fractal and fractional

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Three General Double-Series Identities and Associated Reduction Formulas and Fractional Calculus

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Abstract: In this article, we introduce three general double-series identities using Whipple transformations for terminating generalized hypergeometric ${}_4F_3$ and ${}_5F_4$ functions. Then, by employing the left-sided Riemann–Liouville fractional integral on these identities, we show the ability to derive additional identities of the same nature successively. These identities are used to derive transformation formulas between the Srivastava–Daoust double hypergeometric function (S–D function) and Kampé de Fériet's double hypergeometric function (KDF function) with equal arguments. We also demonstrate reduction formulas from the S–D function or KDF function to the generalized hypergeometric function ${}_pF_q$. Additionally, we provide general summation formulas for the ${}_pF_q$ and S–D function (or KDF function) with specific arguments. We further highlight the connections between the results presented here and existing identities.

Keywords: Bailey quadratic transformation; generalized hypergeometric function; Kampé de Fériet's double hypergeometric function; series rearrangement technique; Srivastava–Daoust double hypergeometric function; Whipple transformations; left-sided Riemann–Liouville fractional integral

MSC: 26A33; 33C05; 33C20

Citation: Qureshi, M.I.; Shah, T.U.R.; Choi, J.; Bhat, A.H. Three General Double-Series Identities and Associated Reduction Formulas and Fractional Calculus. *Fractal Fract.* **2023**, *7*, 700. https://doi.org/10.3390/fractalfract7100700

Academic Editors: Ivanka Stamova, Gheorghe Oros and Georgia Irina Oros

Received: 25 August 2023
Revised: 14 September 2023
Accepted: 15 September 2023
Published: 23 September 2023

1. Introduction and Preliminaries

The generalized hypergeometric series (or function) ${}_pF_q$ $(p,\, q \in \mathbb{Z}_{\geqslant 0})$, which is a natural generalization of the Gaussian hypergeometric series ${}_2F_1$, is defined by (see, e.g., [1–6])

$$\begin{aligned} {}_pF_q\left[\begin{array}{l} \mu_1, \mu_2, \ldots, \mu_p; \\ \nu_1, \nu_2, \ldots, \nu_q; \end{array} z\right] &= {}_pF_q(\mu_1, \ldots, \mu_p; \nu_1, \ldots, \nu_q; z) \\ &= {}_pF_q\left[\begin{array}{l} (\mu_p); \\ (\nu_q); \end{array} z\right] := \sum_{n=0}^{\infty} \frac{\prod\limits_{j=1}^{p} (\mu_j)_n}{\prod\limits_{j=1}^{q} (\nu_j)_n} \frac{z^n}{n!} \end{aligned} \tag{1}$$

$$\big(\mu_k \in \mathbb{C}\,(k = 1, \ldots p),\ \nu_j \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}\,(j = 1, \ldots, q)\big),$$

where $(\xi)_\eta$ $(\xi,\, \eta \in \mathbb{C})$ is the Pochhammer symbol defined in terms of gamma function Γ (see, e.g., [5], pp. 2, 5), by

$$(\xi)_\eta = \frac{\Gamma(\xi+\eta)}{\Gamma(\xi)} = \begin{cases} 1 & (\eta = 0,\, \xi \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}), \\ \xi(\xi+1)\cdots(\xi+n-1) & (\eta = n \in \mathbb{N},\, \xi \in \mathbb{C}), \end{cases} \tag{2}$$

it being understood that $(0)_0 = 1$. Here and elsewhere, an empty product is interpreted as 1, and let $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{Z}$ denote the sets of complex numbers, real numbers, and integers, respectively. Additionally, let $A_{\geqslant \ell}$, $A_{>\ell}$, $A_{\leqslant \ell}$, and $A_{<\ell}$ be the subsets of the set A ($\mathbb{R}$ or $\mathbb{Z}$) whose elements are greater than or equal to, greater than, less than or equal to, and less than some $\ell \in \mathbb{R}$, respectively. In particular, let $\mathbb{N} := \mathbb{Z}_{\geqslant 1}$.

If $\mu_k \in \mathbb{Z}_{\leqslant 0}$ for some $k = 1, \ldots p$, then ${}_pF_q$ series terminates, that is, becomes a polynomial in z, and converges for all $z \in \mathbb{C}$. In case of $\mu_k \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}$ for some $k = 1, \ldots p$, the series ${}_pF_q$ in (1) becomes a polynomial of finite order. For the detailed convergence conditions for ${}_pF_q$ in (1), one can consult, for example, [7], p. 20.

The popularity and usefulness of the hypergeometric function ${}_2F_1$, as well as its generalized versions in one variable ${}_pF_q$, have motivated researchers to explore hypergeometric functions in multiple variables. Appell [8] initiated the study of hypergeometric functions in two variables by introducing the Appell functions F_1, F_2, F_3, and F_4 as generalizations of Gauss's hypergeometric function ${}_2F_1$. Subsequently, Humbert [9] investigated the confluent forms of these functions. A comprehensive list of these functions is available in standard literature, such as [10]. Kampé de Fériet [11] later expanded upon the work of Appell, generalizing the four Appell functions and their confluent forms to more general hypergeometric functions of two variables. Burchnall and Chaundy [12,13] introduced an abbreviation for the notation created by Kampé de Fériet for his double hypergeometric functions of superior order. In a slightly modified notation, Srivastava and Panda [14] (p. 423, Equation (26)) presented the definition of a more comprehensive double hypergeometric function than the one defined by Kampé de Fériet. This convenient generalization of the Kampé de Fériet function is defined as follows (see, for example, [7], p. 27):

$$\begin{aligned} &F^{p:q;k}_{\ell:m;n}\left[\begin{array}{ccc} (a_p): & (b_q); & (c_k); \\ (\alpha_\ell): & (\beta_m); & (\gamma_n); \end{array} x,\, y\right] \\ &\qquad = \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{r+s} \prod_{j=1}^{q}(b_j)_r \prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{\ell}(\alpha_j)_{r+s} \prod_{j=1}^{m}(\beta_j)_r \prod_{j=1}^{n}(\gamma_j)_s} \frac{x^r}{r!} \frac{y^s}{s!}, \end{aligned} \tag{3}$$

where, for convergence,

(i) $p+q < \ell+m+1$, $p+k < \ell+n+1$, $|x| < \infty$, $|y| < \infty$, or

(ii) $p+q = \ell+m+1$, $p+k = \ell+n+1$, and

$$\begin{cases} |x|^{1/(p-\ell)} + |y|^{1/(p-\ell)} < 1, & \text{if } p > \ell, \\ \max\{|x|, |y|\} < 1, & \text{if } p \le \ell. \end{cases}$$

To gain further insight into the convergence properties of the double series in Equation (3), which encompasses conditional convergence as well, one can consult the research conducted by Hái et al. [15].

Lemma 1. *The following formula holds.*

$$F^{p:q;k}_{\ell:m;n}\left[\begin{array}{ccc} (a_p): & (b_q); & (c_k); \\ (\alpha_\ell): & (\beta_m); & (\gamma_n); \end{array} x,\, y\right] = F^{p:k;q}_{\ell:n;m}\left[\begin{array}{ccc} (a_p): & (c_k); & (b_q); \\ (\alpha_\ell): & (\gamma_n); & (\beta_m); \end{array} y,\, x\right]. \tag{4}$$

In particular,

$$F^{p:q;k}_{\ell:m;n}\left[\begin{array}{ccc} (a_p): & (b_q); & (c_k); \\ (\alpha_\ell): & (\beta_m); & (\gamma_n); \end{array} x,\, x\right] = F^{p:k;q}_{\ell:n;m}\left[\begin{array}{ccc} (a_p): & (c_k); & (b_q); \\ (\alpha_\ell): & (\gamma_n); & (\beta_m); \end{array} x,\, x\right]. \tag{5}$$

Proof. Observing the following fact is sufficient: Interchanging the summation indices r and s leaves the quantity on the right-hand side of Equation (3) unchanged. □

Srivastava and Daoust ([16], p. 199) introduced a generalization of the Kampé de Fériet function ([8], p. 150) by means of the double hypergeometric series (see also [17,18]):

$$F^{A:\,B;\,B'}_{C:\,D;\,D'}\left(\begin{array}{l}[(a_A):\vartheta,\,\varphi]:\quad[(b_B):\psi];\quad[(b'_{B'}):\psi'];\\ [(c_C):\delta,\,\varepsilon]:\quad[(d_D):\eta];\quad[(d'_{D'}):\eta'];\end{array}\,x,\,y\right)$$
$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{A}(a_j)_{m\vartheta_j+n\varphi_j}\prod_{j=1}^{B}(b_j)_{m\psi_j}\prod_{j=1}^{B'}(b'_j)_{n\psi'_j}}{\prod_{j=1}^{C}(c_j)_{m\delta_j+n\varepsilon_j}\prod_{j=1}^{D}(d_j)_{m\eta_j}\prod_{j=1}^{D'}(d'_j)_{n\eta'_j}}\frac{x^m}{m!}\frac{y^n}{n!},\tag{6}$$

where the coefficients

$$\begin{aligned}&\vartheta_1,\ldots,\vartheta_A;\ \varphi_1,\ldots,\varphi_A;\ \psi_1,\ldots,\psi_B;\ \psi'_1,\ldots,\psi'_{B'};\ \delta_1,\ldots,\delta_C;\\&\varepsilon_1,\ldots,\varepsilon_C;\ \eta_1,\ldots,\eta_D;\ \eta'_1,\ldots,\eta'_{D'}\end{aligned}\tag{7}$$

are real and positive. Let

$$\Delta_1 := 1+\left(\sum_{j=1}^{C}\delta_j+\sum_{j=1}^{D}\eta_j\right)-\left(\sum_{j=1}^{A}\vartheta_j+\sum_{j=1}^{B}\psi_j\right)$$

and

$$\Delta_2 := 1+\left(\sum_{j=1}^{C}\varepsilon_j+\sum_{j=1}^{D'}\eta'_j\right)-\left(\sum_{j=1}^{A}\varphi_j+\sum_{j=1}^{B'}\psi'_j\right).$$

Then

(i) The double power series in (6) converges for all complex values of x and y when $\Delta_1 > 0$ and $\Delta_2 > 0$.

(ii) The double power series in (6) is convergent for suitably constrained values of $|x|$ and $|y|$ when $\Delta_1 = 0$ and $\Delta_2 = 0$.

(iii) The double power series in (6) would diverge except when, trivially, $x = y = 0$ when $\Delta_1 < 0$ and $\Delta_2 < 0$.

The emergence of extensively generalized special functions, such as (3), has sparked intriguing research into their reducibility. The Kampé de Fériet function, in particular, has been studied extensively by many researchers for its reducibility and transformation formulas. Many reduction and transformation formulas for the Kampé de Fériet function can be found in the literature, as documented in various references, such as [19–42].

Buschman and Srivastava [19] provided insightful remarks on previous studies, specifically [43,44]. They employed a double-series manipulation technique, utilizing Whipple's transformation (see [45], Equation (7.1); see also [10], p. 190, Equation (1), [4], p. 90, Theorem 32):

$$\begin{aligned}&{}_3F_2\left[\begin{array}{r}\alpha,\,\beta,\,\gamma;\\1+\alpha-\beta,\,1+\alpha-\gamma;\end{array}\,z\right]\\&\quad=(1-z)^{-\alpha}{}_3F_2\left[\begin{array}{c}\frac{\alpha}{2},\,\frac{\alpha+1}{2},\,1+\alpha-\beta-\gamma;\\1+\alpha-\beta,\,1+\alpha-\gamma;\end{array}\,-\frac{4z}{(1-z)^2}\right]\\&\left(\alpha-\beta,\,\alpha-\gamma\in\mathbb{C}\setminus\mathbb{Z}_{\leqslant -1};\ |z|<1,\ 4|z|/\{|1-z|^2\}<1,\ |\arg(1-z)|<\pi\right).\end{aligned}\tag{8}$$

Through this approach, they introduced three double-series identities, which incorporated a bounded sequence of complex numbers. In addition, they [19] demonstrated that the application of double-series identities enables the provision of numerous reduction formulas for the Kampé de Fériet function, whether they are already known or newly discovered. Subsequently and concurrently, a number of papers have utilized series manipulation techniques along with, among several others, transformation formulas for ${}_2F_1$ (13) and (19) in Chan et al. [21]; the reduction formula for ${}_2F_1$ (15) in Karlsson [46]; a particular case of Euler's transformation formula for ${}_2F_1$ (12) in Karlsson [47]; terminating summation formulas for ${}_4F_3(1)$ (Equations (2.3), (2.4), (2.6), Tables 1 and 2, there) and the transformation formula for ${}_4F_3(1)$ (10) in Karlsson [30]; transformation formulas for ${}_2F_1$ (12) and (13) in Liu and Wang [48]; transformation formulas for ${}_2F_1$ ([22], Equations (2.8) and (2.9)) (cf. [10], p. 112, Equations (17) and (16), respectively), Whipple's transformation ${}_3F_2$ (8) and summation formula for a terminating ${}_3F_2(1)$ [22], Equation (3.2) (see also [49]), and Dougall's summation theorem for a terminating well-poised ${}_7F_6(1)$ [22], Equation (3.7) (see also [10], p. 189, Equation (4).4(8)), [50], p. 244, Equation (III. 14)) in Chen and Srivastava [22]; and terminating ${}_3F_2(\frac{4}{3})$ [51], Equation (1.3) (see also Gessel–Stanton summation theorem [52], Equation (5.21), and terminating ${}_3F_2(\frac{3}{4})$ [51], Equation (1.4) (see also [53], Equation (1.12)) in Qureshi et al. [51]. These papers have presented multiple or double-series identities, which have been employed to derive a range of reduction formulas for the Kampé de Fériet function and other intriguing identities for the ${}_pF_q$ functions.

Inspired by the aforementioned papers, especially [19], and utilizing Whipple's transformation formulas (refer to [45], p. 266, Equation (6.6))

$$
\begin{aligned}
&{}_5F_4\left[\begin{matrix} -\frac{m}{2},\, \frac{-m+1}{2},\, E,\, 1-m-B-C,\, 1-m-D\,; \\ 1-m-B,\, 1-m-C,\, \frac{1+E-D-m}{2},\, \frac{2+E-D-m}{2}\,; \end{matrix}\; 1\right] \\
&\qquad = \frac{(D)_m}{(D-E)_m}\, {}_4F_3\left[\begin{matrix} -m,\, B,\, C,\, E\,; \\ 1-m-B,\, 1-m-C,\, D\,; \end{matrix}\; 1\right] \\
&\left(m \in \mathbb{Z}_{\geqslant 0};\, B,\, C \in \mathbb{C}\setminus\mathbb{Z}_{\geqslant 1-m};\, D,\, \tfrac{1+E-D-m}{2},\, \tfrac{2+E-D-m}{2} \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0}\right)
\end{aligned}
\tag{9}
$$

and (see [54], p. 537, Equation (10.11); see also [30], Equation (2.5))

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} -m,\, X,\, Y,\, Z\,; \\ U,\, W,\, X+Y+Z+1-U-W-m\,; \end{matrix}\; 1\right] \\
&\qquad = \frac{(U-X)_m(Y+Z+1-U-W-m)_m}{(U)_m(X+Y+Z+1-U-W-m)_m} \\
&\qquad\quad \times {}_4F_3\left[\begin{matrix} -m,\, W-Y,\, W-Z,\, X\,; \\ 1-m+X-U,\, U+W-Y-Z,\, W\,; \end{matrix}\; 1\right] \\
&\left(\begin{matrix} m \in \mathbb{Z}_{\geqslant 0};\, U,\, W,\, X+Y+Z+1-U-W-m, \\ 1-m+X-U,\, U+W-Y-Z,\, W \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0} \end{matrix}\right),
\end{aligned}
\tag{10}
$$

our objective is to introduce three double-series identities. These identities incorporating bounded sequences of complex numbers are derived using series rearrangement techniques and Pochhammer symbol identities. These issues are further discussed in Section 2. In Section 3, we employ these general double-series identities to establish three transformations of Srivastava–Daoust double hypergeometric functions. These transformations are expressed using Kampé de Fériet functions. By utilizing the left-sided Riemann–Liouville fractional integral on these identities in Sections 2 and 4, we demonstrate the capability to iteratively derive further identities of a similar nature. Section 5 further presents various new transformation formulae, such as Bailey's quadratic transformation formula, Clausen reduction formula, Gauss quadratic transformation formula, Karlsson reduction formula, Orr reduction formula, and Whipple quadratic transformation formula. We achieve this by using the following formulas.

Required formulas

Binomial theorem (see, e.g., [6], p. 44, Equation (8)):

$$(1-z)^{-\lambda} = \sum_{n=0}^{\infty} (\lambda)_n \frac{z^n}{n!} = {}_1F_0(\lambda\,;\,-\,;\,z) \\ (|z|<1,\ \lambda \in \mathbb{C},\ |\arg(1-z)|<\pi); \tag{11}$$

Euler's transformation formula (see, e.g., [3], p. 248, Equation (9.5.3), [1], p. 68, Equation (2.2.7)):

$${}_2F_1(\alpha,\,\beta\,;\,\gamma\,;\,z) = (1-z)^{\gamma-\alpha-\beta}\,{}_2F_1(\gamma-\alpha,\,\gamma-\beta\,;\,\gamma\,;\,z) \\ (|z|<1,\ \gamma \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0},\ |\arg(1-z)|<\pi); \tag{12}$$

Pfaff–Kummer transformation formula (see, e.g., [3], p. 247, Equations (9.5.1) and (9.5.2), [1], p. 68, Equation (2.2.6)):

$${}_2F_1(\alpha,\,\beta\,;\,\gamma\,;\,z) = (1-z)^{-\alpha}\,{}_2F_1\left(\alpha,\,\gamma-\beta\,;\,\gamma\,;\,\frac{-z}{1-z}\right) \\ (|z|<1,\ |z|/|1-z|<1,\ \gamma \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0},\ |\arg(1-z)|<\pi) \tag{13}$$

and

$${}_2F_1(\alpha,\,\beta\,;\,\gamma\,;\,z) = (1-z)^{-\beta}\,{}_2F_1\left(\beta,\,\gamma-\alpha\,;\,\gamma\,;\,\frac{-z}{1-z}\right) \\ (|z|<1,\ |z|/|1-z|<1,\ \gamma \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0},\ |\arg(1-z)|<\pi); \tag{14}$$

A reduction formula (see, e.g., [4], p. 70, Equation (10)):

$${}_2F_1\left(\gamma,\,\gamma-\frac{1}{2}\,;\,2\gamma\,;\,z\right) = \left(\frac{2}{1+\sqrt{1-z}}\right)^{2\gamma-1} \\ (|z|<1,\, 2\gamma \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0}); \tag{15}$$

Bailey transformation formula (see, e.g., [55], p. 251, Equation (4.22)):

$${}_2F_1(\alpha,\,\beta\,;\,2\beta\,;\,z) = (1-z)^{-\frac{\alpha}{2}}\,{}_2F_1\left(\tfrac{\alpha}{2},\,\beta-\tfrac{\alpha}{2}\,;\,\beta+\tfrac{1}{2}\,;\,\tfrac{-z^2}{4(1-z)}\right) \\ (|z|<1,\ |z|^2/\{4|1-z|\}<1,\ 2\beta \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0}, \\ \beta+\tfrac{1}{2} \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0},\ |\arg(1-z)|<\pi); \tag{16}$$

Gauss transformation formula (see, e.g., [10], p. 111, Equation (2), and p. 112, Equation (18)):

$${}_2F_1\left(2\alpha,\,2\beta\,;\,\alpha+\beta+\tfrac{1}{2}\,;\,z\right) = {}_2F_1\left(\alpha,\,\beta\,;\,\alpha+\beta+\tfrac{1}{2}\,;\,4z(1-z)\right) \\ \left(|z|<1,\, 4|z(1-z)|<1,\, \alpha+\beta+\tfrac{1}{2} \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0}\right); \tag{17}$$

Bailey product formula (see, e.g., [56], p. 383, Equation (7.4)):

$${}_2F_1(\alpha,\,\beta\,;\,\gamma\,;\,z)\,{}_2F_1(\gamma-\beta,\,1-\beta\,;\,\alpha-\beta+1\,;\,z) \\ = (1-z)^{\beta-\alpha-\gamma}\,{}_4F_3\left[\begin{matrix}\alpha,\gamma-\beta,\,\frac{\alpha+\gamma-\beta}{2},\,\frac{\alpha+\gamma-\beta+1}{2}\,; \\ \gamma,\,\alpha+\gamma-\beta,\,\alpha-\beta+1\,;\end{matrix}\ \frac{-4z}{(1-z)^2}\right] \tag{18}$$

$$\left(|z|<1,\, 4|z|/|1-z|^2;\, \gamma,\, \alpha-\beta+1,\, \alpha+\gamma-\beta \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0};\, |\arg(1-z)|<\pi\right);$$

Letting $\gamma \to \infty$ on both sides of (8) gives the following transformation formula ([10], Equation 2.11 (34)) (see also [21], p. 425, Equation (34)):

$$ {}_2F_1\left[\begin{matrix} \alpha,\ \beta; \\ 1+\alpha-\beta; \end{matrix}\ z\right] = (1+z)^{-\alpha}\,{}_2F_1\left[\begin{matrix} \frac{\alpha}{2},\ \frac{\alpha+1}{2}; \\ 1+\alpha-\beta; \end{matrix}\ \frac{4z}{(1+z)^2}\right] \tag{19} $$

$$ \left(\alpha-\beta \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant -1};\ |z|<1,\ 4|z|/\{|1+z|^2\}<1,\ |\arg(1+z)|<\pi\right), $$

A number of reduction formulae for the Kampé de Fériet function, for example,

$$ \begin{aligned} &F^{0:p;r}_{0:q;s}\left[\begin{matrix} \text{-}: & \alpha_1,\ldots,\alpha_p; & \gamma_1,\ldots,\gamma_r; \\ \text{-}: & \beta_1,\ldots,\beta_q; & \delta_1,\ldots,\delta_s; \end{matrix}\ x,\ y\right] \\ &\quad = {}_pF_q\left[\begin{matrix} \alpha_1,\ldots,\alpha_p; \\ \beta_1,\ldots,\beta_q; \end{matrix}\ x\right] {}_rF_s\left[\begin{matrix} \gamma_1,\ldots,\gamma_r; \\ \delta_1,\ldots,\delta_s; \end{matrix}\ y\right] \end{aligned} \tag{20} $$

(see, e.g., [7], p. 28, Equation (31));

$$ \begin{aligned} F^{p:1;1}_{q:0;0}\left[\begin{matrix} \alpha_1,\ldots,\alpha_p: & \nu; & \sigma; \\ \beta_1,\ldots,\beta_q: & \text{-}; & \text{-}; \end{matrix}\ z,\ z\right] &= {}_{p+1}F_q\left[\begin{matrix} \alpha_1,\ldots,\alpha_p,\ \nu+\sigma; \\ \beta_1,\ldots,\beta_q; \end{matrix}\ z\right] \\ &\left(\beta_1,\ldots,\beta_q \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0};\ |z|<1\ (p=q)\right) \end{aligned} \tag{21} $$

(see, e.g., [8], pp. 23, 155, Equation (25), [57], p. 33, Equation (1.5.1.7));

$$ \begin{aligned} &F^{p:2;1}_{q:1;0}\left[\begin{matrix} \alpha_1,\ldots,\alpha_p: & \lambda^*,\mu^*; & \nu^*-\lambda^*-\mu^*; \\ \beta_1,\ldots,\beta_q: & \nu^*; & -; \end{matrix}\ z,\ z\right] \\ &\quad = {}_{p+2}F_{q+1}\left[\begin{matrix} \alpha_1,\ldots,\alpha_p,\ \nu^*-\lambda^*,\ \nu^*-\mu^*; \\ \beta_1,\ldots,\beta_q,\ \nu^*; \end{matrix}\ z\right] \\ &\left(\beta_1,\ldots,\beta_q,\ \nu^* \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0};\ |z|<1\ (p=q);\ |z|<\infty\ (p\leqslant q-1)\right) \end{aligned} \tag{22} $$

(see, e.g., [7], p. 28, Equation (34));

$$ \begin{aligned} &F^{p:2;2}_{q:1;1}\left[\begin{matrix} \alpha_1,\ldots,\alpha_p: & g,h; & g,h-1; \\ \beta_1,\ldots,\beta_q: & g+h-\frac{1}{2}; & g+h-\frac{1}{2}; \end{matrix}\ z,\ z\right] \\ &\quad = {}_{p+3}F_{q+2}\left[\begin{matrix} \alpha_1,\ldots,\alpha_p,\ 2g,\ 2h-1,\ g+h-1; \\ \beta_1,\ldots,\beta_q,\ g+h-\frac{1}{2},\ 2g+2h-2; \end{matrix}\ z\right] \\ &\left(\beta_1,\ldots,\beta_q,\ g+h-\tfrac{1}{2},\ 2g+2h-2 \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0};\ |z|<1\ (p=q)\right) \end{aligned} \tag{23} $$

(see, e.g., [30], p. 34, Table 3(Ic), [7], p. 29, Equation (38));

A summation formula for ${}_3F_2$ (see, e.g., [58], p. 540, Entry (114)):

$$ {}_3F_2\left[\begin{matrix} -\frac{m}{2},\ \frac{-m+1}{2},\ A; \\ B,\ \frac{3}{2}+A-B-m; \end{matrix}\ 1\right] = 4^{-m}\frac{(2B-2A-1)_m(2B+m-1)_m}{(B)_m(B-A-\frac{1}{2})_m} \tag{24} $$

$$ \left(m\in\mathbb{Z}_{\geqslant 0};\ B,\ \tfrac{3}{2}+A-B-m,\ B-A-\tfrac{1}{2}\in\mathbb{C}\setminus\mathbb{Z}_{\leqslant 0}\right); $$

The following generalized summation formulae for ${}_2F_1$:

Generalized Kummer first summation theorem (see [59], p. 828, Theorem 3):

$$ {}_2F_1\left[\begin{matrix} C,\, D\,; \\ 1+C-D+m\,; \end{matrix}\; 1\right] = \frac{2^{m-2D}\Gamma(D-m)\Gamma(1+C-D+m)}{\Gamma(D)\Gamma(C-2D+m+1)} \times \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{\Gamma\left(\frac{C+k+m+1}{2}-D\right)}{\Gamma\left(\frac{C+k-m+1}{2}\right)} \tag{25} $$

$$ \left(m \in \mathbb{Z}_{\geqslant 0},\ D-C \in \mathbb{C}\setminus\mathbb{Z}_{\geqslant m+1},\ \Re(D) < \tfrac{1+m}{2}\right); $$

Generalized Kummer first summation theorem (see [59], p. 828, Theorem 4):

$$ {}_2F_1\left[\begin{matrix} C,\, D\,; \\ 1+C-D-m\,; \end{matrix}\; -1\right] = \frac{2^{-m-2D}\Gamma(1+C-D-m)}{\Gamma(C-2D-m+1)} \times \sum_{k=0}^{m} \binom{m}{k} \frac{\Gamma\left(\frac{C+k-m+1}{2}-D\right)}{\Gamma\left(\frac{C+k-m+1}{2}\right)} \tag{26} $$

$$ \left(m \in \mathbb{Z}_{\geqslant 0},\ C-D \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant m-1},\ \tfrac{1-m}{2} \leqslant \Re(D) < 1-\tfrac{m}{2}\right); $$

Generalized Kummer second summation theorem (see [59], p. 827, Theorem 1):

$$ {}_2F_1\left[\begin{matrix} C,\, D\,; \\ \frac{1}{2}(1+C+D+m)\,; \end{matrix}\; \frac{1}{2}\right] = \frac{2^{D-1}\Gamma\left(\frac{C+D+m+1}{2}\right)\Gamma\left(\frac{C-D-m+1}{2}\right)}{\Gamma(D)\Gamma\left(\frac{C-D+m+1}{2}\right)} \times \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{\Gamma\left(\frac{D+k}{2}\right)}{\Gamma\left(\frac{C+k-m+1}{2}\right)} \tag{27} $$

$$ \left(m \in \mathbb{Z}_{\geqslant 0},\ \tfrac{1}{2}(1+C+D+m) \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0}\right); $$

Generalized Kummer second summation theorem (see, e.g., [58], p. 491, Entry 7.3.7.2):

$$ {}_2F_1\left[\begin{matrix} C,\, D\,; \\ \frac{1}{2}(1+C+D-m)\,; \end{matrix}\; \frac{1}{2}\right] = \frac{2^{D-1}\Gamma\left(\frac{C+D-m+1}{2}\right)}{\Gamma(D)} \times \sum_{k=0}^{m} \binom{m}{k} \frac{\Gamma\left(\frac{D+k}{2}\right)}{\Gamma\left(\frac{C+k-m+1}{2}\right)} \tag{28} $$

$$ \left(m \in \mathbb{Z}_{\geqslant 0},\ \tfrac{1}{2}(1+C+D-m) \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0}\right); $$

Generalized Kummer third summation theorem (see [59], p. 828, Theorem 5):

$$ {}_2F_1\left[\begin{matrix} C,\, 1-C+m\,; \\ D\,; \end{matrix}\; \frac{1}{2}\right] = \frac{2^{m-C}\Gamma(C-m)\Gamma(D)}{\Gamma(C)\Gamma(D-C)} \times \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{\Gamma\left(\frac{D-C+k}{2}\right)}{\Gamma\left(\frac{D+C+k}{2}-m\right)} \tag{29} $$

$$ (m \in \mathbb{Z}_{\geqslant 0},\ D \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0}); $$

Generalized Kummer third summation theorem (see [60], Equation (20)):

$$ {}_2F_1\left[\begin{matrix} C,\, 1-C-m\,; \\ D\,; \end{matrix}\ \frac{1}{2}\right] = \frac{2^{-m-C}\Gamma(D)}{\Gamma(D-C)} \sum_{k=0}^{m} \binom{m}{k} \frac{\Gamma\left(\frac{D-C+k}{2}\right)}{\Gamma\left(\frac{D+C+k}{2}\right)} \tag{30} $$

$$ (m \in \mathbb{Z}_{\geqslant 0},\ D \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}), $$

which is the corrected version of [59], p. 828, Theorem 6.

Lastly, in Section 6, we derive a set of summation theorems with arguments of 1, -1, $\frac{1}{2}$, $-\frac{1}{4}$, $-\frac{1}{8}$, and $-\frac{1}{16}$.

Remark 1. *It is intriguing to compare Entries 131 and 132 in ([61], p. 583) with the summation Formulas (29) and (30).*

One can find the specific instances of Equations (29) and (30) in [62] (Equation (6) along with Table 3) for the values of m equal to 0, 1, 2, 3, 4, *and* 5.

The numerator parameters $-\frac{m}{2}$ *and* $\frac{-m+1}{2}$ *of the* ${}_5F_4$ *on the left side of Equation (9) both yield negative integers if m is even and odd, respectively. The* ${}_5F_4$ *and* ${}_4F_3$ *on the left and right sides of Equation (9) exhibit properties of being Saalschützian and nearly poised, respectively.*

Wolfram's MATHEMATICA has implemented the ${}_pF_q$ *function as hypergeometric PFQ, which is appropriate for performing both symbolic and numerical computations.*

2. Three General Double-Series Identities

This section demonstrates three general double-series identities that involve bounded sequences by primarily utilizing Whipple transformations (9) and (10).

Theorem 1. *Let* $\{\Psi(\mu)\}_{\mu=0}^{\infty}$ *be a bounded sequence of complex (or real) numbers such that* $\Psi(0) \neq 0$*. Additionally, let* $\alpha+\beta,\ \gamma,\ \delta \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0}$*. Then the following general double-series identity holds true:*

$$ \begin{aligned} &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Psi(2m+n) \frac{(\alpha)_{m+n}(\beta)_{m+n}\left(\frac{\gamma+\delta-1}{2}\right)_{m+n}\left(\frac{\gamma+\delta}{2}\right)_{m+n}}{(\gamma)_{m+n}(\alpha+\beta)_{m+n}(\delta)_{m+n}} \frac{(-4z^2)^m(4z)^n}{m!\,n!} \\ &\quad = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Psi(m+n) \frac{(\gamma+\delta-1)_{m+n}(\alpha)_m(\beta)_m\,(\alpha)_n(\beta)_n}{(\alpha+\beta)_{m+n}(\gamma)_m(\delta)_n} \frac{z^{m+n}}{m!\,n!}, \end{aligned} \tag{31} $$

provided that both sides of the double series are absolutely convergent.

Proof. Let $\Xi_1(z)$ be the left member of (31). By using a double-series manipulation,

$$ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Phi(m,n) = \sum_{n=0}^{\infty}\sum_{m=0}^{[\frac{n}{2}]} \Phi(m,n-2m), \tag{32} $$

where $\Phi : \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0} \to \mathbb{C}$ is a bounded function, and provided that both sides of the double series are absolutely convergent, we obtain

$$ \Xi_1(z) = \sum_{n=0}^{\infty}\sum_{m=0}^{[\frac{n}{2}]} \Psi(n) \frac{(\alpha)_{n-m}(\beta)_{n-m}\left(\frac{\gamma+\delta-1}{2}\right)_{n-m}\left(\frac{\gamma+\delta}{2}\right)_{n-m}(-1)^m\,4^{n-m}\,z^n}{(\gamma)_{n-m}(\alpha+\beta)_{n-m}(\delta)_{n-m}\,m!\,(n-2m)!}. \tag{33} $$

Recall the following Pochhammer symbol identities:

$$ (\lambda)_{n-k} = \frac{(-1)^k\,(\lambda)_n}{(1-\lambda-n)_k} \quad (k = 0,\,1,\,\ldots,\,n). \tag{34} $$

Setting $\lambda = 1$ in (34) gives

$$(n-k)! = \frac{(-1)^k\, n!}{(-n)_k} \quad (k = 0, 1, \ldots, n). \tag{35}$$

Additionally,

$$(\lambda)_{2n} = 2^{2n} \left(\frac{\lambda}{2}\right)_n \left(\frac{\lambda+1}{2}\right)_n \quad (n \in \mathbb{Z}_{\geqslant 0}). \tag{36}$$

Using (34)–(36) in (33), and expressing the inner sum in the resultant double series in terms of ${}_pF_q$ in (1), we have

$$\begin{aligned} \Xi_1(z) = &\sum_{n=0}^{\infty} \Psi(n) \frac{(\alpha)_n(\beta)_n(\gamma+\delta-1)_{2n} z^n}{(\gamma)_n(\alpha+\beta)_n(\delta)_n\, n!} \\ &\times {}_5F_4\left[\begin{matrix} -\frac{n}{2}, \frac{-n+1}{2}, 1-\gamma-n, 1-\alpha-\beta-n, 1-\delta-n; \\ 1-\alpha-n, 1-\beta-n, \frac{3-\delta-\gamma-2n}{2}, \frac{2-\delta-\gamma-2n}{2}; \end{matrix} \; 1\right]. \end{aligned} \tag{37}$$

Applying Whipple transformation (9) to (37), with the aid of

$$(\lambda)_{m+n} = (\lambda)_m\, (\lambda+m)_n \quad (m, n \in \mathbb{Z}_{\geqslant 0}) \tag{38}$$

and (35), we obtain

$$\begin{aligned} \Xi_1(z) &= \sum_{n=0}^{\infty} \Psi(n) \frac{(\alpha)_n(\beta)_n(\gamma+\delta-1)_n\, z^n}{(\alpha+\beta)_n(\delta)_n\; n!} {}_4F_3\left[\begin{matrix} -n, \alpha, \beta, 1-\delta-n; \\ 1-\alpha-n, 1-\beta-n, \gamma; \end{matrix} \; 1\right] \\ &= \sum_{n=0}^{\infty} \Psi(n) \frac{(\alpha)_n(\beta)_n(\gamma+\delta-1)_n\, z^n}{(\alpha+\beta)_n(\delta)_n} \\ &\quad \times \sum_{m=0}^{n} \frac{(-1)^m\, (\alpha)_m(\beta)_m(1-\delta-n)_m}{(1-\alpha-n)_m(1-\beta-n)_m(\gamma)_m\; m!\, (n-m)!}. \end{aligned} \tag{39}$$

Finally, using the following double-series manipulation,

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n} \Phi(m,n) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \Phi(m, n+m), \tag{40}$$

where $\Phi : \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0} \to \mathbb{C}$ is a bounded function, and provided that both sides of the double series are absolutely convergent, and (34) on the right-hand side of (39), we prove (31). $\square$

Theorem 2. *Let $\{\Psi(\mu)\}_{\mu=0}^{\infty}$ be a bounded sequence of complex (or real) numbers such that $\Psi(0) \neq 0$. Additionally, let $\beta+\delta$, $\gamma \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}$. Then the following general double-series identity holds true:*

$$\begin{aligned} &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Psi(2m+n) \frac{(-1)^m\, (\alpha)_{m+n}(\beta)_m(\delta)_m(\gamma-\alpha)_m}{2^{2m}\, (\gamma)_m \left(\frac{\beta+\delta}{2}\right)_m \left(\frac{\beta+\delta+1}{2}\right)_m} \frac{z^{2m+n}}{m!\, n!} \\ &\quad = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Psi(m+n) \frac{(\gamma)_{m+n}(\alpha)_m(\beta)_m\, (\alpha)_n(\delta)_n}{(\beta+\delta)_{m+n}(\gamma)_m(\gamma)_n} \frac{z^{m+n}}{m!\, n!}, \end{aligned} \tag{41}$$

provided that both sides of the double series are absolutely convergent.

Proof. Let $\Xi_2(z)$ be on the right-hand side of (41). A similar process of the proof of Theorem 1 with the aid of the identities (32) and (34)–(36) gives

$$\Xi_2(z) = \sum_{n=0}^{\infty} \Psi(n) \frac{(\alpha)_n\, z^n}{n!}\; {}_5F_4\left[\begin{matrix} \frac{n}{2}, \frac{-n+1}{2}, \beta, \delta, \gamma-\alpha\,; \\ \gamma, 1-\alpha-n, \frac{\beta+\delta}{2}, \frac{\beta+\delta+1}{2}\,; \end{matrix} \; 1\right]. \tag{42}$$

Applying Whipple transformation (9) to the right-hand side of (42) with the aid of (34) ($k = n$), we find

$$\begin{aligned}\Xi_2(z) &= \sum_{n=0}^{\infty} \Psi(n) \frac{(\alpha)_n (\delta)_n\, z^n}{(\delta+\beta)_n n!}\; {}_4F_3\left[\begin{matrix} -n,\, \alpha,\, \beta,\, 1-\gamma-n\,; \\ \gamma,\, 1-\alpha-n,\, 1-\delta-n\,; \end{matrix} 1\right] \\ &= \sum_{n=0}^{\infty} \Psi(n) \frac{(\alpha)_n (\delta)_n\, z^n}{(\delta+\beta)_n\, n!} \sum_{m=0}^{n} \frac{(-n)_m (\alpha)_m (\beta)_m (1-\gamma-n)_m}{(1-\alpha-n)_m (1-\delta-n)_m (\gamma)_m\, m!}.\end{aligned} \tag{43}$$

Employing the double-series manipulation (40) to the last member of (43) and using (34) and (35) in the resultant expression, we obtain the desired identity (41). □

Theorem 3. *Let* $\{\Psi(\mu)\}_{\mu=0}^{\infty}$ *be a bounded sequence of complex (or real) numbers such that* $\Psi(0) \neq 0$*. Additionally, let* $\alpha+\lambda,\, \alpha+\sigma,\, \beta+\lambda,\, \gamma \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}$*. Then the following general double-series identity holds true:*

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Psi(m+n) \frac{(-1)^m\, (\alpha+\beta+\lambda+\sigma-1)_{2m+n} (\alpha)_m (\gamma-\beta)_m}{(\alpha+\sigma)_m (\alpha+\lambda)_m (\gamma)_m} \frac{z^{m+n}}{m!\, n!} \\ &\quad= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Psi(m+n) \frac{(\alpha+\beta+\lambda+\sigma-1)_{m+n} (\lambda+\beta)_{m+n} (\alpha+\beta-\gamma+\sigma)_m}{(\gamma)_{m+n} (\alpha+\lambda)_{m+n} (\alpha+\sigma)_m (\beta+\lambda)_n} \\ &\qquad\qquad \times (\alpha)_m\, (\gamma-\alpha)_n\, (\lambda)_n \frac{z^{m+n}}{m!\, n!},\end{aligned} \tag{44}$$

provided that both sides of the double series are absolutely convergent.

Proof. Let $\Xi_3(z)$ be the left-hand side of (44). Using the following double-series manipulation,

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Phi(m,n) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \Phi(m, n-m), \tag{45}$$

where $\Phi : \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0} \to \mathbb{C}$ is a bounded function, and provided that both sides of the double series are absolutely convergent, we obtain

$$\Xi_3(z) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \Psi(n) \frac{(-1)^m\, (\alpha+\beta+\lambda+\sigma-1)_{m+n} (\alpha)_m (\gamma-\beta)_m\, z^n}{(\alpha+\sigma)_m (\alpha+\lambda)_m (\gamma)_m\, m!\, (n-m)!}. \tag{46}$$

Applying (35) and (36) to (46) and denoting the resultant expression in terms of ${}_pF_q$ in (1), we derive

$$\begin{aligned}\Xi_3(z) &= \sum_{n=0}^{\infty} \Psi(n) \frac{(\alpha+\beta+\lambda+\sigma-1)_n\, z^n}{n!} \\ &\qquad \times {}_4F_3\left[\begin{matrix} -n,\, \alpha+\beta+\lambda+\sigma-1+n,\, \alpha,\, \gamma-\beta; \\ \gamma,\, \alpha+\lambda,\, \alpha+\sigma; \end{matrix} 1\right].\end{aligned} \tag{47}$$

Employing Whipple transformation (10) in (47), we find

$$\begin{aligned}\Xi_3(z) &= \sum_{n=0}^{\infty} \Psi(n) \frac{(\alpha+\beta+\lambda+\sigma-1)_n\, z^n}{n!} \frac{(\gamma-\alpha)_n (\lambda)_n}{(\alpha+\lambda)_n (\gamma)_n} \\ &\qquad \times {}_4F_3\left[\begin{matrix} -n,\, \alpha,\, 1-\beta-\lambda-n,\, \alpha+\beta-\gamma+\sigma; \\ 1+\alpha-\gamma-n,\, 1-\lambda-n,\, \alpha+\sigma; \end{matrix} 1\right] \\ &= \sum_{n=0}^{\infty} \Psi(n) \frac{(\lambda)_n (\gamma-\alpha)_n (\alpha+\beta+\lambda+\sigma-1)_n\, z^n}{(\alpha+\lambda)_n (\gamma)_n} \\ &\qquad \times \sum_{m=0}^{n} \frac{(-n)_m (\alpha)_m (1-\beta-\lambda-n)_m (\alpha+\beta-\gamma+\sigma)_m}{(1+\alpha-\gamma-n)_m (1-\lambda-n)_m (\alpha+\sigma)_m\, m!}.\end{aligned} \tag{48}$$

Using (35) for $(-n)_m$ in the double series in (48) and employing the following double-series manipulation,

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\Phi(m,n)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Phi(m,n+m), \tag{49}$$

where $\Phi:\mathbb{Z}_{\geqslant 0}\times\mathbb{Z}_{\geqslant 0}\to\mathbb{C}$ is a bounded function, and provided that both sides of the double series are absolutely convergent, we prove the desired identity (44). $\square$

3. Transforming Srivastava-Daoust Functions to Kampé de Fériet Function

This section establishes three main transformations between the Srivastava–Daoust function in (6) and Kampé de Fériet function in (3) by utilizing the results in Section 2.

Theorem 4. *The following transformation formulas hold true:*

$$F^{D+4:0;0}_{E+3:0;0}\left(\begin{matrix}[(d_D):2,1],[\alpha:1,1],[\beta:1,1],[\frac{\gamma+\delta-1}{2}:1,1],[\frac{\gamma+\delta}{2}:1,1]:\text{-};\text{-};\\ [(e_E):2,1],[\gamma:1,1],[\alpha+\beta:1,1],[\delta:1,1]:\text{-};\text{-};\end{matrix}-4z^2,4z\right)$$

$$=F^{D+1:2;2}_{E+1:1;1}\left[\begin{matrix}(d_D),\gamma+\delta-1: & \alpha,\beta; & \alpha,\beta;\\ (e_E),\alpha+\beta: & \gamma; & \delta;\end{matrix}\,z,\,z\right]; \tag{50}$$

$$F^{D+1:3;0}_{E:3;0}\left(\begin{matrix}[(d_D):2,1],[\alpha:1,1]: & [\beta:1],[\delta:1],[\gamma-\alpha:1];\text{-};\\ [(e_E):2,1]: & [\gamma:1],[\frac{\beta+\delta}{2}:1],[\frac{\beta+\delta+1}{2}:1];\text{-};\end{matrix}-\frac{z^2}{4},z\right)$$

$$=F^{D+1:2;2}_{E+1:1;1}\left[\begin{matrix}(d_D),\gamma: & \alpha,\beta; & \alpha,\delta;\\ (e_E),\delta+\beta: & \gamma; & \gamma;\end{matrix}\,z,\,z\right]; \tag{51}$$

$$F^{D+1:2;0}_{E:3;0}\left(\begin{matrix}[(d_D):1,1],[\alpha+\beta+\lambda+\sigma-1:2,1]: & [\alpha:1],[\delta:1],[\gamma-\beta:1];\text{-};\\ [(e_E):1,1]: & [\alpha+\lambda:1],[\alpha+\sigma:1],[\gamma:1];\text{-};\end{matrix}-z,z\right)$$

$$=F^{D+2:2;2}_{E+2:1;1}\left[\begin{matrix}(d_D),\alpha+\beta+\lambda+\sigma-1,\lambda+\beta: & \alpha+\beta-\gamma+\sigma,\alpha; & \lambda,\gamma-\alpha;\\ (e_E),\gamma,\alpha+\lambda: & \alpha+\sigma; & \lambda+\beta;\end{matrix}\,z,\,z\right], \tag{52}$$

where

$$z\in\mathbb{R}_{>0},\ \Re(\xi)>0;\ e_1,e_2,\ldots,e_E,\ \delta,\ \gamma,\ \alpha+\beta,\ \alpha+\lambda,\ \alpha+\sigma,\ \beta+\delta,\ \beta+\lambda\in\mathbb{C}\setminus\mathbb{Z}_{\leqslant 0},$$

provided that the other constraints for parameters and variable would follow from those in (3) and (6) so that the identities here are meaningful.

Proof. Setting

$$\Psi(\mu)=\frac{(d_1)_\mu(d_2)_\mu\ldots(d_D)_\mu}{(e_1)_\mu(e_2)_\mu\ldots(e_E)_\mu}=\frac{\prod_{j=1}^{D}(d_j)_\mu}{\prod_{j=1}^{E}(e_j)_\mu}\quad(\mu\in\mathbb{Z}_{\geqslant 0})$$

on both sides of the general double-series identity (31), we obtain

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{D}(d_j)_{2m+n}(\alpha)_{m+n}(\beta)_{m+n}\left(\frac{\gamma+\delta-1}{2}\right)_{m+n}\left(\frac{\gamma+\delta}{2}\right)_{m+n}(-4z^2)^m\,(4z)^n}{\prod_{j=1}^{E}(e_j)_{2m+n}(\gamma)_{m+n}(\alpha+\beta)_{m+n}(\delta)_{m+n}\,m!\,n!}$$
$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{D}(d_j)_{m+n}(\gamma+\delta-1)_{m+n}(\alpha)_m(\beta)_m\,(\alpha)_n(\beta)_n\,z^{m+n}}{\prod_{j=1}^{E}(e_j)_{m+n}(\alpha+\beta)_{m+n}(\gamma)_m(\delta)_n\,m!\,n!},$$

which, upon expressing in terms of the Srivastava–Daoust function (6) for its left side and Kampé de Fériet function (3) for its right side, leads to (50).

Likewise, identities (51) and (52) can be demonstrated, but specific details have been omitted. □

4. Application of Fractional Calculus

This section demonstrates that the identities presented in Sections 2 and 3 can be converted into one another by employing the Riemann–Liouville fractional integrals. To do this, recall the left-sided Riemann–Liouville fractional integral and its related formula (see, e.g., [63], Equations (2.2.1) and (2.2.10), respectively):

$$\left(I_{0+}^{\xi}f\right)(z):=\frac{1}{\Gamma(\xi)}\int_0^z\frac{f(t)}{(z-t)^{1-\xi}}\,\mathrm{d}t\quad(z\in\mathbb{R}_{>0};\ \Re(\xi)>0),\tag{53}$$

and

$$\left(I_{0+}^{\xi}t^{\eta-1}\right)(z)=\frac{\Gamma(\eta)}{\Gamma(\eta+\xi)}\,z^{\eta+\xi-1}\quad(z\in\mathbb{R}_{>0};\ \Re(\xi)>0,\ \Re(\eta)>0).\tag{54}$$

Replacing z by t in the identities in Theorems 1–3, and applying the left-sided Riemann-Liouville fractional integral (53) to both sides of the resultant identities, with the aid of (54), we obtain the following identities, respectively. Here, we provide only a detailed proof of Theorem 5.

Theorem 5. *Let $\{\Psi(\mu)\}_{\mu=0}^{\infty}$ be a bounded sequence of complex (or real) numbers such that $\Psi(0)\neq 0$. Additionally, let $\alpha+\beta,\ \gamma,\ \delta\in\mathbb{C}\setminus\mathbb{Z}_{\leqslant 0};\ z\in\mathbb{R}_{>0},\ \Re(\xi)>0$. Then the following general double-series identity holds true:*

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Psi(2m+n)\frac{(1)_{2m+n}\,(\alpha)_{m+n}(\beta)_{m+n}\left(\frac{\gamma+\delta-1}{2}\right)_{m+n}\left(\frac{\gamma+\delta}{2}\right)_{m+n}}{(\xi+1)_{2m+n}\,(\gamma)_{m+n}(\alpha+\beta)_{m+n}(\delta)_{m+n}}\frac{(-4z^2)^m(4z)^n}{m!\,n!}\\&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Psi(m+n)\frac{(1)_{m+n}\,(\gamma+\delta-1)_{m+n}(\alpha)_m(\beta)_m\,(\alpha)_n(\beta)_n}{(\xi+1)_{m+n}\,(\alpha+\beta)_{m+n}(\gamma)_m(\delta)_n}\frac{z^{m+n}}{m!\,n!},\end{aligned}\tag{55}$$

provided that both sides of the double series are absolutely convergent.

Proof. Replacing z by t in (31), we obtain

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Psi(2m+n)\frac{(\alpha)_{m+n}(\beta)_{m+n}\left(\frac{\gamma+\delta-1}{2}\right)_{m+n}\left(\frac{\gamma+\delta}{2}\right)_{m+n}}{(\gamma)_{m+n}(\alpha+\beta)_{m+n}(\delta)_{m+n}}\frac{(-4)^m4^n\,t^{2m+n}}{m!\,n!}\\&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Psi(m+n)\frac{(\gamma+\delta-1)_{m+n}(\alpha)_m(\beta)_m\,(\alpha)_n(\beta)_n}{(\alpha+\beta)_{m+n}(\gamma)_m(\delta)_n}\frac{t^{m+n}}{m!\,n!}.\end{aligned}\tag{56}$$

Applying the left-sided Riemann-Liouville fractional integral (53) to both sides of (56), we find

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Psi(2m+n)\frac{(\alpha)_{m+n}(\beta)_{m+n}\left(\frac{\gamma+\delta-1}{2}\right)_{m+n}\left(\frac{\gamma+\delta}{2}\right)_{m+n}}{(\gamma)_{m+n}(\alpha+\beta)_{m+n}(\delta)_{m+n}}\frac{(-4)^m4^n\left(I_{0+}^{\xi}t^{2m+n}\right)(z)}{m!\,n!}\\&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Psi(m+n)\frac{(\gamma+\delta-1)_{m+n}(\alpha)_m(\beta)_m\,(\alpha)_n(\beta)_n}{(\alpha+\beta)_{m+n}(\gamma)_m(\delta)_n}\frac{\left(I_{0+}^{\xi}t^{m+n}\right)(z)}{m!\,n!}.\end{aligned}\tag{57}$$

Using (54) in (57), we derive

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Psi(2m+n)\frac{(\alpha)_{m+n}(\beta)_{m+n}\left(\frac{\gamma+\delta-1}{2}\right)_{m+n}\left(\frac{\gamma+\delta}{2}\right)_{m+n}}{(\gamma)_{m+n}(\alpha+\beta)_{m+n}(\delta)_{m+n}}\frac{(-4)^m4^n\,z^{2m+n+\xi}\,\Gamma(2m+n+1)}{m!\,n!\,\Gamma(2m+n+1+\xi)}\\&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Psi(m+n)\frac{(\gamma+\delta-1)_{m+n}(\alpha)_m(\beta)_m\,(\alpha)_n(\beta)_n}{(\alpha+\beta)_{m+n}(\gamma)_m(\delta)_n}\frac{\Gamma(m+n+1)\,z^{m+n+\xi}}{\Gamma(m+n+1+\xi)\,m!\,n!}.\end{aligned}\tag{58}$$

Dividing both sides of (58) by z^{ξ} and using (2) in the resultant identity, we obtain the desired identity (55). $\square$

Theorem 6. *Let $\{\Psi(\mu)\}_{\mu=0}^{\infty}$ be a bounded sequence of complex (or real) numbers such that $\Psi(0)\neq 0$. Additionally, let $\beta+\delta$, $\gamma\in\mathbb{C}\setminus\mathbb{Z}_{\leqslant 0}$; $z\in\mathbb{R}_{>0}$, $\Re(\xi)>0$. Then the following general double-series identity holds true:*

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Psi(2m+n)\frac{(-1)^m\,(1)_{2m+n}\,(\alpha)_{m+n}(\beta)_m(\delta)_m(\gamma-\alpha)_m}{2^{2m}\,(\xi+1)_{2m+n}\,(\gamma)_m\left(\frac{\beta+\delta}{2}\right)_m\left(\frac{\beta+\delta+1}{2}\right)_m}\frac{z^{2m+n}}{m!\,n!}\\&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Psi(m+n)\frac{(1)_{m+n}\,(\gamma)_{m+n}(\alpha)_m(\beta)_m\,(\alpha)_n(\delta)_n}{(\xi+1)_{m+n}\,(\beta+\delta)_{m+n}(\gamma)_m(\gamma)_n}\frac{z^{m+n}}{m!\,n!},\end{aligned}\tag{59}$$

provided that both sides of the double series are absolutely convergent.

Theorem 7. *Let $\{\Psi(\mu)\}_{\mu=0}^{\infty}$ be a bounded sequence of complex (or real) numbers such that $\Psi(0)\neq 0$. Additionally, let $\alpha+\lambda$, $\alpha+\sigma$, $\beta+\lambda$, $\gamma\in\mathbb{C}\setminus\mathbb{Z}_{\leqslant 0}$; $z\in\mathbb{R}_{>0}$, $\Re(\xi)>0$. Then the following general double-series identity holds true:*

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Psi(m+n)\frac{(-1)^m\,(1)_{m+n}\,(\alpha+\beta+\lambda+\sigma-1)_{2m+n}(\alpha)_m(\gamma-\beta)_m}{(\xi+1)_{m+n}\,(\alpha+\sigma)_m(\alpha+\lambda)_m(\gamma)_m}\frac{z^{m+n}}{m!\,n!}\\&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Psi(m+n)\frac{(1)_{m+n}\,(\alpha+\beta+\lambda+\sigma-1)_{m+n}(\lambda+\beta)_{m+n}(\alpha+\beta-\gamma+\sigma)_m}{(\xi+1)_{m+n}\,(\gamma)_{m+n}(\alpha+\lambda)_{m+n}(\alpha+\sigma)_m(\beta+\lambda)_n}\\&\qquad\times(\alpha)_m\,(\gamma-\alpha)_n\,(\lambda)_n\,\frac{z^{m+n}}{m!\,n!},\end{aligned}\tag{60}$$

provided that both sides of the double series are absolutely convergent.

By employing the identical procedure used to derive the identities in Theorem 4, we extend our analysis to the outcomes presented in Theorems 5–7, leading to the subsequent theorem.

Theorem 8. *The following transformation formulas hold true:*

$$F^{D+5:0;0}_{E+4:0;0}\left(\begin{matrix}[(d_D):2,1],[1:2,1],[\alpha:1,1],[\beta:1,1],[\frac{\gamma+\delta-1}{2}:1,1],[\frac{\gamma+\delta}{2}:1,1]:\text{-};\text{-};\\ [\xi+1:2,1],[(e_E):2,1],[\gamma:1,1],[\alpha+\beta:1,1],[\delta:1,1]:\text{-};\text{-};\end{matrix}-4z^2,4z\right)$$

$$= F^{D+2:2;2}_{E+2:1;1}\left[\begin{array}{c} (d_D), 1, \gamma+\delta-1: \quad \alpha,\beta; \quad \alpha,\beta; \\ \xi+1, (e_E), \alpha+\beta: \quad \gamma; \quad \delta; \end{array} z, z\right]; \tag{61}$$

$$F^{D+2:3;0}_{E+1:3;0}\left(\begin{array}{c}[(d_D):2,1],[1:2,1],[\alpha:1,1]: \quad [\beta:1],[\delta:1],[\gamma-\alpha:1]\,;\,\text{-}\,; \\ [\xi+1:2,1],[(e_E):2,1]: \ [\gamma:1],[\frac{\beta+\delta}{2}:1],[\frac{\beta+\delta+1}{2}:1]\,;\,\text{-}\,;\end{array} -\frac{z^2}{4}, z\right)$$

$$= F^{D+2:2;2}_{E+2:1;1}\left[\begin{array}{c} (d_D), 1, \gamma: \quad \alpha,\beta; \quad \alpha,\delta; \\ \xi+1, (e_E), \delta+\beta: \quad \gamma; \quad \gamma; \end{array} z, z\right]; \tag{62}$$

$$F^{D+2:3;0}_{E+1:3;0}\left(\begin{array}{c}[(d_D):1,1],[1:1,1],[\alpha+\beta+\lambda+\sigma-1:2,1]: \quad [\alpha:1],[\delta:1],[\gamma-\beta:1]\,;\,\text{-}\,; \\ [\xi+1:1,1],[(e_E):1,1]: \ [\alpha+\lambda:1],[\alpha+\sigma:1],[\gamma:1]\,;\,\text{-}\,;\end{array} -z, z\right)$$

$$= F^{D+3:2;2}_{E+3:1;1}\left[\begin{array}{c} (d_D), 1, \alpha+\beta+\lambda+\sigma-1, \lambda+\beta: \ \alpha+\beta-\gamma+\sigma, \alpha; \ \lambda, \gamma-\alpha; \\ \xi+1, (e_E), \gamma, \alpha+\lambda: \quad \alpha+\sigma; \quad \lambda+\beta; \end{array} z, z\right], \tag{63}$$

where

$$z \in \mathbb{R}_{>0},\ \Re(\xi) > 0;\ e_1, e_2, \ldots, e_E,\ \delta,\ \gamma,\ \alpha+\beta,\ \alpha+\lambda,\ \alpha+\sigma,\ \beta+\delta,\ \beta+\lambda \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0},$$

provided that the other constraints for parameters and variable would follow from those in (3) *and* (6) *so that the identities here are meaningful.*

5. Certain Instances of Transformations (50)–(52)

This section demonstrates that certain special cases of transformations (50)–(52) result in the Bailey quadratic transformation, Clausen reduction formula, Gauss quadratic transformation, Karlsson reduction formula, Orr reduction formula, Whipple quadratic transformation, and several new transformations, which are given in the following examples.

Example 1. *Putting $D = E = 0$ and $\delta = \alpha+\beta-\gamma+1$ in* (50) *and using the double-series manipulation (see, e.g., [64], p. 4, Equation (12))*

$$\sum_{m,n=0}^{\infty} \Phi(m+n)\frac{x^m y^n}{m!\,n!} = \sum_{p=0}^{\infty} \Phi(p)\frac{(x+y)^p}{p!}, \tag{64}$$

where $\Phi : \mathbb{Z}_{\geqslant 0} \to \mathbb{C}$ is a bounded function, and provided that both sides of the series are absolutely convergent, we obtain a product formula for ${}_pF_q$:

$$\begin{aligned}&{}_4F_3\left[\begin{array}{c}\alpha,\ \beta,\ \frac{\alpha+\beta}{2},\ \frac{\alpha+\beta+1}{2}; \\ \gamma,\ \alpha+\beta,\ \alpha+\beta-\gamma+1;\end{array} 4z(1-z)\right] \\ &\quad = {}_2F_1\left[\begin{array}{c}\alpha,\ \beta; \\ \gamma;\end{array} z\right]{}_2F_1\left[\begin{array}{c}\alpha,\ \beta; \\ \alpha+\beta-\gamma+1;\end{array} z\right]\end{aligned} \tag{65}$$

$$(\alpha+\beta,\ \alpha+\beta-\gamma+1,\ \gamma \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0};\ |z| < 1,\ 4|z(1-z)| < 1).$$

Identity (65) *is due to Bailey ([56], p. 382, Equation (6.1)) (see also ([4], p. 275, Prob. 8)).*

Setting $\gamma = \frac{\alpha+\beta+1}{2}$ in (65) *and replacing α and β by 2α and 2β gives a formula for the square of ${}_2F_1$:*

$${}_3F_2\left[\begin{array}{c}2\alpha,\ 2\beta,\ \alpha+\beta; \\ 2\alpha+2\beta,\ \alpha+\beta+\frac{1}{2};\end{array} 4z(1-z)\right] = \left\{{}_2F_1\left[\begin{array}{c}2\alpha,\ 2\beta; \\ \alpha+\beta+\frac{1}{2};\end{array} z\right]\right\}^2, \tag{66}$$

which, upon using Gauss transformation Formula (17)*, yields*

$${}_3F_2\left[\begin{array}{c}2\alpha,\ 2\beta,\ \alpha+\beta; \\ 2\alpha+2\beta,\ \alpha+\beta+\frac{1}{2};\end{array} 4z(1-z)\right] = \left\{{}_2F_1\left[\begin{array}{c}\alpha,\ \beta; \\ \alpha+\beta+\frac{1}{2};\end{array} 4z(1-z)\right]\right\}^2. \tag{67}$$

Replacing $4z(1-z)$ *by* z *in* (67) *yields the well-known Clausen formula in* [65] *(see, e.g.,* [2]*, p. 86, Equation (4),* [50]*, p. 75, Equation (2.5.7)).*

Putting $\alpha = a$, $\beta = \delta = b$, *and* $\gamma = a + b + \frac{1}{2}$ *in* (88)*, and using the procedure illustrated in Example* 13*, we also acquire the Clausen formula.*

Example 2. *Putting* $D = E = 0$ *and* $\gamma = \delta = \alpha$ *in* (50) *and using* (64)*, we obtain*

$$F^{1:1;1}_{1:0;0}\left[\begin{array}{c} 2\alpha - 1 : \ \beta ; \ \beta ; \\ \alpha + \beta : \ - ; \ - ; \end{array} z, z\right] = {}_2F_1\left[\begin{array}{c} \alpha - \frac{1}{2}, \ \beta ; \\ \alpha + \beta ; \end{array} 4z(1-z)\right] \tag{68}$$

$$(\alpha + \beta \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}; \ |z| < 1, \ 4|z(1-z)| < 1).$$

Applying the reduction Formula (21) *to the left-hand side of* (68)*, we obtain*

$${}_2F_1\left[\begin{array}{c} 2\alpha - 1, \ 2\beta ; \\ \alpha + \beta ; \end{array} z\right] = {}_2F_1\left[\begin{array}{c} \alpha - \frac{1}{2}, \ \beta ; \\ \alpha + \beta ; \end{array} 4z(1-z)\right] \tag{69}$$

$$(\alpha + \beta \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}; \ |z| < 1, \ 4|z(1-z)| < 1),$$

which, upon replacing α *by* $\alpha + \frac{1}{2}$*, corresponds to Gauss transformation Formula* (17)*.*

Example 3. *Putting* $D = E = 0$ *and* $\gamma = \alpha$ $\delta = \beta$ *in* (50) *and using* (64) *gives the first equality of the identity*

$$\begin{aligned} F^{1:1;1}_{1:0;0}\left[\begin{array}{c} \alpha + \beta - 1 : \ \beta ; \ \alpha ; \\ \alpha + \beta : \ - ; \ - ; \end{array} z, z\right] &= {}_2F_1\left[\begin{array}{c} \frac{\alpha+\beta-1}{2}, \ \frac{\alpha+\beta}{2} ; \\ \alpha + \beta ; \end{array} 4z(1-z)\right] \\ &= (1-z)^{1-\alpha-\beta} \end{aligned} \tag{70}$$

$$(\alpha + \beta \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}; \ |z| < 1, \ 4|z(1-z)| < 1).$$

Applying the reduction Formula (21) *to the leftmost member of* (70) *yields the second equality of* (70)*. Interestingly, the identity where* $\beta = \alpha + 1$ *in* [48] *(Equation (2.10)) is equivalent to the second equality of* (70)*.*

Example 4. *Putting* $D = E = 0$ *and* $\delta = 2\alpha + 2\beta - \gamma + 1$ *in* (50) *and using* (64)*, we acquire a reduction formula for the Kampé de Fériet function in* (3)*:*

$$\begin{aligned} F^{1:2;2}_{1:1;1}\left[\begin{array}{c} 2\alpha + 2\beta : \ \alpha, \ \beta ; \qquad \alpha, \ \beta ; \\ \alpha + \beta : \ \gamma ; \ 2\alpha + 2\beta - \gamma + 1 ; \end{array} z, z\right] \\ = {}_3F_2\left[\begin{array}{c} \alpha, \ \beta, \ \alpha + \beta + \frac{1}{2} ; \\ \gamma, \ 2\alpha + 2\beta - \gamma + 1 ; \end{array} 4z(1-z)\right] \end{aligned} \tag{71}$$

$$(\alpha + \beta, \ \gamma, \ 2\alpha + 2\beta - \gamma + 1 \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}; \ |z| < 1, \ 4|z(1-z)| < 1).$$

Example 5. *Putting* $D = E = 0$ *and* $\gamma = \beta$ *in* (50) *and using* (64)*, we attain a reduction formula for the Kampé de Fériet function in* (3)*:*

$$F^{1:1;2}_{1:0;1}\left[\begin{array}{c} \beta + \delta - 1 : \ \alpha ; \ \alpha, \ \beta ; \\ \alpha + \beta : \ - ; \ \delta ; \end{array} z, z\right] = {}_3F_2\left[\begin{array}{c} \alpha, \ \frac{\beta+\delta}{2}, \ \frac{\beta+\delta-1}{2} ; \\ \alpha + \beta, \ \delta ; \end{array} 4z(1-z)\right] \tag{72}$$

$$(\alpha + \beta, \ \delta \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}; \ |z| < 1, \ 4|z(1-z)| < 1).$$

Setting $\delta = \alpha$ *in* (72) *leads to identity* (70)*.*

Example 6. *Putting $D = E = 0$ and $\delta = \gamma + 1$ in (50) and using (64), we gain a reduction formula for the Kampé de Fériet function in (3):*

$$F^{1:2;2}_{1:1;1}\left[\begin{array}{c} 2\gamma: \ \alpha,\ \beta; \quad \alpha,\ \beta; \\ \alpha+\beta: \quad \gamma; \quad \gamma+1; \end{array} z, z\right] = {}_3F_2\left[\begin{array}{c} \alpha,\ \beta,\ \gamma+\frac{1}{2}; \\ \alpha+\beta,\ \gamma+1; \end{array} 4z(1-z)\right] \tag{73}$$

$$(\alpha+\beta,\ \gamma \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0};\ |z|<1,\ 4|z(1-z)|<1).$$

Setting $\gamma = \alpha + \beta - \frac{1}{2}$ in (73) and using a particular case (Table 3, 1a) of the known general reduction formula $F^{p:2;2}_{q:1;1}[z,z] = {}_{p+3}F_{q+2}[z]$ in [30], Equation (3.2), we derive the following transformation formula:

$${}_3F_2\left[\begin{array}{c} 2\alpha,\ 2\beta,\ 2\alpha+\beta-1; \\ \alpha+\beta+\frac{1}{2},\ 2\alpha+2\beta-1; \end{array} z\right] = {}_2F_1\left[\begin{array}{c} \alpha,\ \beta; \\ \alpha+\beta+\frac{1}{2}; \end{array} 4z(1-z)\right] \tag{74}$$

$$\left(\alpha+\beta+\tfrac{1}{2},\ 2\alpha+2\beta-1 \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0};\ |z|<1,\ 4|z(1-z)|<1\right).$$

Example 7. *Putting $D = E = 0$ and $\delta = \gamma - \beta$ and using the binomial theorem (11), we obtain a product formula of ${}_2F_1$'s:*

$$\begin{aligned}(1-z)^{-\alpha}\,{}_4F_3&\left[\begin{array}{c} \alpha,\ \beta,\ \gamma-\alpha,\ \gamma-\beta; \\ \gamma,\ \frac{\gamma}{2},\ \frac{\gamma+1}{2}; \end{array} \frac{-z^2}{4(1-z)}\right] \\ &= {}_2F_1\left[\begin{array}{c} \alpha,\ \beta; \\ \gamma; \end{array} z\right] {}_2F_1\left[\begin{array}{c} \alpha,\ \gamma-\beta; \\ \gamma; \end{array} z\right]\end{aligned} \tag{75}$$

$$\left(\gamma \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0};\ |z|<1,\ |z|^2/\{4|1-z|\}<1\right),$$

which is a known formula due to Bailey [56], p. 382, Equation (6.3).

Applying Pfaff–Kummer transformation Formula (13) to the second ${}_2F_1$ on the right-hand side of (75), we obtain a product formula of ${}_2F_1$'s:

$$\begin{aligned}{}_4F_3&\left[\begin{array}{c} \alpha,\ \beta,\ \gamma-\alpha,\ \gamma-\beta; \\ \gamma,\ \frac{\gamma}{2},\ \frac{\gamma+1}{2}; \end{array} \frac{-z^2}{4(1-z)}\right] \\ &= {}_2F_1\left[\begin{array}{c} \alpha,\ \beta; \\ \gamma; \end{array} z\right] {}_2F_1\left[\begin{array}{c} \alpha,\ \beta; \\ \gamma; \end{array} \frac{-z}{1-z}\right]\end{aligned} \tag{76}$$

$$\left(\gamma \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0};\ |z|<1,\ |z|^2/\{4|1-z|\}<1,\ |z|/|1-z|<1\right),$$

which is another known formula due to Bailey [56], p. 383, Equation (7.2).

Example 8. *Putting $D = E = 0$ and $\gamma = \beta$ in (51) and using the binomial theorem (11), we deduce a reduction formula for the Kampé de Fériet function in (3):*

$$F^{1:1;2}_{1:0;1}\left[\begin{array}{c} \beta: \quad \alpha; \quad \alpha,\ \delta; \\ \beta+\delta: \quad -; \quad \beta; \end{array} z, z\right] = (1-z)^{-\alpha}\,{}_3F_2\left[\begin{array}{c} \alpha,\ \beta-\alpha,\ \delta; \\ \frac{\beta+\delta}{2},\ \frac{\beta+\delta+1}{2}; \end{array} \frac{-z^2}{4(1-z)}\right] \tag{77}$$

$$\left(\beta,\ \beta+\delta \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0};\ |z|<1,\ |z|^2/\{4|1-z|\}<1,\ |\arg(1-z)|<\pi\right).$$

Setting $\delta = \beta - 2\alpha$ in (77) and, via (5), using (22) in the resultant identity, we obtain a transformation formula for ${}_2F_1$:

$${}_2F_1\left[\begin{array}{c} 2\alpha,\ \beta-\alpha; \\ 2\beta-2\alpha; \end{array} z\right] = (1-z)^{-\alpha}\,{}_2F_1\left[\begin{array}{c} \alpha,\ \beta-2\alpha; \\ \beta-\alpha+\frac{1}{2}; \end{array} \frac{-z^2}{4(1-z)}\right] \tag{78}$$

$$\left(2\beta-2\alpha,\ \beta-\alpha+\tfrac{1}{2} \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0};\ |z|<1,\ |z|^2/\{4|1-z|\}<1,\ |\arg(1-z)|<\pi\right),$$

which is a particular case of Bailey transformation Formula (16).

In view of (4), *the reduction Formula* (77) *equals*

$$F^{1:2;1}_{1:1;0}\left[\begin{array}{r} \beta: \\ \beta+\delta: \end{array}\begin{array}{r} \alpha,\ \delta; \\ \beta; \end{array}\begin{array}{r} \alpha; \\ -; \end{array}\ z,z\right] = (1-z)^{-\alpha}\,{}_3F_2\left[\begin{array}{r} \alpha,\ \beta-\alpha,\ \delta; \\ \frac{\beta+\delta}{2},\ \frac{\beta+\delta+1}{2}; \end{array}\ \frac{-z^2}{4(1-z)}\right] \tag{79}$$

$$\Big(\beta,\ \beta+\delta \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0};\ |z|<1,\ |z|^2/\{4|1-z|\}<1,\ |\arg(1-z)|<\pi\Big),$$

which is interesting to compare with the following known formula (see [27], Equation (2.2)):

$$F^{1:2;1}_{1:1;0}\left[\begin{array}{l} \alpha: \\ \beta: \end{array}\begin{array}{r} \beta-\epsilon,\ \gamma; \\ \delta; \end{array}\begin{array}{r} \epsilon; \\ -; \end{array}\ z,z\right] = (1-z)^{-\alpha}\,{}_3F_2\left[\begin{array}{r} \alpha,\ \beta-\epsilon,\ \delta-\epsilon; \\ \beta,\ \delta; \end{array}\ \frac{-z}{1-z}\right] \tag{80}$$

$$(\beta,\ \delta \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0};\ |z|<1,\ |z|/|1-z|<1,\ |\arg(1-z)|<\pi).$$

Example 9. *Putting $D = E = 0$ and $\delta = \beta$ in* (51) *and using the binomial theorem* (11), *we obtain a reduction formula for the Kampé de Fériet function in* (3)*:*

$$F^{1:2;2}_{1:1;1}\left[\begin{array}{r} \gamma: \\ 2\beta: \end{array}\begin{array}{r} \alpha,\ \beta; \\ \gamma; \end{array}\begin{array}{r} \alpha,\ \beta; \\ \gamma; \end{array}\ z,z\right] = (1-z)^{-\alpha}\,{}_3F_2\left[\begin{array}{r} \alpha,\ \gamma-\alpha,\ \beta; \\ \gamma,\ \beta+\frac{1}{2}; \end{array}\ \frac{-z^2}{4(1-z)}\right] \tag{81}$$

$$\Big(2\beta,\ \beta+\tfrac{1}{2},\ \gamma \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0};\ |z|<1,\ |z|^2/\{4|1-z|\}<1,\ |\arg(1-z)|<\pi\Big).$$

Setting $\gamma = \beta$ in (81) *and the reduction formula of the Kampé de Fériet function* (21) *to the right-hand side of transformation* (81) *gives Bailey transformation Formula* (16).

Additionally, as in obtaining (74), *setting $\gamma = \alpha + \beta + \frac{1}{2}$ in* (81), *and using a particular case (Table 3, 1e) of the known general reduction formula ([30], Equation (3.2)), we obtain the following transformation formula:*

$${}_2F_1\left[\begin{array}{r} 2\alpha,\ \alpha+\beta; \\ 2\alpha+2\beta; \end{array}\ z\right] = (1-z)^{-\alpha}\,{}_2F_1\left[\begin{array}{r} \alpha,\ \beta; \\ \alpha+\beta+\frac{1}{2}; \end{array}\ \frac{-z^2}{4(1-z)}\right] \tag{82}$$

$$\Big(\alpha+\beta+\tfrac{1}{2},\ 2\alpha+2\beta \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0};\ |z|<1,\ |z|^2/\{4|1-z|\}<1,\ |\arg(1-z)|<\pi\Big).$$

Example 10. *Putting $D = E = 0$ and $\delta = 2\gamma - 2\alpha - \beta$ and using the binomial theorem* (11), *we attain a reduction formula for the Kampé de Fériet function in* (3)*:*

$$\begin{aligned} &F^{1:2;2}_{1:1;1}\left[\begin{array}{r} \gamma: \\ 2\gamma-2\alpha: \end{array}\begin{array}{r} \alpha,\ \beta; \\ \gamma; \end{array}\begin{array}{r} \alpha,\ 2\gamma-2\alpha-\beta; \\ \gamma; \end{array}\ z,z\right] \\ &\quad = (1-z)^{-\alpha}\,{}_3F_2\left[\begin{array}{r} \alpha,\ \beta,\ 2\gamma-2\alpha-\beta; \\ \gamma,\ \gamma-\alpha+\frac{1}{2}; \end{array}\ \frac{-z^2}{4(1-z)}\right] \end{aligned} \tag{83}$$

$$\Big(2\gamma-2\alpha,\ \gamma-\alpha+\tfrac{1}{2},\ \gamma \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0};\ |z|<1,\ |z|^2/\{4|1-z|\}<1,\ |\arg(1-z)|<\pi\Big).$$

Example 11. *Putting $D = E = 0$ and $\delta = \beta - 1$ in* (51) *and using the binomial theorem* (11), *we acquire a reduction formula for the Kampé de Fériet function in* (3)*:*

$$\begin{aligned} &F^{1:2;2}_{1:1;1}\left[\begin{array}{r} \gamma: \\ 2\beta-1: \end{array}\begin{array}{r} \alpha,\ \beta; \\ \gamma; \end{array}\begin{array}{r} \alpha,\ \beta-1; \\ \gamma; \end{array}\ z,z\right] \\ &\quad = (1-z)^{-\alpha}\,{}_3F_2\left[\begin{array}{r} \alpha,\ \beta-1,\ \gamma-\alpha; \\ \gamma,\ \beta-\frac{1}{2}; \end{array}\ \frac{-z^2}{4(1-z)}\right] \end{aligned} \tag{84}$$

$$\Big(\beta-\tfrac{1}{2},\ \gamma \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0};\ |z|<1,\ |z|^2/\{4|1-z|\}<1,\ |\arg(1-z)|<\pi\Big).$$

Setting $\gamma = 2\beta - 1$ in (84) and using the reduction Formula (20), we obtain a product formula for ${}_2F_1$'s:

$$(1-z)^{-\alpha}\,{}_3F_2\left[\begin{matrix} \alpha,\ \beta-1,\ 2\beta-\alpha-1; \\ 2\beta-1,\ \beta-\frac{1}{2}; \end{matrix}\ \frac{-z^2}{4(1-z)}\right] = {}_2F_1\left[\begin{matrix} \alpha,\ \beta; \\ 2\beta-1; \end{matrix}\ z\right]{}_2F_1\left[\begin{matrix} \alpha,\ \beta-1; \\ 2\beta-1; \end{matrix}\ z\right] \tag{85}$$

$$\left(\beta-\tfrac{1}{2}\in\mathbb{C}\setminus\mathbb{Z}_{\leqslant 0};\ |z|<1,\ |z|^2/\{4|1-z|\}<1,\ |\arg(1-z)|<\pi\right).$$

Example 12. *Putting $D = E = 0$ and $\delta = 2\gamma - 2\alpha - \beta - 1$ in (51) and using the binomial theorem (11), we gain a reduction formula for the Kampé de Fériet function in (3):*

$$F^{1:2;2}_{1:1;1}\left[\begin{matrix} \gamma: & \alpha,\ \beta; & \alpha,\ 2\gamma-2\alpha-\beta-1; \\ 2\gamma-2\alpha-1: & \gamma; & \gamma; \end{matrix}\ z,z\right] = (1-z)^{-\alpha}\,{}_3F_2\left[\begin{matrix} \alpha,\ \beta,\ 2\gamma-2\alpha-\beta-1; \\ \gamma,\ \gamma-\alpha-\frac{1}{2}; \end{matrix}\ \frac{-z^2}{4(1-z)}\right] \tag{86}$$

$$\left(\gamma,\ \gamma-\alpha-\tfrac{1}{2}\in\mathbb{C}\setminus\mathbb{Z}_{\leqslant 0};\ |z|<1,\ |z|^2/\{4|1-z|\}<1,\ |\arg(1-z)|<\pi\right).$$

Setting $\gamma = 2\alpha + 1$ in (86) and using the reduction Formula (20), we deduce a product formula for ${}_2F_1$'s:

$$(1-z)^{-\alpha}\,{}_3F_2\left[\begin{matrix} \alpha,\ \beta,\ 2\alpha-\beta+1; \\ 2\alpha+1,\ \alpha+\frac{1}{2}; \end{matrix}\ \frac{-z^2}{4(1-z)}\right] = {}_2F_1\left[\begin{matrix} \alpha,\ \beta; \\ 2\alpha+1; \end{matrix}\ z\right]{}_2F_1\left[\begin{matrix} \alpha,\ 2\alpha-\beta+1; \\ 2\alpha+1; \end{matrix}\ z\right] \tag{87}$$

$$\left(\alpha+\tfrac{1}{2}\in\mathbb{C}\setminus\mathbb{Z}_{\leqslant 0};\ |z|<1,\ |z|^2/\{4|1-z|\}<1,\ |\arg(1-z)|<\pi\right).$$

Example 13. *Putting $D = E = 1$, $d_1 = \delta + \beta$, and $e_1 = \gamma$ in (51) and using the reduction Formula (20), we obtain a reduction formula for the Srivastava–Daoust function in (6):*

$$F^{2:3;0}_{1:3;0}\left(\begin{matrix} [\delta+\beta:2,1],\ [\alpha:1,1]: & [\beta:1],\ [\delta:1],\ [\gamma-\alpha:1]; & -; \\ [\gamma:2,1]: & [\gamma:1],\ [\frac{\beta+\delta}{2}:1],\ [\frac{\beta+\delta+1}{2}:1]; & -; \end{matrix}\ -\frac{z^2}{4},z\right) = {}_2F_1\left[\begin{matrix} \alpha,\ \beta; \\ \gamma; \end{matrix}\ z\right]{}_2F_1\left[\begin{matrix} \alpha,\ \delta; \\ \gamma; \end{matrix}\ z\right]. \tag{88}$$

Setting $\alpha = a$, $\beta = b$, $\delta = b - 1$, and $\gamma = a + b - \frac{1}{2}$ in (88) gives

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(2b-1)_{2m+n}(a)_{m+n}(b-1)_m(-1)^m(z)^{2m+n}}{\left(a+b-\frac{1}{2}\right)_{2m+n}\left(a+b-\frac{1}{2}\right)_m(4)^m m!n!} = {}_2F_1\left[\begin{matrix} a,\ b; \\ a+b-\frac{1}{2}; \end{matrix}\ z\right]{}_2F_1\left[\begin{matrix} a,\ b-1; \\ a+b-\frac{1}{2}; \end{matrix}\ z\right],$$

which, upon utilizing the double-series manipulation (32) and then the Pochhammer symbol identity (34), yields

$$\sum_{n=0}^{\infty}\frac{(2b-1)_n(a)_n(z)^n}{\left(a+b-\frac{1}{2}\right)_n n!}\,{}_3F_2\left[\begin{matrix} -\frac{n}{2},\ \frac{-n+1}{2},\ b-1; \\ a+b-\frac{1}{2},\ 1-a-n; \end{matrix}\ 1\right] = {}_2F_1\left[\begin{matrix} a,\ b; \\ a+b-\frac{1}{2}; \end{matrix}\ z\right]{}_2F_1\left[\begin{matrix} a,\ b-1; \\ a+b-\frac{1}{2}; \end{matrix}\ z\right]. \tag{89}$$

Applying the summation theorem for $_3F_2(1)$ (24) to the $_3F_2(1)$ in (89), we attain a product formula for $_2F_1$'s:

$$ {}_3F_2\left[\begin{matrix} 2a,\ 2b-1,\ a+b-1; \\ 2a+2b-2,\ a+b-\frac{1}{2}; \end{matrix}\ z\right] = {}_2F_1\left[\begin{matrix} a,\ b; \\ a+b-\frac{1}{2}; \end{matrix}\ z\right] {}_2F_1\left[\begin{matrix} a,\ b-1; \\ a+b-\frac{1}{2}; \end{matrix}\ z\right] \tag{90} $$

$$ \left(2a+2b-2,\ a+b-\tfrac{1}{2} \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0};\ |z|<1\right), $$

which is due to Orr [66] (see also [50], p. 77, Equation (2.5.13)).

Example 14. *Putting $\alpha = a+\frac{1}{2}$, $\beta = b-\frac{1}{2}$, $\delta = b+\frac{1}{2}$, and $\gamma = a+b+\frac{1}{2}$ in (88) and performing the identical procedure as demonstrated in Example 13, we arrive at a known formula (see [30], p. 34, Table 3(Id)):*

$$ {}_3F_2\left[\begin{matrix} 2a,\ 2b+1,\ a+b; \\ 2a+2b,\ a+b+\frac{1}{2}; \end{matrix}\ z\right] = {}_2F_1\left[\begin{matrix} a+\frac{1}{2},\ b-\frac{1}{2}; \\ a+b+\frac{1}{2}; \end{matrix}\ z\right] {}_2F_1\left[\begin{matrix} a+\frac{1}{2},\ b+\frac{1}{2}; \\ a+b+\frac{1}{2}; \end{matrix}\ z\right] \tag{91} $$

$$ \left(2a+2b,\ a+b+\tfrac{1}{2} \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0};\ |z|<1\right). $$

Example 15. *Putting $\alpha = b$, $\beta = a+1$, $\delta = a$, and $\gamma = a+b+\frac{1}{2}$ in (88) and following the same procedure shown in Example 13, we obtain a known formula (see [30], p. 34, Table 3(Ic)):*

$$ {}_3F_2\left[\begin{matrix} 2b,\ 2a+1,\ a+b; \\ 2a+2b,\ a+b+\frac{1}{2}; \end{matrix}\ z\right] = {}_2F_1\left[\begin{matrix} a+1,\ b; \\ a+b+\frac{1}{2}; \end{matrix}\ z\right] {}_2F_1\left[\begin{matrix} a,\ b; \\ a+b+\frac{1}{2}; \end{matrix}\ z\right] \tag{92} $$

$$ \left(2a+2b,\ a+b+\tfrac{1}{2} \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0};\ |z|<1\right). $$

Example 16. *Putting $D = E = 0$, $\beta = \alpha$, and $\gamma = 2\alpha+\lambda+\sigma-1$ in (52) and using the binomial theorem (11), we attain a product formula for $_2F_1$'s:*

$$ \begin{aligned} &(1-z)^{1-2\alpha-\lambda-\sigma}{}_4F_3\left[\begin{matrix} \alpha,\ \alpha+\lambda+\sigma-1,\ \frac{2\alpha+\lambda+\sigma-1}{2},\ \frac{2\alpha+\lambda+\sigma}{2}; \\ \alpha+\sigma,\ \alpha+\lambda,\ 2\alpha+\lambda+\sigma-1; \end{matrix}\ \frac{-4z}{(1-z)^2}\right] \\ &\qquad = {}_2F_1\left[\begin{matrix} 1-\lambda,\ \alpha; \\ \alpha+\sigma; \end{matrix}\ z\right] {}_2F_1\left[\begin{matrix} \lambda,\ \alpha+\lambda+\sigma-1; \\ \alpha+\lambda; \end{matrix}\ z\right] \end{aligned} \tag{93} $$

$$ \begin{aligned} &\big(\alpha+\sigma,\ \alpha+\lambda,\ 2\alpha+\lambda+\sigma-1 \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0}; \\ &\qquad |z|<1,\ |z|^2/\{4|1-z|\}<1,\ |\arg(1-z)|<\pi\big). \end{aligned} $$

Example 17. *Putting $D = E = 0$, $\gamma = \lambda+\beta$, and $\sigma = 1-\beta$ and using the binomial theorem (11), we obtain a product formula for $_2F_1$'s:*

$$ \begin{aligned} &(1-z)^{-\alpha-\lambda}{}_4F_3\left[\begin{matrix} \alpha,\ \lambda,\ \frac{\alpha+\lambda}{2},\ \frac{\alpha+\lambda+1}{2}; \\ \alpha-\beta+1,\ \alpha+\lambda,\ \beta+\lambda; \end{matrix}\ \frac{-4z}{(1-z)^2}\right] \\ &\qquad = {}_2F_1\left[\begin{matrix} \alpha,\ \alpha-\lambda-\beta+1; \\ \alpha-\beta+1; \end{matrix}\ z\right] {}_2F_1\left[\begin{matrix} \lambda,\ \beta-\alpha+\lambda; \\ \beta+\lambda; \end{matrix}\ z\right] \end{aligned} \tag{94} $$

$$ \begin{aligned} &\big(\alpha-\beta+1, \alpha+\lambda,\ \beta+\lambda \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0}; \\ &\qquad |z|<1,\ |z|^2/\{4|1-z|\}<1,\ |\arg(1-z)|<\pi\big). \end{aligned} $$

Setting $\lambda = \gamma - \beta$ in (94) and utilizing Euler transformation (12) in the ${}_2F_1$ on the resultant identity, we obtain a product formula for ${}_2F_1$'s due to Bailey [56], p. 383, Equation (7.4):

$$\begin{aligned}&(1-z)^{\beta-\alpha-\gamma}{}_4F_3\left[\begin{array}{r} \alpha,\, \gamma-\beta,\, \frac{\alpha+\gamma-\beta}{2},\, \frac{\alpha+\gamma-\beta+1}{2};\\ \alpha-\beta+1,\, \alpha+\gamma-\beta,\, \gamma; \end{array} \frac{-4z}{(1-z)^2}\right]\\ &\quad = {}_2F_1\left[\begin{array}{r} \alpha,\, \beta;\\ \gamma; \end{array} z\right] {}_2F_1\left[\begin{array}{r} 1-\beta,\, \gamma-\beta;\\ \alpha-\beta+1; \end{array} z\right]\end{aligned} \tag{95}$$

$$\big(\gamma,\, \alpha-\beta+1,\, \alpha+\gamma-\beta \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0};\ |z|<1,\, |z|^2/\{4|1-z|\}<1,\, |\arg(1-z)|<\pi\big).$$

Example 18. *Putting $D = E = 0$ and $\beta = \alpha$ in (52) and using the binomial theorem (11), we acquire a reduction formula:*

$$\begin{aligned}&F^{1:2;2}_{1:1;1}\left[\begin{array}{r} 2\alpha+\lambda+\sigma-1:\\ \gamma: \end{array} \begin{array}{r} \alpha,\, 2\alpha-\gamma+\sigma;\\ \alpha+\sigma; \end{array} \begin{array}{r} \lambda,\, \gamma-\alpha;\\ \alpha+\lambda; \end{array} z, z\right]\\ &\quad = (1-z)^{1-2\alpha-\lambda-\sigma}{}_4F_3\left[\begin{array}{r} \alpha,\, \gamma-\alpha,\, \frac{2\alpha+\lambda+\sigma-1}{2},\, \frac{2\alpha+\lambda+\sigma}{2};\\ \alpha+\lambda,\, \alpha+\sigma,\, \gamma; \end{array} \frac{-4z}{(1-z)^2}\right]\end{aligned} \tag{96}$$

$$\big(\gamma,\, \alpha+\sigma,\, \alpha+\lambda \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0};\ |z|<1,\, |z|^2/\{4|1-z|\}<1,\, |\arg(1-z)|<\pi\big).$$

Example 19. *Putting $D = E = 0$ and $\gamma = \alpha+\beta+\lambda$ in (52) and using the binomial theorem (11), we gain a reduction formula:*

$$\begin{aligned}&F^{2:2;1}_{2:1;0}\left[\begin{array}{r} \alpha+\beta+\lambda+\sigma-1, \lambda+\beta:\\ \alpha+\beta+\lambda\,, \alpha+\lambda: \end{array} \begin{array}{r} \alpha, \sigma-\lambda;\\ \alpha+\sigma; \end{array} \begin{array}{r} \lambda;\\ -; \end{array} z, z\right]\\ &\quad = (1-z)^{1-\alpha-\beta-\lambda-\sigma}{}_3F_2\left[\begin{array}{r} \alpha,\, \frac{\alpha+\beta+\lambda+\sigma-1}{2},\, \frac{\alpha+\beta+\lambda+\sigma}{2};\\ \alpha+\beta+\lambda,\, \alpha+\sigma; \end{array} \frac{-4z}{(1-z)^2}\right]\end{aligned} \tag{97}$$

$$\big(\alpha+\beta+\lambda,\, \alpha+\lambda,\, \alpha+\sigma \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0};\ |z|<1,\, |z|^2/\{4|1-z|\}<1,\, |\arg(1-z)|<\pi\big).$$

Applying the reduction Formula (22) to the left-hand side of (97), we obtain a transformation formula for ${}_3F_2$:

$$\begin{aligned}&(1-z)^{1-\alpha-\beta-\lambda-\sigma}{}_3F_2\left[\begin{array}{r} \alpha,\, \frac{\alpha+\beta+\lambda+\sigma-1}{2},\, \frac{\alpha+\beta+\lambda+\sigma}{2};\\ \alpha+\beta+\lambda,\, \alpha+\sigma; \end{array} \frac{-4z}{(1-z)^2}\right]\\ &\quad = {}_3F_2\left[\begin{array}{r} \lambda+\beta,\, \alpha+\beta+\lambda+\sigma-1,\, \sigma;\\ \alpha+\beta+\lambda,\, \alpha+\sigma; \end{array} z\right]\end{aligned} \tag{98}$$

$$\big(\alpha+\beta+\lambda,\, \alpha+\sigma \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0};\ |z|<1,\, |z|^2/\{4|1-z|\}<1,\, |\arg(1-z)|<\pi\big),$$

which is a transformation formula of the type in (8) due to Whipple.

Example 20. *Putting $D = E = 0$ and $\sigma = \gamma-\alpha-\beta-\lambda+1$ in (52) and using the binomial theorem (11), we deduce a reduction formula:*

$$\begin{aligned}&F^{1:2;2}_{1:1;1}\left[\begin{array}{r} \beta+\lambda:\\ \alpha+\lambda: \end{array} \begin{array}{r} \alpha,\, 1-\lambda;\\ \gamma-\beta-\lambda+1; \end{array} \begin{array}{r} \lambda,\, \gamma-\alpha;\\ \beta+\lambda; \end{array} z, z\right]\\ &\quad = (1-z)^{-\gamma}{}_4F_3\left[\begin{array}{r} \alpha,\, \frac{\gamma}{2},\, \frac{\gamma+1}{2},\, \gamma-\beta;\\ \alpha+\lambda,\, \gamma-\beta-\lambda+1,\, \gamma; \end{array} \frac{-4z}{(1-z)^2}\right]\end{aligned} \tag{99}$$

$$\big(\alpha+\lambda,\, \gamma-\beta-\lambda+1,\, \gamma,\, \beta+\lambda \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0};\\ |z|<1,\, |z|^2/\{4|1-z|\}<1,\, |\arg(1-z)|<\pi\big).$$

Example 21. *Additionally, we can derive numerous reduction and transformation formulas by specializing the parameters in (52). For instance,*

(i) *Putting $(D=E=0, \lambda=\gamma-\beta)$ or $(D=E=0, \sigma=1-\beta)$ in (52) and using the binomial theorem (11), we can derive reduction formulas for $F^{1:2;2}_{1:1;1}$ of the similar type in (99).*

(ii) *Putting $(D=E=0, \sigma=0)$ or $(D=E=0, \beta=0)$ or $(D=E=0, \lambda=0)$ or $(D=E=0, \gamma=\alpha)$ in (52) and using the binomial theorem (11) and the reduction Formula (22), we can attain certain transformation formulas of the type (8) due to Whipple.*

Example 22. (i) *It is interesting to recall a transformation formula for the Kampé de Fériet function (see [67], Equation (3.3)):*

$$\begin{aligned} &F^{1:2;2}_{1:1;1}\left[\begin{matrix} b: & a,\, c; & a',\, c'; \\ c+c': & b; & b; \end{matrix}\; x,\, y\right] \\ &\quad = (1-x)^{-a}\,(1-y)^{-a'}\,F^{1:2;2}_{1:1;1}\left[\begin{matrix} b: & a,\, c; & a',\, c'; \\ c+c': & b; & b; \end{matrix}\; \tfrac{x}{1-x},\, \tfrac{y}{1-y}\right], \end{aligned} \tag{100}$$

which can be used to provide some suitably altered formulas of (71), (73), (81), (83), (84), (86), (96), and (99). Additionally, Karlsson [46] (see also [30,47], [68], Equation (15)) presented a reduction formula from $F^{p:2;\ldots;2}_{q:1;\ldots;1}$ with equal variables and two more parameters per variable having certain relations to a single variable ${}_{p+2}F_{q+1}$.

(ii) *Liu and Wang provided a number of reduction formulas for $F^{1:1;2}_{1:0;1}[z,z]$ in [48] (Equations (2.4), (2.11), (2.12), (2.13)), which are found to be distinct from (72) and (77).*

6. Summation Formulas for Kampé de Fériet and ${}_{p+1}F_p$

This section demonstrates specific general summation formulas for the Kampé de Fériet and ${}_{p+1}F_p$ with specified parameters and arguments 1, -1, $\frac{1}{2}$, $-\frac{1}{4}$, $-\frac{1}{8}$, and $-\frac{1}{16}$, among many others.

Instance 1. *Putting $\alpha=a$, $\beta=b$, $\gamma=\frac{a+b+j+1}{2}$, and $z=\frac{1}{2}$ in (65) and using (27) and (28), we obtain the following general summation formula for ${}_4F_3(1)$:*

$$\begin{aligned} &{}_4F_3\left[\begin{matrix} a,\, b,\, \frac{a+b}{2},\, \frac{a+b+1}{2}; \\ a+b,\, \frac{a+b+j+1}{2},\, \frac{a+b-j+1}{2}; \end{matrix}\; 1\right] = \left\{\frac{2^{b-1}\Gamma\left(\frac{a+b+j+1}{2}\right)\Gamma\left(\frac{a-b-j+1}{2}\right)}{\Gamma(b)\Gamma\left(\frac{a-b+j+1}{2}\right)}\right. \\ &\quad \left.\times \sum_{r=0}^{j}(-1)^r\binom{j}{r}\frac{\Gamma\left(\frac{b+r}{2}\right)}{\Gamma\left(\frac{a+r-j+1}{2}\right)}\right\}\left\{\frac{2^{b-1}\Gamma\left(\frac{a+b-j+1}{2}\right)}{\Gamma(b)}\sum_{r=0}^{j}\binom{j}{r}\frac{\Gamma\left(\frac{b+r}{2}\right)}{\Gamma\left(\frac{a+r-j+1}{2}\right)}\right\}, \end{aligned} \tag{101}$$

where $j\in\mathbb{Z}_{\geqslant 0}$.

Instance 2. *Putting $\alpha=a$, $\beta=1-a+j$, $\gamma=b$, and $z=\frac{1}{2}$ in (65) and using (29), we gain*

$$\begin{aligned} &{}_4F_3\left[\begin{matrix} a,\, 1-a+j,\, \frac{1+j}{2},\, \frac{2+j}{2}; \\ b,\, 1+j,\, 2+j-b; \end{matrix}\; 1\right] \\ &\quad = \left\{\frac{2^{j-a}\Gamma(a-j)\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\sum_{r=0}^{j}(-1)^r\binom{j}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r}{2}-j\right)}\right\} \\ &\quad\quad \times\left\{\frac{2^{j-a}\Gamma(a-j)\Gamma(2+j-b)}{\Gamma(a)\Gamma(2+j-b-a)}\sum_{r=0}^{j}(-1)^r\binom{j}{r}\frac{\Gamma\left(\frac{2+j-b-a+r}{2}\right)}{\Gamma\left(\frac{2+j-b+a+r}{2}-i\right)}\right\}, \end{aligned} \tag{102}$$

where $j\in\mathbb{Z}_{\geqslant 0}$.

Instance 3. *Putting $\alpha = a$, $\beta = 1-a+j$, $\gamma = 2a-j+k$, and $z = -1$ in (76) and using (25) and (29), we attain*

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} a,\, 1-a+j,\, a-j+k,\, 3a-2j+k-1; \\ 2a-j+k,\, \frac{2a-j+k}{2},\, \frac{1+2a-j+k}{2}; \end{matrix}\; -\frac{1}{8}\right] = \left\{\frac{2^{k-2+2a-2j}}{\Gamma(1-a+j)}\right. \\
&\times \left.\frac{\Gamma(1-a+j-k)\Gamma(2a-j+k)}{\Gamma(3a-2j+k-1)} \sum_{r=0}^{k} (-1)^r \binom{k}{r} \frac{\Gamma\left(\frac{a+r+k+1}{2}-1+a-j\right)}{\Gamma\left(\frac{a+r-k+1}{2}\right)}\right\} \\
&\times \left\{\frac{2^{j-a}\Gamma(a-j)\Gamma(2a-j+k)}{\Gamma(a)\Gamma(a-j+k)} \sum_{r=0}^{j} (-1)^r \binom{j}{r} \frac{\Gamma\left(\frac{a-j+r+k}{2}\right)}{\Gamma\left(\frac{3a-j+r+k}{2}-j\right)}\right\},
\end{aligned}
\tag{103}
$$

where $j, k \in \mathbb{Z}_{\geqslant 0}$.

Instance 4. *Putting $\alpha = a$, $\beta = 1-a-j$, $\gamma = 2a+j+i$, and $z = -1$ in (76) and using (25) and (30), we acquire*

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} a,\, 1-a-j,\, a+j+k,\, 3a+2j+k-1; \\ 2a+j+k,\, \frac{2a+j+k}{2},\, \frac{1+2a+j+k}{2}; \end{matrix}\; -\frac{1}{8}\right] = \left\{\frac{2^{k-2+2a+2j}}{\Gamma(1-a-j)}\right. \\
&\times \left.\frac{\Gamma(1-a-j-k)\Gamma(2a+j+k)}{\Gamma(3a+2j+k-1)} \sum_{r=0}^{k} (-1)^r \binom{k}{r} \frac{\Gamma\left(\frac{a+r+k+1}{2}-1+a+j\right)}{\Gamma\left(\frac{a+r-k+1}{2}\right)}\right\} \\
&\times \left\{\frac{2^{-j-a}\,\Gamma(2a+j+k)}{\Gamma(a+j+k)} \sum_{r=0}^{j} \binom{j}{r} \frac{\Gamma\left(\frac{a+j+r+k}{2}\right)}{\Gamma\left(\frac{3a+j+r+k}{2}\right)}\right\},
\end{aligned}
\tag{104}
$$

where $j, k \in \mathbb{Z}_{\geqslant 0}$.

Instance 5. *Putting $\alpha = a$, $\beta = 1-a+j$, $\gamma = 2a-j-k$, and $z = -1$ in (76) and using (26) and (29), we derive*

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} a,\, 1-a+j,\, a-j-k,\, 3a-2j-k-1; \\ 2a-j-k,\, \frac{2a-j-k}{2},\, \frac{1+2a-j-k}{2}; \end{matrix}\; -\frac{1}{8}\right] \\
&= \left\{\frac{2^{-k-2+2a-2j}\,\Gamma(2a-j-k)}{\Gamma(3a-2j-k-1)} \sum_{r=0}^{k} \binom{k}{r} \frac{\Gamma\left(\frac{a+r-k+1}{2}-1+a-j\right)}{\Gamma\left(\frac{a+r-k+1}{2}\right)}\right\} \\
&\times \left\{\frac{2^{j-a}\Gamma(a-j)\,\Gamma(2a-j-k)}{\Gamma(a-j-k)\,\Gamma(a)} \sum_{r=0}^{j} (-1)^r \binom{j}{r} \frac{\Gamma\left(\frac{a-j+r-k}{2}\right)}{\Gamma\left(\frac{3a-j+r-k}{2}-j\right)}\right\},
\end{aligned}
\tag{105}
$$

where $j, k \in \mathbb{Z}_{\geqslant 0}$.

Instance 6. *Putting $\alpha = a$, $\beta = 1-a-j$, $\gamma = 2a+j-k$, and $z = -1$ in (76) and using (26) and (30), we obtain*

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} a,\, 1-a-j,\, a+j-k,\, 3a+2j-k-1; \\ 2a+j-k,\, \frac{2a+j-k}{2},\, \frac{1+2a+j-k}{2}; \end{matrix}\; -\frac{1}{8}\right] \\
&= \left\{\frac{2^{-k-2+2a+2j}\Gamma(2a+j-k)}{\Gamma(-1+3a+2j-k)} \sum_{r=0}^{k} \binom{k}{r} \frac{\Gamma\left(\frac{a+r-k+1}{2}-1+a+j\right)}{\Gamma\left(\frac{a+r-k+1}{2}\right)}\right\} \\
&\times \left\{\frac{2^{-j-a}\,\Gamma(2a+j-k)}{\Gamma(a+j-k)} \sum_{r=0}^{j} \binom{j}{r} \frac{\Gamma\left(\frac{a+j+r-k}{2}\right)}{\Gamma\left(\frac{3a+j+r-k}{2}\right)}\right\},
\end{aligned}
\tag{106}
$$

where $j,\, k \in \mathbb{Z}_{\geqslant 0}$.

Instance 7. *Putting* $\alpha = a$, $\delta = \beta = b$, $\gamma = 1+a-b+j$, *and* $z = -1$ *in* (88) *and using* (25)*, we obtain*

$$F^{2:2;0}_{1:2;0}\left(\begin{matrix} [2b:2,1],[a:1,1]: & [b:1],\,[1-b+j:1]; & -; \\ [1+a-b+j:2,1]: & [1+a-b+j:1],\,[\frac{2b+1}{2}:1]; & -; \end{matrix}\; -\frac{1}{4}, -1\right)$$
$$= \left\{ \frac{2^{j-2b}\Gamma(b-j)\Gamma(1+a-b+j)}{\Gamma(b)\Gamma(a-2b+j+1)} \sum_{r=0}^{j} (-1)^r \binom{j}{r} \frac{\Gamma\left(\frac{a+r+j+1}{2} - b\right)}{\Gamma\left(\frac{a+r-j+1}{2}\right)} \right\}^2, \tag{107}$$

where $j \in \mathbb{Z}_{\geqslant 0}$.

Instance 8. *Putting* $\alpha = a$, $\delta = \beta = b$, $\gamma = 1+a-b-j$, *and* $z = -1$ *in* (88) *and using* (26)*, we obtain*

$$F^{2:2;0}_{1:2;0}\left(\begin{matrix} [2b:2,1],[a:1,1]: & [b:1],\,[1-b-j:1]; & -; \\ [1+a-b-j:2,1]: & [1+a-b-j:1],\,[\frac{2b+1}{2}:1]; & -; \end{matrix}\; -\frac{1}{4}, -1\right)$$
$$= \left\{ \frac{2^{-j-2b}\Gamma(1+a-b-j)}{\Gamma(a-2b-j+1)} \sum_{r=0}^{j} \binom{j}{r} \frac{\Gamma\left(\frac{a+r-j+1}{2} - b\right)}{\Gamma\left(\frac{a+r-j+1}{2}\right)} \right\}^2, \tag{108}$$

where $j \in \mathbb{Z}_{\geqslant 0}$.

Instance 9. *Putting* $\alpha = a$, $\delta = \beta = b$, $\gamma = \frac{1+a+b+j}{2}$, *and* $z = \frac{1}{2}$ *in* (88) *and using* (27)*, we gain*

$$F^{2:2;0}_{1:2;0}\left(\begin{matrix} [2b:2,1],[a:1,1]: & [b:1],\,[\frac{1-a+b+j}{2}:1]; & -; \\ [\frac{1+a+b+j}{2}:2,1]: & [\frac{1+a+b+j}{2}:1],\,[\frac{2b+1}{2}:1]; & -; \end{matrix}\; -\frac{1}{16}, \frac{1}{2}\right)$$
$$= \left\{ \frac{2^{b-1}\Gamma(\frac{1+a+b+j}{2})\Gamma(\frac{1+a-b-j}{2})}{\Gamma(b)\Gamma(\frac{1+a-b+j}{2})} \sum_{r=0}^{j} (-1)^r \binom{j}{r} \frac{\Gamma\left(\frac{b+r}{2}\right)}{\Gamma\left(\frac{a+r-j+1}{2}\right)} \right\}^2, \tag{109}$$

where $j \in \mathbb{Z}_{\geqslant 0}$.

Instance 10. *Putting* $\alpha = a$, $\delta = \beta = b$, $\gamma = \frac{1+a+b-j}{2}$, *and* $z = \frac{1}{2}$ *in* (88) *and using* (28)*, we attain*

$$F^{2:2;0}_{1:2;0}\left(\begin{matrix} [2b:2,1],[a:1,1]: & [b:1],\,[\frac{1-a+b-j}{2}:1]; & -; \\ [\frac{1+a+b-j}{2}:2,1]: & [\frac{1+a+b-j}{2}:1],\,[\frac{2b+1}{2}:1]; & -; \end{matrix}\; -\frac{1}{16}, \frac{1}{2}\right)$$
$$= \left\{ \frac{2^{b-1}\Gamma(\frac{1+a+b-j}{2})}{\Gamma(b)} \sum_{r=0}^{j} \binom{j}{r} \frac{\Gamma\left(\frac{b+r}{2}\right)}{\Gamma\left(\frac{a+r-j+1}{2}\right)} \right\}^2, \tag{110}$$

where $j \in \mathbb{Z}_{\geqslant 0}$.

Instance 11. *Putting* $\alpha = a$, $\delta = \beta = 1-a+j$, $\gamma = b$, *and* $z = \frac{1}{2}$ *in* (88) *and using* (29)*, we acquire*

$$F^{2:2;0}_{1:2;0}\left(\begin{matrix} [2-2a+2j:2,1],[a:1,1]: & [1-a+j:1],\,[b-a:1]; & -; \\ [b:2,1]: & [b:1],\,[\frac{3-2a+2j}{2}:1]; & -; \end{matrix}\; -\frac{1}{16}, \frac{1}{2}\right)$$
$$= \left\{ \frac{2^{j-a}\Gamma(a-j)\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \sum_{r=0}^{j} (-1)^r \binom{j}{r} \frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r}{2} - j\right)} \right\}^2, \tag{111}$$

where $j \in \mathbb{Z}_{\geqslant 0}$.

Instance 12. *Putting $\alpha = a$, $\delta = \beta = 1 - a - j$, $\gamma = b$, and $z = \frac{1}{2}$ in (88) and using (30), we derive*

$$F^{2:2;0}_{1:2;0}\left(\begin{array}{rrr} [2-2a-2j:2,1],[a:1,1]: & [1-a-j:1],\,[b-a:1]; & -; \\ [b:2,1]: & [b:1],\,[\frac{3-2a-2j}{2}:1]; & -; \end{array} \; -\frac{1}{16}, \frac{1}{2}\right) = \left\{ \frac{2^{-j-a}\Gamma(b)}{\Gamma(b-a)} \sum_{r=0}^{j} \binom{j}{r} \frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r}{2}\right)} \right\}^2, \tag{112}$$

where $j \in \mathbb{Z}_{\geqslant 0}$.

Instance 13. *Putting $\alpha = \frac{a}{2}$, $\beta = \frac{b}{2}$, and $z = \frac{1}{2}$ in (66) and using the classical Kummer second summation theorem (the case $m = 0$ of (28)), with the aid of a duplication formula for the gamma function (see, e.g., [5], p. 6),*

$$\Gamma\left(\tfrac{1}{2}\right)\Gamma(2z) = 2^{2z-1}\,\Gamma(z)\,\Gamma\left(z + \tfrac{1}{2}\right), \tag{113}$$

we obtain

$${}_3F_2\left[\begin{array}{r} a,\, b,\, \frac{a+b}{2}; \\ a+b,\, \frac{a+b+1}{2}; \end{array} \; 1\right] = \left\{ \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{a+b+1}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)} \right\}^2. \tag{114}$$

7. Concluding Remarks

In this article, we introduced three general double-series identities using Whipple transformations for ${}_4F_3$ and ${}_5F_4$ functions. By employing the left-sided Riemann–Liouville fractional integral on those results in Section 2, in Section 4, we showcased the potential to systematically derive additional identities of a similar nature through iterative processes. These identities were then utilized to derive transformation formulas between the Srivastava–Daoust double hypergeometric function (S–D function) and Kampé de Fériet's double hypergeometric function (KDF function) with equal arguments. We also demonstrated reduction formulas from the S–D function or KDF function to the ${}_pF_q$ function. Furthermore, we provided various general summation formulas for the ${}_pF_q$ and S–D function (or KDF function) with specific arguments. By following the steps presented in this article, additional reduction and summation formulas of similar types can be derived. We anticipate that these transformation and summation formulas, as well as those deducible from the same steps, will have applications in diverse fields, such as mathematical physics, statistics, and engineering sciences.

Author Contributions: Writing—original draft, M.I.Q., T.U.R.S., J.C. and A.H.B.; writing—review and editing, M.I.Q., T.U.R.S., J.C. and A.H.B. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Acknowledgments: The authors wish to extend their heartfelt appreciation to the anonymous reviewers for their invaluable feedback. Their constructive and encouraging comments have greatly contributed to enhancing the quality of this paper.

Conflicts of Interest: The authors have no conflict of interest.

References

1. Andrews, G.E.; Askey, R.; Roy, R. *Special Functions, Encyclopedia of Mathematics and its Applications 71*; Cambridge University Press: Cambridge, UK, 1999.
2. Bailey, W.N. *Generalized Hypergeometric Series*; Cambridge Math. Tract No. 32; Cambridge University Press: Cambridge, UK, 1935; Reprinted by Stechert-Hafner: New York, NY, USA, 1964.
3. Lebedev, N.N. *Special Functions and their Applications*; Silverman, R.A., Translator; Prentice-Hall, Inc.: Englewood Cliffs, NJ, USA, 1965.

4. Rainville, E.D. *Special Functions*; The Macmillan Co., Inc.: New York, NY, USA, 1960; Reprinted by Chelsea Publishing Co.: Bronx, NY, USA, 1971.
5. Srivastava, H.M.; Choi, J. *Zeta and q-Zeta Functions and Associated Series Integrals*; Elsevier Science Publishers: Amsterdam, The Netherlands; London, UK; New York, NY, USA, 2012.
6. Srivastava, H.M.; Manocha, H.L. *A Treatise on Generating Functions*; Halsted Press (Ellis Horwood Limited): Chichester, UK; John Wiley and Sons: New York, NY, USA; Chichester, UK; Brisbane, Australia; Toronto, ON, Canada, 1984.
7. Srivastava, H.M.; Karlsson, P.W. *Multiple Gaussian Hypergeometric Series*; Halsted Press (Ellis Horwood Limited): Chichester, UK; John Wiley and Sons: New York, NY, USA; Chichester, UK; Brisbane, Australia; Toronto, ON, Canada, 1985.
8. Appell, P.; de Fériet, J.K. *Fonctions Hypergéométriques et Hypersphériques-Polynômes d' Hermite*; Gauthier-Villars: Paris, France, 1926.
9. Humbert, P. The confluent hypergeometric functions of two variables. *Proc. R. Soc. Edinb.* **1920**, *4*, 73–96. [CrossRef]
10. Erdélyi, A.; Magnus, W.; Oberhettinger, F.; Tricomi, F.G. *Higher Transcendental Functions*; McGraw-Hill Book Co., Inc.: New York, NY, USA; Toronto, ON, Canada; London, UK, 1953; Volume I.
11. de Fériet, J.K. Les Functions hypergéométriques d'ordre supérieur à deux variables. *C. R. Acad. Sci.* **1921**, *173*, 401–404.
12. Burchnall, J.L.; Chaundy, T.W. Expansions of Appell's double hypergeometric functions. *Q. J. Math.* **1940**, *11*, 249–270. [CrossRef]
13. Burchnall, J.L.; Chaundy, T.W. Expansions of Appell's double hypergeometric functions (II). *Q. J. Math.* **1941**, *12*, 112–128. [CrossRef]
14. Srivastava, H.M.; Panda, R. An integral representation for the product of two Jacobi polynomials. *J. Lond. Math. Soc.* **1976**, *12*, 419–425. [CrossRef]
15. Hái, N.T.; Marichev, O.I.; Srivastava, H.M. A note on the convergence of certain families of multiple hypergeometric series. *J. Math. Anal. Appl.* **1992**, *164*, 104–115. [CrossRef]
16. Srivastava, H.M.; Daoust, M.C. On Eulerian integrals associated with Kampé de Fériet function. *Publ. Inst. Math.* **1969**, *9*, 199–202.
17. Srivastava, H.M.; Daoust, M.C. Certain generalized Neumann expansions associated with the Kampé de Fériet's function. *Nederl. Akad. Wetensch. Proc. Ser. A = Indag. Math.* **1969**, *31*, 449–457.
18. Srivastava, H.M.; Daoust, M.C. A note on the convergence of Kampé de Fériet's double hypergeometric series. *Math. Nachr.* **1972**, *53*, 151–159. [CrossRef]
19. Buschman, R.G.; Srivastava, H.M. Series identities and reducibility of Kampé de Fériet functions. *Math. Proc. Camb. Philos. Soc.* **1982**, *91*, 435–440. [CrossRef]
20. Carlitz, L. Summation of a double hypergeometric series. *Mat. Catania* **1967**, *22*, 138–142.
21. Chan, W.-C.C.; Chen, K.-Y.; Chyan, C.-J.; Srivastava, H.M. Some multiple hypergeometric transformations and associated reduction formulas. *J. Math. Anal. Appl.* **2004**, *294*, 418–437. [CrossRef]
22. Chen, K.-Y.; Srivastava, H.M. Series identities and associated families of generating functions. *J. Math. Anal. Appl.* **2005**, *311*, 582–599. [CrossRef]
23. Choi, J.; Rathie, A.K. On the reducibility of Kampé de Fériet function. *Honam Math. J.* **2014**, *36*, 345–355. [CrossRef]
24. Choi, J.; Rathie, A.K. Reducibility of the Kampé de Fériet function. *Appl. Math. Sci.* **2015**, *9*, 4219–4232. [CrossRef]
25. Choi, J.; Rathie, A.K. General summation formulas for the Kampé de Fériet function. *Montes Taures J. Pure Appl. Math.* **2019**, *1*, 107–128.
26. Chu, W.-C.; Srivastava, H.M. Ordinary and basic bivariate hypergeometric transformations associated with the Appell and Kampé de Fériet functions. *J. Comput. Appl. Math.* **2003**, *156*, 355–370. [CrossRef]
27. Cvijović, D.; Miller, A.R. A reduction formula for the Kampé de Fériet function. *Appl. Math. Lett.* **2010**, *23*, 769–771. [CrossRef]
28. Exton, H. On the reducibility of the Kampé de Fériet function. *J. Comput. Appl. Math.* **1997**, *83*, 119–121. [CrossRef]
29. Jain, R.N. Sum of a double hypergeometric series. *Mat. Catania* **1966**, *21*, 300–301.
30. Karlsson, P.W. Some reduction formulae for double power series and Kampé de Fériet functions. *Nederl. Akad. Wetensch. Proc. Ser. A = Indag. Math.* **1984**, *46*, 31–36. [CrossRef]
31. Kim, Y.S. On certain reducibility of Kampé de Fériet function. *Honam Math. J.* **2009**, *31*, 167–176. [CrossRef]
32. Krupnikov, E.D. *A Register of Computer Oriented Reduction of Identities for Kampé de Fériet Function*; Novosibirsk, Russia, 1996.
33. Rakha, M.A.; Awad, M.M.; Rathie, A.K. On a reducibility of the Kampé de Fériet function. *Math. Methods Appl. Sci.* **2015**, *38*, 2600–2605. [CrossRef]
34. Saran, S. Reducibility of generalized Kampé de Fériet function. *Ganita* **1980**, *31*, 89–97.
35. Shanker, O. An integral involving the *G*-function and Kampé de Fériet function. *Proc. Camb. Philos. Soc.* **1968**, *64*, 1041–1044. [CrossRef]
36. Shanker, O.; Saran, S. Reducibility of Kampé de Fériet function. *Ganita* **1970**, *21*, 9–16.
37. Sharma, B.L. Sum of a double series. *Proc. Am. Math. Soc.* **1975**, *52*, 136–138. [CrossRef]
38. Sharma, B.L.; Abiodun, F.A. Some new summation formulae for hypergeometric series of two variable. *Rend. Istit. Mat. Univ. Trieste* **1976**, *8*, 94–100. Available online: http://hdl.handle.net/10077/6500 (accessed on 13 September 2023).
39. Singal, R.P. Transformation formulae for the modified Kampé de Fériet function. *Math. Stud.* **1972**, *40*, 327–330.
40. der Jeugt, J.V. Transformation formula for a double Clausenian hypergeometric series, its q-analogue, and its invariance group. *J. Comput. Appl. Math.* **2002**, *139*, 65–73. [CrossRef]
41. der Jeugt, J.V.; Pitre, S.N.; Rao, K.S. Multiple hypergeometric functions and g-j coefficients. *J. Phys. A Math. Gen.* **1994**, *27*, 5251–5264. [CrossRef]

42. der Jeugt, J.V.; Pitre, S.N.; Rao, K.S. Transformation and summation formulas for double hypergeometric series. *J. Comput. Appl. Math.* **1997**, *83*, 185–193. [CrossRef]
43. Srivastava, H.M. A note on certain summation theorems for multiple hypergeometric series. *Simon Stevin* **1978**, *52*, 97–109.
44. Srivastava, H.M. Some generalizations of Carlson's identity. *Boll. Union Mat. Ital.* **1981**, *18*, 138–143.
45. Whipple, F.J.W. Some transformations of generalized hypergeometric series. *Proc. Lond. Math. Soc.* **1927**, *26*, 257–272. [CrossRef]
46. Karlsson, P.W. Reduction of certain multiple hypergeometric functions. *Nederl. Akad. Wetensch. Proc. Ser. A = Indag. Math.* **1982**, *44*, 285–287. [CrossRef]
47. Karlsson, P.W. Some reducible generalized Kampé de Fériet functions. *J. Math. Anal. Appl.* **1983**, *96*, 546–550. [CrossRef]
48. Liu, H.; Wang, W. Transformation and summation formulae for Kampé de Fériet series. *J. Math. Anal. Appl.* **2014**, *409*, 100–110. [CrossRef]
49. Sheppard, W.F. Summation of the coefficients of some terminating hypergeometric series. *Proc. Lond. Math. Soc.* **1912**, *10*, 469–478. [CrossRef]
50. Slater, L.J. *Generalized Hypergeometric Functions*; Cambridge University Press: Cambridge, UK; London, UK; New York, NY, USA, 1966.
51. Qureshi, M.I.; Paris, R.B.; Malik, S.H.; Shah, T.R. Two reduction formulas for the Srivastava-Daoust double hypergeometric function. *Palest. J. Math.* **2023**, *12*, 181–186.
52. Gessel, I.; Stanton, D. Strange evaluations of hypergeometric series. *SIAM J. Math. Anal.* **1982**, *13*, 295–308. [CrossRef]
53. Andrews, G.E. Connection coefficient problems and partitions. In *AMS Proceedings Symposia in Pure Mathematics 34*; Ray-Chaudhuri, D., Ed.; American Mathematical Society: Providence, RI, USA, 1979; pp. 1–24.
54. Whipple, F.J.W. Well-poised series and other generalized hypergeometric series. *Proc. Lond. Math. Soc.* **1926**, *25*, 525–544. [CrossRef]
55. Bailey, W.N. Products of generalized hypergeometric series. *Proc. Lond. Math. Soc.* **1928**, *28*, 242–254. . [CrossRef]
56. Bailey, W.N. Some theorems concerning products of hypergeometric series. *Proc. Lond. Math. Soc.* **1935**, *38*, 377–384. [CrossRef]
57. Exton, H. *Multiple Hypergeometric Functions and Applications*; Halsted Press (Ellis Horwood Limited): Chichester, UK; John Wiley and Sons: New York, NY, USA; Chichester, UK; Sydney, Australia; Toronto, ON, Canada, 1976.
58. Prudnikov, A.P.; Brychkov, Y.A.; Marichev, O.I. *Integrals and Series*; Gould, G.G., Translator; More Special Functions: Nauka Moscow, Russia, 1986; Volume III. (In Russian); Gordon and Breach Science Publishers: New York, NY, USA; Philadelphia, PA, USA; London, UK; Paris, France; Montreux, Switzerland; Tokyo, Japan; Melbourne, Australia, 1990.
59. Rakha, M.A.; Rathie, A.K. Generalizations of classical summation theorems for the series ${}_2F_1$ and ${}_3F_2$ with applications. *Integral Transform. Spec. Funct.* **2011**, *22*, 823–840. [CrossRef]
60. Choi, J.; Qureshi, M.I.; Bhat, A.H.; Majid, J. Reduction formulas for generalized hypergeometric series associated with new sequences and applications. *Fractal Fract.* **2021**, *5*, 150. [CrossRef]
61. Brychkov, Y.A. *Handbook of Special Functions, Derivatives, Integrals, Series and Other Formulas*; CRC Press: Boca Raton, FL, USA; Taylor & Fancis Group: Abingdon, UK; London, UK; New York, NY, USA, 2008.
62. Lavoie, J.L.; Grondin, F.; Rathie, A.K. Generalizations of Whipple's theorem on the sum of a ${}_3F_2$. *J. Comput. Appl. Math.* **1996**, *72*, 293–300. [CrossRef]
63. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: North-Holland, The Netherlands, 2006.
64. Srivastava, H.M. Certain double integrals involving yypergeometric functions. *Jñānābha Sect. A* **1971**, *1*, 1–10.
65. Clausen, T. Ueber die Fälle, wenn die Reihe von der Form $y =$ etc. ein Quadrat von der Form $z =$ etc. hat. *J. Reine Angew. Math.* **1828**, *3*, 89–91. [CrossRef]
66. Orr, W.M.F. Theorems relating to the product of two hypergeometric series. *Trans. Camb. Philos. Soc.* **1899**, *17*, 1–15.
67. Carlitz, L. A summation theorem for double hypergeometric series. *Rend. Semin. Mat. Univ. Padova* **1967**, *37*, 230–233. Available online: http://www.numdam.org/item?id=RSMUP_1967__37__230_0 (accessed on 13 September 2023).
68. Srivastava, H.M. The sum of a multiple hypergeometric series. *Indag. Math.* **1977**, *80*, 448–452. [CrossRef]