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Local Convergence of Traub's Method and Its Extensions

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Abstract: In this article, we examine the local convergence analysis of an extension of Newton's method in a Banach space setting. Traub introduced the method (also known as the Arithmetic-Mean Newton's Method and Weerakoon and Fernando method) with an order of convergence of three. All the previous works either used higher-order Taylor series expansion or could not derive the desired order of convergence. We studied the local convergence of Traub's method and two of its modifications and obtained the convergence order for these methods without using Taylor series expansion. The radii of convergence, basins of attraction, comparison of iterations of similar iterative methods, approximate computational order of convergence (ACOC), and a representation of the number of iterations are provided.

Keywords: iterative methods; Arithmetic-Mean Newton's method; Weerakoon-Fernando method; order of convergence; Taylor series expansion; Fréchet derivative

1. Introduction

Many problems in engineering and natural sciences can be modeled into an equation of the form

$$F(x) = 0, \tag{1}$$

where $F : \Omega \subseteq X \longrightarrow Y$ is a Fréchet differentiable function on a convex subset $\Omega \subseteq X$; X and Y are Banach spaces [1]. We are interested in finding the local unique solution of Equation (1). Typically, no analytical or closed-form solution exists in general. Therefore, we turn to iterative methods. Newton's method, defined as

$$x_{n+1} = x_n - A_n^{-1}F(x_n), \quad n = 0, 1, 2, \ldots,$$

where $A_n = F'(x_n)$, is very popular. Almost all methods in the literature are some modification or extension of this method; see [2]. Many authors have considered multi-step methods in order to increase efficiency, as well as the order of convergence [3–6].

Among the multi-step methods, the Mean Newton methods are well studied (see [7]). Traub introduced a modification of Newton's method in [5] (see also [8]), defined as follows:

$$\begin{aligned} y_n &= x_n - A_n^{-1}F(x_n) \\ x_{n+1} &= x_n - 2\Delta_n^{-1}F(x_n), \quad n = 0, 1, 2 \ldots, \end{aligned} \tag{2}$$

where $\Delta_n = F'(x_n) + F'(y_n)$. This method, also called the Arithmetic-Mean Newton's method, has an order of convergence of three. Later, Weerakoon and Fernando [9] approached the method using a trapezoidal approximation of the interpolatory quadrature formula. Frontini and Sormani in [10] showed that this method is one of the most efficient

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variants of Newton's method, which uses the quadrature formula. Later, Cordero et al. [11] extended the method defined for $n = 1, 2, \ldots$ by

$$\begin{aligned} y_n &= x_n - A_n^{-1} F(x_n) \\ z_n &= x_n - 2\Delta_n^{-1} F(x_n) \\ x_{n+1} &= z_n - F'(y_n)^{-1} F(z_n), \end{aligned} \tag{3}$$

where $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$. However, the method used Taylor series expansion and assumed the existence of derivatives up to order six. Sharma and Parhi [12] removed these assumptions and studied this method in Banach spaces. Nevertheless, they were unable to obtain the desired order. Many local convergences, as well as semi-local convergence studies, have been conducted on this method [7,13–15].

In this paper, we consider methods (2), (3) and an extension defined for $n = 1, 2, \ldots$ by

$$\begin{aligned} y_n &= x_n - A_n^{-1} F(x_n) \\ z_n &= x_n - 2\Delta_n^{-1} F(x_n) \\ x_{n+1} &= z_n - F'(z_n)^{-1} F(z_n). \end{aligned} \tag{4}$$

This method has an order of six. Parhi [15] has used the above extension to obtain an efficient sixth-order method using a linear interpolation of $F'(z_n)$.

Our paper is divided as follows. In Section 2, each method's preliminary functions, definitions, and auxiliary results are given in order. Some numerical examples to show the radii of convergence, approximate computational order of convergence (ACOC), and an example to illustrate the basins of attractions are given in Section 3. Section 3 also contains a representation of the number of iterations as a heatmap and tables that compare the iterates of methods (2)–(4) with corresponding methods in [16]. Finally, the paper ends with conclusions in Section 4.

2. Main Results

Firstly, we introduce some functions required in the proofs, along with necessary notations and definitions.

Definition 1 ([5] Page 9). *If for a sequence $\{z_n\}$ converging to z^*, there exist some $p \in [0, \infty)$ such that the limit defined as*

$$C_p = \lim_{n \to \infty} \frac{\|z_{n+1} - z^*\|}{\|z_n - z^*\|^p}$$

exists. Then, p is called the order of convergence of sequence $\{z_n\}$.

The above definition is somewhat restrictive. Ortega and Rheinboldt discussed a more general concept of R-order and Q-order in [2]. Nevertheless, with an additional condition $0 < C_p < \infty$ on C_p, there is equivalence between the above definition and $Q-$ and $R-$ orders (see [17,18]).

We use the $R-$ order of convergence defined as follows (see [19,20]).

A sequence $\{z_n\}$ converges to z^* with $R-$order at least p if there are constants $C \in (0, \infty)$ and $\gamma \in (0, 1)$ such that

$$\|z_{n+1} - z^*\| \leq C\gamma^{p^n}, \; n = 0, 1, 2, \ldots. \tag{5}$$

Note that

$$\|z_{n+1} - z^*\| \leq c\|z_n - z^*\|^p, \; c > 0, \; n = 0, 1, 2, \ldots,$$

for $p \in \mathbb{N}$ implies (5) provided $\|z_0 - z^*\| < 1$.

For practical computation of the order of convergence, one may use the Approximate Computational Order of Convergence (ACOC) [21], defined as

$$ACOC = \log\left(\frac{\|z_{k+2}-z_{k+1}\|}{\|z_{k+1}-z_k\|}\right) / \log\left(\frac{\|z_{k+1}-z_k\|}{\|z_k-z_{k-1}\|}\right).$$

Throughout this paper, the open and closed balls centered at $\tilde{u}$ with radius δ are denoted as

$$B(\tilde{u},\delta) = \{u \in X : \|u-\tilde{u}\| < \delta\} \text{ and } B[\tilde{u},\delta] = \{u \in X : \|u-\tilde{u}\| \le \delta\},$$

respectively. Furthermore, $\|x_n - x^*\|$ is denoted by ϵ_{x_n}, $\|y_n - x^*\|$ is denoted by ϵ_{y_n}, and $\|z_n - x^*\|$ is denoted by ϵ_{z_n}. Let K_1, K_2 and M be positive constants in $\mathbb{R}$; our proofs are subject to the following conditions.

Assumption 1. *Assume*

1. *x^* is the root of $F(x) = 0$ and A_*^{-1} exists, where $A_* = F'(x^*)$.*
2. *There exists $K_1 > 0$, such that$\|A_*^{-1}(F'(x) - F'(y))\| \le K_1\|x-y\|$ for all $x, y \in \Omega$.*
3. *There exists $M > 0$, such that $\|A_*^{-1}F''(x)\| \le M$.*
4. *There exists $K_2 > 0$, such that $\|A_*^{-1}(F''(x) - F''(y))\| \le K_2\|x-y\|$ for all $x, y \in \Omega$.*

First, we define function $\zeta_1 : [0, \frac{1}{K_1}) \longrightarrow \mathbb{R}$ as

$$\zeta_1(t) = \frac{K_1}{2(1-K_1 t)}. \tag{6}$$

Let $\rho_1 = \frac{2}{3K_1}$. Note that $h_1(t) := \zeta_1(t)t$ is an increasing function in the interval $[0, \frac{1}{K_1})$. Furthermore, $h_1(0) = 0$ and $h_1(\rho_1) = 1$. That is,

$$0 \le \zeta_1(t)t < 1 \quad \forall t \in [0, \rho_1). \tag{7}$$

Define $\zeta_2 : [0, \rho_1) \longrightarrow \mathbb{R}$ as below.

$$\zeta_2(t) = \frac{K_1}{2}(1 + \zeta_1(t)t)t. \tag{8}$$

Let

$$h_2(t) = \zeta_2(t) - 1.$$

Clearly, since $h_2(0) = -1$ and $h_2(t) \longrightarrow \infty$ as $t \longrightarrow \frac{1}{K_1}^-$, by the intermediate value theorem, there exist a smallest positive root for $h_2(t)$, say ρ_2, in the interval $[0, \frac{1}{K_1})$. So,

$$0 \le \zeta_2(t)t < 1 \quad \forall t \in [0, \rho_2). \tag{9}$$

Consider, $\zeta_3 : [0, \rho_2) \longrightarrow \mathbb{R}$,

$$\zeta_3(t) = \frac{1}{2(1-\zeta_2(t))}\left[M\zeta_1(t) + \frac{K_2}{12}\left(1 + \frac{K_1}{2}t\right)\right]. \tag{10}$$

Let

$$h_3(t) = \zeta_3(t)t^2 - 1.$$

Note that $h_3(0) = -1$ and $h_3(t) \longrightarrow \infty$ as $t \longrightarrow \rho_2^-$. Intermediate value theorem guarantees a smallest positive root ρ_3 in $[0, \rho_2)$ such that

$$0 \le \zeta_3(t) < 1 \quad \forall t \in [0, \rho_3). \tag{11}$$

Let

$$R = \min\{\rho_1, \rho_3, 1\}, \tag{12}$$

then

$$0 \le \zeta_1(t)t < 1,\ 0 \le \zeta_2(t) < 1 \text{ and } 0 \le \zeta_3(t)t^2 < 1 \quad \forall t \in [0, R). \tag{13}$$

Furthermore, one can see that $\zeta_3(t)$ is an increasing function in $[0, \rho_2)$ and hence

$$\zeta_3(t) \le \zeta_3(R) \quad \forall t \in [0, R). \tag{14}$$

Theorem 1. *Let Assumption 1 hold and let R be as in (12); then the sequence $\{x_n\}$ defined by (2) with $x_0 \in B(x^*, R) - \{x^*\}$, converges to x^* such that*

$$\epsilon_{x_{n+1}} \le \zeta_3(R)\epsilon_{x_n}^3, \tag{15}$$

where ζ_3 is as defined in (10).

Proof. First, we will show that $F'(x_0)^{-1}$ is bounded. Using Assumption 1 and (12) ,

$$\|A_*^{-1}(F'(x_0) - A_*)\| \le K_1\epsilon_{x_0} \le K_1 R < 1.$$

Hence, by Banach's lemma on invertible operators [22], $F'(x_0)^{-1}$ is invertible and

$$\|F'(x_0)^{-1}A_*\| \le \frac{1}{1 - K_1\epsilon_{x_0}}. \tag{16}$$

From (2), we have

$$\begin{aligned} y_0 - x^* &= x_0 - x^* - F'(x_0)^{-1}F(x_0) \\ &= x_0 - x^* + F'(x_0)^{-1}(F(x_0) - F(x^*)) \\ &= F'(x_0)^{-1}A_*\left[A_*^{-1}\int_0^1 (F'(x_0) - F'(x^* + t(x_0 - x^*)))dt(x_0 - x^*)\right], \end{aligned}$$

hence, by Assumption 1 , Equations (13) and (16),

$$\begin{aligned} \epsilon_{y_0} &\le \left\|F'(x_0)^{-1}A_*\left[\int_0^1 A_*^{-1}(F'(x_0) - F'(x^* + t(x_0 - x^*)))dt(x_0 - x^*)\right]\right\| \\ &\le \|F'(x_0)^{-1}A_*\| \int_0^1 K_1\|x_0 - x^* - t(x_0 - x^*)\|dt\epsilon_{x_0} \\ &\le \frac{K_1}{2(1 - K_1\epsilon_{x_0})}\epsilon_{x_0}^2 \\ &\le \zeta_1(\epsilon_{x_0})\epsilon_{x_0}^2 \qquad (17) \\ &< \epsilon_{x_0} < R, \end{aligned}$$

so, iterate $y_0 \in B(x^*, R)$. Next, we employ Banach's lemma of invertible operators [22], Equation (13), and Assumption 1 to show that Δ_0^{-1} is bounded.

$$\begin{aligned}
\|(2A_*)^{-1}(\Delta_0 - 2A_*)\| &\leq \frac{1}{2}\|A_*^{-1}(F'(x_0) - A_* + F'(y_0) - A_*)\| \\
&\leq \frac{1}{2}(K_1\epsilon_{x_0} + K_1\epsilon_{y_0}) \\
&\leq \frac{K_1}{2}(1 + \zeta_1(\epsilon_{x_0})\epsilon_{x_0})\epsilon_{x_0} \\
&= \zeta_2(\epsilon_{x_0}) < 1.
\end{aligned}$$

So, Δ_0^{-1} is bounded and

$$\|\Delta_0^{-1}A_*\| \leq \frac{1}{2(1 - \zeta_2(\epsilon_{x_0}))}. \tag{18}$$

From (2),

$$\begin{aligned}
x_1 - x^* &= x_0 - x^* - 2\Delta_0^{-1}F(x_0) \\
&= \Delta_0^{-1}\left[(F'(x_0) + F'(y_0))(x_0 - x^*) - 2(F(x_0) - F(x^*))\right] \\
&= \Delta_0^{-1}\Bigg[\int_0^1 (F'(x_0) - F'(x^* + t(x_0 - x^*)))dt(x_0 - x^*) + \\
&\qquad \int_0^1 (F'(y_0) - F'(x^* + t(x_0 - x^*)))dt(x_0 - x^*)\Bigg] \\
&= \Delta_0^{-1}\Bigg[\int_0^1 \int_0^1 F''(x^* + t(x_0 - x^*) - \theta(x_0 - x^* - t(x_0 - x^*)))d\theta \\
&\qquad \times (x_0 - x^* - t(x_0 - x^*))dt(x_0 - x^*) + \int_0^1 \int_0^1 F''(x^* + t(x_0 - x^*) \\
&\qquad -\theta(y_0 - x^* - t(x_0 - x^*)))d\theta(y_0 - x^* - t(x_0 - x^*))dt \times (x_0 - x^*)\Bigg].
\end{aligned}$$

Let $\theta_{x_0} = \theta(1-t)(x_0 - x^*)$ and $\theta_{y_0} = \theta(y_0 - x^* - t(x_0 - x^*))$. Now, we split up $x_1 - x^*$ as follows.

$$x_1 - x^* = \Delta_0^{-1}A_*[B_1 + B_2 + B_3], \tag{19}$$

where

$$\begin{aligned}
B_1 &= A_*^{-1}\int_0^1 \int_0^1 F''(x^* + t(x_0 - x^*) - \theta_{x_0})d\theta \\
&\quad \times (1 - 2t)(x_0 - x^*)dt(x_0 - x^*), \\
B_2 &= A_*^{-1}\int_0^1 \int_0^1 F''(x^* + t(y_0 - x^*) - \theta_{y_0})d\theta(y_0 - x^*)dt(x_0 - x^*), \\
B_3 &= A_*^{-1}\Bigg[\int_0^1 \int_0^1 F''(x^* + t(x_0 - x^*) - \theta_{x_0})d\theta t dt \\
&\quad - \int_0^1 \int_0^1 F''(x^* + t(y_0 - x^*) - \theta_{y_0})d\theta t dt\Bigg](x_0 - x^*)^2.
\end{aligned}$$

Using Assumption 1, we have

$$\begin{aligned}
\|B_1\| &\leq \left\| \int_0^1 \int_0^1 A_*^{-1} F''(x^* + t(x_0 - x^*) - \theta_{x_0}) d\theta \right. \\
&\quad \times (x_0 - x^* - 2t(x_0 - x^*)) dt (x_0 - x^*) \| \\
&\leq \left\| \sup_{t\in[0,1]} \int_0^1 \|A_*^{-1} F''(x^* + t(x_0 - x^*) - \theta_{x_0})\| d\theta \right. \\
&\quad \left. \times \int_0^1 ((1-2t)(x_0 - x^*)) dt (x_0 - x^*) \right\| \\
&= 0. \qquad (20)
\end{aligned}$$

In addition, by Assumptions 1 and (17), we have

$$\begin{aligned}
\|B_2\| &= \left\| \int_0^1 \int_0^1 A_*^{-1} F''(x^* + t(y_0 - x^*) - \theta_{y_0}) d\theta (y_0 - x^*) dt (x_0 - x^*) \right\| \\
&\leq M\epsilon_{y_0}\epsilon_{x_0} \leq M\zeta_1(\epsilon_{x_0})\epsilon_{x_0}^3, \qquad (21)
\end{aligned}$$

and

$$\begin{aligned}
\|B_3\| &= \left\| \int_0^1 \int_0^1 A_*^{-1} [F''(x^* + t(x_0 - x^*) - \theta_{x_0}) d\theta \right. \\
&\quad \left. - F''(x^* + t(y_0 - x^*) - \theta_{y_0}) d\theta] t dt (x_0 - x^*)^2 \right\| \\
&\leq K_2 \int_0^1 \int_0^1 t(t-\theta) \|x_0 - y_0\| d\theta dt \epsilon_{x_0}^2 \\
&\leq \frac{K_2}{12} \|x_0 - y_0\| \epsilon_{x_0}^2 \\
&\leq \frac{K_2}{12} (\epsilon_{x_0} + \epsilon_{y_0}) \epsilon_{x_0}^2 \\
&\leq \frac{K_2}{12} (1 + \zeta_1(\epsilon_{x_0})\epsilon_{x_0}) \epsilon_{x_0}^3. \qquad (22)
\end{aligned}$$

So, using (19), (20), (21), (22), and (13),

$$\begin{aligned}
\epsilon_{x_1} &\leq \|\Delta_0^{-1} A_*\| \left[M\zeta_1(\epsilon_{x_0})\epsilon_{x_0}^3 + \frac{K_2}{12}(1 + \zeta_1(\epsilon_{x_0})\epsilon_{x_0})\epsilon_{x_0}^3 \right] \\
&\leq \frac{1}{2(1 - \zeta_2(\epsilon_{x_0}))} \left[M\zeta_1(\epsilon_{x_0}) + \frac{K_2}{12}\left(1 + \frac{K_1}{2}\epsilon_{x_0}\right) \right] \epsilon_{x_0}^3 \\
&\leq \zeta_3(\epsilon_{x_0})\epsilon_{x_0}^3 \qquad (23) \\
&\leq \zeta_3(R) R^2 \epsilon_{x_0} < \epsilon_{x_0} < R.
\end{aligned}$$

Hence, $x_1 \in B(x^*, R)$ and (by (23)) (15) holds for $n = 1$. By induction, the proof is complete if x_0, y_0, x_1 are replaced by x_n, y_n, x_{n+1}, respectively. □

Next, to provide the convergence analysis of method (3), we define some functions and parameterss below.

Define $\zeta_4 : [0, \frac{1}{K_1}) \longrightarrow \mathbb{R}$ by

$$\zeta_4(t) = K_1\zeta_1(t)t^2. \qquad (24)$$

and $h_4 : [0, \frac{1}{K_1}) \longrightarrow \mathbb{R}$

$$h_4(t) = \zeta_4(t) - 1.$$

Then, $h_4(0) = -1$ and $h_4(t) \longrightarrow \infty$ as $t \longrightarrow \frac{1}{K_1}^-$. Hence, by the intermediate value theorem, h_4 has a smallest zero $\rho_4 \in [0, \frac{1}{K_1})$, and

$$0 \leq \zeta_4(t) < 1 \quad \forall t \in [0, \rho_4). \tag{25}$$

Define $\zeta_5 : [0, \rho_4) \longrightarrow \mathbb{R}$, by

$$\zeta_5(t) = \frac{K_1}{1 - \zeta_4(t)} \left(\zeta_1(t) + \frac{1}{2}\zeta_3(t)t \right) \zeta_3(t), \tag{26}$$

and $h_5 : [0, \rho_4) \longrightarrow \mathbb{R}$, by

$$h_5(t) = \zeta_5(t)t^4 - 1.$$

Observe that $h_5(0) = -1$ and $h_5(t) \longrightarrow \infty$ as $t \longrightarrow \rho_4^-$. Using the intermediate value theorem, h_5 has a smallest zero ρ_5 in the interval $[0, \rho_4)$. Let

$$R_1 = \min\{R, \rho_5\}, \tag{27}$$

then

$$0 \leq \zeta_4(t) < 1 \text{ and } 0 \leq \zeta_5(t)t^4 < 1 \quad \forall t \in [0, R_1). \tag{28}$$

It is clear that ζ_5 is an increasing function in $[0, \rho_4)$. In particular,

$$\zeta_5(t) \leq \zeta_5(R_1) \quad \forall t \in [0, R_1). \tag{29}$$

Theorem 2. *Let Assumption 1 hold. The sequence $\{x_n\}$ is as in (3). If $x_0 \in B(x^*, R_1) - \{x^*\}$, R_1 is as in (27). Then, the sequence $\{x_n\}$ converges to x^* and*

$$\epsilon_{x_{n+1}} \leq \zeta_5(R_1)\epsilon_{x_n}^5, \tag{30}$$

where ζ_5 is as defined in (26).

Proof. We use induction to prove the theorem. Clearly, one can mimic the proof of Theorem 1 to obtain

$$\epsilon_{z_0} \leq \zeta_3(\epsilon_{x_0})\epsilon_{x_0}^3. \tag{31}$$

Now, we will show that $F'(y_0)^{-1}$ is bounded using Assumption 1 and (28),

$$\begin{aligned} \|A_*^{-1}(F(y_0) - F(x^*))\| &\leq K_1\epsilon_{y_0} \\ &\leq K_1\zeta_1(\epsilon_{x_0})\epsilon_{x_0}^2 \\ &= \zeta_4(\epsilon_{x_0}) \leq \zeta_4(R_1) < 1. \end{aligned}$$

Hence, $F'(y_0)^{-1}$ is invertible and

$$\|F'(y_0)^{-1}A_*\| \leq \frac{1}{1 - \zeta_4(\epsilon_{x_0})}. \tag{32}$$

$$
\begin{aligned}
x_1 - x^* &= z_0 - x^* - F'(y_0)^{-1}F(z_0) \\
&= z_0 - x^* - F'(y_0)^{-1}[F(z_0) - F(x^*)] \\
&= F'(y_0)^{-1}\left[F'(y_0)(z_0 - x^*) - \int_0^1 F'(x^* + t(z_0 - x^*))(z_0 - x^*)dt\right] \\
&= F'(y_0)^{-1}A_*\left[\int_0^1 A_*^{-1}((F'(y_0) - F'(x^* + t(z_0 - x^*))))dt\right](z_0 - x^*).
\end{aligned}
$$

Hence, by using Assumption 1, (28) and (32), we obtain

$$
\begin{aligned}
\epsilon_{x_1} &\leq \|F'(y_0)^{-1}A_*\| \int_0^1 K_1\|y_0 - x^* - t(z_0 - x^*)\|dt\epsilon_{z_0} \\
&\leq \frac{1}{1 - \zeta_4(\epsilon_{x_0})}\int_0^1 K_1(\epsilon_{y_0} + t\epsilon_{z_0})dt\epsilon_{z_0} \\
&\leq \frac{K_1}{1 - \zeta_4(\epsilon_{x_0})}\left(\epsilon_{y_0} + \frac{\epsilon_{z_0}}{2}\right)\epsilon_{z_0} \\
&\leq \frac{K_1}{1 - \zeta_4(\epsilon_{x_0})}\left(\zeta_1(\epsilon_{x_0})\epsilon_{x_0}^2 + \frac{1}{2}\zeta_3(\epsilon_{x_0})\epsilon_{x_0}^3\right) \times \zeta_3(\epsilon_{x_0})\epsilon_{x_0}^3 \\
&\leq \frac{K_1}{1 - \zeta_4(\epsilon_{x_0})}\left(\zeta_1(\epsilon_{x_0}) + \frac{1}{2}\zeta_3(\epsilon_{x_0})\epsilon_{x_0}\right)\zeta_3(\epsilon_{x_0})\epsilon_{x_0}^5 \\
&\leq \zeta_5(\epsilon_{x_0})\epsilon_{x_0}^5 \\
&< \epsilon_{x_0} < R_1.
\end{aligned}
\tag{33}
$$

I.e., $x_1 \in B(x^*, R_1)$, and from (33) and (29),

$$
\begin{aligned}
\epsilon_{x_1} &\leq \frac{K_1}{1 - \zeta_4(\epsilon_{x_0})}\left(\zeta_1(\epsilon_{x_0})\epsilon_{x_0}^2 + \frac{1}{2}\zeta_3(\epsilon_{x_0})\epsilon_{x_0}^3\right)\zeta_3(\epsilon_{x_0})\epsilon_{x_0}^3 \\
&\leq \frac{K_1}{1 - \zeta_4(\epsilon_{x_0})}\left(\zeta_1(\epsilon_{x_0}) + \frac{1}{2}\zeta_3(\epsilon_{x_0})\epsilon_{x_0}\right)\zeta_3(\epsilon_{x_0})\epsilon_{x_0}^5 \\
&\leq \zeta_5(\epsilon_{x_0})\epsilon_{x_0}^5 \leq \zeta_5(R_1)\epsilon_{x_0}^5.
\end{aligned}
\tag{34}
$$

The rest of the proof follows as in Theorem 1. □

To prove the convergence of method (4), we introduce some more functions and parameters. Let $\zeta_6 : [0, \rho_2) \longrightarrow \mathbb{R}$ be defined as

$$\zeta_6(t) = K_1\zeta_3(t)t^3 \tag{35}$$

and $h_6 : [0, \rho_2) \longrightarrow \mathbb{R}$ as

$$h_6(t) = \zeta_6(t) - 1.$$

Then, $h_6(0) = -1$ and $h_6(t) \longrightarrow \infty$ as $t \longrightarrow \rho_2^-$. By the intermediate value theorem there exists a smallest zero ρ_6 in the interval $[0, \rho_2)$ such that

$$0 \leq \zeta_6(t) < 1 \quad \forall t \in [0, \rho_6). \tag{36}$$

Lastly, we define functions $\zeta_7 : [0, \rho_6) \longrightarrow \mathbb{R}$ by

$$\zeta_7(t) = \frac{K_1}{2(1 - \zeta_6(t))}\zeta_3^2(t) \tag{37}$$

and $h_7 : [0, \rho_6) \longrightarrow \mathbb{R}$ by

$$h_7(t) = \zeta_7(t)t^5 - 1.$$

Then, $h_7(0) = -1$ and $h_7(t) \longrightarrow \infty$ as $t \longrightarrow \infty$. The intermediate value theorem gives a smallest root ρ_7 in $[0, \rho_6)$ such that

$$0 \leq \zeta_7(t)t^5 \leq 1 \quad \forall t \in [0, \rho_7). \tag{38}$$

If

$$R_2 = \min\{R, \rho_7\}, \tag{39}$$

then

$$0 \leq \zeta_6(t) < 1 \text{ and } 0 \leq \zeta_7(t)t^5 < 1 \quad \forall t \in [0, R_2). \tag{40}$$

Furthermore, observe that ζ_7 is an increasing function in $[0, \rho_6)$. Specifically,

$$\zeta_7(t) \leq \zeta_7(R_2) \quad \forall t \in [0, R_2). \tag{41}$$

Theorem 3. *Let Assumption 1 hold, and the sequence $\{x_n\}$ defined by (4) with $x_0 \in B(x^*, R_2) - \{x^*\}$, where R_2, as in (39), converges to x^* such that*

$$\epsilon_{x_{n+1}} \leq \zeta_7(R_2)\epsilon_{x_n}^6, \tag{42}$$

where ζ_7 is as in (37).

Proof. In extension (4), only the last step is different from extension (3). So, we can easily repeat the proof to obtain

$$\epsilon_{z_0} \leq \zeta_3(\epsilon_{x_0})\epsilon_{x_0}^3. \tag{43}$$

Now, as in previous case, we will show that $F'(z_0)^{-1}$ exists using Assumptions 1 and (40).

$$\begin{aligned} \|A_*^{-1}(F'(z_0) - A_*)\| &\leq K_1\epsilon_{z_0} \\ &\leq K_1\zeta_3(\epsilon_{x_0})\epsilon_{x_0}^3 \\ &\leq \zeta_6(\epsilon_{x_0}) < \zeta_6(R_2) < 1. \end{aligned}$$

Hence, $F'(z_0)^{-1}$ is invertible and

$$\|F'(z_0)^{-1}A_*\| \leq \frac{1}{1 - \zeta_6(\epsilon_{x_0})}. \tag{44}$$

$$\begin{aligned} x_1 - x^* &= z_0 - x^* - F'(z_0)^{-1}F(z_0) \\ &= F'(z_0)^{-1}\left(F'(z_0)(z_0 - x^*) - \int_0^1 F'(x^* + t(z_0 - x^*))(z_0 - x^*)dt\right) \\ &= F'(z_0)^{-1}A_*\left(\int_0^1 A_*(F'(z_0) - F'(x^* + t(z_0 - x^*)))dt\right)(z_0 - x^*). \end{aligned}$$

Hence, by using (43), Assumptions 1 and (40), it follows that

$$\begin{aligned}
\epsilon_{x_1} &\leq \|F'(z_0)^{-1}F'(x*)\| \left\| \left(\int_0^1 A_*^{-1}(F'(z_0) - F'(x^* + t(z_0 - x^*)))dt \right) \right\| \epsilon_{z_0} \\
&\leq \frac{1}{1 - \zeta_6(\epsilon_{x_0})} \int_0^1 K_1 |(1-t)| \epsilon_{z_0} dt \epsilon_{z_0} \\
&\leq \frac{K_1}{2(1 - \zeta_6(\epsilon_{x_0}))} \zeta_3(\epsilon_{x_0})\epsilon_{x_0}^3 \times \zeta_3(\epsilon_{x_0})\epsilon_{x_0}^3 \\
&\leq \zeta_7(\epsilon_{x_0})\epsilon_{x_0}^6 \qquad (45) \\
&< \epsilon_{x_0} < R_2,
\end{aligned}$$

so, $x_1 \in B(x^*, R_2)$. Furthermore, from (41) and (45), we have

$$\epsilon_{x_1} \leq \zeta_7(R_2)\epsilon_{x_0}^6. \qquad (46)$$

Hence, (42) is satisfied for $n = 1$. Now, the proof follows as in Theorem 2. □

The conditions that guarantee a unique solution are given in the following lemma.

Lemma 1. *Suppose Assumption 1 holds and x^* is a simple solution of the equation $F(x) = 0$. Then, $F(x) = 0$ has a unique solution x^* in $E := \Omega \cap B[x^*, \bar{r}]$ provided*

$$K_1\bar{r} < 2. \qquad (47)$$

Proof. Let $p \in E$ be such that $F(p) = 0$. Define $J = \int_0^1 F'(x^* + u(p - x^*))du$. Then, by Assumption 1, we have, in turn

$$\begin{aligned}
\|A_*^{-1}(J - A_*)\| &\leq K_1 \int_0^1 \|x^* + u(p - x^*) - x^*\| du \\
&\leq K_1 \int_0^1 u\|p - x^*\| du \\
&\leq \frac{K_1\bar{r}}{2} < 1.
\end{aligned}$$

It follows that J is invertible, and hence $p = x^*$ by the identity $0 = F(p) - F(x^*) = J(p - x^*)$. □

3. Illustrations and Numerical Examples

In this section, we will illustrate our results using numerical examples. In the first three examples, we compute the radii of convergence. The next example compares the iterations of methods (2)–(4) with the corresponding methods in [16]. We also compute the ACOC for Examples 2 and 4 (the iterations of Examples 1 and 3 converge within three iterations on almost all initial points, so we have not computed ACOC for these examples). An illustration of the basins of attraction and a representation of the number of iterates as a heatmap follows.

The values of $\rho_i, i \in \{1, 2, \ldots, 7\}$ for the examples (Examples 1–3) are given in Table 1, and the ACOC of Examples 2 and 4 are given in Table 2.

Example 1. *Let $X = Y = \mathbb{R}, \Omega = [r, 2 - r], r \in (2 - \sqrt{2}, 1)$ and $F : \Omega \to Y$ be defined by*

$$F(x) = x^3 - r.$$

Here, $x^* = r^{1/3}$. $M = \frac{2(2-r)}{r^{2/3}}, K_1 = \frac{2(2-r)}{r^{2/3}}$ and $K_2 = \frac{2}{r^{2/3}}$. For instance, if we take $r = 1$, from Table 1, we obtain the values of R, R_1, and R_2 as $0.33196, 0.30365$, and 0.331963, respectively.

Example 2. *Let $X = Y = \mathbb{R}^3$, $\Omega = B[0,1]$. Define function $F(w)$ on Ω for $w = (a_1, a_2, a_3)^T$ by*

$$F(w) = \left(e^{a_1} - 1, a_2^2 \frac{e-1}{2} + a_2, a_3\right)^T.$$

Here, $x^* = (0,0,0)^T$. We have $F'(w) = \begin{pmatrix} e^{a_1} & 0 & 0 \\ 0 & (e-1)a_2 + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Furthermore, $M = e, K_1 = \frac{e-1}{e}$ and $K_2 = e - 1$. Similar to the previous case, we obtain $R = 0.36315$, $R_1 = 0.33616$, and $R_2 = 0.36314$ (see Table 1).

Example 3. *Define F on $\Omega = [-1,1]$ as*

$$F(x) = sinx$$

$x^* = 0$. We obtain $M = 1, K_1 = 1$ and $K_2 = 1$. Consequently, $R = 0.66119$, $R_1 = 0.618403$ and $R_2 = 0.66119$ (see Table 1).

Table 1. The parameters ρ_i of the Examples 1–3.

Example	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7
1	0.3333	0.38197	0.33196	0.36603	0.30365	0.34469	0.33207
2	0.3880	0.44459	0.36315	0.42604	0.33616	0.38418	0.36559
3	0.6667	0.76393	0.66119	0.73205	0.61840	0.68749	0.66163

Table 2. ACOC of Examples 2 and 4.

Example	Root	x_0	Traub's Method	Extension (3)	Extension (4)
2	$(0,0,0)$	$(1, 0.03, 0.03)$	2.98	4.64	5.70
		$(0.5, 0.5, 0.5)$	2.90	4.24	5.32
4	$(0.9, 0.3)$	$(2, -1)$	2.91	4.60	4.43
		$(1.3, 0.4)$	3.01	4.48	5.60

In the next example, we compare the performance of the methods (2)–(4) with that of Noor–Waseem-type methods studied in [16].

Example 4. *The system of equations*

$$\begin{aligned} 3t_1^2 t_2 + t_2^2 &= 1 \\ t_1^4 + t_1 t_2^3 &= 1, \end{aligned}$$

has solutions $(-1, 0.2), (-0.4, -1.3)$, and $(0.9, 0.3)$. The solution $(0.9, 0.3)$ is considered for approximating using the methods (2)–(4) and the corresponding methods studied in [16]. We use the initial point $(2, -1)$ in our computation. Tables 3–5 provide the obtained results.

Table 3. Traub's Method (2) and the Noor–Waseem Method in [16].

k	Traub's Method (2) $x_k = (t_1^k, t_2^k)$	Noor-Waseem Method in [16] $x_k = (t_1^k, t_2^k)$
0	$(2.0000000000000000, -1.0000000000000000)$	$(2.0000000000000000, -1.0000000000000000)$
1	$(1.02074824149820786, 0.25352907082513398)$	$(1.01962359355810994, 0.26538605472406479)$
2	$(0.99287801967429134, 0.30629644813087153)$	$(0.99285365860566110, 0.30634643384624071)$
3	$(0.99277999485546530, 0.30644044650526403)$	$(0.99277999485264400, 0.30644044650915097)$
4	$(0.99277999485112322, 0.30644044651102042)$	$(0.99285365860566110, 0.30634643384624071)$
5	$(0.99277999485112322, 0.30644044651102042)$	$(0.99277999485264400, 0.30644044650915097)$

Table 4. Fifth-Order Method (3) and the Noor–Waseem Fifth-order Extension Method in [16].

k	**Fifth Order Method (3)** $x_k = (t_1^k, t_2^k)$	**Method (3) in [16]** $x_k = (t_1^k, t_2^k)$
0	$(2.0000000000000000, -1.0000000000000000)$	$(2.0000000000000000, -1.0000000000000000)$
1	$(0.99339266265870362, 0.30563908458855637)$	$(0.97999747117802393, 0.31079296183420979)$
2	$(0.99277999485112611, 0.30644044651101687)$	$(0.99252009675815366, 0.30661919359513767)$
3	$(0.99277999485112322, 0.30644044651102042)$	$(0.99277988170910103, 0.30644055554978738)$
4	$(0.99277999485112322, 0.30644044651102042)$	$(0.99277999485110035, 0.30644044651104612)$

Table 5. The Sixth-Order Method (4) and the Noor–Waseem Sixth-order Extension Method in [16].

k	**Sixth Order Method (4)** $x_k = (t_1^k, t_2^k)$	**Method (4) in [16]** $x_k = (t_1^k, t_2^k)$
0	$(2.0000000000000000, -1.0000000000000000)$	$(2.0000000000000000, -1.0000000000000000)$
1	$(0.99278598580223975, 0.30643277796171902)$	$(1.03759994297628344, 0.26149549469920185)$
2	$(0.99277999485112322, 0.30644044651102042)$	$(0.99619799193796287, 0.30257508692302936)$
3	$(0.99277999485112322, 0.30644044651102042)$	$(0.99277999575683006, 0.30644044541552573)$

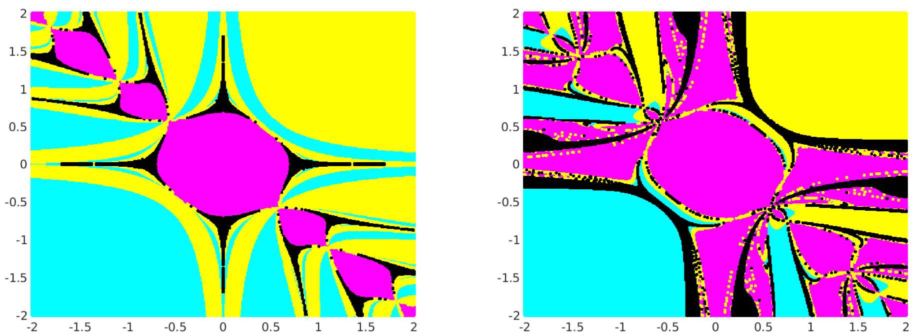
The next example is to compare basins of attraction for each of the discussed methods.

Example 5. *Define F on $\mathbb{R}^2$ by*

$$F(x,y) = (x^3 - y, y^3 - x)$$

with roots $r_1 = (-1,-1), r_2 = (0,0)$ and $r_3 = (1,1)$.

The sub-figures (a), (b), (c), and (d) in the Figure 1 are generated using 400×400 equally spaced grid points from the rectangular region $D = \{(x,y) : x,y \in [-2,2]\}$ as initial points for the iterations. The points that converge to r_1, r_2 and r_3 are colored cyan, magenta, and yellow, respectively. The points that do not converge to any roots after 50 iterations are marked black. The stopping criterion used is $\|x_n - x^*\| < 10^{-8}$. The algorithm used is the same as in [23]. The sub-figures (e), (f), (g), and (h) in Figure 1 are generated with the same grid for the corresponding methods representing the number of iterations required to converge by each point of the grid. It represents the number of iterations required to converge on each grid point. In black, the initial points that did not converge within 50 iterations are represented. The technique used can be found in Ardelean [24].



(a) **(b)**

Figure 1. *Cont.*

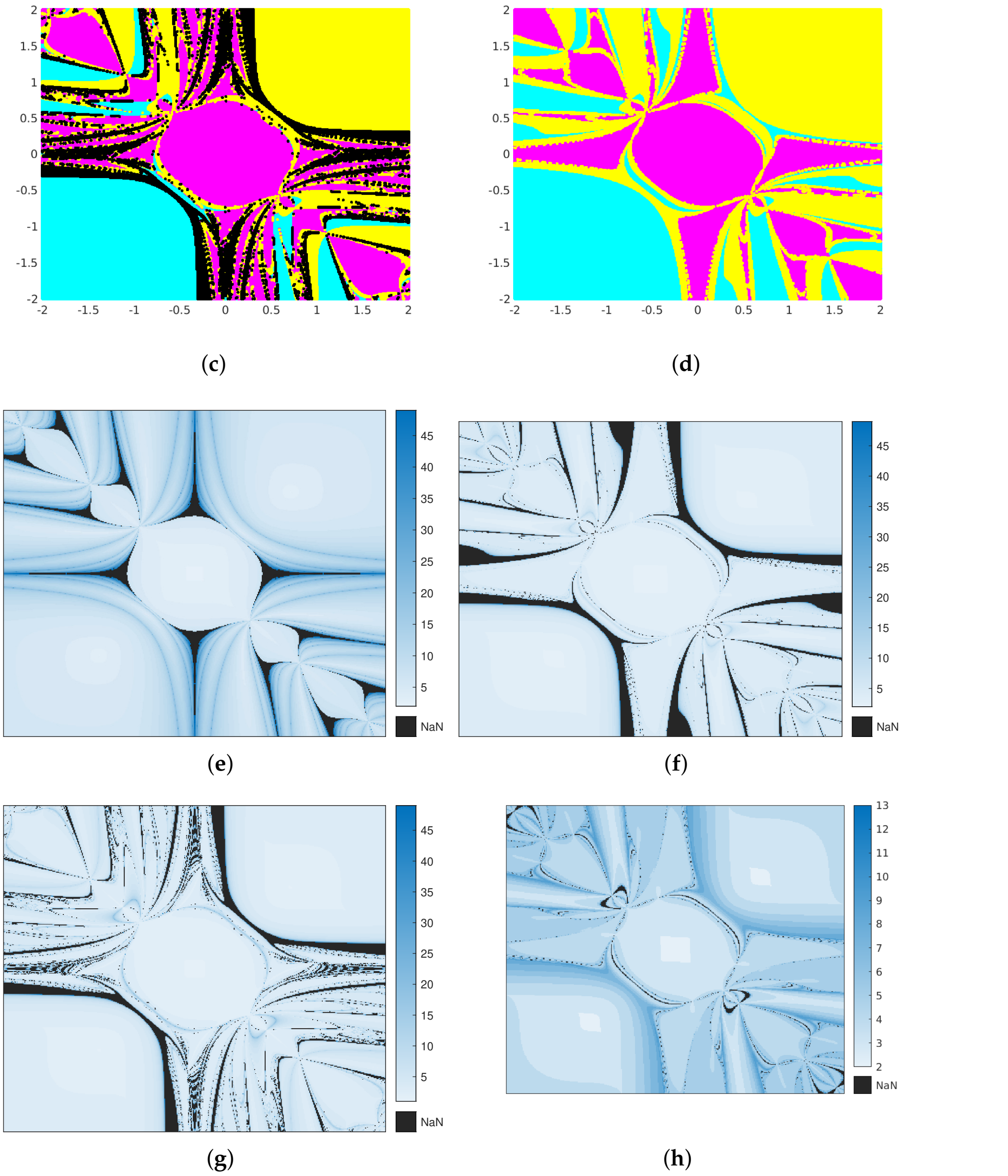


Figure 1. Basins of attraction for the Example 5. (**a**) Newton's Method; (**b**) Weerakoon Method; (**c**) Extension 1 (3); (**d**) Extension 2 (4). Heat maps for (**e**) Newton's Method; (**f**) Weerakoon Method; (**g**) Extension 1 (3); (**h**) Extension 2 (4).

We used a PC with Intel Core i7 processor running Ubuntu 22.4.1 LTS. The programs were executed using MATLAB programming language with version code R2022b.

4. Conclusions

Traub's method (also known as Arithmetic-Mean Newton's Method and Weerakoon and Fernando method) and its two extensions were studied in this paper using assumptions on the derivatives of the operator up to the order two. The theoretical parameters are verified using examples. The dynamics of the methods are also included in this study.

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