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Article

An ε-Approximate Approach for Solving Variable-Order Fractional Differential Equations

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Abstract: As a mathematical tool, variable-order (VO) fractional calculus (FC) was developed rapidly in the engineering field due to it better describing the anomalous diffusion problems in engineering; thus, the research of the solutions of VO fractional differential equations (FDEs) has become a hot topic for the FC community. In this paper, we propose an effective numerical method, named as the ε-approximate approach, based on the least squares theory and the idea of residuals, for the solutions of VO-FDEs and VO fractional integro-differential equations (VO-FIDEs). First, the VO-FDEs and VO-FIDEs are considered to be analyzed in appropriate Sobolev spaces $H_2^n[0,1]$ and the corresponding orthonormal bases are constructed based on scale functions. Then, the space $H_{2,0}^2[0,1]$ is chosen which is just suitable for one of the models the authors want to solve to demonstrate the algorithm. Next, the numerical scheme is given, and the stability and convergence are discussed. Finally, four examples with different characteristics are shown, which reflect the accuracy, effectiveness, and wide application of the algorithm.

Keywords: VO-FDEs; VO-FIDEs; Sobolev space; scale functions; ε-approximate

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1. Introduction

FC refers to the research of integral and differential operators of any fractional order which can describe many real-world phenomena effectively. However, in some fields, such as in viscoelasticity, a time-varying temperature is required to be analyzed. This led researchers to consider the order of FC to be a function of time or some other variable [1]. In fact, early in 1993, the authors Samko et al. in [2] studied the integration and differentiation of functions to a VO $(d/dx)^{\delta(x)}f(x)$, where the order $\delta(x)$ is a function of x. The VO-FC was used as a prospective candidate to provide an efficient mathematical framework [3].

VO-FDEs are very suitable for describing the phenomena in many fields, such as anomalous diffusion and relaxation phenomena [4], electrical networks [5], dynamic system [6], fluid mass balance [7], and lots of other branches of physics and engineering, for example, acoustic wave propagation [8], heat and mass transfer, kinetics, ecology, image and time-series analysis, and signal processing [9]; therefore, it has aroused the interest of many scholars.

Owing to the variable exponent of the VO operator kernel, the exact solutions for VO-FDEs are more difficult to obtain. Then, finding approximate solutions for VO-FDEs is necessary. Liu et al. derived a kind of operational matrix based on the second kind of Chebyshev polynomial to solve a multiterm VO-FDE [10]. Doha et al. proposed the SJ-G-C method (shifted Jacobi–Gauss collocation) to obtain an approximate solution [11]. El-Sayed et al. used shifted Legendre polynomials for constructing the numerical solution [12]. A collocation approach was applied with the aid of shifted Chebyshev polynomials to solve the GVOFDDEs [13]. Li et al. introduced a new reproducing kernel function with a polynomial form for Atangana–Baleanu VO fractional problems [14]. Singh et al. introduced a machine learning framework with the Adams–Bashforth–Moulton method [15]. Chen et al. adopted the Bernstein polynomials basis to solve the equations [16]. Amin et al. developed a collocation method based on the Haar wavelet for solving the linear VO-FIDEs [17].

In the paper, the following VO-FDEs will be considered [18]:

$$\begin{cases} D_C^{\delta(x)}w(x) + a_1(x)w'(x) + a_2(x)w(x) + a_3(x)w\big(\tau(x)\big) = f(x), x \in [0,1], \\ w(0) = \gamma_0, w(1) = \gamma_1. \end{cases} \tag{1}$$

Here, $1 < \delta(x) < 2$, $D_C^{\delta(x)}$ is in a Caputo sense [13], that is,

$$D_C^{\delta(x)}w(x) = \frac{1}{\Gamma(n-\delta(x))}\int_0^x (x-s)^{n-\delta(x)-1}w^{(n)}(s)ds. \tag{2}$$

For Equation (1), $n = 2, a_i(x)(i = 1,2,3) \in C^2[0,1], \delta(x), \tau(x), f(x) \in C[0,1], 0 \le \tau(x) \le 1$, $\gamma_i(i = 0,1)$ is a constant.

In order to show the effectiveness and wide application of the algorithm and basis functions in this paper, we are also concerned with the following VO-FIDEs [19]:

$$\begin{cases} D_C^{\delta(x)}w(x) + a_1(x)\displaystyle\int_0^x w(s)ds + a_2(x)w'(x) + a_3(x)w(x) = f(x), x \in [0,R], R > 0, \\ w(0) = \gamma_0, \end{cases} \tag{3}$$

where $D_C^{\delta(x)}$ is as Equation (2), $0 < \delta(x) < 1, n = 1$, $a_i(x)(i = 1,2,3)$ is smooth enough, $f(x) \in C[0,R]$, and γ_0 is a constant.

The motivation of the present paper is encouraged by orthogonal functions. The original problems can be simplified by using orthogonal functions to reduce the VO-FDEs to an algebraic system. We try to construct standard orthogonal bases for Sobolev spaces $H_2^n[0,1]$ with a different n to obtain better approximate solutions.

The rest of this paper is organized as follows. In Section 2, we first transform the problem into an equivalent one with homogeneous boundary value conditions and then introduce the knowledge of the spaces and bases required for the problems. Section 3 describes the numerical algorithm under the concept of an ε-approximate solution and presents the stability and convergence analysis in space $H_{2,0}^2[0,1]$. Section 4 shows the numerical results for the proposed method. Section 5 provides brief conclusions.

2. Preliminaries

2.1. Homogenization

In order to provide the ε-approximate numerical algorithm for the VO-FDEs and VO-FIDEs, we homogenize (1)–(3) by making a transformation on $w(x)$, and suppose that the solutions of the equations exist and are unique. In the following paper, we still use $w(x)$ to represent the exact solution to avoid too many symbols. Then, (1)–(3) are simplified as (4)–(5):

$$\begin{cases} D_C^{\delta(x)}w(x) + a_1(x)w'(x) + a_2(x)w(x) + a_3(x)w\big(\tau(x)\big) = g(x), x \in [0,1], \\ w(0) = 0, w(1) = 0. \end{cases} \tag{4}$$

$$\begin{cases} D_C^{\delta(x)}w(x) + a_1(x)\displaystyle\int_0^x w(s)ds + a_2(x)w'(x) + a_3(x)w(x) = g(x), x \in [0,R], R > 0, \\ w(0) = 0. \end{cases} \tag{5}$$

2.2. Space Introduction

In this study, we choose to obtain the approximate solutions for different VO-FEDs in Sobolev spaces, such as $H_2^1[0,1], H_{2,0}^2[0,1]$. For Equation (5), $0 < \delta(x) < 1$, $H_2^1[0,1]$ and its subspaces could be considered. In the following definitions, AC represents the space of absolutely continuous functions.

$$H_2^1[0,1] = \{w(x)|w(x) \in AC, w(0) = 0, w'(x) \in L^2[0,1]\},$$

and

$$< w, v >_1 = \int_0^1 w'v'dx, \quad ||w||_1^2 = < w, w >_1 .$$

By [20], H_2^1 is a reproducing kernel space.

Clearly, $H_{2,0}^1[0,1] = \{w(x)|w(x) \in AC, w(0) = w(1) = 0, w'(x) \in L^2[0,1]\}$ is a closed subspace of H_2^1.

Based on the mother wavelet

$$\varphi_{10}(x) = \begin{cases} x, & x \in [0, \frac{1}{2}], \\ 1 - x, & x \in (\frac{1}{2}, 1], \\ 0, & else, \end{cases}$$

The following scale functions are constructed,

$$\varphi_{ik}(x) = 2^{\frac{i-1}{2}} \begin{cases} x - \frac{k}{2^{i-1}}, & x \in [\frac{k}{2^{i-1}}, \frac{k+1/2}{2^{i-1}}], \\ \frac{k+1}{2^{i-1}} - x, & x \in (\frac{k+1/2}{2^{i-1}}, \frac{k+1}{2^{i-1}}], \\ 0, & else, \end{cases} \tag{6}$$

where $i \in N^+, k = 0, 1, \ldots, 2^{i-1} - 1$. Obviously, $\{\varphi_{ik}(x)\}$ is contained in $H_{2,0}^1[0,1]$. Thus, we obtain the following Theorem 1.

Theorem 1. *$\{\varphi_{10}, \varphi_{20}, \varphi_{21}, \ldots, \varphi_{ik}, \ldots\}$ is a set of standard orthogonal bases of $H_{2,0}^1$.*

Proof. It is easy to verify that the scale functions are standard orthogonal. So, the only thing is to prove $\{\varphi_{10}, \varphi_{20}, \varphi_{21}, \ldots, \varphi_{ik}, \ldots\}$ to be complete. For $w \in H_{2,0}^1$, if $\langle w, \varphi_{10}\rangle_1 = 0$, that is, $\int_0^{1/2} w'dx - \int_{1/2}^1 w'dx = 2w\left(\frac{1}{2}\right) - w(0) - w(1) = 2w\left(\frac{1}{2}\right) = 0$, so $w\left(\frac{1}{2}\right) = 0$. If $\langle w, \varphi_{20}\rangle_1 = 0$, that is, $2^{\frac{2-1}{2}}\left(\int_0^{1/4} w'dx - \int_{1/4}^{1/2} w'dx\right) = 0$, so $w(\frac{1}{4}) = 0$. Similarly, if $\langle w, \varphi_{21}\rangle_1 = 0$, then $w(\frac{3}{4}) = 0$. Thus, for all $x_{ik} = \frac{k}{2^{i-1}}$, $w(x_{ik}) = 0$. Because $\{x_{ik}\}$ is dense on $[0,1]$ and $w(x) \in AC$, $w \equiv 0$. □

Then, it is not difficult to prove that $\{x, \varphi_{10}(x), \varphi_{20}(x), \varphi_{21}(x), \ldots, \varphi_{ik}(x), \ldots\}$ is a set of orthonormal bases for H_2^1.

For $1 < \delta(x) < 2$, we give the following spaces with corresponding smoothness.

$$H_2^2[0,1] = \{w(x)|w(x), w'(x) \in AC, w(0) = 0, w''(x) \in L^2[0,1]\},$$

and

$$< w, v >_2 = w'(0)v'(0) + \int_0^1 w''v''dx, \quad ||w||_2^2 = < w, w >_2 .$$

Moreover, the notations below are needed for simplifying the problem expression

$$J_0 w(x) \triangleq \int_0^x w(s)ds, \qquad J_{ik}(x) \triangleq J_0\varphi_{ik}(x) - x\int_0^1 \varphi_{ik}(x)dx.$$

Theorem 2. *$\{\beta_j(x)\}_{j=1}^\infty = \left\{x, x^2/2, J_0\varphi_{10}(x), J_0\varphi_{20}(x), J_0\varphi_{21}(x), \ldots, J_0\varphi_{ik}(x), \ldots\right\}$ is a group of standard orthogonal bases of H_2^2.*

Proof. Similar to Theorem 1, we need to verify the completeness of $\{\beta_j(x)\}_{j=1}^\infty$. For $w \in H_2^2$, if $\langle w, x\rangle_2 = \langle w, \frac{x^2}{2}\rangle_2 = 0$, then $w'(0) = w'(1) = 0$. With the fact that $w' \in AC, w'' \in L^2$, we have $w' \in H^1_{2,0}$. If $\langle w, J_0\varphi_{ik}\rangle_2 = 0$, then $\langle w, J_0\varphi_{ik}\rangle_2 = \int_0^1 w''\varphi'_{ik}dx = 0$. So, $\langle w', \varphi_{ik}\rangle_{H^1_{2,0}} = \int_0^1 w''\varphi'_{ik}dx = 0$, by the completeness of the bases of $H^1_{2,0}$, $w' = 0$. That is, $w = c$(c is a constant), because $w \in AC, w(0) = 0$, so $w \equiv 0$. □

Further, for Equation (4), we give a specific space $H^2_{2,0}[0,1]$ as below:

$$H^2_{2,0}[0,1] = \{w(x)|w(x), w'(x) \in AC, w(0) = w(1) = 0, w''(x) \in L^2[0,1]\},$$

and

$$< w, v >_{H^2_{2,0}} = \int_0^1 w''v''dx, \quad ||w||^2_{H^2_{2,0}} =< w, w >_{H^2_{2,0}} .$$

Obviously, $H^2_{2,1}[0,1] = \{w(x)|w(x), w'(x) \in AC, w(0) = w(1) = 0, w'(0) = w'(1), w''(x) \in L^2[0,1]\}$ is a subspace of $H^2_{2,0}[0,1]$. At the same time, similar to the proof process of Theorem 2, $\{\eta_j(x)\}_{j=1}^{\infty} = \{J_{10}(x), J_{20}(x), J_{21}(x), \ldots, J_{ik}(x), \ldots\}$ are the orthonormal bases of $H^2_{2,1}$.

Theorem 3. *$\{\mu_j(x)\}_{j=1}^{\infty} = \{x(1-x)/2, J_{10}(x), J_{20}(x), J_{21}(x), \ldots, J_{ik}(x), \ldots\}$ is a set of standard orthogonal bases of $H^2_{2,0}$.*

Proof. In a similar manner, for $w \in H^2_{2,0}$, if $\langle w, \frac{x(1-x)}{2}\rangle_{H^2_{2,0}} = 0$, then $\int_0^1 w''(-1)dx = -[w'(1) - w'(0)] = 0$, that is, $w'(1) = w'(0)$, so $w \in H^2_{2,1}$, $\langle w, J_{ik}(x)\rangle_{H^2_{2,1}} = 0$. As $\{\eta_j(x)\}_{j=1}^{\infty}$ is a complete system, $w \equiv 0$, the theorem holds. □

3. Algorithm and Convergence Analysis

In this part, we propose a numerical algorithm to solve VO-FDEs in the spaces introduced in Section 2. Specifically, for the simplified (4), $1 < \delta(x) < 2$, we choose the space $H^2_{2,0}[0,1]$. As for (5), $H^1_2[0,1]$ is suitable, due to searching for approximate solutions, $H^2_2[0,1]$ can also be considered. In the following part, we show the algorithm and theoretical analysis for (4). In addition, for (5), it can be analyzed by the same idea.

3.1. The ε-Approximate Approach

Let $\mathcal{A} : H^2_{2,0} \to L^2$ be a linear operator, for $w \in H^2_{2,0}$, we have

$$\mathcal{A}w = \frac{1}{\Gamma(2-\delta(x))}\int_0^x (x-s)^{1-\delta(x)}w''(s)ds + a_1(x)w'(x) + a_2(x)w(x) + a_3(x)w\big(\tau(x)\big). \tag{7}$$

Then, (4) can be replaced by

$$\mathcal{A}w = g. \tag{8}$$

Theorem 4. *The operator $\mathcal{A}$ is bounded.*

Proof.

$$||\mathcal{A}w||_{L^2} = \left|\left|\frac{1}{\Gamma(2-\delta(x))}\int_0^x (x-s)^{1-\delta(x)}w''(s)ds + a_1(x)w'(x) + a_2(x)w(x) + a_3(x)w\big(\tau(x)\big)\right|\right|_{L^2}. \tag{9}$$

For $1 < \delta(x) < 2$, then $1 < \delta_1 \le \delta(x) \le \delta_2 < 2$ holds $\big(\delta_i$ is a constant$(i = 1,2)\big)$. Obviously, the following inequalities are true:

$$\begin{gathered} 0 < 2-\delta_2 \le 2-\delta(x) \le 2-\delta_1 < 1, \\ -1 < 1-\delta_2 \le 1-\delta(x) \le 1-\delta_1 < 0. \end{gathered}$$

In $(0,1)$, the $\Gamma(x)$ is monotonically decreasing, $\Gamma(x) > 0$, and in Equation (7) , $0 < x - s < 1$, the following inequalities hold.

$$0 < \frac{1}{\Gamma(2-\delta_2)} \leq \frac{1}{\Gamma(2-\delta(x))} \leq \frac{1}{\Gamma(2-\delta_1)},$$
$$0 < (x-s)^{1-\delta_1} \leq (x-s)^{1-\delta(x)} \leq (x-s)^{1-\delta_2}.$$

The boundedness of $\left|\left|\frac{1}{\Gamma(2-\delta(x))}\int_0^x (x-s)^{1-\delta(x)} w''(s)ds\right|\right|_{L^2}$ can be proved by *Hölder* inequality as below.

$$\begin{aligned}
&\left|\left|\frac{1}{\Gamma(2-\delta(x))}\int_0^x (x-s)^{1-\delta(x)} w''(s)ds\right|\right|_{L^2}^2 \\
&\leq \frac{1}{\Gamma^2(2-\delta_1)}\int_0^1 \left[\int_0^x (x-s)^{1-\delta(x)} w''(s)ds\right]^2 dx \\
&\leq \frac{1}{\Gamma^2(2-\delta_1)}\int_0^1 \left[\int_0^x (x-s)^{1-\delta_2} w''(s)ds\right]^2 dx \\
&\leq \frac{1}{\Gamma^2(2-\delta_1)}\int_0^1 \left[\int_0^x (x-s)^{1-\delta_2} ds \int_0^x (x-s)^{1-\delta_2} w''^2(s)ds\right] dx \\
&\leq K_1 \int_0^1 dx \int_0^x (x-s)^{1-\delta_2} w''^2(s)ds = K_1 \int_0^1 w''^2(s)ds \int_s^1 (x-s)^{1-\delta_2} dx \\
&= K_2 \int_0^1 (1-s)^{2-\delta_2} w''^2(s)ds \leq K_2 \int_0^1 w''^2(s)ds = K_2||w||^2_{H^2_{2,0}}.
\end{aligned}$$

In Equation (9), by the reproducing property of kernel function R_x of $H^2_{2,0}$, $|w'(x)| = | < w, R'_x >_{H^2_{2,0}} | \leq ||R'_x||_{H^2_{2,0}}||w||_{H^2_{2,0}}$, so $w'^2(x) \leq ||R'_x||^2_{H^2_{2,0}}||w||^2_{H^2_{2,0}}$. It implies that $||a_1(x)w'(x)||_{L^2} \leq K_3||w||_{H^2_{2,0}}$.

Similarly, $||a_2(x)w(x)||_{L^2} \leq K_4||w||_{H^2_{2,0}}$.

As for $|w(\tau(x))| = | < w, R_{\tau(x)} >_{H^2_{2,0}} |$, applying the Cauchy–Schwartz inequality, it yields that

$$|w(\tau(x))| \leq ||w||_{H^2_{2,0}}||R_{\tau(x)}||_{H^2_{2,0}}.$$

Because $||R_{\tau(x)}||_{H^2_{2,0}}$ is bounded on $[0,1]$, $|w(\tau(x))| \leq K_5||w||_{H^2_{2,0}}$, so $||a_3(x)w(\tau(x))||_{L^2} \leq K_6||w||_{H^2_{2,0}}$. Finally, one obtains

$$||\mathcal{A}w||_{L^2} \leq (\sqrt{K_2} + K_3 + K_4 + K_6)||w||_{H^2_{2,0}},$$

where $K_i (i = 1, 2, \ldots, 6)$ is a positive constant, the theorem holds true. □

Next, we give the definition of ε-approximate solution for the problem.

Definition 1. *For* $\forall\varepsilon > 0$, *if* $||\mathcal{A}w - g||^2_{L^2} < \varepsilon^2$, *$w$ is called ε-approximate solution of* (8).

Numerical Scheme: To obtain the ε-approximate solution, we provide the following method. Let $w_n = \sum_{k=1}^{n} d_k \mu_k$, and

$$J(d_1, d_2, \ldots, d_n) = ||\mathcal{A}w_n - g||^2_{L^2} = \left|\left|\sum_{k=1}^{n} d_k \mathcal{A}\mu_k - g\right|\right|^2_{L^2}. \tag{10}$$

Then, the ε-approximate approach reads as: seeking for $d_k^*(k = 1, 2, \ldots, n)$ so that

$$J(d_1^*, d_2^*, \ldots, d_n^*) = \left\| \sum_{k=1}^{n} d_k^* \mathcal{A}\mu_k - g \right\|_{L^2}^2 \triangleq \min_{d_k} \left\| \sum_{k=1}^{n} d_k \mathcal{A}\mu_k - g \right\|_{L^2}^2 .$$

Here, $\sum_{k=1}^{n} d_k^* \mu_k$ is the ε-approximate solution. For the theoretical support, please see Theorem 2.1 in [21].

To obtain the coefficient d_k^*, we minimize $J(d_1, d_2, \ldots, d_n)$ by applying $\frac{\partial J(d_1, d_2, \ldots, d_n)}{\partial d_j} = 0$. Then, we obtain

$$\sum_{k=1}^{n} d_k \langle \mathcal{A}\mu_k, \mathcal{A}\mu_j \rangle_{L^2} = \langle \mathcal{A}\mu_j, g \rangle_{L^2} \quad (j = 1, 2, \ldots, n). \tag{11}$$

Thus, $d_1^*, d_2^*, \ldots, d_n^*$ can be determined by

$$Bd = c, \tag{12}$$

where

$$B = \left(\langle \mathcal{A}\mu_k, \mathcal{A}\mu_j \rangle_{L^2} \right)_{n \times n}, \quad c = \left(\langle \mathcal{A}\mu_j, g \rangle_{L^2} \right)_{n \times 1}^T, \quad d = (d_1, d_2, \ldots, d_n)^T.$$

Theorem 5. *The solution for Equation (12) is unique provided that $\mathcal{A}$ is reversible.*

Proof. It is equivalent to proving that $Bd = 0$ has only zero solution. That is,

$$\sum_{k=1}^{n} d_k \langle \mathcal{A}\mu_k, \mathcal{A}\mu_j \rangle_{L^2} = 0, j = 1, 2, \ldots, n. \tag{13}$$

Multiply d_j on the two sides of Equation (13) and make a cumulative sum by j from 1 to n which yields that $\langle \sum_{k=1}^{n} d_k \mathcal{A}\mu_k, \sum_{j=1}^{n} d_j \mathcal{A}\mu_j \rangle_{L^2} = 0$, which means $\left\| \sum_{k=1}^{n} d_k \mathcal{A}\mu_k \right\|_{L^2}^2 = 0$. As $\mathcal{A}$ is reversible, then $d_k = 0(k = 1, 2, \ldots, n)$, so $d = 0$. $\square$

3.2. Stability Results for the Problem

In this part, we will prove that the ε-approximate approach is stable.

Lemma 1. *The eigenvalue λ of B is bounded by $||\mathcal{A}||^2$.*

Proof. For λ, there exists a unit vector $x = (x_1, x_2, \ldots, x_n)^T$, such that $Bx = \lambda x$.

$$\lambda x_k = \sum_{j=1}^{n} \left(\langle \mathcal{A}\mu_k, \mathcal{A}\mu_j \rangle_{L^2} \right) x_j = \sum_{j=1}^{n} \left(\langle \mathcal{A}\mu_k, x_j \mathcal{A}\mu_j \rangle_{L^2} \right) = \langle \mathcal{A}\mu_k, \sum_{j=1}^{n} x_j \mathcal{A}\mu_j \rangle_{L^2}. \tag{14}$$

Multiply x_k on both sides of (14) and sum x_k from 1 to n, we obtain

$$\lambda \sum_{k=1}^{n} x_k^2 = \langle \sum_{k=1}^{n} x_k \mathcal{A}\mu_k, \sum_{j=1}^{n} x_j \mathcal{A}\mu_j \rangle_{L^2} = \left\| \sum_{k=1}^{n} x_k \mathcal{A}\mu_k \right\|_{L^2}^2 \leq ||\mathcal{A}||^2 \sum_{k=1}^{n} x_k^2.$$

So, $\lambda = \lambda \sum_{k=1}^{n} x_k^2 \leq ||\mathcal{A}||^2$. $\square$

Lemma 2. $||\mathcal{A}w||_{L^2} \geq \frac{1}{||\mathcal{A}^{-1}||}$ *for* $w \in H_{2,0}^2$ *and* $||w||_{H_{2,0}^2} = 1$.

Proof. Denote $h = \mathcal{A}w$, then $w = \mathcal{A}^{-1}h$. Thus, $1 = ||w||_{H^2_{2,0}} = ||\mathcal{A}^{-1}h||_{H^2_{2,0}} \leq ||\mathcal{A}^{-1}||||h||_{L^2}$, so $||\mathcal{A}w||_{L^2} = ||h||_{L^2} \geq \frac{1}{||\mathcal{A}^{-1}||}$. □

Theorem 6. *The algorithm for obtaining the approximate solution $w_n = \sum\limits_{k=1}^{n} d_k^* \mu_k$ by ε-approximate approach is stable.*

Proof. By Lemmas 1 and 2, it yields $\lambda = \left|\left|\sum\limits_{k=1}^{n} x_k \mathcal{A}\mu_k\right|\right|^2_{L^2} = \left|\left|\mathcal{A}\sum\limits_{k=1}^{n} x_k \mu_k\right|\right|^2_{L^2} \geq \frac{1}{||\mathcal{A}^{-1}||^2}$. Then, $cond(B)_2 = |\frac{\lambda_{max}}{\lambda_{min}}| \leq \frac{||\mathcal{A}||^2}{\frac{1}{||\mathcal{A}^{-1}||^2}} = ||\mathcal{A}||^2||\mathcal{A}^{-1}||^2$, which means the condition number of matrix B is bounded, so the algorithm is stable. □

3.3. Convergence Analysis

Theorem 7. *The approximate solution w_n converges to the exact solution w of Equation (8) uniformly.*

Proof. By the property of R_x which is the reproducing kernel of $H^2_{2,0}$, we have $|w_n(x) - w(x)| = |\langle w_n - w, R_x\rangle_{H^2_{2,0}}| \leq ||w_n - w||_{H^2_{2,0}}||R_x||_{H^2_{2,0}}$. In addition, for $||w_n - w||_{H^2_{2,0}}$, there is $||w_n - w||_{H^2_{2,0}} = ||\mathcal{A}^{-1}\mathcal{A}(w_n - w)||_{H^2_{2,0}} \leq ||\mathcal{A}^{-1}||||\mathcal{A}(w_n - w)||_{L^2} = ||\mathcal{A}^{-1}||||\mathcal{A}w_n - g||_{L^2}$. Because w_n is determined by the ε-approximate approach, then $||\mathcal{A}w_n - g||_{L^2} < \varepsilon$, so $||w_n - u||_{H^2_{2,0}} \to 0$. In addition, then $|w_n(x) - w(x)| \leq ||w_n - w||_{H^2_{2,0}}||R_x||_{H^2_{2,0}} \leq M||w_n - w||_{H^2_{2,0}} \to 0$, M is a constant. □

Then, by Theorem 3, $w(x)$ may be represented as $w(x) = d_0\frac{x(1-x)}{2} + \sum\limits_{i=1}^{\infty}\sum\limits_{k=0}^{2^{i-1}-1} d_{ik}J_{ik}(x)$, where $d_0 = \left\langle w, \frac{x(1-x)}{2}\right\rangle$, $d_{ik} = \langle w, J_{ik}(x)\rangle$.

Theorem 8. $||w - w_n||^2_{H^2_{2,0}} \leq 2^{-2n}P$ *if $w'''(x)$ is bounded in [0,1], P is a constant.*

Proof.

$$
\begin{aligned}
\|w - w_n\|^2_{H^2_{2,0}} &= \left\|w - d_0\frac{x(1-x)}{2} - \sum_{i=1}^{n}\sum_{k=0}^{2^{i-1}-1} d_{ik}J_{ik}(x)\right\|^2_{H^2_{2,0}} = \left\|\sum_{i=n+1}^{\infty}\sum_{k=0}^{2^{i-1}-1} d_{ik}J_{ik}(x)\right\|^2_{H^2_{2,0}} \\
&= \sum_{i=n+1}^{\infty}\sum_{k=0}^{2^{i-1}-1} d^2_{ik}\langle J_{ik}(x), J_{ik}(x)\rangle_{H^2_{2,0}} = \sum_{i=n+1}^{\infty}\sum_{k=0}^{2^{i-1}-1} d^2_{ik}.
\end{aligned}
$$

Then, the problem is to prove $\sum_{i=n+1}^{\infty}\sum_{k=0}^{2^{i-1}-1} d^2_{ik} \leq 2^{-2n}P$.
It is easy to get that

$$
J''_{ik}(x) = \varphi'_{ik}(x) = \begin{cases} 2^{\frac{i-1}{2}}, & x \in \left[\frac{k}{2^{i-1}}, \frac{k+1/2}{2^{i-1}}\right], \\ -2^{\frac{i-1}{2}}, & x \in \left(\frac{k+1/2}{2^{i-1}}, \frac{k+1}{2^{i-1}}\right], \\ 0, & else, \end{cases} \tag{15}
$$

and $w''(x) = w''(\frac{k}{2^{i-1}}) + w'''(\xi)(x - \frac{k}{2^{i-1}})$.

So, $|d_{ik}| = \left|\int_0^1 w''(x)J''_{ik}(x)dx\right| = \left|\int_0^1 w''(x)\varphi'_{ik}(x)dx\right| \leq \left|\int_{\frac{k}{2^{i-1}}}^{\frac{k+1}{2^{i-1}}} w''(\frac{k}{2^{i-1}})\varphi'_{ik}(x)dx\right| + \left|\int_{\frac{k}{2^{i-1}}}^{\frac{k+1}{2^{i-1}}} w'''(\xi)(x - \frac{k}{2^{i-1}})\varphi'_{ik}(x)dx\right|$.

Applying Equation (15), $\left|\int_{\frac{k}{2^{i-1}}}^{\frac{k+1}{2^{i-1}}} w''(\frac{k}{2^{i-1}})\varphi'_{ik}(x)dx\right| = 0$,

$$|d_{ik}| \leq \left|\int_{\frac{k}{2^{i-1}}}^{\frac{k+1/2}{2^{i-1}}} w'''(\xi)(x-\frac{k}{2^{i-1}})2^{\frac{i-1}{2}}dx\right| + \left|\int_{\frac{k+1/2}{2^{i-1}}}^{\frac{k+1}{2^{i-1}}} w'''(\xi)(x-\frac{k}{2^{i-1}})(-2^{\frac{i-1}{2}})dx\right| \leq 2^{-\frac{3}{2}i}P_1.$$

Then, $d_{ik}^2 \leq 2^{-3i}P_2$, so $\|w-w_n\|^2_{H^2_{2,0}} = \sum_{i=n+1}^{\infty}\sum_{k=0}^{2^{i-1}-1} d_{ik}^2 \leq \sum_{i=n+1}^{\infty}\sum_{k=0}^{2^{i-1}-1} 2^{-3i}P_2 = 2^{-2n}P$, where $P_i(i=1,2)$ is a constant. The theorem holds true. □

4. Numerical Results

Example 1. *Consider the VO-FDE ([18,22]) as follows,*

$$\begin{cases} D_C^{\delta(x)}w(x) + cos(x)w'(x) + 4w(x) + 5w(x^2) = f(x), x \in [0,1], \\ w(0) = 0, w(1) = 1, \end{cases}$$

where $\delta(x) = \big(5+sin(x)\big)/4, f(x) = 2x^{2-\delta(x)}/\Gamma\big(3-\delta(x)\big) + 5x^4 + 4x^2 + 2xcos(x)$. *The exact solution is* $w(x) = x^2$.

We solve this problem in $H^2_{2,0}[0,1]$ with the ε-approximate approach presented in Section 3, taking $x_i = \frac{i}{10}(i = 1,2,\ldots,10)$. The value of $|w_{10}(x) - w(x)|$ at x_i is calculated compared with ([18,22]) in Table 1. $MAE'_{10}(x) \triangleq |w'_{10}(x) - w'(x)|_{max}$ and $MAE^{\frac{3}{2}}_{10}(x) \triangleq |D^{\frac{3}{2}}w_{10}(x) - D^{\frac{3}{2}}w(x)|_{max}$ are also calculated as [18] and the results are shown in Table 2. MAE indicates the maximum absolute error. Note that we compared with their best results.

Table 1. Absolute errors for Example 1.

x_i	Ref. [22]	Ref. [18]	Our Approach
0.1	1.17×10^{-8}	1.53037×10^{-14}	4.16334×10^{-17}
0.2	1.77×10^{-8}	1.13937×10^{-14}	8.32667×10^{-17}
0.3	2.17×10^{-8}	7.66054×10^{-15}	8.32667×10^{-17}
0.4	2.39×10^{-8}	4.08007×10^{-15}	1.11022×10^{-16}
0.5	2.45×10^{-8}	5.27356×10^{-16}	1.66533×10^{-16}
0.6	2.34×10^{-8}	2.66454×10^{-15}	1.66533×10^{-16}
0.7	2.07×10^{-8}	6.21725×10^{-15}	8.32667×10^{-17}
0.8	1.59×10^{-8}	9.65894×10^{-15}	2.77556×10^{-17}
0.9	1.11×10^{-8}	1.28786×10^{-14}	1.38778×10^{-16}

Table 2. MAEs for Example 1.

	Ref. [22]	Ref. [18]	Our Approach
$MAE'_{10}(x)$	1.15×10^{-6}	3.90643×10^{-12}	2.22045×10^{-16}
$MAE^{\frac{3}{2}}_{10}(x)$	3.50×10^{-6}	1.78113×10^{-12}	0

Example 2. *Consider Example 1 in [23],*

$$\begin{cases} D_C^{\delta(x)}w(x) - 10w'(x) + w(x) = f(x), x \in [0,1), \\ w(0) = 5, \end{cases}$$

where $\delta(x) = \frac{x+2e^x}{7}$, $f(x) = 10\big(\frac{x^{2-\delta(x)}}{\Gamma(3-\delta(x))} + \frac{x^{1-\delta(x)}}{\Gamma(2-\delta(x))}\big) + 5x^2 - 90x - 95$, *and the exact solution is* $w(x) = 5(1+x)^2$. *The numerical errors for* $\delta(x) = \frac{x+2e^x}{7}$ *are shown in Table 3 as well as* $\delta_1(x) = \frac{x+2e^x}{3}, \delta_2(x) = \frac{x+2e^x}{5}, \delta_3(x) = \frac{x+2e^x}{9}$. *Table 3 indicates that the numerical solutions are in good agreement with the exact values.*

Table 3. The absolute errors of different $\delta(x)$ for Example 2.

x_i	$\delta(x)$		$\delta_1(x)$		$\delta_2(x)$		$\delta_3(x)$	
	[23]	**Ours**	**[23]**	**Ours**	**[23]**	**Ours**	**[23]**	**Ours**
0.2	2.524647×10^{-13}	3.10862×10^{-15}	1.993516×10^{-11}	7.54952×10^{-15}	1.436628×10^{-13}	5.77316×10^{-15}	7.636113×10^{-13}	5.77316×10^{-15}
0.4	2.577937×10^{-13}	0	2.035882×10^{-11}	9.76996×10^{-15}	1.467714×10^{-13}	4.44089×10^{-15}	7.769340×10^{-13}	6.21725×10^{-15}
0.6	4.085176×10^{-12}	8.88178×10^{-16}	8.341771×10^{-12}	9.76996×10^{-15}	9.983125×10^{-13}	1.06581×10^{-14}	6.705747×10^{-14}	2.66454×10^{-15}
0.8	4.176659×10^{-12}	3.55271×10^{-15}	8.530065×10^{-12}	3.55271×10^{-15}	1.019850×10^{-12}	5.68434×10^{-14}	6.927791×10^{-14}	2.66454×10^{-14}
1.0	4.270361×10^{-12}	7.10543×10^{-15}	8.622436×10^{-12}	1.442109×10^{-14}	1.040945×10^{-12}	1.38556×10^{-13}	7.105427×10^{-14}	8.52651×10^{-14}

Example 3. *Consider the following VO-FIDE in [19],*

$$\begin{cases} D_C^{\delta(x)} w(x) + 6\int_0^x w(s)ds + 2xw'(x) + w(x) = g(x), x \in [0, R], \\ w(0) = 0, \end{cases}$$

where $\delta(x) = \frac{3(sinx+cosx)}{5}$, $g(x) = \frac{10}{\Gamma(3-\delta(x))}x^{2-\delta(x)} + \frac{15}{\Gamma(2-\delta(x))}x^{1-\delta(x)} + 10x^3 + 70x^2 + 45x$. *The exact solution is* $w(x) = 5x^2 + 15x$. *The errors for different* R *are shown in Table 4 where* $\varepsilon = \max_{i=0,1,\ldots,n} |w(x_i) - w_n(x_i)|$ *and* $x_i = R((2i+1)/2(n+1))$.

Table 4. Error ε of Example 3 for different R.

R	**Method in [19]**			**The Proposed Method**		
	n **= 3**	n **= 4**	n **= 5**	n **= 3**	n **= 4**	n **= 5**
1	3.952×10^{-14}	3.524×10^{-13}	-	7.10543×10^{-15}	1.06581×10^{-14}	1.77636×10^{-14}
2	1.7408×10^{-13}	5.2847×10^{-13}	6.9678×10^{-13}	2.66454×10^{-14}	2.13163×10^{-14}	2.84217×10^{-14}
4	2.6148×10^{-12}	8.7255×10^{-12}	7.5460×10^{-12}	2.84217×10^{-13}	8.73968×10^{-13}	7.81597×10^{-13}
8	1.3301×10^{-11}	4.0711×10^{-10}	3.6262×10^{-9}	2.27374×10^{-13}	4.26326×10^{-14}	1.36424×10^{-12}

Finally, we consider an example with a non-smooth solution. Here, the fractional-order derivative is in Riemann–Liouville (R.L) sense.

$$D_{R.L}^{\delta(x)} w(x) = \frac{1}{\Gamma(n-\delta(x))} \frac{d^n}{dx^n} \int_0^x (x-s)^{n-\delta(x)-1} w(s)ds, x > 0, n-1 < \delta(x) \le n,$$

where n denotes the ceiling of $\delta(x)$.

Example 4. *Consider the following equation with* $\delta(x) = x$,

$$\begin{cases} D_{R.L}^{\delta(x)} w(x) = f(x), x \in [0, 1], \\ w(0) = 0, \end{cases}$$

where the exact solution is

$$w(x) = \begin{cases} x^{\frac{5}{4}}, & x \in [0, 0.5], \\ \frac{1}{2}x^{\frac{1}{4}}, & x \in (0.5, 1]. \end{cases}$$

AEs in Table 5 are calculated by $|w_n(x_i) - w(x_i)|$; here, we picked $n = 32$ which is the number of bases. Figure 1 shows the exact solution $w(x)$ and the ε-approximate solution $w_n(x)$.

Table 5. The AEs for Example 4.

x_i	AEs	x_i	AEs
0.1	1.31108×10^{-4}	0.6	8.80766×10^{-8}
0.2	5.00746×10^{-5}	0.7	1.36829×10^{-5}
0.3	1.62240×10^{-5}	0.8	9.01757×10^{-6}
0.4	2.72771×10^{-6}	0.9	2.53482×10^{-7}
0.5	7.63714×10^{-6}	1.0	8.74362×10^{-6}

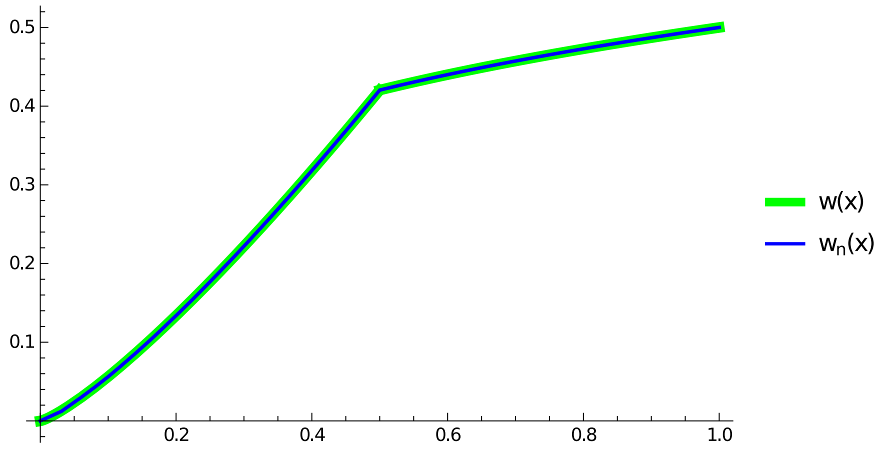

Figure 1. The ε-approximate solution $w_n(x)$ and the exact solution $w(x)$ of Example 4.

5. Conclusions

In the paper, we studied the VO-FDEs which have been a hot topic in recent years and proposed the ε-approximate algorithm to obtain the approximate solutions based on the scale functions which are used for constructing the bases for Sobolev spaces $H_2^n[0,1]$. From a theoretical analysis to numerical experiments, it can be seen that this algorithm is stable, more effective, and widely used. In future work, we intend to focus on the numerical algorithm research and application of variable fractional-order partial differential equations. Further, we consider combining deep learning algorithms with fractional-order problems to obtain better results.

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