell) \\
&\quad -\wp_1^{1-\rho+\rho\nu-\nu\vartheta}\int_0^{\wp_1}(\wp_1-\ell)^{\nu-1}\mathbf{Q}_\nu(\wp_2-\ell)\mathscr{G}(\ell)dW(\ell)\bigg\|^2 \\
&\le 3E\bigg\|\int_0^{\wp_1}\Big[\wp_2^{1-\rho+\rho\nu-\nu\vartheta}(\wp_2-\ell)^{\nu-1}-\wp_1^{1-\rho+\rho\nu-\nu\vartheta}(\wp_1-\ell)^{\nu-1}\Big]\mathbf{Q}_\nu(\wp_2-\ell)\mathscr{G}(\ell)dW(\ell)\bigg\|^2 \\
&\le 3Tr(Q)\int_0^{\wp_1}E\bigg\|\wp_2^{1-\rho+\rho\nu-\nu\vartheta}(\wp_2-\ell)^{\nu-1}-\wp_1^{1-\rho+\rho\nu-\nu\vartheta}(\wp_1-\ell)^{\nu-1}\bigg\|^2 \\
&\qquad \|\mathbf{Q}_\nu(\wp_2-\ell)\|^2E\|\mathscr{G}(\ell)\|^2d\ell \\
&\le 3Tr(Q)\beta_p^2\int_0^{\wp_1}(\wp_2-\ell)^{2\nu(\vartheta-1)}E\bigg\|\wp_2^{1-\rho+\rho\nu-\nu\vartheta}(\wp_2-\ell)^{\nu-1} \\
&\quad -\wp_1^{1-\rho+\rho\nu-\nu\vartheta}(\wp_1-\ell)^{\nu-1}\bigg\|^2L_{\mathfrak{G},r}(\ell)d\ell,
\end{aligned}
$$

consider

$$
\begin{aligned}
(\wp_2-\ell)^{2\nu(\vartheta-1)}&E\bigg\|\wp_2^{1-\rho+\rho\nu-\nu\vartheta}(\wp_2-\ell)^{\nu-1}-\wp_1^{1-\rho+\rho\nu-\nu\vartheta}(\wp_1-\ell)^{\nu-1}\bigg\|^2L_{\mathfrak{G},r}(\ell) \\
&\le\Big[2\wp_2^{2(1-\rho+\rho\nu-\nu\vartheta)}(\wp_2-\ell)^{2\nu(\vartheta-1)} \\
&\qquad +2\wp_1^{2(1-\rho+\rho\nu-\nu\vartheta)}(\wp_1-\ell)^{2\nu-2}(\wp_2-\ell)^{2\nu(\vartheta-1)}\Big]L_{\mathfrak{G},r}(\ell)
\end{aligned}
$$

$$
\begin{aligned}
&\le\Big[2\wp_2^{2(1-\rho+\rho\nu-\nu\vartheta)}(\wp_2-\ell)^{2\nu(\vartheta-1)}+2\wp_1^{2(1-\rho+\rho\nu-\nu\vartheta)}(\wp_1-\ell)^{2\nu(\vartheta-1)}\Big]L_{\mathfrak{G},r}(\ell) \\
&\le 4\wp_1^{2(1-\rho+\rho\nu-\nu\vartheta)}(\wp_1-\ell)^{2\nu(\vartheta-1)}L_{\mathfrak{G},r}(\ell),
\end{aligned}
$$

and $\int_0^{\wp_1}4\wp_1^{2(1-\rho+\rho\nu-\nu\vartheta)}(\wp_1-\ell)^{2\nu(\vartheta-1)}L_{\mathfrak{G},r}(\ell)d\ell$ exists $\ell\in(0,\wp_1]$, so by Lebesgue's dominated convergence theorem, we have

$$
\int_0^{\wp_1}(\wp_2-\ell)^{2\nu(\vartheta-1)}E\bigg\|\wp_2^{1-\rho+\rho\nu-\nu\vartheta}(\wp_2-\ell)^{\nu-1}-\wp_1^{1-\rho+\rho\nu-\nu\vartheta}(\wp_1-\ell)^{\nu-1}\bigg\|^2L_{\mathfrak{G},r}(\ell)d\ell\to 0
$$

as $\wp_2\to\wp_1$, so we conclude $\lim_{\wp_2\to\wp_1}I_2=0$.
For all $\epsilon>0$, we have

$$
\begin{aligned}
I_3 &= 3E\left\| \int_0^{\wp_1} \wp_1^{1-\rho+\rho\nu-\nu\vartheta}\big[\mathbf{Q}_\nu(\wp_2-\ell)-\mathbf{Q}_\nu(\wp_1-\ell)\big](\wp_1-\ell)^{\nu-1}\mathscr{G}(\ell)dW(\ell)\right\|^2 \\
&\le 3Tr(Q)\int_0^{\wp_1} \wp_1^{2(1-\rho+\rho\nu-\nu\vartheta)}\big\|\mathbf{Q}_\nu(\wp_2-\ell)-\mathbf{Q}_\nu(\wp_1-\ell)\big\|^2(\wp_1-\ell)^{2(\nu-1)}E\|\mathscr{G}(\ell)\|^2 d\ell \\
&\le 3Tr(Q)\int_0^{\wp_1} \wp_1^{2(1-\rho+\rho\nu-\nu\vartheta)}\big\|\mathbf{Q}_\nu(\wp_2-\ell)-\mathbf{Q}_\nu(\wp_1-\ell)\big\|^2(\wp_1-\ell)^{2(\nu-1)}L_{\mathfrak{G},r}(\ell)d\ell \\
&\le 3Tr(Q)\bigg\{\int_0^{\wp_1-\epsilon} \wp_1^{2(1-\rho+\rho\nu-\nu\vartheta)}\big\|\mathbf{Q}_\nu(\wp_2-\ell)-\mathbf{Q}_\nu(\wp_1-\ell)\big\|^2(\wp_1-\ell)^{2(\nu-1)}L_{\mathfrak{G},r}(\ell)d\ell \\
&\quad+\int_{\wp_1-\epsilon}^{\wp_1} \wp_1^{2(1-\rho+\rho\nu-\nu\vartheta)}\big\|\mathbf{Q}_\nu(\wp_2-\ell)-\mathbf{Q}_\nu(\wp_1-\ell)\big\|^2(\wp_1-\ell)^{2(\nu-1)}L_{\mathfrak{G},r}(\ell)d\ell\bigg\} \\
&\le 3Tr(Q)\bigg\{\wp_1^{2(1-\rho+\rho\nu-\nu\vartheta)}\int_0^{\wp_1-\epsilon}(\wp_1-\ell)^{2(\nu-1)}L_{\mathfrak{G},r}(\ell)d\ell \\
&\qquad\qquad \sup_{\ell\in[0,\wp_1-\epsilon]}\big\|\mathbf{Q}_\nu(\wp_2-\ell)-\mathbf{Q}_\nu(\wp_1-\ell)\big\|^2 \\
&\quad+\beta_p^2\int_{\wp_1-\epsilon}^{\wp_1}\wp_1^{2(1-\rho+\rho\nu-\nu\vartheta)}[2(\wp_2-\ell)^{2\nu(\vartheta-1)}+2(\wp_1-\ell)^{2\nu(\vartheta-1)}]L_{\mathfrak{G},r}(\ell)d\ell\bigg\} \\
&\le 3Tr(Q)\bigg\{\wp_1^{2(1-\rho+\rho\nu-\nu\vartheta)-2\nu(\vartheta-1)}\int_0^{\wp_1-\epsilon}(\wp_1-\ell)^{2(\nu\vartheta-1)}L_{\mathfrak{G},r}(\ell)d\ell\sup_{\ell\in[0,\wp_1-\epsilon]}\big\|\mathbf{Q}_\nu(\wp_2-\ell) \\
&\quad-\mathbf{Q}_\nu(\wp_1-\ell)\big\|^2+4\beta_p^2\int_{\wp_1-\epsilon}^{\wp_1}\wp_1^{2(1-\rho+\rho\nu-\nu\vartheta)}(\wp_1-\ell)^{2(\nu\vartheta-1)}L_{\mathfrak{G},r}(\ell)d\ell\bigg\}.
\end{aligned}
$$

From Theorem 2 and $\lim_{\wp_2\to\wp_1} I_1 = 0$, we obtain $I_3 \to 0$ independently of $z \in \mathscr{B}_r(\mathcal{J})$ as $\wp_2 \to \wp_1$, $\epsilon \to 0$. Hence $\|\Psi_2 z(\wp_2) - \Psi_2 z(\wp_1)\|^2 \to 0$ independently of $z \in \mathscr{B}_r(\mathcal{J})$ as $\wp_2 \to \wp_1$. This implies that $\{\Psi_2 z(\wp) : z \in \mathscr{B}_r(\mathcal{J})\}$ is equicontinuous on $\mathcal{J}$.

Claim 4: To prove, Ψ_2 is completely continuous.

For $\alpha \in (0,\wp)$ and $\eta > 0$, assume the operator Ψ_2' on $\mathscr{B}_r(\mathcal{J})$ by

$$
\begin{aligned}
(\Psi_2)_{\alpha,\eta} z(\wp) &= \wp^{1-\rho+\rho\nu-\nu\vartheta}\int_0^{\wp-\alpha}(\wp-\ell)^{\nu-1}\mathbf{Q}_\nu(\wp-\ell)\mathscr{G}(\ell)dW(\ell) \\
&= \nu\wp^{1-\rho+\rho\nu-\nu\vartheta}\int_0^{\wp-\alpha}\int_\eta^\infty \theta M_\nu(\theta)(\wp-\ell)^{\nu-1}T((\wp-\ell)^\nu\theta)\mathscr{G}(\ell)d\theta dW(\ell) \\
&= \nu\wp^{1-\rho+\rho\nu-\nu\vartheta}T(\alpha^\nu\eta)\int_0^{\wp-\eta}\int_\eta^\infty \theta M_\nu(\theta)(\wp-\ell)^{\nu-1}T((\wp-\ell)^\nu\theta-\alpha^\nu\eta) \\
&\qquad \mathscr{G}(\ell)d\theta dW(\ell).
\end{aligned}
$$

Hence $V_{\alpha,\eta}(\wp) = \{(\Psi_2)_{\alpha,\eta} z(\wp) : z \in \mathscr{B}_r(\mathcal{J})\}$ is precompact in $\mathcal{Z}$ for all $\alpha \in (0,\wp)$ and $\eta > 0$ because of the compactness of $T(\alpha^\nu\eta)$. For all $z \in \mathscr{B}_r(\mathcal{J})$, we get

$$
\begin{aligned}
E\left\|\Psi_2 z(\wp)-(\Psi_2)_{\alpha,\eta} z(\wp)\right\|^2 \leq\; & E\bigg\|\wp^{1-\rho+\rho\nu-\nu\vartheta}\int_0^{\wp}(\wp-\ell)^{\nu-1}\mathbf{Q}_\nu(\wp-\ell)\mathscr{G}(\ell)dW(\ell) \\
& -\wp^{1-\rho+\rho\nu-\nu\vartheta}\int_0^{\wp-\alpha}(\wp-\ell)^{\nu-1}\mathbf{Q}_\nu(\wp-\ell)\mathscr{G}(\ell)dW(\ell)\bigg\|^2 \\
\leq\; & 2E\bigg\|\nu\wp^{1-\rho+\rho\nu-\nu\vartheta}\int_0^{\wp}\int_0^{\eta}\theta M_\nu(\theta)(\wp-\ell)^{\nu-1}T((\wp-\ell)^\nu\theta) \\
& \quad \mathscr{G}(\ell)d\theta dW(\ell)\bigg\|^2 \\
& +2E\bigg\|\nu\wp^{1-\rho+\rho\nu-\nu\vartheta}\int_{\wp-\alpha}^{\wp}\int_{\eta}^{\infty}(\wp-\ell)^{\nu-1}\theta M_\nu(\theta)T((\wp-\ell)^\nu\theta) \\
& \quad \mathscr{G}(\ell)d\theta dW(\ell)\bigg\|^2 \\
\leq\; & 2Tr(Q)\nu^2\wp^{2(1-\rho+\rho\nu-\nu\vartheta)}\int_0^{\wp}\int_0^{\eta}\theta^2 M_\nu^2(\theta)(\wp-\ell)^{2(\nu-1)} \\
& \|T((\wp-\ell)^\nu\theta)\|^2 E\|\mathscr{G}(\ell)\|^2 d\theta d\ell \\
& +2Tr(Q)\nu^2\wp^{2(1-\rho+\rho\nu-\nu\vartheta)}\int_{\wp-\alpha}^{\wp}\int_{\eta}^{\infty}(\wp-\ell)^{2(\nu-1)}\theta^2 M_\nu^2(\theta) \\
& \|T((\wp-\ell)^\nu\theta)\|^2 E\|\mathscr{G}(\ell)\|^2 d\theta d\ell \\
\leq\; & 2Tr(Q)\nu^2\beta_0^2\wp^{2(1-\rho+\rho\nu-\nu\vartheta)}\int_0^{\wp}\int_0^{\eta}\theta^2 M_\nu^2(\theta)(\wp-\ell)^{2(\nu-1)} \\
& (\wp-\ell)^{2\nu(\vartheta-1)}\theta^{2(\vartheta-1)}L_{\mathfrak{G},r}(\ell)d\theta d\ell \\
& +2Tr(Q)\nu^2\beta_0^2\wp^{2(1-\rho+\rho\nu-\nu\vartheta)}\int_{\wp-\alpha}^{\wp}\int_{\eta}^{\infty}(\wp-\ell)^{2(\nu-1)}\theta^2 M_\nu^2(\theta) \\
& (\wp-\ell)^{2\nu(\vartheta-1)}\theta^{2(\vartheta-1)}L_{\mathfrak{G},r}(\ell)d\theta d\ell \\
\leq\; & 2Tr(Q)\nu^2\beta_0^2\wp^{2(1-\rho+\rho\nu-\nu\vartheta)}\int_0^{\wp}(\wp-\ell)^{2(\nu\vartheta-1)}L_{\mathfrak{G},r}(\ell)d\ell \\
& \int_0^{\eta}\theta^{2\vartheta}M_\nu^2(\theta)d\theta \\
& +2Tr(Q)\nu^2\beta_0^2\wp^{2(1-\rho+\rho\nu-\nu\vartheta)}\int_{\wp-\alpha}^{\wp}(\wp-\ell)^{2(\nu\vartheta-1)}L_{\mathfrak{G},r}(\ell)d\ell \\
& \int_0^{\infty}\theta^{2\vartheta}M_\nu^2(\theta)d\theta \\
\leq\; & 2Tr(Q)\nu^2\beta_0^2\wp^{2(1-\rho+\rho\nu-\nu\vartheta)}\int_0^{\wp}(\wp-\ell)^{2(\nu\vartheta-1)}L_{\mathfrak{G},r}(\ell)d\ell \\
& \int_0^{\eta}\theta^{2\vartheta}M_\nu^2(\theta)d\theta \\
& +2Tr(Q)\nu^2\beta_0^2\left(\frac{\Gamma(1+\vartheta)}{\Gamma(1+\nu\vartheta)}\right)^2\wp^{2(1-\rho+\rho\nu-\nu\vartheta)}\int_{\wp-\alpha}^{\wp}(\wp-\ell)^{2(\nu\vartheta-1)} \\
& L_{\mathfrak{G},r}(\ell)d\ell \\
\to\; & 0 \text{ as } \alpha\to 0, \wp\to 0.
\end{aligned}
$$

Because $V_{\alpha,\eta}(\wp) = \{(\Psi_2)_{\alpha,\eta}z(\wp) : z \in \mathscr{B}_r(\mathcal{J})\}$ are arbitrary closed to $\{\Psi_2 z(\wp) : \wp \in \mathscr{B}_r(\mathcal{J})\}$. Due to the Arzela–Ascoli Theorem, $\{\Psi_2 z(\wp) : \wp \in \mathscr{B}_r(\mathcal{J})\}$ is relatively compact. Hence, Ψ_2 is a perfectly continuous operator, as evidenced by the connectedness of Ψ_2 and relatively compactness of $\{\Psi_2 z(\wp) : \wp \in \mathscr{B}_r(\mathcal{J})\}$ imply this fact.

Claim 5: Ψ_2 is closed graph.

Assume $z_n \to z_*$ as $n \to \infty$, $\Psi_2^n \in \Gamma(z_n)$ and $\Psi_2^n \to \Psi_2^*$ as $n \to \infty$, we must demonstrate that $\Psi_2^* \in \Gamma(z_*)$. Because $\Psi_2^n \in \Gamma(z_n)$ then there exists a function $\mathscr{G}_n \in \mathcal{S}_{\mathfrak{G},z_n}$ such that

$$\Psi_2^n z(\wp) = \wp^{1-\rho+\rho\nu-\nu\vartheta} \int_0^{\wp} (\wp-\ell)^{\nu-1} \mathbf{Q}_\nu(\wp-\ell) \mathscr{G}_n(\ell) dW(\ell).$$

We have to prove there exists $\mathscr{G}_0 \in \mathcal{S}_{\mathfrak{G},z_0}$ such that

$$\Psi_2^* z(\wp) = \wp^{1-\rho+\rho\nu-\nu\vartheta} \int_0^{\wp} (\wp-\ell)^{\nu-1} \mathbf{Q}_\nu(\wp-\ell) \mathscr{G}_*(\wp) dW(\ell).$$

Clearly,

$$\begin{aligned} &\Big\| \big(\Psi_2^n z(\wp) - \int_0^{\wp} (\wp-\ell)^{\nu-1} \mathbf{Q}_\nu(\wp-\ell) \mathscr{G}_n(\ell) dW(\ell)\big) \\ &\qquad - \big(\Psi_2^* z(\wp) - \int_0^{\wp} (\wp-\ell)^{\nu-1} \mathbf{Q}_\nu(\wp-\ell) \mathscr{G}_*(\ell) dW(\ell)\big) \Big\| \to 0 \text{ as } n \to \infty. \end{aligned}$$

Now, we will examine an operator $\Upsilon : L^2(\mathcal{J}, \mathcal{Z}) \to \mathcal{C}(\mathcal{J}, \mathcal{Z})$,

$$\Upsilon(\mathscr{G})(\wp) = \int_0^{\wp} (\wp-\ell)^{\nu-1} \mathbf{Q}_\nu(\wp-\ell) \mathscr{G}(\ell) dW(\ell).$$

We know that $\Upsilon \circ \mathcal{S}_{\mathfrak{G},z}$ is closed graph operator because of Lemma 2. From Υ, we have

$$\Psi_2^n z(\wp) - \wp^{1-\rho+\rho\nu-\nu\vartheta} \int_0^{\wp} (\wp-\ell)^{\nu-1} \mathbf{Q}_\nu(\wp-\ell) \mathscr{G}_n(\ell) dW(\ell) \in \Upsilon(\mathcal{S}_{\mathfrak{G},z_n}),$$

because $\mathscr{G}_n \to \mathscr{G}_*$, follows from Lemma 2, indicating that

$$\Psi_2^* z(\wp) - \wp^{1-\rho+\rho\nu-\nu\vartheta} \int_0^{\wp} (\wp-\ell)^{\nu-1} \mathbf{Q}_\nu(\wp-\ell) \mathscr{G}_*(\ell) dW(\ell) \in \Upsilon(\mathcal{S}_{\mathfrak{G},z_*}).$$

Therefore, Ψ_2 is a closed graph. As a result, from **Step 1–3**, we finalise (1) and (2) of Lemma 3.

Step 4: Consider the case where $\mathscr{B}_r(\mathcal{J}) = \{z \in \mathcal{C} \text{ such that } \|z\|^2 \le r\}$. We must demonstrate that a number r such that $\Psi(\mathscr{B}_r(\mathcal{J})) \subseteq \mathscr{B}_r(\mathcal{J})$ exists. Suppose there is a function $z^r \in \mathscr{B}_r(\mathcal{J})$ that does not belong to $\Psi(\mathscr{B}_r(\mathcal{J}))$,

$$\begin{aligned} E\Big\|\Psi z^r(\wp)\Big\|^2 = E\Big\| &\wp^{1-\rho+\rho\nu-\nu\vartheta} \mathbf{S}_{\nu,\rho}(\wp) z_0 \\ &+ \wp^{1-\rho+\rho\nu-\nu\vartheta} \int_0^{\wp} (\wp-\ell)^{\nu-1} \mathbf{Q}_\nu(\wp-\ell) \mathscr{G}(\ell) dW(\ell)\Big\|^2 > r, \end{aligned}$$

from the hypotheses (H_3)–(H_4), we get

$$r \leq 2E\left\|\wp^{1-\rho+\rho\nu-\nu\vartheta}\mathbf{S}_{\nu,\rho}(\wp)z_0\right\|^2 + 2E\left\|\wp^{1-\rho+\rho\nu-\nu\vartheta}\int_0^{\wp}(\wp-\ell)^{\nu-1}\mathbf{Q}_\nu(\wp-\ell)\mathscr{G}(\ell)dW(\ell)\right\|^2$$
$$\leq 2\wp^{2(1-\rho+\rho\nu-\nu\vartheta)}\left\|\mathbf{S}_{\nu,\rho}(\wp)z_0\right\|^2$$
$$+ 2Tr(Q)\wp^{2(1-\rho+\rho\nu-\nu\vartheta)}\int_0^{\wp}(\wp-\ell)^{2(\nu-1)}\left\|\mathbf{Q}_\nu(\wp-\ell)\right\|^2 E\left\|\mathscr{G}(\ell)\right\|^2 d\ell$$
$$\leq 2\wp^{2(1-\rho+\rho\nu-\nu\vartheta)}\left(\frac{\Gamma(\vartheta)}{\Gamma(\rho(1-\nu)+\nu\vartheta)}\right)^2\beta_p^2\wp^{2(-1+\rho-\rho\nu+\nu\vartheta)}\|z_0\|^2$$
$$+ 2Tr(Q)\wp^{2(1-\rho+\rho\nu-\nu\vartheta)}\beta_p^2\int_0^{\wp}(\wp-\ell)^{2(\nu\vartheta-1)}L_{\mathfrak{G},r}(\ell)d\ell.$$

Both sides are divided by r and $r\to\infty$, and we have

$$1 \leq \lim_{r\to\infty}\inf\frac{2Tr(Q)\wp^{2(1-\rho+\rho\nu-\nu\vartheta)}\beta_p^2\int_0^{\wp}(\wp-\ell)^{2(\nu\vartheta-1)}L_{\mathfrak{G},r}(\ell)d\ell}{r} = 2Tr(Q)\beta_p^2 M_r.$$

This is a contradiction to our assumption. As a result, for every $z\in\mathscr{B}_r(\mathcal{J})$, $\left\|\Psi z(\wp)\right\|^2 \leq r$. As a result, $z(\wp)$ is a fixed point of Ψ, the mild solution.

Hence, we have completed the proof. □

4. Example

Let us assume that the proceeding *HF* stochastic differential inclusions

$$\begin{aligned} D_{0+}^{\frac{1}{2},\frac{1}{3}}z(\wp,\ell) &\in \partial_\ell^2 z(\wp,\ell) + \mathfrak{G}(\wp,z(\wp))\frac{dW(\wp)}{d\wp}, && \wp\in(0,b],\ \ell\in[0,\pi],\\ z(\wp,0) &= z(\wp,\pi) = 0, && \wp\geq 0,\\ I_{0+}^{(1-\frac{1}{2})(1-\frac{1}{3})} &= z_0(\ell), && \ell\in[0,\pi], \end{aligned} \tag{9}$$

in the Hilbert space $\mathcal{Z} = C^\nu([0,\pi])(0<\nu<1)$ of all Hölder continuous functions, and $\nu=\frac{1}{2}$, $\rho=\frac{1}{3}$, $\mathfrak{G}(\wp,z(\wp)) = \wp^{\frac{-1}{3}}\sin z(\wp)$ is a multivalued function fulfills the hypothesis (H_1)–(H_4). Assume $U=\mathcal{Z}=L^2([0,\pi],\mathbb{R})$ is a space. The one-dimensional conventional Brownian motion $W(\wp)$ stands on the filtered probability space $(\Omega,\mathscr{E},P)$. From [9], we have the abstract expression

$$\begin{aligned} D_{0+}^{\nu,\rho}z(\wp) &\in Az(\wp) + \mathfrak{G}(\wp,z(\wp))\frac{dW(\wp)}{d\wp}, \quad \wp\in(0,b],\\ I_{0+}^{(1-\nu)(1-\rho)}z(0) &= z_0. \end{aligned}$$

The almost sectorial operator $A=\partial_\ell^2$ is used here, and $D(A)=\{z\in C^{2+\nu}([0,\pi])$, such that $z(\wp,0)=z(\wp,\pi)=0\}$. From [26] $\delta,\epsilon>0$ be the constants, we write $A+\delta\in\Theta_{\frac{\pi}{2}-\epsilon}^{\frac{\nu}{2}-1}(\mathcal{Z})$. We have $D(A)\subset C^{2+p}([0,\pi])$ and $C^{2+p}([0,\pi])$ is compactly connected in the Hilbert space $\mathcal{Z}$. According to [4] (Lemma 4.66), the operator $R(p,-(A+\delta))$ is compact for all $p>0$, so the semigroup operator $T(\wp)$ is compact for all $\wp>0$. We choose $L_{\mathfrak{G},r}(\ell)=\ell^{\frac{-1}{3}}$ and

$$\begin{aligned} M_r &= \lim_{r\to\infty}\inf\frac{\int_0^{\wp}(\wp-\ell)^{\frac{1}{2}-1}\ell^{\frac{2}{3}-1}d\ell}{r}\\ &= \lim_{r\to\infty}\inf\wp^{\frac{1}{6}}\frac{\frac{\Gamma(\frac{2}{3})\Gamma(\frac{1}{2})}{\Gamma(\frac{7}{6})}}{r}. \end{aligned}$$

Then, the hypotheses (H_1)–(H_4) are fulfilled. According to Theorem 3, the systems (1) and (2) have the mild solution on $\mathcal{J}$.

5. Conclusions

The existence of HF stochastic differential systems with almost sectorial operators was the subject of our research. The major conclusions are established by utilising the concepts and ideas from fractional calculus, multivalued maps, semigroup theory, sectorial operators, and fixed-point technique. We started by confirming the existence of the mild solution. Then, an illustration is given to explain the principle. The exact and approximate controllability of HF stochastic differential systems via almost sectorial operators will be examined by using a fixed-point method, as well as we can develop with the reference of [13–17].

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Abbreviations

The following abbreviations are used in this manuscript:

HF	Hilfer Fractional
R-L	Riemann Liouville

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