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Article

The Iterative Properties for Positive Solutions of a Tempered Fractional Equation

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Abstract: In this article, we investigate the iterative properties of positive solutions for a tempered fractional equation under the case where the boundary conditions and nonlinearity all involve tempered fractional derivatives of unknown functions. By weakening a basic growth condition, some new and complete results on the iterative properties of the positive solutions to the equation are established, which include the uniqueness and existence of positive solutions, the iterative sequence converging to the unique solution, the error estimate of the solution and convergence rate as well as the asymptotic behavior of the solution. In particular, the iterative process is easy to implement as it can start from a known initial value function.

Keywords: iterative properties; uniqueness; tempered fractional equation; asymptotic behavior

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1. Introduction

In this article, we consider the iterative properties of positive solutions for the following tempered fractional equation

$$\begin{cases} {}_0^R\mathbb{D}_t{}^{\vartheta,\mu}x(t) = f(t, e^{\mu t}x(t), {}_0^R\mathbb{D}_t{}^{\delta,\mu}x(t)), \\ {}_0^R\mathbb{D}_t{}^{\delta,\mu}x(0) = 0, \ {}_0^R\mathbb{D}_t{}^{\delta,\mu}x(1) = \int_0^1 e^{-\mu(1-t)}{}_0^R\mathbb{D}_t{}^{\delta,\mu}x(t)dt, \end{cases} \tag{1}$$

where $1 < \vartheta \leq 2, 0 < \delta < 1$, and $\vartheta - \delta > 1$, μ is a positive constant, $f : (0,1) \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is a continuous and nondecreasing function with respect to the two space variables, and ${}_0^R\mathbb{D}_t{}^{\vartheta,\mu}x(t)$ and ${}_0^R\mathbb{D}_t{}^{\delta,\mu}x(t)$ are tempered fractional derivatives, which are related to the Riemann–Liouville fractional derivative $\mathscr{D}_t{}^{\vartheta}x(t)$ by

$${}_0^R\mathbb{D}_t{}^{\vartheta,\mu}x(t) = e^{-\mu t}\mathscr{D}_t{}^{\vartheta}(e^{\mu t}x(t)), \tag{2}$$

where $\mathscr{D}_t{}^{\vartheta}x(t)$ is defined by

$$\mathscr{D}_t{}^{\vartheta}x(t) = \frac{d^n}{dt^n}\left(I^{n-\vartheta}x(t)\right) = \frac{1}{\Gamma(n-\vartheta)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\vartheta-1}x(s)ds,$$

and

$$I^{\vartheta}x(t) = \frac{1}{\Gamma(\vartheta)}\int_0^t (t-s)^{\vartheta-1}x(s)ds$$

denotes the Riemann–Liouville fractional integral operator [1–4].

Since fractional order differential equations can describe many natural phenomena with long-time behavior such as abnormal dispersion, analytical chemistry, biological sciences, artificial neural network, time-frequency analysis, and so on, the theories of fractional calculus have attracted the attention of a large number of mathematical researchers [5–20]. The classical fractional integrals and derivatives are only convolutions by using a power law, such as the Riemann–Liouville fractional derivatives [21–23], the Caputo fractional derivatives [24,25], the Hilfer fractional derivative [26], the Atangana-Baleanu-Caputo fractional derivative [27], Hadamard fractional derivatives [28,29], and so on, which fail to model the limits of random walk if they have an exponentially tempered jump distribution [30] exhibiting the semi-heavy tails or semi-long range dependence. Thus, to describe semi-heavy tails or semi-long range dependence, it is necessary to multiply by an exponential factor leading to tempered fractional integrals and derivatives [31,32]. Recently, Mali, Kucche, Fernandez, and Fahad [33] developed some theories, properties, and applications for tempered fractional calculus. In [1], by employing a fixed point theorem, Zhou et al. considered a tempered fractional differential equation with a Riemann–Stieltjes integral boundary condition by using the sum-type mixed monotone operator fixed point theorem, and some results on the existence and uniqueness of positive solutions and successively sequences for approximating the unique positive solution were obtained. In a recent work [4], the authors considered a singular tempered p-Laplacian fractional equation

$$\begin{cases} {}^R_0\mathbb{D}_t^{\alpha,\mu}\left(\varphi_p\left({}^R_0\mathbb{D}_t^{\beta,\mu}u(t)\right)\right) = f(t,u(t)), \\ u(0)=0,\ {}^R_0\mathbb{D}_t^{\beta,\mu}u(0)=0,\ u(1)=\int_0^1 e^{-\mu(1-t)}u(t)dt, \end{cases} \tag{3}$$

where $\alpha \in (0,1], \beta \in (1,2]$, $\mu > 0$ is a constant, $\varphi_p(t) = |t|^{p-2}t$ denotes a p-Laplacian operator satisfying the conjugate index $\frac{1}{p}+\frac{1}{q}=1, p>1$, and $f(\cdot,\cdot)$ is a continuous function and decreasing only on the second variable. By adopting the method of upper and lower solutions, it has been proven that Equation (3) has at least one positive solution regardless of singular or nonsingular cases. In addition, under the case where the boundary conditions and nonlinearities all involve the Riemann–Liouville fractional derivative, Rehman and Khan [34] focused on the following fractional order differential equation

$$\mathscr{D}_t{}^{\alpha}u(t) = f(t,u(t),\mathscr{D}_t{}^{\beta}u(t)),\ \ t\in(0,1), \tag{4}$$

subject to the multipoint boundary condition

$$u(0)=0,\ \mathscr{D}_t{}^{\beta}u(1) - \sum_{i=1}^{m-2}\theta_i\mathscr{D}_t{}^{\beta}u(\xi_i) = u_0,$$

where $\alpha \in (1,2]$, $\theta_i \geq 0$, and $0 < \beta, \xi_i < 1$ satisfy $\sum_{i=1}^{m-2}\theta_i\xi_i^{\vartheta-\delta-1} \in (0,1)$. By adopting several fixed point theorems, it was proven that Equation (4) possesses a unique nontrivial solution provided that f satisfies some suitable growth conditions.

Benefiting from the development of the nonlinear analysis theories, in recent years, many powerful tools, such as function spaces theories [35–41], regularity theories [42–46], the operator technique [47–52], the method of upper and lower solutions [53,54], the method of moving sphere [55], variational theories [56–60], and so on, have been developed to solve various partial or ordinary differential equations. Thus, inspired by the above results, in this paper, we investigate the iterative properties of positive solutions for Equation (1) by using the space and operator theory as well as some analytical techniques. Although Equation (1) and similar types of equations have been studied before, further results are still required. For example, the work [4] only obtained the existence of positive solutions for Equation (3), and so the result of the uniqueness of the solution is still unknown. In [1], Zhou et al. employed the fixed point theorem of the sum-type mixed monotone operator to establish the existence and uniqueness of a positive solution and construct iterative sequences for approximating the unique positive solution for a tempered fractional equation with more complex boundary conditions. However the method based on the fixed point theorem of the sum-type mixed monotone operator cannot yield the results of error estimates and the convergence rate between the exact and approximate solutions as well as the asymptotic behavior of the solution. Also, the method in [1] cannot handle equations for the case where nonlinear terms contain derivatives of unknown function such as Equation (1). Moreover, the nonlinear term of the work in [1] must be mixed monotone and cannot handle the cases where the nonlinear term of the equation is increasing only or decreasing only. In addition, in [1], a strong growth condition (H_2) is applied:

(H_2) For $\forall t \in [0,1], \gamma \in (0,1), u, v \in [0,+\infty)$, there exists a constant $\xi \in (0,1)$, such that

$$f(t, \gamma u, \gamma^{-1} v) \geq \varphi_p^{\xi}(\gamma) f(t,u,v) = \gamma^{(p-1)\xi} f(t,u,v).$$

Clearly, (H_2) is a strong growth condition limiting $\varphi_p^{\xi}(\gamma)$ to a power function only, rather than a more general function. For example, it cannot handle the case $\varphi(\gamma) = e^{\gamma} + 1$ given in our condition $(\mathbf{G})$. Hence in this paper, a new method is developed to study Equation (1), yielding many new results that have not been obtained previously.

The main contribution of this paper is that we not only weaken a basic growth condition, which was used in some previous work [1,61–65] (see Remark 5 given in Section 3) but also establish some new and complete results on the iterative properties of positive solutions, including the uniqueness and existence of positive solutions, a sequence of iterations that converges to the unique solution, the error estimate of the solution and convergence rate, as well as the asymptotic behavior of the solution. The iterative properties of solutions are crucial for one to understand the natural phenomena governed by the equations. In addition, the nonlinearity and boundary conditions of Equation (1) include a tempered fractional derivative of unknown functions. In particular, the boundary conditions of Equation (1) are nonlocal, which allows the equation to describe natural phenomena over a wide range and long time. To the best of our knowledge, no work has been reported for the cases where boundary conditions and nonlinearities all include the tempered fractional derivative of unknown functions, and so this work also contributes in this area.

2. Preliminaries and Lemmas

In order to facilitate the reader's understanding, it is necessary to introduce some preliminaries and lemmas.

Lemma 1 ([4])**.** *Suppose $\vartheta > \delta > 0$. One has*

(i) *If $x(t) \in C[0,1] \cap L^1[0,1]$, then*

$$I^{\delta} \mathscr{D}_t^{\delta} x(t) = x(t) + c_1 t^{\delta-1} + c_2 t^{\delta-2} + \cdots + c_n t^{\delta-n},$$

where $c_i \in R$, and $i = 1, 2, \cdots, n, (n = [\delta] + 1)$;

(ii) *If $x \in L^1(0,1)$, the following equalities hold:*

$$I^{\vartheta} I^{\delta} x(t) = I^{\vartheta+\delta} x(t), \;\; \mathscr{D}_t^{\delta} I^{\vartheta} x(t) = I^{\vartheta-\delta} x(t), \;\; \mathscr{D}_t^{\delta} I^{\delta} x(t) = x(t).$$

The following conclusion can be found in [1,4].

Lemma 2. *Suppose $h : [0,1] \to [0,+\infty)$ is continuous, and $1 < \vartheta - \delta \leq 2$. Then, the positive solution of the following linear equation,*

$$\begin{cases} {}_0^R\mathbb{D}_t{}^{\vartheta-\delta,\mu} x(t) = h(t), \\ x(0) = 0, \;\; x(1) = \int_0^1 e^{-\mu(1-t)} x(t) dt, \end{cases} \tag{5}$$

is unique, which can be expressed by

$$x(t) = \int_0^1 G(t,s) h(s) ds, \tag{6}$$

where

$$G(t,s)=$$
$$\begin{cases} \dfrac{[(\vartheta-\delta)(1-s)^{\vartheta-\delta-1}(\vartheta-\delta-1+s)e^{\mu s}t^{\vartheta-\delta-1}-(\vartheta-\delta)(\vartheta-\delta-1)e^{\mu s}(t-s)^{\vartheta-\delta-1}]e^{-\mu t}}{(\vartheta-\delta-1)\Gamma(\vartheta-\delta+1)}, s\le t; \\ \\ \dfrac{(\vartheta-\delta)(1-s)^{\vartheta-\delta-1}(\vartheta-\delta-1+s)e^{\mu s}}{(\vartheta-\delta-1)\Gamma(\vartheta-\delta+1)}t^{\vartheta-\delta-1}e^{-\mu t}, t\le s. \end{cases} \tag{7}$$

is the Green function of (5).

Lemma 3 ([4])**.** *The function $G(t,s)$ possesses the following characteristics:*

(1) $G(t,s)$ is a continuous and nonnegative function in $[0,1]\times[0,1]$;

(2) Let

$$M(s)=\frac{(\vartheta-\delta)(1-s)^{\vartheta-\delta-1}(\vartheta-\delta-1+s)e^{\mu s}}{(\vartheta-\delta-1)\Gamma(\vartheta-\delta+1)}, m(s)=\frac{(\vartheta-\delta)s(1-s)^{\vartheta-\delta-1}e^{\mu s}}{(\vartheta-\delta-1)\Gamma(\vartheta-\delta+1)}.$$

Then

$$m(s)e^{-\mu t}t^{\vartheta-\delta-1}\le G(t,s)\le M(s)e^{-\mu t}t^{\vartheta-\delta-1}, (t,s)\in[0,1]\times[0,1]. \tag{8}$$

In what follows, we focus on the following mixed integro-differential tempered fractional equation

$$\begin{cases} {}^R_0\mathbb{D}_t^{\vartheta-\delta,\mu}z(t)=f(t,I^\delta(e^{\mu t}z(t)),z(t)), \\ z(0)=0,\ \ z(1)=\displaystyle\int_0^1 e^{-\mu(1-t)}z(t)dt. \end{cases} \tag{9}$$

Lemma 4. *Assume z is a continuous function in* $[0,1]$*. Let*

$$x(t)=e^{-\mu t}I^\delta(e^{\mu t}z(t)).$$

We have

(i) Equation (1) is equivalent to the mixed integro-differential tempered fractional Equation (9).

(ii) Suppose z is a positive solution of Equation (9). Then, $x(t)=e^{-\mu t}I^\delta(e^{\mu t}z(t))$ is a positive solution of Equation (1).

Proof. Firstly, notice $x(t)=e^{-\mu t}I^\delta(e^{\mu t}z(t))$; so, it follows from (2) and Lemma 1 that

$$\begin{aligned} {}^R_0\mathbb{D}_t^{\vartheta,\mu}x(t)) &= e^{-\mu t}\boldsymbol{\mathscr{D}_t}^{\vartheta}(e^{\mu t}x(t)) \\ &= e^{-\mu t}\frac{d^n}{dt^n}I^{n-\vartheta}(e^{\mu t}x(t)) \\ &= e^{-\mu t}\frac{d^n}{dt^n}I^{n-\vartheta}(I^\delta(e^{\mu t}z(t))) \\ &= e^{-\mu t}\frac{d^n}{dt^n}I^{n-\vartheta+\delta}(e^{\mu t}z(t))) \\ &= e^{-\mu t}\boldsymbol{\mathscr{D}_t}^{\vartheta-\delta}(e^{\mu t}z(t))) \\ &= {}^R_0\mathbb{D}_t^{\vartheta-\delta,\mu}z(t), \end{aligned} \tag{10}$$

and

$$\begin{aligned} {}^R_0\mathbb{D}_t^{\delta,\mu}x(t)) &= e^{-\mu t}\boldsymbol{\mathscr{D}_t}^{\delta}(e^{\mu t}x(t)) \\ &= e^{-\mu t}\boldsymbol{\mathscr{D}_t}^{\delta}(I^\delta(e^{\mu t}z(t))) \\ &= z(t). \end{aligned} \tag{11}$$

Thus, (10) and (11) imply that

$$ {}_{0}^{R}\mathbb{D}_{t}^{\vartheta-\delta,\mu} z(t) = f(t, I^{\delta}(e^{\mu t} z(t)), z(t)), \tag{12} $$

and

$$ \begin{aligned} z(0) &= {}_{0}^{R}\mathbb{D}_{t}^{\delta,\mu} x(0) = 0, \\ z(1) &= {}_{0}^{R}\mathbb{D}_{t}^{\delta,\mu} x(1) = \int_{0}^{1} e^{-\mu(1-t)} {}_{0}^{R}\mathbb{D}_{t}^{\delta,\mu} x(t) dt = \int_{0}^{1} e^{-\mu(1-t)} z(t) dt. \end{aligned} \tag{13} $$

Thus, (12) and (13) indicate that Equation (1) is transformed into (9).

Conversely, suppose z is a positive solution of Equation (9). Then, for any $0 < t < 1$, by (2) and Lemma 1, one has

$$ \begin{aligned} {}_{0}^{R}\mathbb{D}_{t}^{\vartheta,\mu} x(t)) &= e^{-\mu t} \boldsymbol{\mathscr{D}_t}^{\vartheta} (e^{\mu t} x(t)) \\ &= e^{-\mu t} \frac{d^n}{dt^n} I^{n-\vartheta} (e^{\mu t} x(t)) \\ &= e^{-\mu t} \frac{d^n}{dt^n} I^{n-\vartheta} I^{\delta} (e^{\mu t} z(t)) \\ &= e^{-\mu t} \frac{d^n}{dt^n} I^{n-\vartheta+\delta} (e^{\mu t} z(t)) \\ &= e^{-\mu t} \boldsymbol{\mathscr{D}_t}^{\vartheta-\delta} (e^{\mu t} z(t)) \\ &= {}_{0}^{R}\mathbb{D}_{t}^{\vartheta-\delta,\mu} z(t) \\ &= f(t, I^{\delta}(e^{\mu t} z(t)), z(t)) \\ &= f(t, e^{\mu t} x(t), {}_{0}^{R}\mathbb{D}_{t}^{\delta,\mu} x(t)). \end{aligned} $$

Notice

$$ {}_{0}^{R}\mathbb{D}_{t}^{\delta,\mu} x(t) = e^{-\mu t} \boldsymbol{\mathscr{D}_t}^{\delta} (e^{\mu t} x(t)) = e^{-\mu t} \boldsymbol{\mathscr{D}_t}^{\delta} I^{\delta} (e^{\mu t} z(t)) = z(t); $$

then, one has

$$ {}_{0}^{R}\mathbb{D}_{t}^{\delta,\mu} x(0) = 0, \ {}_{0}^{R}\mathbb{D}_{t}^{\delta,\mu} x(1) = \int_{0}^{1} e^{-\mu(1-t)} {}_{0}^{R}\mathbb{D}_{t}^{\delta,\mu} x(t) dt. $$

Since $z \in C([0,1],[0,+\infty))$, from the monotonicity and property of I^{δ}, we see that $I^{\delta}(e^{\mu t} z(t))$ is a positive continuous function. Therefore, $x(t) = e^{-\mu t} I^{\delta}(e^{\mu t} z(t))$ is a positive solution of Equation (1). □

In order to obtain the positive solution of Equation (1), we choose a Banach space $X = C[0,1]$ as our working space, which has norm $||z|| = \max_{t\in[0,1]} |z(t)|$. We define a cone of X by

$$ P = \{z \in X : z(t) \geq 0, t \in [0,1]\} $$

and a sub-set of P by

$$ \begin{aligned} P_0 = \{z(t) \in P; \text{ there exists } 0 < k_z < 1, \text{ such that} \\ k_z e^{-\mu t} t^{\vartheta-\delta-1} \leq z(t) \leq k_z^{-1} e^{-\mu t} t^{\vartheta-\delta-1}, \ t \in [0,1]\}. \end{aligned} $$

Now, let us define an integral operator T in X,

$$ (Tz)(t) = \int_{0}^{1} G(t,s) f(s, I^{\delta}(e^{\mu t} z(s)), z(s)) ds. \tag{14} $$

By Lemma 2, finding the solution of the mixed integro-differential tempered fractional Equation (9) is equivalent to finding the fixed point of the operator T.

Now, we give the hypotheses used in this paper.

($\mathbf{G}$) There exists a real value function $\varphi : [0,1] \to [0,+\infty)$ and a constant $0 < \theta < 1$ with $\varphi(\lambda) > \lambda^{\theta},\ \lambda \in (0,1)$, such that for $\lambda \in (0,1)$, f satisfies

$$f(t,\lambda x,\lambda y) \geq \varphi(\lambda) f(t,x,y), (t,x,y) \in (0,1) \times [0,+\infty) \times [0,+\infty). \tag{15}$$

($\mathbf{H}$)

$$0 < \int_0^1 f\Big(s, s^{\vartheta-1}, e^{-\mu s} s^{\vartheta-\delta-1}\Big) ds < +\infty. \tag{16}$$

Remark 1. *Suppose* ($\mathbf{G}$) *holds. Then, for any $\lambda \in [1,+\infty)$, the following inequality holds*

$$f(t,\lambda x,\lambda y) \leq \varphi^{-1}(\lambda^{-1}) f(t,x,y),\ \ (t,x,y) \in (0,1) \times [0,+\infty) \times [0,+\infty). \tag{17}$$

Proof. Indeed, for any $0 < t < 1, x \geq 0, y \geq 0$, and $\lambda \geq 1$, by (15), we have

$$f(t,x,y) = f(t,\lambda^{-1}\lambda x, \lambda^{-1}\lambda y) \geq \varphi(\lambda^{-1}) f(t,\lambda x,\lambda y).$$

Thus, inequality (17) is true. □

3. Main Results

In this section, we first evaluate the Riemann–Liouville fractional integral for the function $t^{\vartheta-\delta-1}$,

$$\begin{aligned} I^{\delta}(t^{\vartheta-\delta-1}) &= \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} s^{\vartheta-\delta-1} ds = \frac{1}{\Gamma(\delta)} \int_0^1 (t-tx)^{\delta-1} (tx)^{\vartheta-\delta-1} t dx \\ &= \frac{B(\delta,\vartheta-\delta)}{\Gamma(\delta)} t^{\vartheta-1} = \frac{\Gamma(\vartheta-\delta)}{\Gamma(\vartheta)} t^{\vartheta-1}. \end{aligned} \tag{18}$$

Theorem 1. *If* ($\mathbf{G}$) *and* ($\mathbf{H}$) *hold, then we have the following results:*
(i) $(Existence\ and\ uniquness)$ The tempered fractional Equation (1) has a unique positive solution $x^*(t)$.
(ii) $(Iterative\ sequence)$ Let $z^*(t) = {}_0^R\mathbb{D}_t^{\delta,\mu} x^*(t)$. Then,

$$z^* \in P_0.$$

Moreover, choose any $z_0 \in P_0$ as an initial value, and construct the function iterative sequence

$$z_n = \int_0^1 G(t,\tau) f(\tau, I^{\delta}(e^{\mu\tau} z_{n-1}(\tau)), z_{n-1}(\tau)) d\tau,\ \ n = 1,2,\cdots. \tag{19}$$

Then,

$$z_n(t) \rightrightarrows z^*(t) = {}_0^R\mathbb{D}_t^{\delta,\mu} x^*(t);$$

i.e., $\{z_n(t)\}_{n\geq 1}$ is a function sequence of uniform convergence on t.
(iii) $(Error\ estimate)$ The error between the exact value z^* and the iterative value z_n may be expressed by

$$||z_n - z^*|| \leq 2\sigma^{\frac{\theta}{2}} \Big(1 - \sigma^{\theta^{n+1}}\Big) ||z_0||,$$

where $0 < \sigma < 1$ is constant dependent only on the initial value z_0.
(iv) $(Convergence\ rate)$ The convergence rate of the iterative solutions can be calculated by

$$||z_n - z^*|| = o\Big(1 - \sigma^{\theta^{n+1}}\Big).$$

(v) (*Asymptotic behavior*) *The unique positive solution x^* of the Equation (1) satisfies the following asymptotic property*

$$\kappa^* e^{-\mu t} t^{\vartheta-1} \leq x^*(t) \leq \frac{1}{\kappa^*} e^{-\mu t} t^{\vartheta-1}, \ \ t \in [0,1],$$

where $\kappa^* \in (0,1)$ is a constant dependent only on z^*.

Proof. In order to establish the iterative properties of positive solutions for the tempered fractional Equation (1), we consider the operator defined by (14)

$$(Tz)(t) = \int_0^1 G(t,s) f(s, I^\delta(e^{\mu t} z(s)), z(s)) ds.$$

By Lemma 4, to determine the positive solution to the mixed integro-differential tempered fractional Equation (9) is equivalent to determining the positive fixed point z^* of operator T, and the function $x^*(t) = e^{-\mu t} I^\delta(e^{\mu t} z^*(t))$ is a positive solution of the Equation (1).

Firstly, we shall show that $T : P_0 \to P_0$ is well defined. Indeed, for any $z \in P_0$, in view of the definition of P_0, it is easy to find a constant $0 < k_z < 1$, such that

$$k_z e^{-\mu t} t^{\vartheta-\delta-1} \leq z(t) \leq k_z^{-1} e^{-\mu t} t^{\vartheta-\delta-1}, \ t \in [0,1]. \tag{20}$$

Consequently, (18) and (20) imply that

$$\frac{k_z \Gamma(\vartheta-\delta)}{\Gamma(\vartheta)} t^{\vartheta-1} \leq I^\delta(k_z t^{\vartheta-\delta-1}) \leq I^\delta(e^{\mu s} z(s)) \leq I^\delta(k_z^{-1} t^{\vartheta-\delta-1}) \leq \frac{k_z^{-1}\Gamma(\vartheta-\delta)}{\Gamma(\vartheta)} t^{\vartheta-1}. \tag{21}$$

Now, by using (8), (20), (21), and (17), one has

$$\begin{aligned}
(Tz)(t) &= \int_0^1 G(t,s) f(s, I^\delta(e^{\mu s} z(s)), z(s)) ds \\
&\leq e^{-\mu t} t^{\vartheta-\delta-1} \int_0^1 M(s) f\left(s, \frac{k_z^{-1}\Gamma(\vartheta-\delta)}{\Gamma(\vartheta)} s^{\vartheta-1}, k_z^{-1} e^{-\mu s} s^{\vartheta-\delta-1}\right) ds \\
&\leq e^{-\mu t} t^{\vartheta-\delta-1} \frac{(\vartheta-\delta)^2 e^\mu}{(\vartheta-\delta-1)\Gamma(\vartheta-\delta+1)} \int_0^1 f\left(s, k_z^{-1} s^{\vartheta-1}, k_z^{-1} e^{-\mu s} s^{\vartheta-\delta-1}\right) ds \\
&\leq e^{-\mu t} t^{\vartheta-\delta-1} \frac{(\vartheta-\delta)^2 e^\mu \varphi^{-1}(k_z)}{(\vartheta-\delta-1)\Gamma(\vartheta-\delta+1)} \int_0^1 f\left(s, s^{\vartheta-1}, e^{-\mu s} s^{\vartheta-\delta-1}\right) ds < \infty.
\end{aligned} \tag{22}$$

On the other hand, one also has

$$\begin{aligned}
(Tz)(t) &= \int_0^1 G(t,s) f(s, I^\delta(e^{\mu s} z(s)), z(s)) ds \\
&\geq e^{-\mu t} t^{\vartheta-\delta-1} \int_0^1 m(s) f\left(s, \frac{k_z \Gamma(\vartheta-\delta)}{\Gamma(\vartheta)} s^{\vartheta-1}, k_z e^{-\mu s} s^{\vartheta-\delta-1}\right) ds \\
&\geq e^{-\mu t} t^{\vartheta-\delta-1} \int_0^1 m(s) f\left(s, \frac{k_z \Gamma(\vartheta-\delta)}{\Gamma(\vartheta)} s^{\vartheta-1}, \frac{k_z \Gamma(\vartheta-\delta)}{\Gamma(\vartheta)} e^{-\mu s} s^{\vartheta-\delta-1}\right) ds \\
&\geq e^{-\mu t} t^{\vartheta-\delta-1} \varphi\left(\frac{k_z \Gamma(\vartheta-\delta)}{\Gamma(\vartheta)}\right) \int_0^1 m(s) f\left(s, s^{\vartheta-1}, e^{-\mu s} s^{\vartheta-\delta-1}\right) ds.
\end{aligned} \tag{23}$$

Choose

$$\begin{aligned}
k_z^* = \min\Bigg\{ &\frac{1}{3}, \left(\frac{(\vartheta-\delta)^2 e^\mu \varphi^{-1}(k_z)}{(\vartheta-\delta-1)\Gamma(\vartheta-\delta+1)} \int_0^1 f\left(s, s^{\vartheta-1}, e^{-\mu s} s^{\vartheta-\delta-1}\right) ds\right)^{-1}, \\
&\varphi\left(\frac{k_z \Gamma(\vartheta-\delta)}{\Gamma(\vartheta)}\right) \int_0^1 m(s) f\left(s, s^{\vartheta-1}, e^{-\mu s} s^{\vartheta-\delta-1}\right) ds \Bigg\};
\end{aligned}$$

then, by (22) and (23), we have

$$k_z^* e^{-\mu t} t^{\vartheta-\delta-1} \leq (Tz)(t) \leq (k_z^*)^{-1} e^{-\mu t} t^{\vartheta-\delta-1},\ t \in [0,1].$$

Thus, $T(P_0) \subset P_0$ is well defined.

Now, we proceed to the iterative process. For any given $z_0 \in P_0$, there are two constants $0 < k_{z_0}, \widetilde{k}_{z_0} < 1$, such that

$$k_{z_0} e^{-\mu t} t^{\vartheta-\delta-1} \leq z_0(t) \leq (k_{z_0})^{-1} e^{-\mu t} t^{\vartheta-\delta-1},\ \widetilde{k}_{z_0} e^{-\mu t} t^{\vartheta-\delta-1} \leq (Tz_0)(t) \leq (\widetilde{k}_{z_0})^{-1} e^{-\mu t} t^{\vartheta-\delta-1},$$

which yields

$$\widetilde{k}_{z_0} k_{z_0} z_0 \leq Tz_0 \leq \frac{1}{k_{z_0}\widetilde{k}_{z_0}} z_0.$$

Now, we denote

$$l = \widetilde{k}_{z_0} k_{z_0};$$

then, $l \in (0,1)$, and

$$l z_0 \leq T z_0 \leq \frac{1}{l} z_0. \tag{24}$$

Noticing $\varphi(\lambda) > \lambda^\theta, 0 < \lambda < 1$, let us select a sufficiently large real number $m > 0$, such that

$$\left(\frac{\varphi(l)}{l^\theta}\right)^m > \frac{1}{l}. \tag{25}$$

Thus, we take

$$\phi_0 = l^{\theta m} z_0,\ \ \psi_0 = \frac{1}{l^{\theta m}} z_0,\ l \in (0,1). \tag{26}$$

Obviously, $\phi_0 \leq \psi_0$. In what follows, we define two iterative sequences

$$\phi_n = T\phi_{n-1}, \psi_n = T\psi_{n-1},\ (n = 1, 2, \cdots). \tag{27}$$

Noticing the monotonicity of f with respect to the two space variables, we have

$$\begin{aligned} T(rz) &= \int_0^1 G(t,s) f(s, rI^\delta(e^{\mu s} z(s)), rz(s)) ds \\ &\geq \varphi(r) \int_0^1 G(t,s) f(s, I^\delta(e^{\mu s} z(s)), z(s)) ds \\ &= \varphi(r) Tz,\ 0 < r < 1, \end{aligned} \tag{28}$$

$$\begin{aligned} T(rz) &= \int_0^1 G(t,s) f(s, rI^\delta(e^{\mu s} z(s)), rz(s)) ds \\ &\leq \varphi^{-1}\left(\frac{1}{r}\right) \int_0^1 G(t,s) f(s, I^\delta(e^{\mu s} z(s)), z(s)) ds \\ &= \varphi^{-1}\left(\frac{1}{r}\right) Tz,\ r \geq 1. \end{aligned} \tag{29}$$

By (24)–(29) and $0 < \theta < 1$, one has

$$
\begin{aligned}
\phi_1 = T\phi_0 &\geq \int_0^1 G(t,s) f(s, l^m I^\delta(e^{\mu t} z_0(s)), l^m z_0(s)) ds \\
&\geq \varphi(l) \int_0^1 G(t,s) f(s, l^{m-1} I^\delta(e^{\mu t} z_0(s)), l^{m-1} z_0(s)) ds \\
&\geq \varphi^2(l) \int_0^1 G(t,s) f(s, l^{m-2} I^\delta(e^{\mu t} z_0(s)), l^{m-2} z_0(s)) ds \\
&\geq \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
&\geq \varphi^m(l) \int_0^1 G(t,s) f(s, I^\delta(e^{\mu t} z_0(s)), z_0(s)) ds \\
&= \varphi^m(l) T z_0 \geq \varphi^m(l) l z_0 \geq l^{m\theta} z_0 = \phi_0.
\end{aligned} \tag{30}
$$

In the same way, we also have $\psi_0 \geq \psi_1$. Moreover, it follows from (26) that

$$\phi_0 = l^{\theta m} z_0 \leq \frac{1}{l^{\theta m}} z_0 = \psi_0, \ l \in (0,1);$$

thus, from the increasing property of T, we have

$$\phi_0 \leq \phi_1 \leq \cdots \leq \phi_n \leq \cdots \leq \psi_n \leq \cdots \leq \psi_1 \leq \psi_0. \tag{31}$$

On the other hand, by (28), we have

$$T(rz) \geq \varphi(r) T z \geq r^\theta T z, \ 0 < r < 1, \tag{32}$$

which implies that

$$\phi_1 \geq \phi_0 = l^{2\theta m} \psi_0,$$

$$\phi_2 = T\phi_1 = T(T\phi_0) = T(T(l^{2m\theta}\psi_0)) \geq T((l^{2m\theta})^\theta T(\psi_0)) \geq (l^{2m})^{\theta^3} T(T(\psi_0)) \geq (l^{2m})^{\theta^3} \psi_2.$$

By induction, one has

$$\phi_n \geq (l^{2m})^{\theta^{n+1}} \psi_n, \ (n = 0, 1, 2, \cdots). \tag{33}$$

Thus, for any $p, n \in \mathbb{N}$, in view of (31) and (33), we have

$$\phi_{n+p} - \phi_n \leq \psi_n - \phi_n \leq \left(1 - (l^{2m})^{\theta^{n+1}}\right) \psi_n \leq \left(1 - (l^{2m})^{\theta^{n+1}}\right) \psi_0.$$

Noticing that the cone P is normal, and the normality constant is 1, one has

$$||\phi_{n+p} - \phi_n|| \leq ||\psi_n - \phi_n|| \leq \left(1 - (l^{2m})^{\theta^{n+1}}\right) ||\psi_0|| \to 0, \ \text{as } n \to +\infty; \tag{34}$$

hence, $\{\phi_n\}_{n \geq 1}$ is a monotonically increasing Cauchy sequence with upper bound ψ_0. Thus, ϕ_n converges to some $z^* \in P_0$. On the other hand, by using (34), we also have

$$||\psi_n - z^*|| \leq ||\psi_n - \phi_n|| + ||\phi_n - z^*|| \to 0, \ \text{as } n \to +\infty;$$

i.e., $\psi_n \to z^*$. By (27) and (31), $z^* \in P_0$ is a positive fixed point of T with

$$\phi_0 \leq \phi_n \leq z^* \leq \psi_n \leq \psi_0,$$

and then z^* is a positive solution to Equation (9).

Thus, for any given initial value $z_0 \in P_0$, we construct the iterative sequence

$$z_n = \int_0^1 G(t,s) f(s, I^\delta(e^{\mu t} z_{n-1}(s)), z_{n-1}(s)) ds, \ \ n = 1, 2, \cdots. \tag{35}$$

Noticing that

$$\phi_0 = l^{\theta m} z_0 \leq z_0 \leq \frac{1}{l^{\theta m}} z_0 = \psi_0,\ l \in (0,1),$$

we have

$$\phi_n \leq z_n \leq \psi_n,\ (n = 1, 2, \cdots),$$

which implies

$$\begin{aligned} ||z_n - z^*|| &\leq ||z_n - \phi_n|| + ||\phi_n - z^*|| \leq 2||\psi_n - \phi_n|| \\ &\leq 2\Big(1 - (l^{2m})^{\theta^{n+1}}\Big)||\psi_0||, \text{as } n \to +\infty. \end{aligned} \tag{36}$$

Thus, by (36), the sequence of functions defined by (35) satisfies

$$z_n \rightrightarrows z_0, \text{ as } n \to +\infty.$$

Furthermore, let $\sigma = l^{2m}$; in view of (36), the error estimation can be expressed by

$$||z_n - z^*|| \leq 2\Big(1 - (l^{2m})^{\theta^{n+1}}\Big)||\psi_0|| = 2\sigma^{\frac{\theta}{2}}\Big(1 - \sigma^{\theta^{n+1}}\Big)||z_0||,$$

and the corresponding convergence rate can be determined by

$$||z_n - z^*|| = o\Big(1 - \sigma^{\theta^{n+1}}\Big),$$

where the constant $0 < \sigma = l^{2m} < 1$ is determined by z_0.

In what follows, we show that the positive solution to the mixed integro-differential tempered fractional Equation (9) is unique. Suppose $\bar{z} \in P_0$ is any other positive solution of Equation (9), then we can find a number $0 < l_{\bar{z}} < 1$ such that

$$l_{\bar{z}} e^{-\mu t} t^{\vartheta - \delta - 1} \leq \bar{z}(t) \leq l_{\bar{z}}^{-1} e^{-\mu t} t^{\vartheta - \delta - 1},\ t \in [0,1]. \tag{37}$$

Noticing $e^{-\mu t} t^{\vartheta - \delta - 1} \in P_0$, in particular, in the above iterative process, we take the initial value $z_0 = e^{-\mu t} t^{\vartheta - \delta - 1}$ and let the m defined by (25) be large enough such that $l^{\theta m} < l_{\bar{z}}$. Then, for any $t \in [0,1]$, one has

$$\phi_0(t) \leq \bar{z}(t) \leq \psi_0(t). \tag{38}$$

Consequently, by the monotonicity of T, $T\bar{z} = \bar{z}$, and (38), one has

$$\phi_n \leq \bar{z} \leq \psi_n, n = 1, 2, \cdots. \tag{39}$$

Take the limit for (39); then, one obtains $z^* = \bar{z}$; that is, the fixed point z^* of T in P_0 is unique. Consequently, the positive solution to the mixed integro-differential tempered fractional Equation (9) is also unique.

Let $x^*(t) = e^{-\mu t} I^\delta(e^{\mu t} z^*(t))$; from Lemma 4, one sees that x^* is the unique positive solution for Equation (1). Notice $0 < \delta < 1$; then, by Lemma 1 (1), one has $z^*(t) = {}_0^R\mathbb{D}_t^{\delta,\mu} x^*(t)$. Thus, for any initial value $z_0 \in P_0$, construct a sequence of functions

$$z_n = \int_0^1 G(t,s) f(s, I^\delta(e^{\mu t} z_{n-1}(s)), z_{n-1}(s)) ds,\ \ n = 1, 2, \cdots.$$

Then,

$$z_n(t) \rightrightarrows z^*(t) = {}_0^R\mathbb{D}_t^{\delta,\mu} x^*(t), t \in [0,1].$$

Moreover, since $z^* \in P_0$, there exists a positive constant $0 < \kappa < 1$, such that

$$\kappa e^{-\mu t} t^{\vartheta - \delta - 1} \leq {}_0^R\mathbb{D}_t^{\delta,\mu} x^*(t) \leq \kappa^{-1} e^{-\mu t} t^{\vartheta - \delta - 1},$$

which implies that
$$\kappa t^{\vartheta-\delta-1} \le \mathscr{D}_t{}^{\delta}(e^{\mu t}x^*(t)) \le \kappa^{-1}t^{\vartheta-\delta-1}.$$

Noticing that $0 < \delta < 1$, by using Lemma 1(i) and (18), we have
$$\frac{\kappa\Gamma(\vartheta-\delta)}{\Gamma(\vartheta)}t^{\vartheta-1} = \kappa I^{\delta}(t^{\vartheta-\delta-1}) \le I^{\delta}\mathscr{D}_t{}^{\delta}(e^{\mu t}x^*(t)) = e^{\mu t}x^*(t) \le \kappa^{-1}I^{\delta}t^{\vartheta-\delta-1} = \frac{\Gamma(\vartheta-\delta)}{\kappa\Gamma(\vartheta)}t^{\vartheta-1};$$
that is,
$$\kappa^* e^{-\mu t}t^{\vartheta-1} \le x^*(t) \le \frac{1}{\kappa^*}e^{-\mu t}t^{\vartheta-1},$$
where
$$\kappa^* = \frac{\kappa\Gamma(\vartheta-\delta)}{\Gamma(\vartheta)}.$$

□

Remark 2. *Theorem 1 gives a comprehensive result on the iterative properties of a positive solution for a tempered fractional equation, which includes the uniqueness and existence of solutions, the sequence of iterations that converges to the unique solution, the error estimate of the solution and convergence rate, as well as the asymptotic behavior of the solution.*

Remark 3. *The iterative process has the advantage of simplification, noticing that the initial value of the starting iteration may be selected arbitrarily in P_0; thus, in particular, according to the actual needs of calculation, we can select $z_0 = e^{-\mu t}t^{\vartheta-\delta-1}$ as the initial value of the iteration.*

Remark 4. *In the case of low accuracy requirements, we can choose a sub-cone of P as P_0, i.e.,*

$$P_0 = \{z(t) \in P : \text{there exists a real number } k_z > 1 \text{ satisfying } z(t) \le k_z e^{-\mu t}t^{\vartheta-\delta-1},\ t \in [0,1]\}.$$

In this case, the initial value of the iteration can be selected as $z_0 = 0$ or $z_0 = e^{-\mu t}t^{\vartheta-\delta-1}$.

Remark 5. *The main condition* $(\mathbf{G})$ *is weaker than the following hypotheses.*

*(**A**) f is a positive continuous function in $(0,1) \times [0,+\infty) \times [0,+\infty)$ and is nondecreasing with respect to the two space variables, and there exist two constants $0 < \theta_1 \le \theta_2 < 1$, such that for any $\lambda \in (0,1)$,*

$$\lambda^{\theta_1}f(t,x,y) \le f(t,\lambda x,\lambda y) \le \lambda^{\theta_2}f(t,x,y),\ (t,x,y) \in (0,1) \times [0,+\infty) \times [0,+\infty). \tag{40}$$

Condition (40) was employed by Wei [61] to study the necessary and sufficient condition of a positive solution for super-linear singular higher order multiple points boundary value problems.

*(**B**) f is a positive and continuous function in $(0,1) \times [0,+\infty) \times [0,+\infty)$ and is nondecreasing with respect to the two space variables, and there exists a constant $0 < \theta < 1$, such that for any $\lambda \in (0,1)$,*

$$f(t,\lambda x,\lambda y) \ge \lambda^{\theta}f(t,x,y),\ (t,x,y) \in (0,1) \times [0,+\infty) \times [0,+\infty). \tag{41}$$

In a recent work [62–64], Zhang et al. established some convergence properties of entire large solutions for some k-Hessian equations and systems by using condition (41) when f does not involve a third variable.

*(**C**) f is a positive and continuous function in $(0,1) \times [0,+\infty) \times [0,+\infty)$ and is nondecreasing with respect to the two space variables, and for any $0 < \lambda < 1$, there exists a function $\eta(\lambda) = m(\lambda^{-\kappa} - 1)$, such that for all $m \in (0,1], \kappa \in (0,1)$,*

$$f(t,\lambda x,\lambda y) \ge \lambda\Big[1+\eta(\lambda)\Big]f(t,x,y), (t,x,y) \in (0,1) \times [0,+\infty) \times [0,+\infty). \tag{42}$$

In the work [65], the authors adopted (42) to obtain a necessary and sufficient condition for the existence of positive solutions of singular boundary value problems.

Obviously, condition $(\mathbf{G})$ includes condition $(\mathbf{A})$, condition $(\mathbf{B})$, condition $(\mathbf{C})$, and (H_2) in [1] as special cases. Thus, if $(\mathbf{G})$ is replaced by any one of $(\mathbf{A})$, $(\mathbf{B})$, or $(\mathbf{C})$, Theorem 1 still holds. Here, we refer the reader to some work on similar conditions for nondecreasing or decreasing nonlinearities [66–68].

4. Numerical Results

The following numerical example is designed to show the effectiveness of our main results.

Example 1. *Consider the iterative properties of positive solutions for the following tempered fractional equation*

$$\begin{cases} {}_0^R\mathbb{D}_t{}^{\frac{3}{4},2}x(t) = \dfrac{e^{\frac{2}{3}t}x^{\frac{1}{7}}(t) + \left({}_0^R\mathbb{D}_t{}^{\frac{1}{4},2}x(t)\right)^{\frac{1}{5}}}{\sqrt{t}}, \\ {}_0^R\mathbb{D}_t{}^{\frac{1}{4},2}x(0) = 0,\ {}_0^R\mathbb{D}_t{}^{\frac{1}{4},2}x(1) = \displaystyle\int_0^1 e^{-\mu(1-t)}{}_0^R\mathbb{D}_t{}^{\frac{1}{4},2}x(t)dt, \end{cases} \tag{43}$$

where

$$\vartheta = \frac{3}{2},\ \delta = \frac{1}{4},\ \mu = 2.$$

In what follows, we verify that Equation (43) satisfies the conditions $(\mathbf{G})$ and $(\mathbf{H})$. In fact, let

$$f(t,x,y) = \frac{x^{\frac{1}{7}} + y^{\frac{1}{5}}}{\sqrt{t}};$$

then, $f(t,x,y)$ is a positive and continuous function in $(0,1) \times [0,+\infty) \times [0,\infty)$, and for any fixed $t \in (0,1)$, it is also nondecreasing with respect to the space variables x, y.

Choose $\theta = \frac{1}{3}, \varphi(\lambda) = \lambda^{\frac{1}{5}}, \lambda \in (0,1)$; then, we have

$$\varphi(\lambda) = \lambda^{\frac{1}{5}} > \lambda^{\frac{1}{3}} = \lambda^{\theta}, \lambda \in (0,1),$$

and for any $(t,x,y) \in (0,1) \times [0,+\infty) \times [0,+\infty)$ and $0, \lambda < 1$,

$$f(t,\lambda x,\lambda y) = \frac{\lambda^{\frac{1}{7}}x^{\frac{1}{7}} + \lambda^{\frac{1}{5}}y^{\frac{1}{5}}}{\sqrt{t}} \geq \frac{\lambda^{\frac{1}{5}}x^{\frac{1}{7}} + \lambda^{\frac{1}{5}}y^{\frac{1}{5}}}{\sqrt{t}} = \varphi(\lambda)f(t,x,y), \tag{44}$$

which indicates that $(\mathbf{G})$ holds.

In what follows, we verify the condition $(\mathbf{H})$. Indeed,

$$0 < \int_0^1 f(s, s^{\vartheta-1}, e^{-\mu s}s^{\vartheta-\delta-1})ds = \int_0^1 \left(s^{-\frac{3}{7}} + e^{-\frac{2}{5}s}s^{-\frac{9}{20}}\right)ds < 3.5682 < +\infty. \tag{45}$$

Thus, condition $(\mathbf{H})$ also holds.

Let us take the initial value $z_0 = e^{-2t}t^{\frac{1}{4}}$; then, we have $k_{z_0} = 1$, and

$$\begin{aligned}
\widetilde{k}_{z_0} &= \min\left\{\frac{1}{3}, \left(\frac{(\vartheta-\delta)^2 e^{\mu}\varphi^{-1}(k_{z_0})}{(\vartheta-\delta-1)\Gamma(\vartheta-\delta+1)}\int_0^1 f\left(s, s^{\vartheta-1}, e^{-\mu s}s^{\vartheta-\delta-1}\right)ds\right)^{-1},\right. \\
&\qquad \left.\varphi\left(\frac{k_{z_0}\Gamma(\vartheta-\delta)}{\Gamma(\vartheta)}\right)\int_0^1 m(s) f\left(s, s^{\vartheta-1}, e^{-\mu s}s^{\vartheta-\delta-1}\right)ds\right\} \\
&= \min\left\{\frac{1}{3}, \left(\frac{(\frac{5}{4})^2 e^2}{\frac{1}{4}\Gamma(\frac{9}{4})}\int_0^1 \left(s^{-\frac{3}{7}} + e^{-\frac{2}{5}s}s^{-\frac{9}{20}}\right)ds\right)^{-1},\right. \\
&\qquad \left.\left(\frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{2})}\right)^{\frac{1}{5}}\int_0^1 \frac{5s(1-s)^{\frac{1}{4}}e^{2s}}{\Gamma(\frac{9}{4})}\left(s^{-\frac{3}{7}} + e^{-\frac{2}{5}s}s^{-\frac{9}{20}}\right)ds\right\} \\
&= \min\{0.3333, 0.0069, 4.2268\} = 0.0069.
\end{aligned}$$

Thus, $l = 0.0069$. Take $m = 8$; then,

$$\left(\frac{\varphi(l)}{l^{\theta}}\right)^m = \left(0.0069^{-\frac{2}{15}}\right)^8 = 201.9659 > 144.9275 = \frac{1}{l}. \tag{46}$$

Consequently, we have $\sigma = l^{2m} = 2.6399 \times 10^{-35}$. So, it follows from Theorem 1 that the following results are valid:

(i) (Existence and uniquness) The tempered fractional Equation (43) has a unique positive solution $x^*(t)$.

(ii) (Iterative sequence) Let $z^*(t) = {}_0^R\mathbb{D}_t^{\frac{1}{4},2}x^*(t)$; then, we have $z^* \in P_0$, where

$$\begin{aligned}
P_0 = \{z(t) \geq 0 : &\text{ there exists a real number } 0 < k_z < 1 \text{ satisfying} \\
&k_z e^{-2t}t^{\frac{1}{4}} \leq z(t) \leq k_z^{-1}e^{-2t}t^{\frac{1}{4}},\ t \in [0,1]\}.
\end{aligned}$$

Moreover, choose any $z_0 \in P_0$ as an initial value, and construct the function iterative sequence

$$\begin{aligned}
z_n =& \frac{5t^{\frac{1}{4}}e^{-2t}}{\Gamma(\frac{9}{4})}\int_0^1 \frac{(1-s)^{\frac{1}{4}}(\frac{1}{4}+s)e^{2s}\left(\Gamma^{-\frac{1}{7}}(\frac{1}{4})\left(\int_0^s (s-\tau)^{-\frac{3}{4}}z_{n-1}(\tau)d\tau\right)^{\frac{1}{7}} + z_{n-1}^{\frac{1}{5}}(s)\right)}{\sqrt{s}}ds \\
&- \frac{5e^{-2t}}{4\Gamma(\frac{9}{4})}\int_0^t \frac{(t-s)^{\frac{1}{4}}e^{2s}\left(\Gamma^{-\frac{1}{7}}(\frac{1}{4})\left(\int_0^s (s-\tau)^{-\frac{3}{4}}z_{n-1}(\tau)d\tau\right)^{\frac{1}{7}} + z_{n-1}^{\frac{1}{5}}(s)\right)}{\sqrt{s}}ds, \quad n = 1, 2, \cdots.
\end{aligned} \tag{47}$$

Then, the sequence converges to $z^*(t) = {}_0^R\mathbb{D}_t^{\frac{1}{4},2}x^*(t)$ on $[0,1]$ uniformly as $n \to +\infty$.

(iii) (Error estimate) If the initial value is taken as $z_0 = e^{-2t}t^{\frac{1}{4}}$, then the error estimate between the iterative value z_n and the exact value z^* can be calculated by

$$||z_n - z^*|| \leq \frac{2}{\sqrt[4]{8}e^{\frac{1}{4}}}(2.6399 \times 10^{-35})^{\frac{1}{6}}\left(1 - (2.6399 \times 10^{-35})^{(\frac{1}{3})^{n+1}}\right).$$

(iv) (Convergence rate) The convergence rate of the iterative process can be expressed by

$$||z_n - z^*|| = o\left(1 - (2.6399 \times 10^{-35})^{(\frac{1}{3})^{n+1}}\right).$$

(v) (Asymptotic behavior) The unique positive solution x^* to Equation (43) satisfies the following asymptotic property

$$\kappa^* e^{-2t} t^{\frac{1}{2}} \leq x^*(t) \leq \frac{1}{\kappa^*} e^{-2t} t^{\frac{1}{2}}, \ \ t \in [0,1],$$

where $\kappa^* \in (0,1)$ is a constant.

In the following, using the iterative Formula (47), we derive a graphical simulation (Figure 1) and numerical tables (Tables 1 and 2) to show the iterative sequences of the approximate solutions converging to the exact solution.

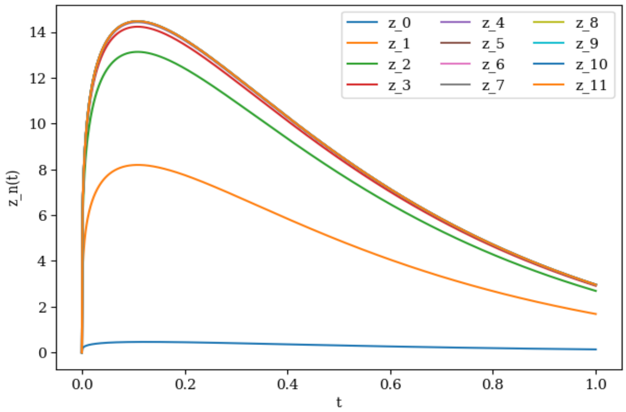

Figure 1. The iterative solution of Equation (43) at iterations $n = 0, 1, 2, 3, 4, \ldots, 10, 11$.

Table 1. The numerical approximation of the solution for Equation (43), $z_n(t)$, for $n = 0, 1, \ldots, 5$.

t	z_0	z_1	z_2	z_3	z_4	z_5
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.052632	0.431118	7.678863	12.287098	13.313274	13.497462	13.529278
0.105263	0.461465	8.081921	12.923475	14.001940	14.195515	14.228953
0.157895	0.459670	7.929976	12.673741	13.730705	13.920422	13.953193
0.210526	0.444596	7.559823	12.076544	13.083165	13.263846	13.295057
0.263158	0.423134	7.092271	11.324893	12.268412	12.437767	12.467021
0.315789	0.398619	6.584625	10.510231	11.385499	11.542603	11.569741
0.368421	0.372889	6.067706	9.681678	10.487626	10.632287	10.657275
0.421053	0.347026	5.559109	8.867222	9.605101	9.737544	9.760422
0.473684	0.321688	5.069048	8.083070	8.755471	8.876161	8.897008
0.526316	0.297276	4.603461	7.338626	7.948915	8.058455	8.077376
0.578947	0.274026	4.165377	6.638626	7.190553	7.289617	7.306728
0.631579	0.252072	3.756296	5.985416	6.482920	6.572215	6.587639
0.684211	0.231473	3.376366	5.379152	5.826177	5.906410	5.920269
0.736842	0.212242	3.025093	4.818990	5.219404	5.291270	5.303683
0.789474	0.194360	2.701592	4.303489	4.661037	4.725208	4.736292
0.842105	0.177786	2.404518	3.830444	4.148679	4.205794	4.215660
0.894737	0.162467	2.132438	3.397539	3.679820	3.730481	3.739232
0.947368	0.148339	1.883824	3.002303	3.251778	3.296551	3.304285
1.000000	0.135335	1.657098	2.642194	2.861796	2.901208	2.908016

Table 2. The numerical approximation of the solution for Equation (43), $z_n(t)$, for $n = 6, 7, ...11$.

t	z_6	z_7	z_8	z_9	z_{10}	z_{11}
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.052632	13.534738	13.535674	13.535834	13.535861	13.535866	13.535867
0.105263	14.234691	14.235674	14.235843	14.235871	14.235876	14.235877
0.157895	13.958817	13.959780	13.959946	13.959974	13.959979	13.959980
0.210526	13.300412	13.301330	13.301487	13.301514	13.301519	13.301520
0.263158	12.472041	12.472901	12.473049	12.473074	12.473078	12.473079
0.315789	11.574398	11.575196	11.575333	11.575356	11.575360	11.575361
0.368421	10.661563	10.662298	10.662424	10.662446	10.662449	10.662450
0.421053	9.764347	9.765020	9.765136	9.765155	9.765159	9.765159
0.473684	8.900585	8.901198	8.901303	8.901321	8.901325	8.901325
0.526316	8.080623	8.081180	8.081275	8.081291	8.081294	8.081295
0.578947	7.309665	7.310168	7.310254	7.310269	7.310271	7.310272
0.631579	6.590285	6.590739	6.590817	6.590830	6.590832	6.590833
0.684211	5.922647	5.923055	5.923125	5.923137	5.923139	5.923139
0.736842	5.305813	5.306179	5.306241	5.306252	5.306254	5.306254
0.789474	4.738194	4.738520	4.738576	4.738586	4.738588	4.738588
0.842105	4.217353	4.217643	4.217693	4.217701	4.217703	4.217703
0.894737	3.740734	3.740991	3.741035	3.741043	3.741044	3.741044
0.947368	3.305612	3.305840	3.305879	3.305885	3.305887	3.305887
1.000000	2.909184	2.909385	2.909419	2.909425	2.909426	2.909426

Remark 6. *In Example 1, if we take the initial value $z_0 = e^{-2t}t^{\frac{1}{4}}$, Figure 1 shows that the iterative solution to Equation (43) has converged almost to the exact solution of Equation (43) after four iterations ($n = 4$). This implies that the error between the exact value z^* and the approximate value z_4 is already very small; that is, the convergence rate of the iteration is very high.*

5. Conclusions

In this work, by weakening a growth condition of the existing results, we establish some comprehensive results on the iterative properties of positive solutions for a tempered fractional equation, which include the uniqueness and existence of solutions, the sequence of iterations that converges to the unique solution, the error estimate of the approximal solution and convergence rate, as well as the asymptotic behavior of the solution. In particular, the iterative process is simple and can start from a known initial value, and the convergence rate of the iteration is very high. In addition, in this work, we only solve the uniqueness of the positive solution to the tempered fractional equation; so based on this work, some interesting problems such as the existence of multiple solutions and the changing sign problem are still worth future study.

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