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Article

The Right Equivalent Integral Equation of Impulsive Caputo Fractional-Order System of Order $\epsilon \in (1,2)$

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Abstract: For the impulsive fractional-order system (IFrOS) of order $\epsilon \in (1,2)$, there have appeared some conflicting equivalent integral equations in existing studies. However, we find two fractional-order properties of piecewise function and use them to verify that these given equivalent integral equations have some defects to not be the equivalent integral equation of the IFrOS. For the IFrOS, its limit property shows the linear additivity of the impulsive effects. For the IFrOS, we use the limit analysis and the linear additivity of the impulsive effects to find its correct equivalent integral equation, which is a combination of some piecewise functions with two arbitrary constants; that is, the solution of the IFrOS is a general solution. Finally, a numerical example is given to show the equivalent integral equation and the non-uniqueness of the solution of the IFrOS.

Keywords: impulsive fractional differential equations; equivalent integral equations; general solution; non-uniqueness of solution

MSC: 26A33; 34A08; 34A37; 34A12

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1. Introduction

The subject of the impulsive fractional-order system (IFrOS) has been a research hotspot, and more than 800 articles can be searched on the Web of Science on the topic of impulsive fractional differential equations. For the IFrOS, its equivalent integral equation is an important tool to discuss some properties (such as the existence of solution, numerical solution, stability, and controllability).

However, in the mainstream research of the IFrOS, the fractional derivative in the IFrOS was independently considered from the whole interval or each subinterval respectively, which caused that there appeared two conflicting equivalent integral equations for the same IFrOS (For details, see [1–13]). Up to now, the two conflicting thoughts regarding the equivalent integral equation of the IFrOS are used to study numerical solution [14,15], oscillation behavior of solution [16,17], solvability [18], stability [19–21], asymptotic behavior of solution [22] and the existence of solution [23–32]. Moreover, by combining the two previous conflicting thoughts regarding the equivalent integral equation of IFrOS, the third equivalent integral equation of the same IFrOS was presented in [33,34].

However, recently, we found two fractional-order properties of piecewise function to uncover that the above three thoughts about the equivalent integral equation of IFrOS are incorrect. To support our viewpoint, we will give the fractional order properties of piecewise function and restudy the equivalent integral equation of the first IFrOS that was proposed in [35] by

$$\begin{cases} {}_{t_0}^{C}\mathcal{D}_t^{\epsilon}x(t) = h(t, x(t)), & t \in [t_0, T] \text{ and } t \neq t_k \ (k = 1, 2, \ldots, N), \\ x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), & k = 1, 2, \ldots, N, \\ x'(t_k^+) - x'(t_k^-) = J_k(x(t_k^-)), & k = 1, 2, \ldots, N, \\ x(t_0) = x_0, x'(t_0) = x_1, \end{cases} \tag{1}$$

here ${}_{t_0}^{C}\mathcal{D}_t^{\epsilon}$ ($\epsilon \in (1,2)$) denotes the left-sided Caputo fractional derivative of order ϵ, $h : [t_0, T] \times \mathbb{R} \to \mathbb{R}$, $-\infty < t_0 < t_1 < \ldots < t_N < t_{N+1} = T < +\infty$, $I_k : \mathbb{R} \to \mathbb{R}$ and $J_k : \mathbb{R} \to \mathbb{R}$ $(k = 1, 2, \ldots, N)$.

Next three conflicting equivalent integral equations of (1) in existing papers will be given. The results in [1–6,35] show that the equivalent integral equation of (1) is

$$x(t) = \begin{cases} x_0 + x_1(t - t_0) + \int_{t_0}^{t} \frac{(t-s)^{\epsilon-1}}{\Gamma(\epsilon)} hds, & t \in [t_0, t_1], \\ x(t_k^+) + x'(t_k^+)(t - t_k) + \int_{t_k}^{t} \frac{(t-s)^{\epsilon-1}}{\Gamma(\epsilon)} hds, & t \in (t_k, t_{k+1}], k = 1, 2, \ldots, N, \end{cases} \tag{2}$$

or equivalently,

$$x(t) = \begin{cases} x_0 + x_1(t - t_0) + \int_{t_0}^{t} \frac{(t-s)^{\epsilon-1}}{\Gamma(\epsilon)} hds, \quad t \in [t_0, t_1], \\ x_0 + x_1(t - t_0) + \sum_{i=1}^{k} I_i(x(t_i^-)) + \sum_{i=1}^{k} J_i(x(t_i^-))(t - t_i) + \int_{t_k}^{t} \frac{(t-s)^{\epsilon-1}}{\Gamma(\epsilon)} hds \\ \quad + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\epsilon-1}}{\Gamma(\epsilon)} hds + \sum_{i=1}^{k} (t - t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\epsilon-2}}{\Gamma(\epsilon-1)} hds, \quad t \in (t_k, t_{k+1}], \\ \quad k = 1, 2, \ldots, N. \end{cases} \tag{3}$$

Remark 1. *For simplicity, let $hds = h(s, x(s))ds$ of all integrals in the whole paper. However, by Lemma 1, (2) (or (3)) is actually the equivalent integral equation of the following hybrid system*

$$\begin{cases} {}_{t_i}^{C}\mathcal{D}_t^{\epsilon}x(t) = h(t, x(t)), & t \in (t_i, t_{i+1}],\ i = 0, 1, \ldots, N, \\ x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), & k = 1, 2, \ldots, N, \\ x'(t_k^+) - x'(t_k^-) = J_k(x(t_k^-)), & k = 1, 2, \ldots, N, \\ x(t_0) = x_0, x'(t_0) = x_1. \end{cases} \tag{4}$$

On the other hand, the results in [7–13] show that the equivalent integral equation of (1) is

$$x(t) = \begin{cases} x_0 + x_1(t - t_0) + \int_{t_0}^{t} \frac{(t-s)^{\epsilon-1}}{\Gamma(\epsilon)} hds, \quad t \in [t_0, t_1], \\ x_0 + x_1(t - t_0) + \sum_{i=1}^{k} I_i(x(t_i^-)) + \sum_{i=1}^{k} J_i(x(t_i^-))(t - t_i) + \int_{t_0}^{t} \frac{(t-s)^{\epsilon-1}}{\Gamma(\epsilon)} hds, \\ \quad t \in (t_k, t_{k+1}], k = 1, 2, \ldots, N. \end{cases} \tag{5}$$

Moreover, the author in [33] thought that the equivalent integral equation of (1) is an integral equation with two arbitrary constants:

$$
x(t)=\begin{cases} x_0+x_1(t-t_0)+\int_{t_0}^{t}\frac{(t-s)^{\epsilon-1}}{\Gamma(\epsilon)}hds, \ \ t\in[t_0,t_1],\\ x_0+x_1(t-t_0)+\sum_{i=1}^{k}I_i(x(t_i^-))+\sum_{i=1}^{k}J_i(x(t_i^-))(t-t_i)+\int_{t_0}^{t}\frac{(t-s)^{\epsilon-1}}{\Gamma(\epsilon)}hds\\ \quad+\sum_{i=1}^{k}\left[\xi I_i(x(t_i^-))+\eta J_i(x(t_i^-))\right]\Bigg[\int_{t_0}^{t_i}\frac{(t_i-s)^{\epsilon-1}}{\Gamma(\epsilon)}hds+\int_{t_i}^{t}\frac{(t-s)^{\epsilon-1}}{\Gamma(\epsilon)}hds\\ \quad-\int_{t_0}^{t}\frac{(t-s)^{\epsilon-1}}{\Gamma(\epsilon)}hds+(t-t_i)\int_{t_0}^{t_i}\frac{(t_i-s)^{\epsilon-2}}{\Gamma(\epsilon-1)}hds\Bigg], \ \ t\in(t_k,t_{k+1}],\\ \quad k=1,2,\dots,N, \text{here } \xi \text{ and } \eta \text{ are two arbitrary constants.} \end{cases} \tag{6}
$$

We will illuminate that four equations ((2), (3), (5), and (6)) are not the equivalent integral equation of (1) and find the correct equivalent integral equation of (1).

The rest of this paper is organized as follows. In Section 2, some preliminaries are presented. In Section 3, two fractional-order properties of piecewise function are given. In Section 4, we verify that four equations ((2), (3), (5), and (6)) are not the equivalent integral equation of (1) and search the correct equivalent integral equation of (1) by using the fractional order properties of piecewise function and some limit properties of (1).

2. Preliminaries

We can find the basic definitions and conclusions of fractional calculus in the monographs [36–38], and we briefly review several definitions and properties of fractional derivatives in this section.

Definition 1. *Let $x\in L^p(t_0,T)$ $(p\geq 1)$. The left-sided Riemann–Liouville fractional integral ${}_{t_0}^{RL}\mathcal{I}_t^{\beta}x$ $(\beta>0)$ is defined as*

$$
{}_{t_0}^{RL}\mathcal{I}_t^{\beta}x(t)=\int_{t_0}^{t}\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}x(s)ds, \ \ t>t_0, \ \textit{where } \ \Gamma(\cdot) \textit{ is the gamma function }.
$$

Definition 2. *The left-sided Riemann–Liouville fractional derivative ${}_{t_0}^{RL}\mathcal{D}_t^{\alpha}x$ ($\alpha\in(n-1,n)$ and $n\in\mathbb{N}^+$) is defined as*

$$
{}_{t_0}^{RL}\mathcal{D}_t^{\alpha}x(t)=\frac{d^n}{dt^n}\int_{t_0}^{t}\frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)}x(s)ds, \ \ t>t_0.
$$

Definition 3. *Let $x\in C^n[t_0,T]$. The left-sided Caputo fractional derivative ${}_{t_0}^{C}\mathcal{D}_t^{\alpha}x$ ($\alpha\in(n-1,n)$ and $n\in\mathbb{N}^+$) is defined by*

$$
{}_{t_0}^{C}\mathcal{D}_t^{\alpha}x(t)=\int_{t_0}^{t}\frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)}x^{(n)}(s)ds, \ \ t>t_0.
$$

Lemma 1. *If $h\in AC[t_0,T]$ or $h\in C^1[t_0,T]$, then the initial value problem*

$$
\begin{cases} {}_{t_0}^{C}\mathcal{D}_t^{\alpha}x(t)=h(t,x(t)), \ \ \alpha\in(n-1,n) \textit{ and } t\in[t_0,T],\\ x^{(j)}(t_0)=b_j, \qquad j=0,1,2,\dots,n-1, \end{cases} \tag{7}
$$

is equivalent to the following nonlinear Volterra integral equation of the second kind,

$$
x(t)=\sum_{j=0}^{n-1}\frac{b_j}{j!}(t-t_0)^j+\int_{t_0}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s,x(s))ds \ \textit{ for } t\in[t_0,T]. \tag{8}
$$

3. Two Fractional Order Properties of Piecewise Function

For the piecewise function

$$f(t) = \begin{cases} f_0(t), & t \in [t_0, t_1], \\ f_1(t), & t \in (t_1, t_2], \\ \vdots \\ f_N(t), & t \in (t_N, T], \end{cases}$$
$$= \begin{cases} f_0(t), & t \in [t_0, t_1], \\ 0, & t \in (t_1, T], \end{cases} + \begin{cases} 0, & t \in [t_0, t_1], \\ f_1(t), & t \in (t_1, t_2], \\ 0, & t \in (t_2, T], \end{cases} + \ldots + \begin{cases} 0, & t \in [t_0, t_N], \\ f_N(t), & t \in (t_N, T], \end{cases} \tag{9}$$

its fractional derivative and fractional integral have respectively two expressions.

Property 1. *Let $\epsilon \in (1,2)$ and $f_i(t) \in C^2[t_i, t_{i+1}]$ $(i = 0, 1, \ldots, N)$, then the left-sided Caputo fractional derivative of (9) can be expressed by*

$$\begin{aligned} {}_{t_0}^{C}\mathcal{D}_t^{\epsilon} f(t)\Big|_{t\in[t_0,t_1]} &= \int_{t_0}^{t} \frac{(t-s)^{1-\epsilon} f_0''(s)}{\Gamma(2-\epsilon)} ds \quad \textit{for } t \in [t_0, t_1], \\ {}_{t_0}^{C}\mathcal{D}_t^{\epsilon} f(t)\Big|_{t\in(t_k,t_{k+1}]} &= \int_{t_0}^{t} \frac{(t-s)^{1-\epsilon} f''(s)}{\Gamma(2-\epsilon)} ds \ \textit{ for } t \in (t_k, t_{k+1}] \ (k = 1, 2, \ldots, N) \\ &= \int_{t_0}^{t_1} \frac{(t-s)^{1-\epsilon} f_0''(s)}{\Gamma(2-\epsilon)} ds + \int_{t_1}^{t_2} \frac{(t-s)^{1-\epsilon} f_1''(s)}{\Gamma(2-\epsilon)} ds + \ldots + \int_{t_k}^{t} \frac{(t-s)^{1-\epsilon} f_k''(s)}{\Gamma(2-\epsilon)} ds, \end{aligned} \tag{10}$$

and

$$\begin{aligned} {}_{t_0}^{C}\mathcal{D}_t^{\epsilon} f(t) = &\begin{cases} {}_{t_0}^{C}\mathcal{D}_t^{\epsilon} f_0(t), & t \in [t_0, t_1], \\ \displaystyle\int_{t_0}^{t_1} \frac{(t-s)^{1-\epsilon} f_0''(s)}{\Gamma(2-\epsilon)} ds, & t \in (t_1, T], \end{cases} + \begin{cases} 0, & t \in [t_0, t_1], \\ {}_{t_1}^{C}\mathcal{D}_t^{\epsilon} f_1(t), & t \in (t_1, t_2], \\ \displaystyle\int_{t_1}^{t_2} \frac{(t-s)^{1-\epsilon} f_1''(s)}{\Gamma(2-\epsilon)} ds, & t \in (t_2, T], \end{cases} \\ &+ \ldots + \begin{cases} 0, & t \in [t_0, t_{N-1}], \\ {}_{t_{N-1}}^{C}\mathcal{D}_t^{\epsilon} f_{N-1}(t), & t \in (t_{N-1}, t_N], \\ \displaystyle\int_{t_{N-1}}^{t_N} \frac{(t-s)^{1-\epsilon} f_{N-1}''(s)}{\Gamma(2-\epsilon)} ds, & t \in (t_N, T], \end{cases} + \begin{cases} 0, & t \in [t_0, t_N], \\ {}_{t_N}^{C}\mathcal{D}_t^{\epsilon} f_N(t), & t \in (t_N, T]. \end{cases} \end{aligned} \tag{11}$$

Property 2. *Let $\epsilon \in (1,2)$ and $f_i(t) \in C[t_i, t_{i+1}]$ $(i = 0, 1, \ldots, N)$, then the left-sided Riemann–Liouville fractional integral of (9) can be expressed by*

$$\begin{aligned} {}_{t_0}^{RL}\mathcal{I}_t^{\epsilon} f(t)\Big|_{t\in[t_0,t_1]} &= \int_{t_0}^{t} \frac{(t-s)^{\epsilon-1} f_0(s)}{\Gamma(\epsilon)} ds \quad \textit{for } t \in [t_0, t_1], \\ {}_{t_0}^{RL}\mathcal{I}_t^{\epsilon} f(t)\Big|_{t\in(t_k,t_{k+1}]} &= \int_{t_0}^{t} \frac{(t-s)^{\epsilon-1} f(s)}{\Gamma(\epsilon)} ds \ \textit{ for } t \in (t_k, t_{k+1}] \ (k = 1, 2, \ldots, N) \\ &= \int_{t_0}^{t_1} \frac{(t-s)^{\epsilon-1} f_0(s)}{\Gamma(\epsilon)} ds + \int_{t_1}^{t_2} \frac{(t-s)^{\epsilon-1} f_1(s)}{\Gamma(\epsilon)} ds + \ldots + \int_{t_k}^{t} \frac{(t-s)^{\epsilon-1} f_k(s)}{\Gamma(\epsilon)} ds, \end{aligned} \tag{12}$$

and

$$
{}_{t_0}^{RL}\mathcal{I}_t^{\epsilon} f(t) = \begin{cases} {}_{t_0}^{RL}\mathcal{I}_t^{\epsilon} f_0(t), & t \in [t_0, t_1], \\ \displaystyle\int_{t_0}^{t_1} \frac{(t-s)^{\epsilon-1} f_0(s)}{\Gamma(\epsilon)} ds, & t \in (t_1, T], \end{cases} + \begin{cases} 0, & t \in [t_0, t_1], \\ {}_{t_1}^{RL}\mathcal{I}_t^{\epsilon} f_1(t), & t \in (t_1, t_2], \\ \displaystyle\int_{t_1}^{t_2} \frac{(t-s)^{\epsilon-1} f_1(s)}{\Gamma(\epsilon)} ds, & t \in (t_2, T], \end{cases}
$$
$$
+ \ldots + \begin{cases} 0, & t \in [t_0, t_{N-1}], \\ {}_{t_{N-1}}^{RL}\mathcal{I}_t^{\epsilon} f_{N-1}(t), & t \in (t_{N-1}, t_N], \\ \displaystyle\int_{t_{N-1}}^{t_N} \frac{(t-s)^{\epsilon-1} f_{N-1}(s)}{\Gamma(\epsilon)} ds, & t \in (t_N, T], \end{cases} + \begin{cases} 0, & t \in [t_0, t_N], \\ {}_{t_N}^{RL}\mathcal{I}_t^{\epsilon} f_N(t), & t \in (t_N, T]. \end{cases} \tag{13}
$$

4. The Equivalent Integral Equation of (1)

To discuss the equivalent integral equation of (1), we give some limit properties of (1):

$$
\lim_{J_k(x(t_k^-))\to 0 \text{ for all } k\in\{1,2,\ldots,N\}} \{\text{system (1)}\}
$$
$$
= \begin{cases} {}_{t_0}^{C}\mathcal{D}_t^{\epsilon} x(t) = h(t, x(t)), & t \in [t_0, T] \text{ and } t \neq t_k \ (k = 1, 2, \ldots, N), \\ x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \ k = 1, 2, \ldots, N, \\ x(t_0) = x_0, x'(t_0) = x_1, \end{cases} \tag{14}
$$

$$
\lim_{I_k(x(t_k^-))\to 0 \text{ for all } k\in\{1,\ldots,N\}} \{\text{system (1)}\}
$$
$$
= \begin{cases} {}_{t_0}^{C}\mathcal{D}_t^{\epsilon} x(t) = h(t, x(t)), & t \in [t_0, T] \text{ and } t \neq t_k \ (k = 1, 2, \ldots, N), \\ x'(t_k^+) - x'(t_k^-) = J_k(x(t_k^-)), \ k = 1, 2, \ldots, N, \\ x(t_0) = x_0, x'(t_0) = x_1, \end{cases} \tag{15}
$$

$$
\lim_{\substack{I_k(x(t_k^-))\to 0 \text{ for all } k\in\{1,\ldots,N\} \\ J_k(x(t_k^-))\to 0 \text{ for all } k\in\{1,\ldots,N\}}} \{\text{system (1)}\}
$$
$$
= \begin{cases} {}_{t_0}^{C}\mathcal{D}_t^{\epsilon} x(t) = h(t, x(t)), & t \in [t_0, T], \\ x(t_0) = x_0, x'(t_0) = x_1, \end{cases} \tag{16}
$$
$$
\Leftrightarrow x(t) = x_0 + x_1(t - t_0) + \int_{t_0}^{t} \frac{(t-s)^{\epsilon-1}}{\Gamma(\epsilon)} h ds, \ \ t \in [t_0, T],
$$

$$
\lim_{t_k\to t_j \text{ for all } k\in\{1,2,\ldots,N\} \ (\text{here } j\in\{1,2,\ldots,N\})} \{\text{system (1)}\}
$$
$$
= \begin{cases} {}_{t_0}^{C}\mathcal{D}_t^{\epsilon} x(t) = h(t, x(t)), \ t \in [t_0, T] \text{ and } t \neq t_j, \\ x(t_j^+) - x(t_j^-) = \displaystyle\sum_{i=1}^{N} I_i(x(t_j^-)) \text{ and } x'(t_j^+) - x'(t_j^-) = \sum_{i=1}^{N} J_i(x(t_j^-)), \\ x(t_0) = x_0, x'(t_0) = x_1. \end{cases} \tag{17}
$$

Moreover, the limit property (17) shows the linear additivity of the impulsive effects in (1).

4.1. Some Defects in These Equivalent Integral Equations ((2), (3), (5), and (6))

In this subsection, we will verify that four equations ((2), (3), (5), and (6)) are not the equivalent integral equation of (1). We use (10) to compute the fractional derivative of (2):

$$
\begin{aligned}
&{}_{t_0}^{C}\mathcal{D}_t^{\epsilon}x(t)\Big|_{t\in[t_0,t_1]} = h(t,x(t)) \qquad\qquad \text{for } t\in[t_0,t_1],\\
&{}_{t_0}^{C}\mathcal{D}_t^{\epsilon}x(t)\Big|_{t\in(t_k,t_{k+1}]} = \int_{t_0}^{t}\frac{(t-r)^{1-\epsilon}}{\Gamma(2-\epsilon)}x''(r)dr \ \text{ for } t\in(t_k,t_{k+1}]\ (k=1,2,\ldots,N)\\
&\quad = \int_{t_0}^{t_1}\frac{(t-r)^{1-\epsilon}}{\Gamma(2-\epsilon)}\left[x_0+x_1(r-t_0)+\int_{t_0}^{r}\frac{(r-s)^{\epsilon-1}}{\Gamma(\epsilon)}hds\right]''dr+\ldots\\
&\quad + \int_{t_k}^{t}\frac{(t-r)^{1-\epsilon}}{\Gamma(2-\epsilon)}\left[x(t_k^+)+x'(t_k^+)(r-t_k)+\int_{t_k}^{r}\frac{(r-s)^{\epsilon-1}}{\Gamma(\epsilon)}hds\right]''dr\\
&\quad = \int_{t_0}^{t_1}\frac{(t-r)^{1-\epsilon}}{\Gamma(2-\epsilon)}\left[\int_{t_0}^{r}\frac{(r-s)^{\epsilon-1}}{\Gamma(\epsilon)}hds\right]''dr+\ldots+\int_{t_k}^{t}\frac{(t-r)^{1-\epsilon}}{\Gamma(2-\epsilon)}\left[\int_{t_k}^{r}\frac{(r-s)^{\epsilon-1}}{\Gamma(\epsilon)}hds\right]''dr\\
&\quad \neq h(t,x(t)) \ \text{ for } t\in(t_k,t_{k+1}]\ (k=1,2,\ldots,N).
\end{aligned}
\tag{18}
$$

Therefore, (2) does not satisfy the condition of the fractional derivative of (1). Moreover, with similarity to computation of (18), both of (3) and (6) do not satisfy the condition of the fractional derivative of (1). Thus, none of (2), (3), and (6) is the equivalent integral equation of (1).

Moreover, it is obvious that (5) satisfies the condition of the fractional derivative of (1) and the fractional derivative of (5) is

$$
\begin{aligned}
&{}_{t_0}^{C}\mathcal{D}_t^{\epsilon}x(t)\Big|_{t\in[t_0,t_1]} = h(t,x(t)) \qquad\qquad \text{for } t\in[t_0,t_1],\\
&{}_{t_0}^{C}\mathcal{D}_t^{\epsilon}x(t)\Big|_{t\in(t_k,t_{k+1}]} = \int_{t_0}^{t}\frac{(t-r)^{1-\epsilon}}{\Gamma(2-\epsilon)}x''(r)dr \ \text{ for } t\in(t_k,t_{k+1}]\ (k=1,2,\ldots,N)\\
&= \int_{t_0}^{t_1}\frac{(t-r)^{1-\epsilon}}{\Gamma(2-\epsilon)}\left[x_0+x_1(r-t_0)+\int_{t_0}^{r}\frac{(r-s)^{\epsilon-1}}{\Gamma(\epsilon)}hds\right]''dr+\ldots\\
&+ \int_{t_k}^{t}\frac{(t-r)^{1-\epsilon}}{\Gamma(2-\epsilon)}\left[x_0+x_1(r-t_0)+\sum_{i=1}^{k}I_i(x(t_i^-))+\sum_{i=1}^{k}J_i(x(t_i^-))(r-t_i)+\int_{t_0}^{r}\frac{(r-s)^{\epsilon-1}}{\Gamma(\epsilon)}hds\right]''dr\\
&= \int_{t_0}^{t_1}\frac{(t-r)^{1-\epsilon}}{\Gamma(2-\epsilon)}\left[\int_{t_0}^{r}\frac{(r-s)^{\epsilon-1}}{\Gamma(\epsilon)}hds\right]''dr+\ldots+\int_{t_k}^{t}\frac{(t-r)^{1-\epsilon}}{\Gamma(2-\epsilon)}\left[\int_{t_0}^{r}\frac{(r-s)^{\epsilon-1}}{\Gamma(\epsilon)}hds\right]''dr\\
&= \int_{t_0}^{t}\frac{(t-r)^{1-\epsilon}}{\Gamma(2-\epsilon)}\left[\int_{t_0}^{r}\frac{(r-s)^{\epsilon-1}}{\Gamma(\epsilon)}hds\right]''dr\\
&= h(t,x(t)) \ \text{ for } t\in(t_k,t_{k+1}]\ (k=1,2,\ldots,N).
\end{aligned}
\tag{19}
$$

Next, we will show that (5) is not the unique piecewise function to satisfy the condition of the fractional derivative in (1) and illuminate that (5) is not the equivalent integral solution of (1).

By using (2) and Properties 1 and 2, we can reconstruct a new piecewise function:

$$
\begin{aligned}
x(t) = &\begin{cases} x_0 + x_1(t - t_0) + \int_{t_0}^{t} \frac{(t-s)^{\epsilon-1}}{\Gamma(\epsilon)} h ds, & t \in [t_0, t_1], \\ 0, & t \in (t_1, T], \end{cases} \\
&+ \sum_{k=1}^{N-1} \begin{cases} 0, & t \in [t_0, t_k], \\ x(t_k^+) + x'(t_k^+)(t - t_k) + \int_{t_k}^{t} \frac{(t-s)^{\epsilon-1}}{\Gamma(\epsilon)} h ds, & t \in (t_k, t_{k+1}], \\ 0, & t \in (t_{k+1}, T], \end{cases} \\
&+ \begin{cases} 0, & t \in [t_0, t_N], \\ x(t_N^+) + x'(t_N^+)(t - t_N) + \int_{t_N}^{t} \frac{(t-s)^{\epsilon-1}}{\Gamma(\epsilon)} h ds, & t \in (t_N, T], \end{cases} \\
&- \sum_{i=0}^{N-1} \begin{cases} 0, & t \in [t_0, t_{i+1}], \\ \int_{t_{i+1}}^{t} \frac{(t-u)^{\epsilon-1}}{\Gamma(\epsilon)} \int_{t_i}^{t_{i+1}} \frac{(u-r)^{1-\epsilon}}{\Gamma(2-\epsilon)} \left[\int_{t_i}^{r} \frac{(r-s)^{\epsilon-1}}{\Gamma(\epsilon)} h ds \right]'' dr du, & t \in (t_{i+1}, T]. \end{cases}
\end{aligned} \tag{20}
$$

In addition, we compute the fractional derivative of (20):

$$
\begin{aligned}
{}_{t_0}^{C}\mathcal{D}_t^{\epsilon} x(t) = &\begin{cases} h(t, x(t)), & t \in [t_0, t_1], \\ \int_{t_0}^{t_1} \frac{(t-r)^{1-\epsilon}}{\Gamma(2-\epsilon)} \left[\int_{t_0}^{r} \frac{(r-s)^{\epsilon-1}}{\Gamma(\epsilon)} h ds \right]'' dr, & t \in (t_1, T], \end{cases} \\
&+ \sum_{k=1}^{N-1} \begin{cases} 0, & t \in [t_0, t_k], \\ h(t, x(t)), & t \in (t_k, t_{k+1}], \\ \int_{t_k}^{t_{k+1}} \frac{(t-r)^{1-\epsilon}}{\Gamma(2-\epsilon)} \left[\int_{t_k}^{r} \frac{(r-s)^{\epsilon-1}}{\Gamma(\epsilon)} h ds \right]'' dr, & t \in (t_{k+1}, T], \end{cases} \\
&+ \begin{cases} 0, & t \in [t_0, t_N], \\ h(t, x(t)), & t \in (t_N, T], \end{cases} \\
&- \sum_{i=0}^{N-1} \begin{cases} 0, & t \in [t_0, t_{i+1}], \\ \int_{t_i}^{t_{i+1}} \frac{(t-r)^{1-\epsilon}}{\Gamma(2-\epsilon)} \left[\int_{t_i}^{r} \frac{(r-s)^{\epsilon-1}}{\Gamma(\epsilon)} h ds \right]'' dr, & t \in (t_{i+1}, T], \end{cases} \\
= &\, h(t, x(t)), \quad t \in \left([t_0, t_1] \cup \cup_{k=1}^{N} (t_k, t_{k+1}] \right).
\end{aligned} \tag{21}
$$

Therefore, both (5) and (20) satisfy these conditions (including fractional derivative, impulses, and initial value) in (1).

Thus, (5) is only **a special case** of the equivalent integral equation of (1) because it does not contain the important part $\int_{t_k}^{t} \frac{(t-s)^{\epsilon-1}}{\Gamma(\epsilon)} h ds$ which can satisfy ${}_{t_0}^{C}\mathcal{D}_t^{\epsilon} x(t) = h(t, x(t))$ as $t \in (t_k, t_{k+1}]$ $(k = 1, 2, \ldots, N)$. Moreover, (20) does not satisfy the limit property (16) to be not the equivalent integral equation of (1).

By the above discussion, none of the four equations ((2), (3), (5), and (6)) is the equivalent integral equation of (1).

4.2. The Correct Equivalent Integral Equation of (1)

In this subsection, we will use Properties 1 and 2 and (16) and (17) to search for the correct equivalent integral equation of (1) and give the equivalent integral equations of (14) and (15).

Define the space of function

$$IC([t_0,T],\mathbb{R}) := \Big\{x : [t_0,T] \to \mathbb{R}, x \in C^2\Big([t_0,t_1] \cup \cup_{k=1}^{N}(t_k,t_{k+1}]\Big) \text{ and}$$
$$x''(t_k^-) = \lim_{t\uparrow t_k} x''(t) = x''(t_k) < \infty,\ x''(t_k^+) = \lim_{t\downarrow t_k} x''(t) < \infty\Big\}.$$

Theorem 1. *Let ξ and η be two arbitrary constants, and let function $h(\cdot,x(\cdot))$ satisfy*

$$|h(t,y) - h(s,z)| \le L|t-s| + M|y-z| \text{ for } \forall s,t \in [t_0,T] \text{ and } \forall y,z \in \mathbb{R}$$

where L and M are two positive constants.

Let $x(t) \in IC([t_0,T],\mathbb{R})$, then $x(t)$ satisfies (1) iff $x(t)$ satisfies

$$x(t) = \begin{cases} x_0 + x_1(t-t_0) + \displaystyle\int_{t_0}^{t} \frac{(t-s)^{\epsilon-1}}{\Gamma(\epsilon)} h ds, \quad t \in [t_0,t_1], \\ x_0 + x_1(t-t_0) + \displaystyle\sum_{i=1}^{k} I_i(x(t_i^-)) + \sum_{i=1}^{k} J_i(x(t_i^-))(t-t_i) + \int_{t_0}^{t} \frac{(t-s)^{\epsilon-1}}{\Gamma(\epsilon)} h ds \\ \quad + \displaystyle\sum_{i=1}^{k} [\xi I_i(x(t_i^-)) + \eta J_i(x(t_i^-))] \Bigg[\int_{t_0}^{t_i} \frac{(t_i-s)^{\epsilon-1}}{\Gamma(\epsilon)} h ds + \int_{t_i}^{t} \frac{(t-s)^{\epsilon-1}}{\Gamma(\epsilon)} h ds \\ \quad - \displaystyle\int_{t_0}^{t} \frac{(t-s)^{\epsilon-1}}{\Gamma(\epsilon)} h ds - \int_{t_i}^{t} \frac{(t-u)^{\epsilon-1}}{\Gamma(\epsilon)} \int_{t_0}^{t_i} \frac{(u-r)^{1-\epsilon}}{\Gamma(2-\epsilon)} \left(\int_{t_0}^{r} \frac{(r-s)^{\epsilon-1}}{\Gamma(\epsilon)} h ds \right)'' dr du \\ \quad + (t-t_i) \displaystyle\int_{t_0}^{t_i} \frac{(t_i-s)^{\epsilon-2}}{\Gamma(\epsilon-1)} h ds \Bigg], \quad t \in (t_k,t_{k+1}], k = 1,2,\ldots,N. \end{cases} \tag{22}$$

Remark 2. *To verify that* (22) *satisfies the condition of the fractional derivative in* (1), *we transform* (22) *into*

$$\begin{aligned} x(t) = &\begin{cases} \phi(t), \ t \in [t_0,t_1], \\ \phi(t), \ t \in (t_k,t_{k+1}], k = 1,\ldots,N, \end{cases} + \sum_{k=1}^{N} \begin{cases} 0, & t \in [t_0,t_k], \\ I_k(x(t_k^-)) + J_k(x(t_k^-))(t-t_k), & t \in (t_k,T], \end{cases} \\ &+ \left[\xi I_1(x(t_1^-)) + \eta J_1(x(t_1^-))\right] \begin{cases} 0, \ t \in [t_0,t_1], \\ \Phi_1(t) - \phi(t) - \displaystyle\int_{t_1}^{t} \frac{(t-r)^{\epsilon-1}}{\Gamma(\epsilon)} \int_{t_0}^{t_1} \frac{(r-s)^{1-\epsilon}\phi''(s)}{\Gamma(2-\epsilon)} ds dr, \\ \quad t \in (t_1,T], \end{cases} \\ &+ \ldots + \\ &+ \left[\xi I_N(x(t_N^-)) + \eta J_N(x(t_N^-))\right] \begin{cases} 0, \ t \in [t_0,t_N], \\ \Phi_N(t) - \phi(t) - \displaystyle\int_{t_N}^{t} \frac{(t-r)^{\epsilon-1}}{\Gamma(\epsilon)} \int_{t_0}^{t_N} \frac{(r-s)^{1-\epsilon}\phi''(s)}{\Gamma(2-\epsilon)} ds dr, \\ \quad t \in (t_N,T], \end{cases} \end{aligned} \tag{23}$$

where

$$\phi(t) = x_0 + x_1(t-t_0) + \int_{t_0}^{t} \frac{(t-s)^{\epsilon-1}}{\Gamma(\epsilon)} h ds, \tag{24}$$

and

$$\begin{aligned} \Phi_k(t) = x_0 + x_1(t-t_0) + \int_{t_0}^{t_k} \frac{(t_k-s)^{\epsilon-1}}{\Gamma(\epsilon)} h ds + \int_{t_k}^{t} \frac{(t-s)^{\epsilon-1}}{\Gamma(\epsilon)} h ds \\ + (t-t_k) \int_{t_0}^{t_k} \frac{(t_k-s)^{\epsilon-2}}{\Gamma(\epsilon-1)} h ds, \text{ where } k = 1,2,\ldots,N. \end{aligned} \tag{25}$$

Proof. 'Sufficiency'. Because of

$$
\begin{aligned}
&{}_{t_0}^{C}\mathcal{D}_t^{\epsilon}\begin{cases} 0, \ \ t \in [t_0, t_i], \\ \Phi_i(t) - \phi(t) - \int_{t_i}^{t} \frac{(t-r)^{\epsilon-1}}{\Gamma(\epsilon)} \int_{t_0}^{t_i} \frac{(r-s)^{1-\epsilon}\phi''(s)}{\Gamma(2-\epsilon)} ds dr, \ \ t \in (t_i, T], \end{cases} \\
&= \begin{cases} 0, \ \ t \in [t_0, t_i], \\ h(t, x(t)) - \int_{t_i}^{t} \frac{(t-s)^{1-\epsilon}\phi''(s)}{\Gamma(2-\epsilon)} ds - \int_{t_0}^{t_i} \frac{(t-s)^{1-\epsilon}\phi''(s)}{\Gamma(2-\epsilon)} ds, \ \ t \in (t_i, T], \end{cases} \\
&= 0, \ \text{where } i = 1, 2, \ldots, N,
\end{aligned} \tag{26}
$$

the fractional derivative of (23) satisfies

$$
{}_{t_0}^{C}\mathcal{D}_t^{\epsilon} x(t) = h(t, x(t)), \ \ t \in \Big([t_0, t_1] \cup \cup_{k=1}^{N} (t_k, t_{k+1}]\Big). \tag{27}
$$

Thus, (23) satisfies the condition of the fractional derivative in (1). Moreover, the first order derivative of (23) is

$$
\begin{aligned}
x'(t) = &\begin{cases} x_1 + \int_{t_0}^{t} \frac{(t-s)^{\epsilon-2}}{\Gamma(\epsilon-1)} h ds, \ t \in [t_0, t_1], \\ x_1 + \int_{t_0}^{t} \frac{(t-s)^{\epsilon-2}}{\Gamma(\epsilon-1)} h ds, \ t \in (t_k, t_{k+1}], k = 1, \ldots, N, \end{cases} + \sum_{k=1}^{N} \begin{cases} 0, & t \in [t_0, t_k], \\ J_k(x(t_k^-)), & t \in (t_k, T], \end{cases} \\
&+ \big[\xi I_1(x(t_1^-)) + \eta J_1(x(t_1^-))\big] \\
&\times \begin{cases} 0, \ \ t \in [t_0, t_1], \\ \int_{t_0}^{t_1} \frac{(t_1-s)^{\epsilon-2}}{\Gamma(\epsilon-1)} h ds + \int_{t_1}^{t} \frac{(t-s)^{\epsilon-2}}{\Gamma(\epsilon-1)} h ds - \int_{t_0}^{t} \frac{(t-s)^{\epsilon-2}}{\Gamma(\epsilon-1)} h ds \\ \quad - \int_{t_1}^{t} \frac{(t-r)^{\epsilon-2}}{\Gamma(\epsilon-1)} \int_{t_0}^{t_1} \frac{(r-s)^{1-\epsilon}\phi''(s)}{\Gamma(2-\epsilon)} ds dr, \ \ t \in (t_1, T], \end{cases} \\
&+ \ldots + \big[\xi I_N(x(t_N^-)) + \eta J_N(x(t_N^-))\big] \\
&\times \begin{cases} 0, \ \ t \in [t_0, t_N], \\ \int_{t_0}^{t_N} \frac{(t_N-s)^{\epsilon-2}}{\Gamma(\epsilon-1)} h ds + \int_{t_N}^{t} \frac{(t-s)^{\epsilon-2}}{\Gamma(\epsilon-1)} h ds - \int_{t_0}^{t} \frac{(t-s)^{\epsilon-2}}{\Gamma(\epsilon-1)} h ds \\ \quad - \int_{t_N}^{t} \frac{(t-r)^{\epsilon-2}}{\Gamma(\epsilon-1)} \int_{t_0}^{t_N} \frac{(r-s)^{1-\epsilon}\phi''(s)}{\Gamma(2-\epsilon)} ds dr, \ \ t \in (t_N, T]. \end{cases}
\end{aligned} \tag{28}
$$

Therefore, it easily verify that (23) satisfies $x(t_0) = x_0$, $x'(t_0) = x_1$, $x(t_k^+) - x(t_k^-) = I_k(x(t_k^-))$ and $x'(t_k^+) - x'(t_k^-) = J_k(x(t_k^-))$ $(k = 1, 2, \ldots, N)$ and these limit properties (14)–(17). Then (23) satisfies all conditions of (1).

'Necessity'. For $\forall i \in \{1, 2, \ldots, N\}$, consider the special case of (1):

$$
\begin{aligned}
&\lim_{I_k(x(t_k^-)) \to 0 \text{ and } J_k(x(t_k^-)) \to 0 \text{ for all } k \in \{1,2,\ldots,N\}/i} \{\text{system } (1)\} \\
&= \begin{cases} {}_{t_0}^{C}\mathcal{D}_t^{\epsilon} x(t) = h(t, x(t)), & t \in [t_0, T] \text{ and } t \neq t_i, \\ x(t_i^+) - x(t_i^-) = I_i(x(t_i^-)), \\ x'(t_i^+) - x'(t_i^-) = J_i(x(t_i^-)), \\ x(t_0) = x_0, \ x'(t_0) = x_1. \end{cases}
\end{aligned} \tag{29}
$$

Next, we search the solution of (29). When $t \in [t_0, t_i]$, the solution of (29) satisfies

$$
x(t) = \phi(t) = x_0 + x_1(t - t_0) + \int_{t_0}^{t} \frac{(t-s)^{\epsilon-1}}{\Gamma(\epsilon)} h ds, \ \ t \in [t_0, t_i], \tag{30}
$$

with $x(t_i^-) = x_0 + x_1(t_i - t_0) + \int_{t_0}^{t_i} \frac{(t_i-s)^{\epsilon-1}}{\Gamma(\epsilon)} hds$ and $x'(t_i^-) = x_1 + \int_{t_0}^{t_i} \frac{(t_i-s)^{\epsilon-2}}{\Gamma(\epsilon-1)} hds$. Then, we use Properties 1 and 2 to construct the approximate solution of (29):

$$\begin{aligned}
\widetilde{x}(t) &= \begin{cases} \phi(t), & t \in [t_0, t_i], \\ 0, & t \in (t_i, T], \end{cases} + \begin{cases} 0, \quad t \in [t_0, t_i], \\ x(t_i^+) + x'(t_i^+)(t - t_i) + \int_{t_i}^{t} \frac{(t-s)^{\epsilon-1}}{\Gamma(\epsilon)} hds \\ \quad - \int_{t_i}^{t} \frac{(t-r)^{\epsilon-1}}{\Gamma(\epsilon)} \int_{t_0}^{t_i} \frac{(r-s)^{1-\epsilon}\phi''(s)}{\Gamma(2-\epsilon)} dsdr, \ t \in (t_i, T], \end{cases} \\
&= \begin{cases} \phi(t), & t \in [t_0, t_i], \\ 0, & t \in (t_i, T], \end{cases} + \begin{cases} 0, \quad t \in [t_0, t_i], \\ \Phi_i(t) + I_i(x(t_i^-)) + J_i(x(t_i^-))(t - t_i) \\ \quad - \int_{t_i}^{t} \frac{(t-r)^{\epsilon-1}}{\Gamma(\epsilon)} \int_{t_0}^{t_i} \frac{(r-s)^{1-\epsilon}\phi''(s)}{\Gamma(2-\epsilon)} dsdr, \ \ t \in (t_i, T], \end{cases}
\end{aligned} \tag{31}$$

with the error $e(t) = \begin{cases} 0, & t \in [t_0, t_i], \\ x(t) - \widetilde{x}(t), & t \in [t_i, T], \end{cases}$, here $x(t)$ represents the exact solution of (29).

Consider the special case of $e(t)$:

$$\begin{aligned}
\lim_{\substack{I_i(x(t_i^-)) \to 0 \\ J_i(x(t_i^-)) \to 0}} e(t) &= \lim_{\substack{I_i(x(t_i^-)) \to 0 \\ J_i(x(t_i^-)) \to 0}} \{x(t)\} - \lim_{\substack{I_i(x(t_i^-)) \to 0 \\ J_i(x(t_i^-)) \to 0}} \{\widetilde{x}(t)\} \\
&= \begin{cases} 0, \ \ t \in [t_0, t_i], \\ \phi(t) - \Phi_i(t) + \int_{t_i}^{t} \frac{(t-r)^{\epsilon-1}}{\Gamma(\epsilon)} \int_{t_0}^{t_i} \frac{(r-s)^{1-\epsilon}\phi''(s)}{\Gamma(2-\epsilon)} dsdr, \ t \in (t_i, T], \end{cases}
\end{aligned} \tag{32}$$

and assume

$$\begin{aligned}
e(t) &= g\big(I_i(x(t_i^-)), J_i(x(t_i^-))\big) \lim_{\substack{I_i(x(t_i^-)) \to 0 \\ J_i(x(t_i^-)) \to 0}} e(t) \\
&= g\big(I_i(x(t_i^-)), J_i(x(t_i^-))\big) \begin{cases} 0, \ \ t \in [t_0, t_i], \\ \phi(t) - \Phi_i(t) + \int_{t_i}^{t} \frac{(t-r)^{\epsilon-1}}{\Gamma(\epsilon)} \int_{t_0}^{t_i} \frac{(r-s)^{1-\epsilon}\phi''(s)}{\Gamma(2-\epsilon)} dsdr, \\ \quad t \in (t_i, T], \end{cases}
\end{aligned} \tag{33}$$

where $g(\cdot,\cdot)$ is an undetermined function with $g(0,0) = 1$. Thus, by (31) and (33), the solution of (29) is

$$\begin{aligned}
x(t) &= \widetilde{x}(t) + e(t), \ \ t \in [t_0, T] \\
&= \begin{cases} \phi(t), \ \ t \in [t_0, t_i], \\ \phi(t), \ \ t \in (t_i, T], \end{cases} + \begin{cases} 0, & t \in [t_0, t_i], \\ I_i(x(t_i^-)) + J_i(x(t_i^-))(t - t_i), & t \in (t_i, T], \end{cases} \\
&\quad + \big[1 - g\big(I_i(x(t_i^-)), J_i(x(t_i^-))\big)\big] \\
&\quad \times \begin{cases} 0, \ \ t \in [t_0, t_i], \\ \Phi_i(t) - \phi(t) - \int_{t_i}^{t} \frac{(t-r)^{\epsilon-1}}{\Gamma(\epsilon)} \int_{t_0}^{t_i} \frac{(r-s)^{1-\epsilon}\phi''(s)}{\Gamma(2-\epsilon)} dsdr, \ \ t \in (t_i, T]. \end{cases}
\end{aligned} \tag{34}$$

Because (29) is the special case of (1), (34) is a part of the solution of (1). Moreover, the limit property (17) shows the linear additivity of the impulsive effects in (1). Thus, we combine (34) and the particular solution (5) to obtain the solution of (1):

$$
\begin{aligned}
x(t) = &\begin{cases} \phi(t),\ t \in [t_0, t_1], \\ \phi(t),\ t \in (t_k, t_{k+1}], k = 1, \ldots, N, \end{cases} + \sum_{k=1}^{N} \begin{cases} 0, & t \in [t_0, t_k], \\ I_k(x(t_k^-)) + J_k(x(t_k^-))(t - t_k), & t \in (t_k, T], \end{cases} \\
&+ \left[1 - g\left(I_1(x(t_1^-)), J_1(x(t_1^-))\right)\right] \\
&\times \begin{cases} 0,\ \ t \in [t_0, t_1], \\ \Phi_1(t) - \phi(t) - \displaystyle\int_{t_1}^{t} \frac{(t-r)^{\epsilon-1}}{\Gamma(\epsilon)} \int_{t_0}^{t_1} \frac{(r-s)^{1-\epsilon}\phi''(s)}{\Gamma(2-\epsilon)} ds dr,\ \ t \in (t_1, T], \end{cases} \\
&+ \ldots + \left[1 - g\left(I_N(x(t_N^-)), J_N(x(t_N^-))\right)\right] \\
&\times \begin{cases} 0,\ \ t \in [t_0, t_N], \\ \Phi_N(t) - \phi(t) - \displaystyle\int_{t_N}^{t} \frac{(t-r)^{\epsilon-1}}{\Gamma(\epsilon)} \int_{t_0}^{t_N} \frac{(r-s)^{1-\epsilon}\phi''(s)}{\Gamma(2-\epsilon)} ds dr,\ \ t \in (t_N, T]. \end{cases}
\end{aligned} \tag{35}
$$

On the other hand, (35) need satisfy (17) such that

$$
\begin{aligned}
&\left[1 - g\left(I_i(x(t_i^-)), J_i(x(t_i^-))\right)\right] + \left[1 - g\left(I_j(x(t_j^-)), J_j(x(t_j^-))\right)\right] \\
&\qquad = 1 - g\left(I_i(x(t_i^-)) + I_j(x(t_j^-)), J_i(x(t_i^-)) + J_j(x(t_j^-))\right)
\end{aligned} \tag{36}
$$

where $\forall I_i(x(t_i^-)),\ \forall I_j(x(t_j^-)), \forall J_i(x(t_i^-)),\ \forall J_j(x(t_j^-)) \in \mathbb{R}$. Thus

$$
1 - g\left(I_i(x(t_i^-)), J_i(x(t_i^-))\right) = \xi I_i(x(t_i^-)) + \eta J_i(x(t_i^-)) \text{ here } \xi \text{ and } \eta \text{ are two arbitrary reals}, \tag{37}
$$

then (35) is (23). The proof is completed. □

Next, by (14), (15), and Theorem 1, we can draw the following conclusions.

Corollary 1. *Let ξ be an arbitrary constant, and let function $h(\cdot, x(\cdot))$ satisfy*

$$
|h(t,y) - h(s,z)| \le L|t-s| + M|y-z| \text{ for } \forall s,t \in [t_0, T] \text{ and } \forall y,z \in \mathbb{R}
$$

where L and M are two positive constants.

Let $x(t) \in IC([t_0, T], \mathbb{R})$, then $x(t)$ satisfies (14) iff $x(t)$ satisfies

$$
x(t) = \begin{cases}
x_0 + x_1(t - t_0) + \displaystyle\int_{t_0}^{t} \frac{(t-s)^{\epsilon-1}}{\Gamma(\epsilon)} h ds,\ \ t \in [t_0, t_1], \\
x_0 + x_1(t - t_0) + \displaystyle\sum_{i=1}^{k} I_i(x(t_i^-)) + \int_{t_0}^{t} \frac{(t-s)^{\epsilon-1}}{\Gamma(\epsilon)} h ds \\
\quad + \xi \displaystyle\sum_{i=1}^{k} I_i(x(t_i^-)) \Bigg[\int_{t_0}^{t_i} \frac{(t_i-s)^{\epsilon-1}}{\Gamma(\epsilon)} h ds + \int_{t_i}^{t} \frac{(t-s)^{\epsilon-1}}{\Gamma(\epsilon)} h ds - \int_{t_0}^{t} \frac{(t-s)^{\epsilon-1}}{\Gamma(\epsilon)} h ds \\
\quad - \displaystyle\int_{t_i}^{t} \frac{(t-u)^{\epsilon-1}}{\Gamma(\epsilon)} \int_{t_0}^{t_i} \frac{(u-r)^{1-\epsilon}}{\Gamma(2-\epsilon)} \left(\int_{t_0}^{r} \frac{(r-s)^{\epsilon-1}}{\Gamma(\epsilon)} h ds \right)'' dr du \\
\quad + (t - t_i) \displaystyle\int_{t_0}^{t_i} \frac{(t_i-s)^{\epsilon-2}}{\Gamma(\epsilon-1)} h ds \Bigg],\ \ t \in (t_k, t_{k+1}], k = 1, 2, \ldots, N.
\end{cases} \tag{38}
$$

Remark 3. (38) *can be rewritten as*

$$x(t)=\begin{cases}\phi(t), & t\in[t_0,t_1],\\ \phi(t), & t\in(t_k,t_{k+1}], k=1,2,\ldots,N,\end{cases}+\sum_{k=1}^{N}\begin{cases}0, & t\in[t_0,t_k],\\ I_k(x(t_k^-)), & t\in(t_k,T],\end{cases}$$
$$+\xi I_1(x(t_1^-))\begin{cases}0, \ t\in[t_0,t_1],\\ \Phi_1(t)-\phi(t)-\int_{t_1}^{t}\frac{(t-r)^{\epsilon-1}}{\Gamma(\epsilon)}\int_{t_0}^{t_1}\frac{(r-s)^{1-\epsilon}\phi''(s)}{\Gamma(2-\epsilon)}dsdr, \ t\in(t_1,T],\end{cases}$$
$$+\ldots$$
$$+\xi I_N(x(t_N^-))\begin{cases}0, \ t\in[t_0,t_N],\\ \Phi_N(t)-\phi(t)-\int_{t_N}^{t}\frac{(t-r)^{\epsilon-1}}{\Gamma(\epsilon)}\int_{t_0}^{t_N}\frac{(r-s)^{1-\epsilon}\phi''(s)}{\Gamma(2-\epsilon)}dsdr, \ t\in(t_N,T].\end{cases} \tag{39}$$

Corollary 2. *Let η be an arbitrary constant, and let function $h(\cdot,x(\cdot))$ satisfy*

$$|h(t,y)-h(s,z)|\leq L|t-s|+M|y-z| \text{ for } \forall s,t\in[t_0,T] \text{ and } \forall y,z\in\mathbb{R}$$

where L and M are two positive constants.

Let $x(t)\in IC([t_0,T],\mathbb{R})$, then $x(t)$ satisfies (15) iff $x(t)$ satisfies

$$x(t)=\begin{cases}x_0+x_1(t-t_0)+\int_{t_0}^{t}\frac{(t-s)^{\epsilon-1}}{\Gamma(\epsilon)}hds, \ t\in[t_0,t_1],\\ x_0+x_1(t-t_0)+\sum_{i=1}^{k}J_i(x(t_i^-))(t-t_i)+\int_{t_0}^{t}\frac{(t-s)^{\epsilon-1}}{\Gamma(\epsilon)}hds\\ \quad+\eta\sum_{i=1}^{k}J_i(x(t_i^-))\Big[\int_{t_0}^{t_i}\frac{(t_i-s)^{\epsilon-1}}{\Gamma(\epsilon)}hds+\int_{t_i}^{t}\frac{(t-s)^{\epsilon-1}}{\Gamma(\epsilon)}hds-\int_{t_0}^{t}\frac{(t-s)^{\epsilon-1}}{\Gamma(\epsilon)}hds\\ \quad-\int_{t_i}^{t}\frac{(t-u)^{\epsilon-1}}{\Gamma(\epsilon)}\int_{t_0}^{t_i}\frac{(u-r)^{1-\epsilon}}{\Gamma(2-\epsilon)}\Big(\int_{t_0}^{r}\frac{(r-s)^{\epsilon-1}}{\Gamma(\epsilon)}hds\Big)''drdu\\ \quad+(t-t_i)\int_{t_0}^{t_i}\frac{(t_i-s)^{\epsilon-2}}{\Gamma(\epsilon-1)}hds\Big], \ t\in(t_k,t_{k+1}], k=1,2,\ldots,N.\end{cases} \tag{40}$$

Remark 4. (40) *can be rewritten as*

$$x(t)=\begin{cases}\phi(t), & t\in[t_0,t_1],\\ \phi(t), & t\in(t_k,t_{k+1}], k=1,2,\ldots,N,\end{cases}+\sum_{k=1}^{N}\begin{cases}0, & t\in[t_0,t_k],\\ J_k(x(t_k^-))(t-t_k), & t\in(t_k,T],\end{cases}$$
$$+\eta J_1(x(t_1^-))\begin{cases}0, \ t\in[t_0,t_1],\\ \Phi_1(t)-\phi(t)-\int_{t_1}^{t}\frac{(t-r)^{\epsilon-1}}{\Gamma(\epsilon)}\int_{t_0}^{t_1}\frac{(r-s)^{1-\epsilon}\phi''(s)}{\Gamma(2-\epsilon)}dsdr, \ t\in(t_1,T],\end{cases}$$
$$+\ldots$$
$$+\eta J_N(x(t_N^-))\begin{cases}0, \ t\in[t_0,t_N],\\ \Phi_N(t)-\phi(t)-\int_{t_N}^{t}\frac{(t-r)^{\epsilon-1}}{\Gamma(\epsilon)}\int_{t_0}^{t_N}\frac{(r-s)^{1-\epsilon}\phi''(s)}{\Gamma(2-\epsilon)}dsdr, \ t\in(t_N,T].\end{cases} \tag{41}$$

5. Applications

In this section, we use Theorem 1 to consider the equivalent integral equation of an IFrOS and draw three solution trajectories of the IFrOS by the numerical simulation to show the non-uniqueness of the solution of the IFrOS.

Example 1. *Consider the following IFrOS*

$$\begin{cases} {}_0^C\mathcal{D}_t^{\frac{3}{2}}x(t) = t, & t \in [0,2] \text{ and } t \neq 1, \\ x(1^+) - x(1^-) = 1, \\ x'(1^+) - x'(1^-) = 1, \\ x(0) = 1, x'(0) = 1, \end{cases} \tag{42}$$

By Theorem 1, the equivalent integral equation of (42) *is*

$$x(t) = \begin{cases} 1 + t + \frac{1}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}}, \ \ t \in [0,1], \\ 1 + 2t + \frac{1}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}} + [\xi + \eta]\Bigg[\frac{1}{\Gamma(\frac{7}{2})} + \frac{1}{\Gamma(\frac{5}{2})}(t-1)^{\frac{3}{2}} + \frac{1}{\Gamma(\frac{7}{2})}(t-1)^{\frac{5}{2}} - \frac{1}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}} \\ \quad + \frac{t-1}{\Gamma(\frac{5}{2})} - \int_1^t \frac{(t-r)^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}\int_0^1 \frac{(r-s)^{-\frac{1}{2}}s^{\frac{1}{2}}}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})}dsdr\Bigg], \ \ t \in (1,2], \end{cases} \tag{43}$$

where ξ and η are two arbitrary reals. Moreover we can transform (43) *into*

$$\begin{aligned} x(t) = &\begin{cases} 1 + t + \frac{1}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}}, \ \ t \in [0,1], \\ 1 + t + \frac{1}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}}, \ \ t \in (1,2], \end{cases} + \begin{cases} 0, \ \ t \in [0,1], \\ t, \ \ t \in (1,2], \end{cases} \\ &+ [\xi + \eta]\begin{cases} 0, \ \ t \in [0,1], \\ \frac{1}{\Gamma(\frac{7}{2})} + \frac{1}{\Gamma(\frac{5}{2})}(t-1)^{\frac{3}{2}} + \frac{1}{\Gamma(\frac{7}{2})}(t-1)^{\frac{5}{2}} - \frac{1}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}} + \frac{t-1}{\Gamma(\frac{5}{2})} \\ \quad - \int_1^t \frac{(t-r)^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}\int_0^1 \frac{(r-s)^{-\frac{1}{2}}s^{\frac{1}{2}}}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})}dsdr, \ \ t \in (1,2]. \end{cases} \end{aligned} \tag{44}$$

We compute the first order derivative and the fractional derivative of (44):

$$\begin{aligned} x'(t) = &\begin{cases} 1 + \frac{1}{\Gamma(\frac{5}{2})}t^{\frac{3}{2}}, \ \ t \in [0,1], \\ 1 + \frac{1}{\Gamma(\frac{5}{2})}t^{\frac{3}{2}}, \ \ t \in (1,2], \end{cases} + \begin{cases} 0, \ \ t \in [0,1], \\ 1, \ \ t \in (1,2], \end{cases} \\ &+ [\xi + \eta]\begin{cases} 0, \ \ t \in [0,1], \\ \frac{1}{\Gamma(\frac{5}{2})} + \frac{1}{\Gamma(\frac{3}{2})}(t-1)^{\frac{1}{2}} + \frac{1}{\Gamma(\frac{5}{2})}(t-1)^{\frac{3}{2}} - \frac{1}{\Gamma(\frac{5}{2})}t^{\frac{3}{2}} \\ \quad - \int_1^t \frac{(t-r)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}\int_0^1 \frac{(r-s)^{-\frac{1}{2}}s^{\frac{1}{2}}}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})}dsdr, \ \ t \in (1,2]. \end{cases} \end{aligned} \tag{45}$$

and

$$
\begin{aligned}
{}_0^C\mathcal{D}_t^{\frac{3}{2}}x(t) &= \begin{cases} t, & t \in [0,1], \\ t, & t \in (1,2], \end{cases} + \begin{cases} 0, & t \in [0,1], \\ 0, & t \in (1,2], \end{cases} \\
&+ [\xi+\eta] \begin{cases} 0, \quad t \in [0,1], \\ \displaystyle\int_1^t \frac{(t-s)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}\left[\frac{1}{\Gamma(\frac{1}{2})}(s-1)^{-\frac{1}{2}} + \frac{1}{\Gamma(\frac{3}{2})}(s-1)^{\frac{1}{2}} - \frac{1}{\Gamma(\frac{3}{2})}s^{\frac{1}{2}}\right]ds \\ \quad -\displaystyle\int_0^1 \frac{(t-s)^{-\frac{1}{2}}s^{\frac{1}{2}}}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})}ds, \quad t \in (1,2]. \end{cases} \\
&= \begin{cases} t, \ t \in [0,1], \\ t, \ t \in (1,2], \end{cases} + [\xi+\eta] \begin{cases} 0, \quad t \in [0,1], \\ t - \displaystyle\int_1^t \frac{(t-s)^{-\frac{1}{2}}s^{\frac{1}{2}}}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})}ds - \int_0^1 \frac{(t-s)^{-\frac{1}{2}}s^{\frac{1}{2}}}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})}ds, \ t \in (1,2]. \end{cases} \\
&= \begin{cases} t, & t \in [0,1], \\ t, & t \in (1,2]. \end{cases}
\end{aligned} \tag{46}
$$

By (45) and (46), (44) satisfies the condition of the fractional derivative, the impulsive conditions, and the initial value in (42), that is, (44) is the general solution of (42). Moreover, three curves of solution of (42) in Figure 1 are drawn by using equation (43) with $\xi + \eta = 0,\ 10,\ -10$, respectively, and the numerical algorithm is with the step size $l = 0.01$.

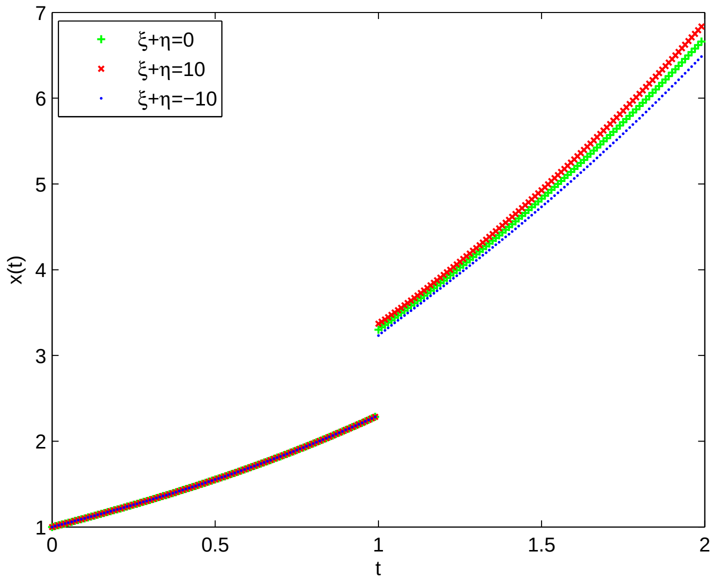

Figure 1. Three solution trajectories of (42).

6. Conclusions

We find that the fractional derivative and the fractional integral of the piecewise function have two different expressions, respectively, and then we use the fractional order properties of the piecewise function to verify that four equations ((2), (3), (5), and (6)) are not the equivalent integral equation of (1). Next, we combine the fractional order properties of piecewise function, some limit properties, the particular solution (5), and the linear additivity of the impulsive effects in (1) to find that the correct equivalent integral equation of (1) is a combination of two functions ($\phi(t)$ and $\Phi_k(t)$) with two arbitrary constants to uncover the non-uniqueness of the solution of (1).

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