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Brief Report

Fractional Complex Euler–Lagrange Equation: Nonconservative Systems

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Abstract: Classical forbidden processes paved the way for the description of mechanical systems with the help of complex Hamiltonians. Fractional integrals of complex order appear as a natural generalization of those of real order. We propose the complex fractional Euler-Lagrange equation, obtained by finding the stationary values associated with the fractional integral of complex order. The complex Hamiltonian obtained from the Lagrangian is suitable for describing nonconservative systems. We conclude by presenting the conserved quantities attached to Noether symmetries corresponding to complex systems. We illustrate the theory with the aid of the damped oscillatory system.

Keywords: complex fractional integral; complex Hamiltonian dynamic; symmetries

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1. Introduction

The reality we can observe and measure is limited to rational numbers. Complex numbers are especially used in physics because of their beautiful properties, which translate into a powerful mathematical tool designed to solve many real-world problems. For example, the evaluation of a large class of real integrals is most easily performed by embedding the problem in the complex plane [1].

To be able to take into account classical forbidden processes [2], it was necessary to introduce complex Hermitian Hamiltonians. In the paper [3], the authors show that the Dirac hermicity condition is not necessary and that it can be replaced by the physical condition of space–time reflection symmetry. Since this work, the study of complex Hamiltonians, both in the quantum and the classical system, has become of considerable theoretical interest, without noticing much of its necessity on the experimental front. Recently, in the work [4], the authors developed the Struckmeier and Riedel formalism in a complex phase space and constructed invariants for some physical systems. Considering the importance of complex dynamical systems, in the paper [5], the authors investigated the classical invariants for some non-Hermitian anharmonic potentials in one dimension.

To study physical behavior, we need a suitable mathematical apparatus. Probably the most well-known fractional differential operator is the Riemann–Liouville. One of the remarkable properties of this operator is that the functions on which it acts do not necessarily have to be continuous at the origin and do not have to be differentiable [6]. Being the first defined fractional operator, it was also a source of continuous inspiration for other fractional operators. Due to the fact that when applied to a constant, it does not produce the result zero, its applications in the real world are limited.

By far the most popular fractional differential operator is the Caputo one, and this is mainly due to the fact that it presents initial conditions [7], making it perfect for modeling physical phenomena. Unlike the Riemann–Liouville derivative, the Caputo derivative of a constant is zero. The Caputo derivative is suitable for problems with nonlocal properties, being able to preserve the history of the analyzed phenomenon.

Both the derivatives in the Riemann–Liouville sense and those in the Caputo sense have singular kernels. Wanting to introduce a class of fractional derivatives built on nonsingular kernels, Fabrizio proposed a new fractional differential [8]. The Caputo–Fabrizio operator presents two of the most important properties of the Caputo operator: it presents initial conditions, and the derivative of the constants is always zero [9].

Using the generalized Mittag–Leffler function, Atangana and Baleanu [10] introduced a fractional derivative, using as in the case of the Caputo–Fabrizio derivative a nonsingular kernel. The derivative also has nonlocal properties and preserves the good properties of Caputo derivatives. Being based on Mittag–Leffler functions, the differential equations based on these derivatives generalize the exponential behavior. By analytical continuity, the Atangana–Baleanu derivatives can be extended to complex values.

All derivatives have pluses and minuses. For example, the Caputo derivative requires that all the functions on which it operates are continuous and differentiable. On the other hand, nonsingular kernels are not useful in describing problems that have nonzero initial conditions. According to the latest studies, the differential equations described with the help of nonsingular kernels can be equated with ordinary differential equations [11].

In this paper, we use fractional derivatives in the Caputo sense. The fractional differential of complex order appeared as a natural generalization of those with real order and opened a way to fractional differential equations of complex order [12]. In this paper, we propose the complex fractional Euler–Lagrange equation, obtained by finding the stationary values associated with the fractional integral of complex order, intended for the description of nonconservative systems. We further introduce the Hamiltonian for the situation where the Lagrangian is not explicitly time dependent and conclude that it is not conservative. We conclude the paper with the calculation of the conserved quantities associated with nonconservative systems.

The paper is structured as follows: In Section 2, we introduce the definition of complex fractional integrals and derive the complex fractional Euler–Lagrange equation, by finding the stationary values associated with the fractional integral of complex order. We also study the dynamics of damped oscillatory systems. In Section 3, we introduce the Hamiltonian for the situation where the Lagrangian has no explicit time dependence, and in Section 4, we establish the transformation criteria of the Noether symmetries and introduce the conserved quantities.

2. Fractional Complex Euler–Lagrange Equation

For a function f that accepts the conditions $f(x) = 0$ for $x \leq 0$, for a complex number α with $\Re(\alpha) \geq 0$, we define the α-order integral of the function f by the relation [13]

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(\tau)(t-\tau)^{\alpha-1} d\tau \,. \tag{1}$$

The integral $I^{\alpha} f(t)$ is absolutely convergent for $\alpha \in \mathbb{C}$ and $\Re(\alpha) > 0$.

In what follows, we aim to find the Euler–Lagrange equation by finding the stationary values associated with Equation (1). For this, we define the following action

$$S(q) \equiv \frac{1}{\Gamma(\alpha)} \int_0^t L(\dot{q}, q, \tau)(t-\tau)^{\alpha-1} d\tau \,, \tag{2}$$

where $L(\dot{q}, q, \tau)$ is the Lagrangian of a system with N degrees of freedom, and $q = (q_1, \ldots, q_N)$.

It is easy to verify that

$$\frac{d}{d\tau} \frac{t^{\alpha} - (t-\tau)^{\alpha}}{\Gamma(1+\alpha)} \equiv \frac{d}{d\tau} g_t(\tau) = \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \,,$$

and then we can define the action as the Riemann–Stieltjes integral

$$S(q) = \frac{1}{\Gamma(\alpha)} \int_0^t L(\dot{q}, q, \tau) dg_t(\tau) \,.$$

Proposition 1. *If the equations of motion support solutions with fixed boundary values, then the Euler–Lagrange equation is written as*

$$\frac{\partial L}{\partial q_k} - \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{1-\alpha}{t-\tau}\frac{\partial L}{\partial \dot{q}_k} \,, \quad \alpha \in \mathbb{C} \,. \tag{3}$$

Proof. We start by writing $q(\tau) = q_0(\tau) + Q(\tau)$, with $Q = (Q_1, \ldots, Q_N)$, and then

$$S(q(\tau)) = \frac{1}{\Gamma(\alpha)} \int_0^t L(\dot{q_0}(\tau) + \dot{Q}(\tau), q_0(\tau) + Q(\tau), \tau)(t-\tau)^{\alpha-1} d\tau \,.$$

Keeping in the above expression only the first order of the Taylor expansion, we obtain

$$\begin{aligned} S(q(\tau)) \; &= \; \frac{1}{\Gamma(\alpha)} \int_0^t L(\dot{q}_0, q_0, \tau)(t-\tau)^{\alpha-1} d\tau \\ &+ \; \frac{1}{\Gamma(\alpha)} \sum_k \int_0^t \left(\frac{\partial L}{\partial \dot{q}_k} \dot{Q}_k(\tau) + \frac{\partial L}{\partial q_k} Q_k(\tau) \right)(t-\tau)^{\alpha-1} d\tau \,. \end{aligned}$$

Using

$$\dot{Q}_k(t-\tau)^{\alpha-1} = \frac{d}{d\tau}\Big(Q_k(t-\tau)^{\alpha-1}\Big) + (\alpha-1)Q_k(t-\tau)^{\alpha-2}$$

and integrating by parts we obtain

$$\delta S = \frac{1}{\Gamma(\alpha)} \sum_k \int_0^t Q_k \left(\frac{\partial L}{\partial q_k} - \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{q}_k}\right) + \frac{\alpha-1}{t-\tau}\frac{\partial L}{\partial \dot{q}_k} \right)(t-\tau)^{\alpha-1} d\tau \,,$$

where we used the fact that the physical system is subject to fixed boundary conditions. According to the principle of least action, $\delta S = 0$, the desired result is obtained. □

Example 1. *(Damped oscillatory system) We consider as a one-dimensional dynamical system a pendulum of length l and mass m. For small oscillations around the equilibrium point, we have the kinetic energy and potential energy given by the formulas [14]*

$$K = \frac{1}{2} m l^2 \dot{\theta}^2 \,, \quad V = \frac{1}{2} m g l \theta^2 \,,$$

where θ is the angular coordinate and g the gravitational acceleration, and the derivative is obtained with respect to τ. The Lagrangian is defined as $L = K - V$; i.e.,

$$L = \frac{1}{2} m l^2 \dot{\theta}^2 - \frac{1}{2} m g l \theta^2 \,,$$

and making the notation $T = t - \tau$ and $\omega^2 = g/l$, from Equation (3), we obtain the differential equation

$$\ddot{\theta}(\tau) + \frac{\alpha-1}{T}\dot{\theta}(\tau) + \omega^2 \theta(\tau) = 0 \,, \tag{4}$$

where the derivative is made in relation to T.

To solve this system of equations, we look for solutions of the form $\theta(\tau) = T^{\rho}\varphi(T)$. Since ω is constant for a given problem, taking conveniently chosen units of measure, we set $\omega = 1$ and obtain

$$T^2\ddot{\varphi}(T) + T\dot{\varphi}(T)(2\rho + \alpha - 1) + \varphi(T)\left(\rho(\rho + \alpha - 2) + T^2\right) = 0\,.$$

By choosing $2\rho + \alpha - 1 = 1$, we obtain the Bessel differential equation [15]

$$T^2\ddot{\varphi}(T) + T\dot{\varphi}(T) + \varphi(T)\left(T^2 - \rho^2\right) = 0\,.$$

This equation admits two classes of solutions, called the Bessel function of the first kind $J_\rho(T)$ and the Bessel function of the second kind $Y_\rho(T)$, with $\rho = (2-\alpha)/2$. The two solutions of Equation (4) are

$$\theta(\tau) = T^{\rho} J_\rho(T)\,, \tag{5}$$

and

$$\chi(\tau) = T^{\rho} Y_\rho(T)\,. \tag{6}$$

The complex contribution of α appears in ρ and therefore in θ.

The most general solution is the linear combination of the two solutions. We can choose, for example, the Hankel functions of the first $H^1_\rho(T)$ and the second kind $H^2_\rho(T)$, and we can write the solutions of Equation (4) as $\pi^1(\tau) = T^\rho H^1_\rho(T)$ and $\pi^2(\tau) = T^\rho H^2_\rho(T)$.

In Figure 1, we have represented the solution (5) for the case where $\alpha \in \mathbb{R}$, and in Figure 2, we have represented the same solution in the case of $\alpha \in \mathbb{C}$. In Figure 2, we plotted $\theta(\tau)$ for the situation where $\alpha = 0.5 + i\alpha_2$, with different values of α_2, concluding that the graph of the function depends on the imaginary part of α. On the other hand, compared to Figure 1, in Figure 2, all graphs for $\alpha = 0.5 + i\alpha_2$ intersect the graph corresponding to the real case, also outside the point $T = 0$.

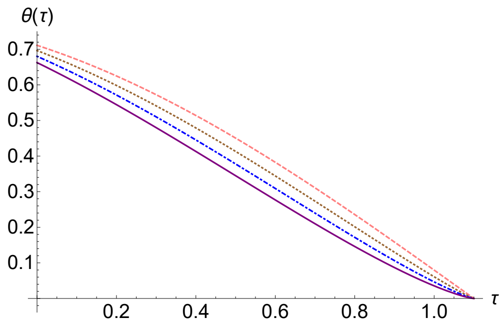

Figure 1. $\theta(\tau)$ for $t = 1.1$, $\alpha_2 = 0$, with $\alpha_1 = 0.7$ (dashed, opal), $\alpha_1 = 0.8$ (dotted, brown), $\alpha_1 = 0.9$ (dashed–dotted, blue), and $\alpha_1 = 1$ (continuous, purple).

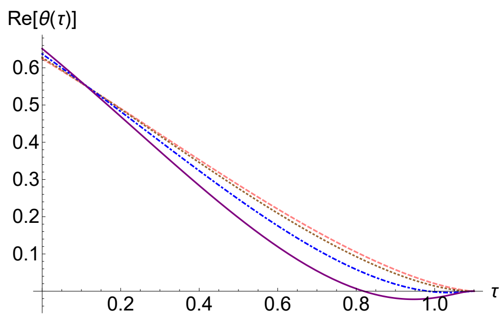

Figure 2. $\Re(\theta(\tau))$ for $t = 1.1$, $\alpha_1 = 0.5$, with $\alpha_2 = 0.9$ (dashed, opal), $\alpha_2 = 0.6$ (dotted, brown), $\alpha_2 = 0.3$ (dashed–dotted, blue), and $\alpha_2 = 0$ (continuous, purple).

In Figure 3, we have represented the solution (6) for the case where $\alpha \in \mathbb{R}$, and in Figure 4, we have represented the same solution in the case of $\alpha \in \mathbb{C}$. In Figure 2, we plotted $\chi(\tau)$ for the situation where $\alpha = 0.5 + i\alpha_2$, with different values of α_2, concluding that the graph of the function depends on the imaginary part of α. The behavior of this solution is very different from that of the solution (5), which is negative and not zero in $T = 0$. Also, regardless of whether α is real real or complex, all graphs intersect the ordinary case where $\alpha = 1$.

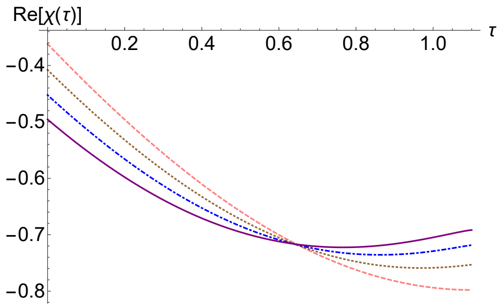

Figure 3. $\chi(\tau)$ for $t = 1.1$, $\alpha_2 = 0$, with $\alpha_1 = 0.7$ (dashed, opal), $\alpha_1 = 0.8$ (dotted, brown), $\alpha_1 = 0.9$ (dashed–dotted, blue), and $\alpha_1 = 1$ (continuous, purple).

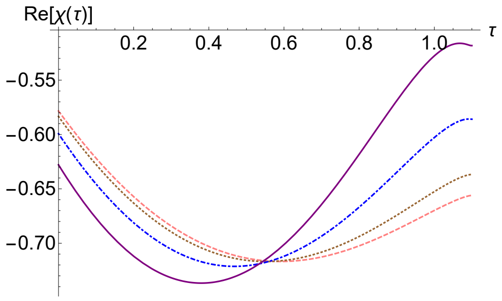

Figure 4. $\Re(\chi(\tau))$ for $t = 1.1$, $\alpha_1 = 0.5$, with $\alpha_2 = 0.9$ (dashed, opal), $\alpha_2 = 0.6$ (dotted, brown), $\alpha_2 = 0.3$ (dashed–dotted, blue), and $\alpha_2 = 0$ (continuous, purple).

Next, we analyze the solutions expressed with the help of Hankel functions. When $\alpha \in \mathbb{R}$, both the function $\pi^1(\tau)$ and $\pi^2(\tau)$ have graphs similar to the one represented in Figure 1. When $\alpha \in \mathbb{C}$, the situation changes. For the same setup as the one in Figure 1, in Figure 5 and in Figure 6, we have represented the solutions $\Re(\pi^1(\tau))$ and $\Re(\pi^2(\tau))$, respectively. In Figure 5, all graphs intersect the ordinary situation $\alpha = 1$ but in Figure 6 none. We can conclude that with the help of the linear combination of the two solutions (5) and (6) very different behaviors can be obtained, which can correspond to certain situations existing in nature.

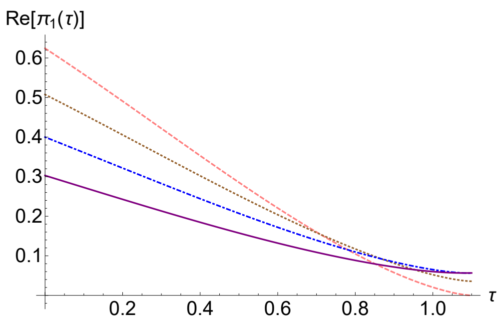

Figure 5. $\Re(\pi^1(\tau))$ for $t = 1.1$, $\alpha_1 = 0.5$, with $\alpha_2 = 0.9$ (dashed, opal), $\alpha_2 = 0.6$ (dotted, brown), $\alpha_2 = 0.3$ (dashed–dotted, blue), and $\alpha_2 = 0$ (continuous, purple).

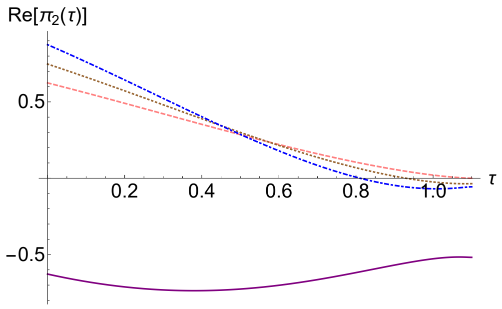

Figure 6. $\Re(\pi^2(\tau))$ for $t = 1.1$, $\alpha_1 = 0.5$, with $\alpha_2 = 0.9$ (dashed, opal), $\alpha_2 = 0.6$ (dotted, brown), $\alpha_2 = 0.3$ (dashed–dotted, blue), and $\alpha_2 = 0$ (continuous, purple).

Remark 1. *To see how the nature of α influences Equation (4), we approach another way of solving. We make the notations $\alpha = \alpha_1 + i\alpha_2$, with $\{\alpha_1, \alpha_2\} \in \mathbb{R}$ and $\theta = \theta_1 + i\theta_2$, with $\{\theta_1, \theta_2\}$ defined on $\mathbb{R}$, obtaining the system of equations*

$$\begin{cases} \ddot{\theta}_1 + \frac{\alpha_1 - 1}{T}\dot{\theta}_1 - \frac{\alpha_2}{T}\dot{\theta}_2 + \omega^2\theta_1 = 0\,, \\ \ddot{\theta}_2 + \frac{\alpha_1 - 1}{T}\dot{\theta}_2 + \frac{\alpha_2}{T}\dot{\theta}_1 + \omega^2\theta_2 = 0\,, \end{cases}$$

in which the solutions θ_1 and θ_2 are coupled. If in Equation (4) $\alpha \in \mathbb{R}$, ($\alpha_2 = 0$), we obtain a system where the equations of motion for θ_1 and θ_2 are decoupled. Consequently, only the complex nature of α connects this system of equations.

Remark 2. *In Equation (4), the second term is the dissipative one and is a consequence of the fractional calculation. If $\alpha = 1$, the dissipative term is canceled. If in Equation (4), without a dissipative term, we add a small imaginary contribution $i\epsilon$, i.e.,*

$$\ddot{\theta} + \frac{i\epsilon}{T}\dot{\theta} + \omega^2\theta = 0\,,$$

then the solution of the equation coincides with the solution of the complex fractional theory where $\epsilon = \alpha_2$. This means that the Euler–Lagrange equation that is associated with a fractional derivative with complex order corresponds to a complex situation.

3. Hamiltonian Dynamics

In this section, we analyze the situation where the Lagrangian does not have an explicit time dependence, being able to define the Hamiltonian as

$$H(q, \dot{q}) = \sum_k \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L(q, \dot{q})\,. \tag{7}$$

Theorem 1. *The Hamiltonian defined above is not a constant of motion; its derivative with respect to time has the form*

$$\frac{dH}{d\tau} = \frac{\alpha - 1}{t - \tau} \sum_k \dot{q}_k \frac{\partial L}{\partial \dot{q}_k}\,. \tag{8}$$

We are interested in the real part of the Hamiltonian derivative above being zero, since the imaginary part is not measurable. We obtain the condition

$$\Re\left(\sum_k (\alpha - 1)\dot{q}_k \frac{\partial L}{\partial \dot{q}_k}\right) = 0\,. \tag{9}$$

Proof. Starting from Equation (7), we can write

$$\frac{dH}{d\tau} = \sum_k \left(\ddot{q}_k \frac{\partial L}{\partial \dot{q}_k} + \dot{q}_k \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{q}_k} \right) \right) - \frac{dL(q,\dot{q})}{d\tau} \,,$$

or

$$\frac{dH}{d\tau} = \sum_k \dot{q}_k \left(\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} \right),$$

which together with Equation (3), concludes the demonstration.

If we write the Hamiltonian of the system in the form $H = H_1 + iH_2$ and ask that the time derivative of the real part of the Hamiltonian be canceled, we obtain Equation (9). □

Remark 3. *It is observed from Equation (8) that only if $\alpha \in \mathbb{C}$ is condition (9) obtained. If $\alpha \in \mathbb{R}$, then the only possibility for Equation (8) to be an equation of motion is the nonfractional situation corresponding to $\alpha = 1$.*

Example 2. *(Damped oscillatory system) For this case, we have only one variable, and Equation (9) is written*

$$\Re\left((\alpha - 1)\dot{\theta} \frac{\partial L}{\partial \dot{\theta}} \right) = \Re\left((\alpha_1 + i\alpha_2 - 1)(\dot{\theta}_1 + i\dot{\theta}_2)^2 \right) = 0 \,,$$

obtaining the condition

$$(\alpha_1 - 1)\left(\dot{\theta}_1^2 - \dot{\theta}_2^2 \right) - 2\alpha_2 \dot{\theta}_1 \dot{\theta}_2 = 0 \,. \tag{10}$$

The above equation together with solution (5) determines a spectrum of values for τ. The easiest way to solve this equation is geometrically. From Equation (8), we know that we have an infinity of solutions for $\alpha = 1$, and from Equation (10), we obtain a discrete spectrum of solutions if $\alpha \in \mathbb{C}$. So the joint spectrum of solutions is given by the intersection of the graphs $\alpha = 1$ with $\alpha = \alpha_1 + i\alpha_2$, for a given α_1 and α_2.

In Figure 7, for the case where $t = 1.1$, we have represented with continuous (purple) $\alpha = 1$ and with dashed–dotted (blue) $\alpha = 1 + i\,0.1$. The point of intersection gives us the solution of Equation (10). Equation (10) does not always have a solution. In Figure 8, we have represented the graphs of $\theta(\tau)$ for $\alpha = 1$ (continuous, purple) and for $\alpha = 0.5 + i\,0.1$ (dashed–dotted, blue), and it can be seen that the two graphs do not intersect.

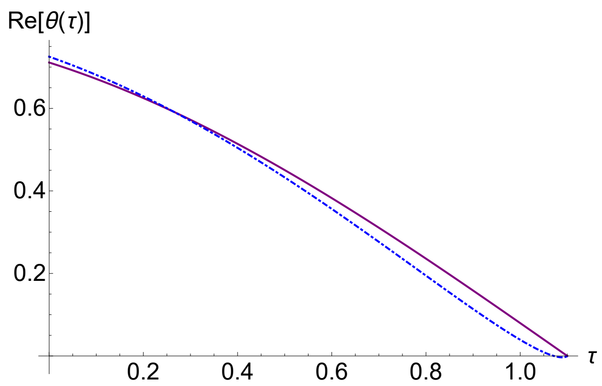

Figure 7. $\Re(\theta(\tau))$ for $t = 1.1$, with $\alpha = 1 + i\,0.1$ (dashed–dotted, blue), and $\alpha = 1$ (continuous, purple).

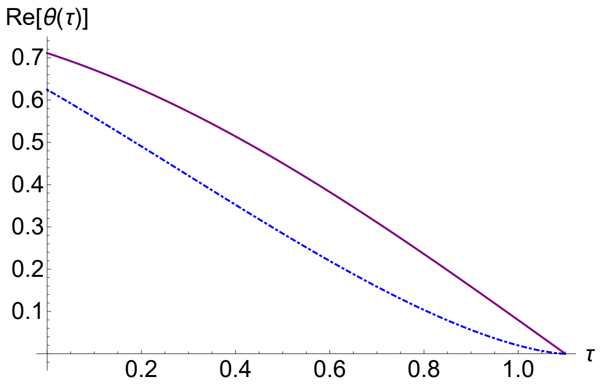

Figure 8. $\Re(\theta(\tau))$ for $t = 1.1$, with $\alpha = 0.5 + i\,0.1$ (dashed–dotted, blue), and $\alpha = 1$ (continuous, purple).

Example 3. *If the Lagrangian does not explicitly depend on the coordinates* q_k*, we have* $\frac{\partial L}{\partial q_k} = 0$*, and then Equation (3) becomes*

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{q}_k}\right) + \frac{1-\alpha}{t-\tau}\frac{\partial L}{\partial \dot{q}_k} = 0\,,$$

from which we obtain

$$\ln\left(\frac{\partial L}{\partial \dot{q}_k}\right) = (1-\alpha)\ln(\tau - t) + \ln C\,,$$

which gives

$$\frac{\partial L}{\partial \dot{q}_k} = C\ln(\tau - t)^{(1-\alpha)}\,.$$

By entering this result in Equation (9), we obtain

$$\Re\Big(C(\alpha-1)^2\ln(\tau - t)\Big) = 0\,,$$

which translates to $\alpha_1 = 1$*, which is an ordinary derivative. In conclusion, if the Lagrangian does not explicitly depend on the coordinates* q_k*, for the Hamiltonian to be a constant of motion, we must have the condition* $\Re(\alpha) = 1$*.*

Example 4. *In the situation where the Lagrangian does not explicitly depend on the coordinates* $\dot{q}_k$*, it is directly seen from Equation (8) that* $\frac{dH}{d\tau} = 0$ *, and therefore the Hamiltonian is a constant of motion.*

4. Complex Noether Symmetries

In this section, we establish the criteria for the transformation of Noether symmetries [16,17]. The first observation we make is that, according to Equation (3), we define the generalized nonpotential force

$$F_k = \frac{1-\alpha}{t-\tau}\frac{\partial L}{\partial \dot{q}_k}\,.$$

Definition 1. *We say that Equation (2) is invariant to generalized quasi-infinitesimal transformations of the finite transformation group* G_r

$$\tau^* = \tau\,, \quad q_k^*(\tau) = q_k(\tau) + \epsilon_i \xi_k^i(\tau, q)\,,$$

if

$$\int_0^t L(\dot{q},q,\tau)(t-\tau)^{\alpha-1}d\tau = \int_0^t L(\dot{q}^*,q^*,\tau)(t-\tau)^{\alpha-1}d\tau - \int_0^t \left(\frac{d(\epsilon_i G^i)}{dt} + F_k\delta q_k\right)(t-\tau)^{\alpha-1}d\tau\,, \tag{11}$$

where ϵ_i, $i = \overline{1,r}$, $G^i = G^i(\tau,q,\dot{q})$, *and where* $\xi^i_k \in \mathbb{C}$ *are called the complex infinitesimal generators of the transformation.*

Proposition 2. *The generalized quasi-invariant tranformations satisfy the following equation*

$$\frac{\partial L}{\partial \dot{q}_k}\dot{\xi}^i_k + \frac{\partial L}{\partial q_k}\xi^i_k - \frac{1-\alpha}{t-\tau}\frac{\partial L}{\partial \dot{q}_k}\xi^i_k - \dot{G}^i = 0\,. \tag{12}$$

Proof. Considering the fact that

$$L(\dot{q}^*,q^*,\tau) = L(\dot{q}_k + \epsilon_i\dot{\xi}^i_k, q_k + \epsilon_i\xi^i_k, \tau)\,,$$

differentiating both parts of Equation (11) to ϵ_i, and then making $\epsilon_i = 0$, the required result is obtained. □

Definition 2. *We call a function* $I(\dot{q},q,\tau)$ *a conserved quantity if*

$$\frac{\partial I(\dot{q},q,\tau)}{\partial \tau} = 0\,.$$

Proposition 3. *The generalized quasi-invariant tranformations given by Equation (12) preserve the following quantity*

$$I^\alpha = \frac{\partial L}{\partial \dot{q}_k}\xi^i_k - G^i\,. \tag{13}$$

Proof. Equation (12) together with Equation (3)

$$\frac{\partial L}{\partial \dot{q}_k}\dot{\xi}^i_k + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\xi^i_k - \dot{G}^i = 0\,,$$

or

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\xi^i_k - G^i\right) = 0\,,$$

and the demonstration is completed. □

Example 5. *(Damped oscillatory system) To calculate the conserved quantities associated with the damped oscillatory system, we start from Equation (12)*

$$ml^2\dot{\theta}\dot{\xi}^i_k + ml^2\ddot{\theta}\xi^i_k - \dot{G} = 0\,,$$

or after rearrangement

$$\ddot{\theta} + \frac{\dot{\xi}}{\xi}\dot{\theta} - \frac{1}{ml^2\xi}\dot{G} = 0\,.$$

Comparing this equation with Equation (4), we can make the identifications

$$\frac{\dot{\xi}}{\xi} = \frac{\alpha - 1}{t - \tau}, \quad \dot{G} = -ml^2\omega^2\xi\theta\,.$$

It is easy to calculate

$$\xi(\tau) = (t - \tau)^{(1-\alpha)}\,,$$

a result that allows the analytical calculation of $G(\tau)$ in the case of $\alpha = 1$, which is a result that can be expressed with the help of the hypergeometric function. If $\alpha \neq 1$, the integral can be solved numerically. By inserting the values of $\xi(\tau)$ and $G(\tau)$ into Equation (13), the conserved quantity is determined.

5. Conclusions

In this work we applied complex fractional calculus to obtain complex Euler–Lagrange equations, specific to nonconservative systems. Next, we introduced the Hamiltonian for the situation in which the Lagrangian does not explicitly depend on time, and we concluded that when the Hamiltonian derivative with respect to time is zero, the fractional trajectory intersects the classical trajectory, i.e., the one corresponding to $\alpha = 1$. For example, we considered the damped oscillatory system and learned that this behavior is specific to the complex case described in Figure 8, and in the real case the trajectories do not intersect, as illustrated in Figure 7. Finally, we calculated the quantities associated with the complex fractional action.

Chaos is understood as the sensitivity of the solutions of a dynamic system to the initial conditions. There is a close connection between chaos and fractional derivatives in the Caputo sense, primarily due to the fact that the fractional derivatives in the Caputo sense are sensitive to the initial conditions. Differential equations that use differential operators in the Caputo sense have the advantage that they obtain ordinary dynamics when the fractional parameter is one. There is a rule, called order, which clearly states that we cannot obtain chaotic behavior with a system consisting of less than three differential equations. The exception to this rule appears when the system contains fractional derivatives [18]. There is also the opposite procedure, in which the use of Caputo derivatives stabilizes the chaotic behavior [19], for example, in optimal control problems. One of the basic reasons for the introduction of new fractional derivatives consists in building operators that preserve the history of interactions. On the other hand, in the paper [11], it was demonstrated that only fractional operators containing singular kernels satisfy this property. Last but not least, it should be stated that the solutions of the fractional equations are generally expressed using the Mittag–Leffler functions $E_\alpha(t)$, where α is the fractional coefficient. These functions are very sensitive to the variation of the dynamic parameter t. For example, $E_{0.25}(3) \sim 10^{35}$, $E_{0.25}(4) \sim 10^{111}$, and $E_{0.25}(5) \sim 10^{272}$. In this work, we use the Riemann–Liouville integral operator, because it is the inverse operator of the fractional derivative in the Caputo sense.

In general, the complexity seems to take into account new features of the data, and its role is to introduce constraints into the analyzed problems. We are also familiar with the fact that the laws of physics work well when studying closed systems, but the functionality of the theory is questioned in the case of open systems. The complexity of a dynamic system is a step forward in open systems accounting. Complex fractional calculus can also be seen as a new tool for the study of open chaotic systems.

In general, complexification is conducted by rewriting the real differential equations in complex form, and in the end, to describe the physical solution, the real part is considered. Differential equations of motion can be modified by replacing ordinary derivatives with complex fractional derivatives, characterized by a number of free parameters. In the method proposed in this paper, a single free parameter $\alpha \in \mathbb{C}$ appears.

Mathematical models assume simplifications, and crucial factors can be found among the omitted factors. In this context, the fractional calculus by means of the factional parameter can bring significant corrections in the description of the dynamics of physical systems. Considering the chaotic behavior around the equilibrium points, a similar analysis can be extended beyond damped oscillatory systems. But we learned from this lesson that the complex part of alpha consistently changes the dynamics of solutions, and the existence of this wide range of behaviors can allow the identification of concrete situations existing in nature. We also saw that there are situations where the fractional trajectory intersects the classical trajectory.

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