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Article

Solving a Nonlinear Fractional Differential Equation Using Fixed Point Results in Orthogonal Metric Spaces

Afrah Ahmad Noman Abdou

Department of Mathematics, University of Jeddah, P.O. Box 80327, Jeddah 21589, Saudi Arabia; aabdou@uj.edu.sa

Abstract: This research article aims to solve a nonlinear fractional differential equation by fixed point theorems in orthogonal metric spaces. To achieve our goal, we define an orthogonal Θ-contraction and orthogonal (α, Θ)-contraction in the setting of complete orthogonal metric spaces and prove fixed point theorems for such contractions. In this way, we consolidate and amend innumerable celebrated results in fixed point theory. We provide a non-trivial example to show the legitimacy of the established results.

Keywords: nonlinear fractional differential equation; fixed point; orthogonal $(\alpha; \Theta)$-contraction; orthogonal metric space

MSC: 46S40; 54H25; 47H10

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1. Introduction

In fixed point theory, the Banach contraction principle [1] is one of the most prominent and substantial results that was first introduced and established by Stefan Banach in 1922. Based on the intelligibility, adequacy, and applications of this result, it has become a very famous tool in solving existence problems in numerous branches of mathematical analysis. So several researchers have boosted, broadened, and elongated this theorem in various directions. In 2014, Jleli et al. [2] introduced a new variant of contractions in the setting of generalized metric spaces, which is known as Θ-contraction. As a consequence, they obtained a fixed point result in complete metric space, which is a generalization of Banach's fixed point theorem. Hussain et al. [3] introduced a different condition in the notion of Θ-contraction and proved a result that is an extension of the result of Jleli et al. [2]. Ahmad et al. [4] changed the third postulate of Θ-contraction with an easy one. Later on, Imdad et al. [5] gave the notion of weak Θ-contraction by omitting some conditions of Θ-contraction and established some related theorems in the framework of complete metric spaces. Subsequently, Ameer et al. [6,7] presented Ćirić type α_*-η_*-Θ-contractions and Suzuki-type Θ-contractions and obtained a fixed point theorem for multivalued mappings. For further details in this field, we refer the researchers to [8–11].

Gordji et al. [12] innovated the concept of orthogonality in metric spaces and set up the fixed point result for self-mappings in the background of orthogonal metric spaces. Baghani et al. [13] improved the leading result of Gordji et al. [12] by proving some new fixed point theorems. They also investigated the existence and uniqueness of a solution to a Volterra-type integral equation in L^p space as application of their main theorem. Afterward, Baghani et al. [14,15] manifested fixed and coinciding point results for multivalued mappings. Hazarika et al. [16] discussed the general convergence methods in the setting of orthogonal metric spaces and studied the applications of fixed point results to obtain the existence of a solution of differential and integral equations. For more achievements in this direction, we refer researchers to [17–20].

On the other hand, abstract spaces like metric spaces, normed spaces, and inner product spaces are all examples of "topological spaces", which are more general spaces.

These spaces have been specified in order of increasing structure; that is, every inner product space is a normed space, and in turn, every normed space is a metric space. Two vectors are said to be orthogonal if and only if their inner product is zero, i.e., they make an angle of 90° ($\pi/2$ radians), or one of the vectors is zero in the context of inner product spaces. The complete inner product space is called a Hilbert space. Some fixed point theorems for contractive and nonexpansive mappings in the setting of Hilbert spaces are given in the literature [21–23]. However, no one has obtained fixed point theorems for Θ-contraction mappings in Hilbert spaces.

In this research, we introduce the notion of Θ-contraction mappings in orthogonally complete metric spaces and obtain some fixed point results for these mappings. Also, we give an example to illustrate the validity of our results. Moreover, we apply our results to investigate the solution to a differential equation. As a consequence of our leading result, we deduce the prime theorem of Jleli et al. [2] and several well-known results from the literature.

2. Preliminaries

In this article, we represent by $\mathbb{N}$ and $\mathbb{R}^+$ the set of natural numbers and the set of positive real numbers, respectively.

Jleli et al. [2] initiated the notion of Θ-contraction along the following lines.

Definition 1. *Let $\Theta : (0,\infty) \to (1,\infty)$ be a function such that*

($\mathcal{J}_1$) Θ is non-decreasing; i.e., $0 < \xi < \varsigma$ implies $\Theta(\xi) < \Theta(\varsigma)$;
($\mathcal{J}_2$) For every sequence $\{\xi_J\} \subseteq (0,\infty)$, we have $\lim_{J\to\infty} \Theta(\xi_J) = 1$ if and only if $\lim_{J\to\infty}(\xi_J) = 0$;
($\mathcal{J}_3$) There exists $0 < r < 1$ and $\sigma \in (0,\infty]$ such that $\lim_{\xi\to 0^+} \frac{\Theta(\xi)-1}{\xi^r} = \sigma$.

A mapping $\mathcal{V} : (\mathcal{X},\tau) \to (\mathcal{X},\tau)$ is said to be Θ-contraction if there exists some function $\Theta : \mathbb{R}^+ \to (1,\infty)$ satisfying ($\mathcal{J}_1$)-($\mathcal{J}_3$) and a constant $\lambda \in (0,1)$ such that for all $\xi,\varsigma \in \mathcal{X}$,

$$\tau(\mathcal{V}\xi,\mathcal{V}\varsigma) \neq 0 \Longrightarrow \Theta(\tau(\mathcal{V}\xi,\mathcal{V}\varsigma)) \leq [\Theta(\tau(\xi,\varsigma))]^\lambda.$$

Theorem 1 ([2])**.** *Let $(\mathcal{X},\tau)$ be a complete metric space and $\mathcal{V} : (\mathcal{X},\tau) \to (\mathcal{X},\tau)$ be a Θ-contraction; then, there exists a unique point $\xi^* \in \mathcal{X}$ such that $\xi^* = \mathcal{V}\xi^*$.*

Hussain et al. [3] introduced the following condition
($\mathcal{J}_4$): $\Theta(\xi+\varsigma) \leq \Theta(\xi)\Theta(\varsigma)$,
of the function $\Theta : (0,\infty) \to (1,\infty)$ and generalized the above theorem of Jleli et al. [2] in complete metric spaces. Inspired by Hussain et al. [3], we express by Ψ the class of all mappings $\Theta : (0,\infty) \to (1,\infty)$ fulfilling ($\mathcal{J}_1$)–($\mathcal{J}_4$).

Ahmad et al. [4] replaced the condition ($\mathcal{J}_3$) with a simple condition ($\mathcal{J}_3^{/}$).

($\mathcal{J}_3^{/}$)Θ is continuous on $(0,\infty)$.

We represent by Ω the class of all mappings satisfying ($\mathcal{J}_1$), ($\mathcal{J}_2$), and ($\mathcal{J}_3'$).

Gordji et al. [12] present the concept of the orthogonal set ($\mathcal{O}$-set, for short) in this way.

Definition 2 ([12])**.** *Let $\mathcal{X}$ be a non-empty set and $\perp \subseteq \mathcal{X}\times\mathcal{X}$ be a binary relation. Then $(\mathcal{X},\perp)$ is said to be an $\mathcal{O}$-set if there exists $\xi_0 \in \mathcal{X}$ such that*

$$\varsigma \perp \xi_0 \text{ or } \xi_0 \perp \varsigma$$

for all $\varsigma \in \mathcal{X}$. The element ξ_0 is said to be an orthogonal element.

Example 1 ([12])**.** *Let $\mathcal{X} = \mathbb{Z}$. Define $\perp$ on $\mathcal{X}$ by $l \perp m$ if there exists $\aleph \in \mathbb{Z}$ such that $l = \aleph m$. Then, $0 \perp m$, for all $m \in \mathbb{Z}$. Thus, $(\mathcal{X},\perp)$ is an O-set.*

Example 2 ([12]). *Let $(\mathcal{X}, < .,. >)$ be an inner product space. Define $\perp$ on $\mathcal{X}$ by $\xi \perp \varsigma$ if $< \xi, \varsigma >= 0$. Then, $0 \perp m$, for all $m \in \mathbb{Z}$. Thus, $(\mathcal{X}, \perp)$ is an $\mathcal{O}$-set.*

Definition 3 ([12]). *Let $(\mathcal{X}, \perp)$ be $\mathcal{O}$-set. A sequence $\{\xi_\jmath\}$ is called an $\mathcal{O}$-sequence if*

$$\xi_\jmath \perp \xi_{\jmath+1} \ or\ \xi_{\jmath+1} \perp \xi_\jmath$$

for all $\jmath \in \mathbb{N}$.

Definition 4 ([12]). *The triplet $(\mathcal{X}, \perp, \tau)$ is said to be an orthogonal metric space if the pair $(\mathcal{X}, \perp)$ is an orthogonal set and the pair $(\mathcal{X}, \tau)$ is a metric space.*

Definition 5 ([12]). *A set $\mathcal{X}$ of $(\mathcal{X}, \perp, \tau)$ is claimed to be $\mathcal{O}$-complete if each Cauchy $\mathcal{O}$-sequence is convergent.*

Definition 6 ([12]). *Let $(\mathcal{X}, \perp, \tau)$ be an orthogonal metric space. A mapping $\mathcal{V} : (\mathcal{X}, \perp, \tau) \to (\mathcal{X}, \perp, \tau)$ is said to be orthogonally continuous ($\perp$-continuous) at a point $\xi \in \mathcal{X}$ if for $\mathcal{O}$-sequence $\{\xi_\jmath\}$ in $\mathcal{X}$ converging to ξ implies $\mathcal{V}\xi_\jmath \to \mathcal{V}\xi$. If $\mathcal{V}$ is $\perp$-continuous on each of its points $\xi \in \mathcal{X}$, then $\mathcal{V}$ is said to be $\perp$-continuous on $\mathcal{X}$.*

Definition 7 ([12]). *Let $(\mathcal{X}, \perp)$ be an $\mathcal{O}$-set. A mapping $\mathcal{V} : \mathcal{X} \to \mathcal{X}$ is called $\perp$-preserving if $\mathcal{V}\xi \perp \mathcal{V}\varsigma$ whenever $\xi \perp \varsigma$.*

The authors [12] established the following result as a generalization of Banach's fixed point theorem in this way.

Theorem 2 ([12]). *Let $(\mathcal{X}, \perp, \tau)$ be an $\mathcal{O}$-COMS and $\mathcal{V} : (\mathcal{X}, \perp, \tau) \to (\mathcal{X}, \perp, \tau)$ be a self mapping. If there there exists $\lambda \in (0, 1)$ such that*

$$\tau(\mathcal{V}\xi, \mathcal{V}\varsigma) \leq \lambda\tau(\xi, \varsigma)$$

for all $\xi, \varsigma \in \mathcal{X}$ and the mapping $\mathcal{V}$ is $\perp$-preserving and $\perp$-continuous, then $\mathcal{V}$ has a unique fixed point.

Samet et al. [24] introduced the notion of α-admissible mapping as follows:

Definition 8 ([24]). *A mapping $\mathcal{V} : \mathcal{X} \to \mathcal{X}$ is called α-admissible if there exists a function $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ such that*

$$\alpha(\xi, \varsigma) \geq 1 \quad implies \quad \alpha(\mathcal{V}\xi, \mathcal{V}\varsigma) \geq 1.$$

Ramezani [25] presented the idea of orthogonal α admissibility in the following way.

Definition 9 ([25]). *A mapping $\mathcal{V} : \mathcal{X} \to \mathcal{X}$ is called an orthogonally α-admissible if there exists a function $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ such that*

$$\xi \perp \varsigma \ \ and \ \ \alpha(\xi, \varsigma) \geq 1 \ implies \ \ \alpha(\mathcal{V}\xi, \mathcal{V}\varsigma) \geq 1.$$

We give the following property (JH), which is required to prove the uniqueness of fixed points in our main theorem.

Definition 10. *Let $(\mathcal{X}, \perp, \tau)$ be an $\mathcal{O}$-COMS and $\mathcal{V} : (\mathcal{X}, \perp, \tau) \to (\mathcal{X}, \perp, \tau)$. We say that the function $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ satisfies the property (JH) if $\alpha(\xi, \varsigma) \geq 1$, for all $\xi, \varsigma \in \{\rho \in \mathcal{X} : \rho = \mathcal{V}\rho\}$ and $\xi \perp \varsigma$.*

In this manuscript, we prove some fixed point results for orthogonal Θ-contraction and orthogonal (α,Θ)-contraction in the context of $\mathcal{O}$-COMS. The established results will combine and modify many celebrated results from the literature.

3. Main Results

Definition 11. *Let $(\mathcal{X},\perp,\tau)$ be a $\mathcal{O}$-COMS. A mapping $\mathcal{V}:(\mathcal{X},\perp,\tau)\to(\mathcal{X},\perp,\tau)$ is said to be an orthogonal (α,Θ)-contraction if there exist the functions $\alpha:\mathcal{X}\times\mathcal{X}\to[0,\infty)$, $\Theta\in\Psi$ and non-negative real numbers $\lambda_1,\lambda_2,\lambda_3$ and λ_4 with $\lambda_1+\lambda_2+\lambda_3+2\lambda_4<1$ such that for all $\xi,\varsigma\in\mathcal{X}$ with $\xi\perp\varsigma$, $\tau(\mathcal{V}\xi,\mathcal{V}\varsigma)>0$ implies*

$$\begin{aligned}\alpha(\xi,\varsigma)\Theta(\tau(\mathcal{V}\xi,\mathcal{V}\varsigma)) \quad &\le \quad [\Theta(\tau(\xi,\varsigma))]^{\lambda_1}\cdot[\Theta(\tau(\xi,\mathcal{V}\xi))]^{\lambda_2} \\ &\cdot[\Theta(\tau(\varsigma,\mathcal{V}\varsigma))]^{\lambda_3}\cdot[\Theta(\tau(\xi,\mathcal{V}\varsigma)+\tau(\varsigma,\mathcal{V}\xi))]^{\lambda_4}. \end{aligned} \tag{1}$$

Theorem 3. *Let $(\mathcal{X},\perp,\tau)$ be a $\mathcal{O}$-COMS and $\mathcal{V}:(\mathcal{X},\perp,\tau)\to(\mathcal{X},\perp,\tau)$ be an orthogonal (α,Θ)-contraction. Suppose that these conditions hold:*

(i) $\mathcal{V}$ is $\perp$-preserving,

(ii) $\mathcal{V}$ is orthogonally α-admissible mapping,

(iii) There exists $\xi_0\in\mathcal{X}$ such that $\xi_0\perp\mathcal{V}\xi_0$ and $\alpha(\xi_0,\mathcal{V}\xi_0)\ge 1$,

(iv) $\mathcal{V}$ is $\perp$-continuous.

Then, $\mathcal{V}$ has a fixed point. Furthermore, if the function $\alpha:\mathcal{X}\times\mathcal{X}\to[0,\infty)$ satisfies the property (JH), then $\mathcal{V}$ has a unique fixed point.

Proof. From the hypothesis (iii), there exists $\xi_0\in\mathcal{X}$ such that $\xi_0\perp\mathcal{V}\xi_0$ and $\alpha(\xi_0,\mathcal{V}\xi_0)\ge 1$. Let the sequence $\{\xi_\jmath\}$ be defined as

$$\xi_1=\mathcal{V}\xi_0,\cdots,\xi_{\jmath+1}=\mathcal{V}\xi_\jmath=\mathcal{V}^{\jmath+1}\xi_0,$$

for all $\jmath\ge 0$. As $\mathcal{V}$ is $\perp$-preserving, so $\{\xi_\jmath\}$ is an $\mathcal{O}$-sequence in $\mathcal{X}$. As $\mathcal{V}$ is orthogonally α-admissible, we obtain $\alpha(\mathcal{V}\xi_\jmath,\mathcal{V}\xi_{\jmath+1})>1$, for all $\jmath\ge 0$. If $\xi_\jmath=\xi_{\jmath+1}$, for any $\jmath\in\mathbb{N}\cup\{0\}$, then it is very clear that $\xi_\jmath$ is a fixed point of $\mathcal{V}$. Now, we consider that $\xi_\jmath\ne\xi_{\jmath+1}$, for all $\jmath\in\mathbb{N}\cup\{0\}$. Thus we obtain $\tau(\mathcal{V}\xi_\jmath,\mathcal{V}\xi_{\jmath+1})>0$, for all $\jmath\ge 0$. As $\mathcal{V}$ is $\perp$-preserving, we obtain

$$\xi_\jmath\perp\xi_{\jmath+1} \text{ or } \xi_{\jmath+1}\perp\xi_\jmath$$

for all $\jmath\in\mathbb{N}\cup\{0\}$. Thus $\{\xi_\jmath\}$ is an $\mathcal{O}$-sequence. Now, suppose that

$$0<\tau(\xi_\jmath,\mathcal{V}\xi_\jmath)=\tau(\mathcal{V}\xi_{\jmath-1},\mathcal{V}\xi_\jmath),$$

for all $\jmath\in\mathbb{N}\cup\{0\}$. Now, from (1) and $(\mathcal{J}_4)$, we have

$$\begin{aligned} 1 \quad &< \quad \Theta\big(\tau(\xi_\jmath,\xi_{\jmath+1})\big)=\Theta\big(\tau(\mathcal{V}\xi_{\jmath-1},\mathcal{V}\xi_\jmath)\big) \\ &\le \quad \big[\Theta\big(\tau(\xi_{\jmath-1},\xi_\jmath)\big)\big]^{\lambda_1}\cdot\big[\Theta\big(\tau(\xi_{\jmath-1},\mathcal{V}\xi_{\jmath-1})\big)\big]^{\lambda_2}\cdot\big[\Theta\big(\tau(\xi_\jmath,\mathcal{V}\xi_\jmath)\big)\big]^{\lambda_3} \\ &\quad\ \cdot\big[\Theta\big(\tau(\xi_{\jmath-1},\mathcal{V}\xi_\jmath)+\tau(\xi_\jmath,\mathcal{V}\xi_{\jmath-1})\big)\big]^{\lambda_4} \\ &= \quad \big[\Theta\big(\tau(\xi_{\jmath-1},\xi_\jmath)\big)\big]^{\lambda_1}\cdot\big[\Theta\big(\tau(\xi_{\jmath-1},\xi_\jmath)\big)\big]^{\lambda_2}\cdot\big[\Theta\big(\tau(\xi_\jmath,\xi_{\jmath+1})\big)\big]^{\lambda_3} \\ &\quad\ \cdot\big[\Theta\big(\tau(\xi_{\jmath-1},\xi_{\jmath+1})+\tau(\xi_\jmath,\xi_\jmath)\big)\big]^{\lambda_4} \\ &= \quad \big[\Theta\big(\tau(\xi_{\jmath-1},\xi_\jmath)\big)\big]^{\lambda_1}\cdot\big[\Theta\big(\tau(\xi_{\jmath-1},\xi_\jmath)\big)\big]^{\lambda_2}\cdot\big[\Theta\big(\tau(\xi_\jmath,\xi_{\jmath+1})\big)\big]^{\lambda_3} \\ &\quad\ \cdot\big[\Theta\big(\tau(\xi_{\jmath-1},\xi_{\jmath+1})\big)\big]^{\lambda_4}. \end{aligned}$$

From the triangle inequality and $(\mathcal{J}_1)$, we have

$$\begin{aligned} 1 \quad < \quad &\Theta\big(\tau(\xi_\jmath,\xi_{\jmath+1})\big)\le\big[\Theta\big(\tau(\xi_{\jmath-1},\xi_\jmath)\big)\big]^{\lambda_1}\cdot\big[\Theta\big(\tau(\xi_{\jmath-1},\xi_\jmath)\big)\big]^{\lambda_2}\cdot\big[\Theta\big(\tau(\xi_\jmath,\xi_{\jmath+1})\big)\big]^{\lambda_3} \\ &\cdot\big[\Theta\big(\tau(\xi_{\jmath-1},\xi_\jmath)+\tau(\xi_\jmath,\xi_{\jmath+1})\big)\big]^{\lambda_4}. \end{aligned}$$

Using $(\mathcal{J}_4)$, we obtain

$$\begin{aligned} 1 &< \Theta\big(\tau(\xi_\jmath,\xi_{\jmath+1})\big) \leq \big[\Theta\big(\tau(\xi_{\jmath-1},\xi_\jmath)\big)\big]^{\lambda_1} \\ &\quad \cdot \big[\Theta\big(\tau(\xi_{\jmath-1},\xi_\jmath)\big)\big]^{\lambda_2} \cdot \big[\Theta\big(\tau(\xi_\jmath,\xi_{\jmath+1})\big)\big]^{\lambda_3} \\ &\quad \cdot \big[\Theta\big(\tau(\xi_{\jmath-1},\xi_\jmath)\big) \cdot \Theta\big(\tau(\xi_\jmath,\xi_{\jmath+1})\big)\big]^{\lambda_4} \\ &= \big[\Theta\big(\tau(\xi_{\jmath-1},\xi_\jmath)\big)\big]^{\lambda_1} \cdot \big[\Theta\big(\tau(\xi_{\jmath-1},\xi_\jmath)\big)\big]^{\lambda_2} \cdot \big[\Theta\big(\tau(\xi_\jmath,\xi_{\jmath+1})\big)\big]^{\lambda_3} \\ &\quad \cdot \big[\Theta\big(\tau(\xi_{\jmath-1},\xi_\jmath)\big)\big]^{\lambda_4} \cdot \big[\Theta\big(\tau(\xi_\jmath,\xi_{\jmath+1})\big)\big]^{\lambda_4} \\ &= \big[\Theta\big(\tau(\xi_{\jmath-1},\xi_\jmath)\big)\big]^{\lambda_1+\lambda_2+\lambda_4} \cdot \big[\Theta\big(\tau(\xi_\jmath,\xi_{\jmath+1})\big)\big]^{\lambda_3+\lambda_4} \end{aligned}$$

which implies that

$$\big[\Theta\big(\tau(\xi_\jmath,\xi_{\jmath+1})\big)\big]^{1-\lambda_3-\lambda_4} \leq \big[\Theta\big(\tau(\xi_{\jmath-1},\xi_\jmath)\big)\big]^{\lambda_1+\lambda_2+\lambda_4}$$

for all $\jmath \in \mathbb{N} \cup \{0\}$; that is,

$$1 < \Theta\big(\tau(\xi_\jmath,\xi_{\jmath+1})\big) \leq \big[\Theta\big(\tau(\xi_{\jmath-1},\xi_\jmath)\big)\big]^{\frac{\lambda_1+\lambda_2+\lambda_4}{1-\lambda_3-\lambda_4}}.$$

Let $\frac{\lambda_1+\lambda_2+\lambda_4}{1-\lambda_3-\lambda_4} = \mu < 1$. Consequently,

$$1 < \Theta\big(\tau(\xi_\jmath,\xi_{\jmath+1})\big) \leq \big[\Theta\big(\tau(\xi_{\jmath-1},\xi_\jmath)\big)\big]^{\mu} \tag{2}$$

for all $\jmath \in \mathbb{N} \cup \{0\}$. This implies

$$\begin{aligned} 1 &< \Theta\big(\tau(\xi_\jmath,\xi_{\jmath+1})\big) \leq \big[\Theta\big(\tau(\xi_{\jmath-1},\xi_\jmath)\big)\big]^{\mu} \\ &\leq \big[\Theta\big(\tau(\xi_{\jmath-2},\xi_{\jmath-1})\big)\big]^{\mu^2} \\ &\leq \ldots \leq [\Theta(\tau(\xi_0,\xi_1))]^{\mu^\jmath} \end{aligned}$$

for all $\jmath \in \mathbb{N} \cup \{0\}$. Taking $\jmath \to \infty$ and by using $(\mathcal{J}_2)$, we obtain

$$\lim_{\jmath\to\infty} \Theta\big(\tau(\xi_\jmath,\xi_{\jmath+1})\big) = 1 \;\Leftrightarrow\; \lim_{\jmath\to\infty} \tau(\xi_\jmath,\xi_{\jmath+1}) = 0. \tag{3}$$

From the condition $(\mathcal{J}_3)$, there exists $0 < r < 1$ and $\sigma \in (0,\infty]$ such that

$$\lim_{\jmath\to\infty} \frac{\Theta(\tau(\xi_\jmath,\xi_{\jmath+1})) - 1}{\tau(\xi_\jmath,\xi_{\jmath+1})^r} = \sigma.$$

Assume that $\sigma < \infty$ and let $\wp_2 = \frac{\sigma}{2} > 0$. From the concept of the limit, there exists $\jmath_0 \in \mathbb{N}$ such that

$$\Big|\frac{\Theta(\tau(\xi_\jmath,\xi_{\jmath+1})) - 1}{\tau(\xi_\jmath,\xi_{\jmath+1})^r} - \sigma\Big| \leq \wp_2$$

for all $\jmath > \jmath_0$. This implies that

$$\frac{\Theta(\tau(\xi_\jmath,\xi_{\jmath+1})) - 1}{\tau(\xi_\jmath,\xi_{\jmath+1})^r} \geq \sigma - \wp_2 = \frac{\sigma}{2} = \wp_2$$

for all $\jmath > \jmath_0$. Then

$$\jmath\tau(\xi_\jmath,\xi_{\jmath+1})^r \leq \wp_1 \jmath[\Theta(\tau(\xi_\jmath,\xi_{\jmath+1})) - 1]$$

for all $\jmath > \jmath_0$, where $\wp_1 = \frac{1}{\wp_2}$. Now, we suppose that $\sigma = \infty$. Let $\wp_2 > 0$ be an arbitrary positive number. From the concept of the limit, there exists $\jmath_0 \in \mathbb{N}$ such that

$$\wp_2 \leq \frac{\Theta(\tau(\xi_\jmath,\xi_{\jmath+1})) - 1}{\tau(\xi_\jmath,\xi_{\jmath+1})^r}$$

for all $\jmath > \jmath_0$. This implies that

$$\jmath\tau(\xi_\jmath, \xi_{\jmath+1})^r \le \wp_1 \jmath[\Theta(\tau(\xi_\jmath, \xi_{\jmath+1})) - 1]$$

for all $\jmath > \jmath_0$, where $\wp_1 = \frac{1}{\wp_2}$. Thus, in all cases, there exist $\wp_1 > 0$ and $\jmath_0 \in \mathbb{N}$ such that

$$\jmath\tau(\xi_\jmath, \xi_{\jmath+1})^r \le \wp_1 \jmath[\Theta(\tau(\xi_\jmath, \xi_{\jmath+1})) - 1] \tag{4}$$

for all $\jmath > \jmath_0$. Thus, from (3) and (4), we obtain

$$\jmath\tau(\xi_\jmath, \xi_{\jmath+1})^r \le \wp_1 \jmath([\Theta(\tau(\xi_0, \xi_1))]^{r^\jmath} - 1). \tag{5}$$

Taking the limit $\jmath \to \infty$ in the inequality (5) and using the fact that

$$\lim_{\jmath\to\infty} [(\Theta(\tau(\xi_0, \xi_1))]^{r^\jmath} = 1,$$

because $0 < r < 1$, we get

$$\lim_{\jmath\to\infty} \jmath\tau(\xi_\jmath, \xi_{\jmath+1})^r = 0.$$

Thus, there exists $\jmath_1 \in \mathbb{N}$ such that

$$\tau(\xi_\jmath, \xi_{\jmath+1}) \le \frac{1}{\jmath^{1/r}} \tag{6}$$

for all $\jmath > \jmath_1$. Now, for $m > \jmath > \jmath_1$, we have

$$\tau(\xi_\jmath, \xi_m) \le \sum_{i=\jmath}^{m-1} \tau(\xi_i, \xi_{i+1}) \le \sum_{i=\jmath}^{m-1} \frac{1}{i^{1/r}} \le \sum_{i=1}^{\infty} \frac{1}{i^{1/r}}.$$

As $0 < r < 1$, $\sum_{i=1}^{\infty} \frac{1}{i^{1/r}}$ converges. Therefore, $\tau(\xi_\jmath, \xi_m) \to 0$ as $m, \jmath \to \infty$. Thus, we have $\{\xi_\jmath\}$ is a Cauchy $\mathcal{O}$-sequence in $(\mathcal{X}, \perp, \tau)$. From the $\mathcal{O}$-completeness of $(\mathcal{X}, \perp, \tau)$, there is $\xi^* \in \mathcal{X}$ such that, $\lim_{\jmath\to\infty} \xi_\jmath \to \xi^*$. We show that ξ^* is a fixed point of $\mathcal{V}$. As $\mathcal{V}$ is a $\perp$-continuous mapping, so

$$\xi_{\jmath+1} = \mathcal{V}\xi_\jmath \to \mathcal{V}\xi^*$$

as $\jmath \to \infty$, that is, $\xi^* = \mathcal{V}\xi^*$. Lastly, we suppose that $\xi' = \mathcal{V}\xi'$ such that $\xi' \ne \xi^*$. Now, since the function $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ satisfies the property (JH), we have $\xi^* \perp \xi'$ and $\alpha\left(\xi^*, \xi'\right) \ge 1$. Thus, from (1), we have

$$\begin{aligned}
\Theta\left(\tau(\xi^*, \xi')\right) &= \Theta\left(\tau(\mathcal{V}\xi^*, \mathcal{V}\xi')\right) \le \alpha\left(\xi^*, \xi'\right)\Theta\left(\tau(\mathcal{V}\xi^*, \mathcal{V}\xi')\right) \\
&\le \left[\Theta\left(\tau(\xi', \xi^*)\right)\right]^{\lambda_1} \cdot \left[\Theta\left(\tau(\xi', \mathcal{V}\xi')\right)\right]^{\lambda_2} \\
&\quad \cdot [\Theta(\tau(\xi^*, \mathcal{V}\xi^*))]^{\lambda_3} \\
&\quad \cdot \left[\Theta\left(\tau(\xi', \mathcal{V}\xi^*) + \tau(\xi^*, \mathcal{V}\xi')\right)\right]^{\lambda_4} \\
&= \left[\Theta\left(\tau(\xi', \xi^*)\right)\right]^{\lambda_1} \cdot \left[\Theta\left(\tau(\xi', \xi^*) + \tau(\xi^*, \xi')\right)\right]^{\lambda_4} \\
&\le \left[\Theta\left(\tau(\xi', \xi^*)\right)\right]^{\lambda_1} \cdot \left[\Theta\left(\tau(\xi', \xi^*)\right)\right]^{2\lambda_4} \\
&= \left[\Theta\left(\tau(\xi', \xi^*)\right)\right]^{\lambda_1 + 2\lambda_4} < \Theta\left(\tau(\xi', \xi^*)\right),
\end{aligned}$$

which is a contradiction. Thus $\xi' = \xi^*$. $\square$

Remark 1. *Let us consider $\Theta(t) = e^{\sqrt{t}}$ in (1); then Θ satisfies $(\mathcal{J}_1)$–$(\mathcal{J}_4)$, and we obtain*

$$\begin{aligned} \ln(\alpha(\xi,\varsigma))\sqrt{\tau(\mathcal{V}\xi,\mathcal{V}\varsigma)} &\leq \lambda_1\sqrt{\tau(\xi,\varsigma)} + \lambda_2\sqrt{\tau(\xi,\mathcal{V}\xi)} \\ &+\lambda_3\sqrt{\tau(\varsigma,\mathcal{V}\varsigma)} + \lambda_4\sqrt{\tau(\xi,\mathcal{V}\varsigma)+\tau(\varsigma,\mathcal{V}\xi)} \end{aligned} \tag{7}$$

which is Ćirić-type contraction ([26]). *Now, if we take square in both terms of* (7), *then we have*

$$\begin{aligned} [\ln(\alpha(\xi,\varsigma))]^2\tau(\mathcal{V}\xi,\mathcal{V}\varsigma) &\leq \lambda_1^2\tau(\xi,\varsigma) + \lambda_2^2\tau(\xi,\mathcal{V}\xi) + \lambda_3^2\tau(\varsigma,\mathcal{V}\varsigma) \\ &+\lambda_4^2(\tau(\xi,\mathcal{V}\varsigma)+\tau(\varsigma,\mathcal{V}\xi)) \\ &+2\lambda_1\lambda_2\sqrt{\tau(\xi,\varsigma)\tau(\xi,\mathcal{V}\xi)} \\ &+2\lambda_1\lambda_3\sqrt{\tau(\xi,\varsigma)\tau(\varsigma,\mathcal{V}\varsigma)} \\ &+2\lambda_1\lambda_4\sqrt{\tau(\xi,\varsigma)(\tau(\xi,\mathcal{V}\varsigma)+\tau(\varsigma,\mathcal{V}\xi))} \\ &+2\lambda_2\lambda_3\sqrt{\tau(\xi,\mathcal{V}\xi)\tau(\varsigma,\mathcal{V}\varsigma)} \\ &+2\lambda_2\lambda_4\sqrt{\tau(\xi,\mathcal{V}\xi)(\tau(\xi,\mathcal{V}\varsigma)+\tau(\varsigma,\mathcal{V}\xi))} \\ &+2\lambda_3\lambda_4\sqrt{\tau(\varsigma,\mathcal{V}\varsigma)(\tau(\xi,\mathcal{V}\varsigma)+\tau(\varsigma,\mathcal{V}\xi))}. \end{aligned} \tag{8}$$

Now, if we consider some particular values for λ_i, $i = 1,2,3,4$ in (8), *then we shall obtain some generalizations of some well-known conditions*

- For $\lambda_1 = \lambda_4 = 0$, then, (8) becomes

$$\begin{aligned} [\ln(\alpha(\xi,\varsigma))]^2\tau(\mathcal{V}\xi,\mathcal{V}\varsigma) &\leq \lambda_2^2\tau(\xi,\mathcal{V}\xi) + \lambda_3^2\tau(\varsigma,\mathcal{V}\varsigma) \\ &+2\lambda_2\lambda_3\sqrt{\tau(\xi,\mathcal{V}\xi)\tau(\varsigma,\mathcal{V}\varsigma)} \end{aligned} \tag{9}$$

  which represents a Kannan-type contraction ([27]),
- For $\lambda_1 = \lambda_2 = \lambda_3 = 0$, then, (8) becomes

$$[\ln(\alpha(\xi,\varsigma))]^2\tau(\mathcal{V}\xi,\mathcal{V}\varsigma) \leq \lambda_4^2(\tau(\xi,\mathcal{V}\varsigma)+\tau(\varsigma,\mathcal{V}\xi)) \tag{10}$$

  which represents a Chatterjea-type contraction ([28]),
- For $\lambda_4 = 0$, then, (8) becomes

$$\begin{aligned} [\ln(\alpha(\xi,\varsigma))]^2\tau(\mathcal{V}\xi,\mathcal{V}\varsigma) &\leq \lambda_1^2\tau(\xi,\varsigma) + \lambda_2^2\tau(\xi,\mathcal{V}\xi) + \lambda_3^2\tau(\varsigma,\mathcal{V}\varsigma) \\ &+2\lambda_1\lambda_2\sqrt{\tau(\xi,\varsigma)\tau(\xi,\mathcal{V}\xi)} \\ &+2\lambda_1\lambda_3\sqrt{\tau(\xi,\varsigma)\tau(\varsigma,\mathcal{V}\varsigma)} \\ &+2\lambda_2\lambda_3\sqrt{\tau(\xi,\mathcal{V}\xi)\tau(\varsigma,\mathcal{V}\varsigma)} \end{aligned} \tag{11}$$

  which represents a Reich-type contraction ([29]).

Corollary 1. *Let $(\mathcal{X},\perp,\tau)$ be an $\mathcal{O}$-COMS and let $\mathcal{V} : (\mathcal{X},\perp,\tau) \to (\mathcal{X},\perp,\tau)$ be an orthogonally α-admissible, $\perp$-continuous and $\perp$-preserving mapping. If there exists $\xi_0 \in \mathcal{X}$ such that $\xi_0 \perp \mathcal{V}\xi_0$ with $\alpha(\xi_0,\mathcal{V}\xi_0) \geq 1$ and any one of the inequalities* (9), (10) *or* (11) *hold, for all $\xi,\varsigma \in \mathcal{X}$ with $\xi \perp \varsigma$, $\tau(\mathcal{V}\xi,\mathcal{V}\varsigma) > 0$. Then, there exists $\xi^* \in \mathcal{X}$ such that $\xi^* = \mathcal{V}\xi^*$. Moreover, if the function $\alpha : \mathcal{X}\times\mathcal{X} \to [0,\infty)$ satisfies the property (JH), then ξ^* is unique.*

In what follows, we shall present another result in which we replace $(\mathcal{J}_3)$ and $(\mathcal{J}_4)$ with the general condition $(\mathcal{J}_3^{/})$.

Definition 12. *Let $(\mathcal{X}, \perp, \tau)$ be an $\mathcal{O}$-COMS. A mapping $\mathcal{V} : (\mathcal{X}, \perp, \tau) \to (\mathcal{X}, \perp, \tau)$ is said to be a orthogonal Θ-contraction if there exist $\Theta \in \Omega$ and $\lambda \in (0,1)$ such that, for all $\xi, \varsigma \in \mathcal{X}$ with $\xi \perp \varsigma$, $[\tau(\mathcal{V}\xi, \mathcal{V}\varsigma)] > 0$ implies*

$$\Theta(\tau(\mathcal{V}\xi, \mathcal{V}\varsigma)) \leq [\Theta(\tau(\xi, \varsigma))]^{\lambda}. \tag{12}$$

Theorem 4. *Let $(\mathcal{X}, \perp, \tau)$ be an $\mathcal{O}$-COMS and $\mathcal{V} : (\mathcal{X}, \perp, \tau) \to (\mathcal{X}, \perp, \tau)$ a generalized Θ-contraction, and let $\mathcal{V}$ be $\perp$-preserving. Then, there exists a unique $\xi^* \in \mathcal{X}$ such that $\xi^* = \mathcal{V}\xi^*$.*

Proof. Let $\xi_0 \in \mathcal{X}$. Since $(\mathcal{X}, \perp)$ is an $\mathcal{O}$-set,

$$\xi_0 \perp \xi, \text{ for all } \xi \in \mathcal{X}$$

or

$$\xi \perp \xi_0, \text{ for all } \xi \in \mathcal{X}.$$

It follows that $\xi_0 \perp \mathcal{V}\xi_0$ or $\mathcal{V}\xi_0 \perp \xi_0$. Now, we define the sequence $\{\xi_\jmath\}$ as

$$\xi_1 = \mathcal{V}\xi_0, \cdots, \xi_{\jmath+1} = \mathcal{V}\xi_\jmath = \mathcal{V}^{\jmath+1}\xi_0,$$

for all $\jmath \geq 0$. If $\xi_\jmath = \xi_{\jmath+1}$, for any $\jmath \in \mathbb{N} \cup \{0\}$, then it is very clear that $\xi_\jmath$ is a fixed point of $\mathcal{V}$. Thus, we consider that $\xi_\jmath \neq \xi_{\jmath+1}$, for all $\jmath \in \mathbb{N} \cup \{0\}$. Hence, we have $\tau(\mathcal{V}\xi_\jmath, \mathcal{V}\xi_{\jmath+1}) > 0$, for all $\jmath \geq 0$. As $\mathcal{V}$ is $\perp$-preserving, we obtain

$$\xi_\jmath \perp \xi_{\jmath+1} \text{ or } \xi_{\jmath+1} \perp \xi_\jmath$$

for all $\jmath \in \mathbb{N} \cup \{0\}$. It implies that $\{\xi_\jmath\}$ is an $\mathcal{O}$-sequence. Thus, we suppose that

$$0 < \tau(\xi_\jmath, \mathcal{V}\xi_\jmath) = \tau(\mathcal{V}\xi_{\jmath-1}, \mathcal{V}\xi_\jmath),$$

for all $\jmath \in \mathbb{N} \cup \{0\}$. Now, from (12) and $(\mathcal{J}_1)$, we have

$$\begin{aligned} 1 &< \Theta\big(\tau(\xi_\jmath, \xi_{\jmath+1})\big) = \Theta\big(\tau(\mathcal{V}\xi_{\jmath-1}, \mathcal{V}\xi_\jmath)\big) \\ &\leq \big[\Theta\big(\tau(\xi_{\jmath-1}, \xi_\jmath)\big)\big]^{\lambda}. \end{aligned}$$

which implies that

$$\begin{aligned} 1 &< \Theta\big(\tau(\xi_\jmath, \xi_{\jmath+1})\big) \leq \big[\Theta\big(\tau(\xi_{\jmath-1}, \xi_\jmath)\big)\big]^{\lambda} \\ &\leq \big[\Theta\big(\tau(\xi_{\jmath-2}, \xi_{\jmath-1})\big)\big]^{\lambda^2} \\ &\leq \ldots \leq [\Theta(\tau(\xi_0, \xi_1))]^{\lambda^\jmath} \end{aligned}$$

for all $\jmath \in \mathbb{N} \cup \{0\}$. Now, taking the limit as $\jmath \to \infty$ and from using $(\mathcal{J}_2)$, we obtain

$$\lim_{\jmath\to\infty} \Theta\big(\tau(\xi_\jmath, \xi_{\jmath+1})\big) = 1 \text{ if and only if } \lim_{\jmath\to\infty} \tau(\xi_\jmath, \xi_{\jmath+1}) = 0. \tag{13}$$

Now, we say that $\{\xi_\jmath\}_{\jmath=1}^{\infty}$ is an $\mathcal{O}$-Cauchy sequence. Then, suppose, on the contrary, that $\{\xi_\jmath\}_{\jmath=1}^{\infty}$ is not $\mathcal{O}$-Cauchy sequence; then, we suppose that there exist $\varepsilon > 0$ and sequences $\{p(\jmath)\}_{\jmath=1}^{\infty}$ and $\{q(\jmath)\}_{\jmath=1}^{\infty}$ of natural numbers such that for $p(\jmath) > q(\jmath) > \jmath$, we have

$$\tau\big(\xi_{p(\jmath)}, \xi_{q(\jmath)}\big) \geq \varepsilon.$$

Then

$$\tau\big(\xi_{p(\jmath)-1}, \xi_{q(\jmath)}\big) < \varepsilon \tag{14}$$

for all $\jmath \in \mathbb{N}$. Hence, using the triangle inequality and (14), we obtain

$$\begin{aligned} \varepsilon &\leq \tau(\xi_{p(\jmath)}, \xi_{q(\jmath)}) \leq \tau(\xi_{p(\jmath)}, \xi_{p(\jmath)-1}) + \tau(\xi_{p(\jmath)-1}, \xi_{q(\jmath)}) \\ &\leq \tau(\xi_{p(\jmath)-1}, \xi_{p(\jmath)}) + \varepsilon. \end{aligned}$$

Taking $\jmath \to \infty$ in the above inequality and using the inequality (13), we obtain

$$\lim_{\jmath \to \infty} \tau(\xi_{p(\jmath)}, \xi_{q(\jmath)}) = \varepsilon. \tag{15}$$

From (13), we can choose $\jmath_0 \in \mathbb{N}$ such that

$$\tau(\xi_{p(\jmath)}, \xi_{p(\jmath)+1}) < \frac{\varepsilon}{4} \text{ and } \tau(\xi_{q(\jmath)}, \xi_{q(\jmath)+1}) < \frac{\varepsilon}{4} \tag{16}$$

for all $\jmath \geq \jmath_0$. Next, we claim that $\mathcal{V}\xi_{p(\jmath)} \neq \mathcal{V}\xi_{q(\jmath)}$ for all $\jmath \geq \jmath_0$; i.e.,

$$\tau(\xi_{p(\jmath)+1}, \xi_{q(\jmath)+1}) = \tau(\mathcal{V}\xi_{p(\jmath)}, \mathcal{V}\xi_{q(\jmath)}) > 0. \tag{17}$$

Arguing by contradiction, there exists $\jmath \geq \jmath_{0,}$ such that $\tau(\xi_{p(\jmath)+1}, \xi_{q(\jmath)+1}) = 0$. It follows from (13), (16), and (17) that

$$\begin{aligned} \varepsilon &\leq \tau(\xi_{p(\jmath)}, \xi_{q(\jmath)}) \leq \tau(\xi_{p(\jmath)}, \xi_{p(\jmath)+1}) \\ &\quad + \tau(\xi_{p(\jmath)+1}, \xi_{q(\jmath)+1}) + \tau(\xi_{p(\jmath)+1}, \xi_{q(\jmath)}) \\ &\leq \frac{\varepsilon}{4} + 0 + \frac{\varepsilon}{4} = \frac{\varepsilon}{2} \end{aligned}$$

a contradiction. Hence, (16) holds. Then, from the supposition, we have

$$\Theta(\tau(\mathcal{V}\xi_{p(\jmath)}, \mathcal{V}\xi_{q(\jmath)})) \leq [\Theta(\tau(\xi_{p(\jmath)}, \xi_{q(\jmath)}))]^{\lambda}. \tag{18}$$

Letting $\jmath \to +\infty$ and using $(\mathcal{J}_3')$, (15) and (18), we have

$$\Theta(\varepsilon) \leq [\Theta(\varepsilon)]^{\lambda}$$

which is a contradiction. Thus, $\{\xi_\jmath\}$ is a Cauchy $\mathcal{O}$-sequence. Since $(\mathcal{X}, \perp, \tau)$ is a complete orthogonal metric space, there exists $\xi^* \in \mathcal{X}$ such that, $\xi_\jmath \to \xi^*$ as $\jmath \to \infty$. Next, we prove that ξ^* is a fixed point of $\mathcal{V}$. Otherwise, $\mathcal{V}\xi^* \neq \xi^*$.

$$\tau(\xi^*, \mathcal{V}\xi^*) = \lim_{\jmath \to \infty} \tau(\xi_\jmath, \mathcal{V}\xi_\jmath) = \lim_{\jmath \to \infty} \tau(\xi_\jmath, \xi_{\jmath+1}) = \tau(\xi^*, \xi^*) = 0.$$

Hence, ξ^* is a fixed point of $\mathcal{V}$. Now, we assume on the contrary that there is another fixed point $\xi^{/} \in \mathcal{X}$ of $\mathcal{V} : (\mathcal{X}, \perp, \tau) \to (\mathcal{X}, \perp, \tau)$ such that

$$\mathcal{V}\xi^* = \xi^* \neq \xi^{/} = \mathcal{V}\xi^{/}, \text{ that is, } \mathcal{V}\xi^* \neq \mathcal{V}\xi^{/}.$$

Then, from the supposition, we obtain

$$\begin{aligned} \Theta(\tau(\xi^*, \xi^{/})) &= \Theta(\tau(\mathcal{V}\xi^*, \mathcal{V}\xi^{/})) \\ &\leq [\Theta(\tau(\xi^*, \xi^{/}))]^{\lambda} \end{aligned}$$

which is contradiction because $\lambda \in (0,1)$. Thus, ξ^* is unique. $\square$

Example 3. *Let* $\mathcal{X} = [0, \infty)$ *be a set equipped with the metric*

$$\tau(\xi, \varsigma) = |\xi - \varsigma|$$

for all $\xi, \varsigma \in \mathcal{X}$. *Define the sequence* $\{x_\jmath\}$ *as follows:*

$x_1 = 1$
$x_2 = 1 + 4$
$\ldots$
$x_\jmath = 1 + 4 + 7 + \ldots + (3\jmath - 2) = \frac{\jmath(3\jmath-1)}{2}$
for all $\jmath \in \mathbb{N}$. *Define the orthogonality relation* $\perp$ *on* $\mathcal{X}$ *by*

$$\xi \perp \varsigma \text{ if and only if } \xi\varsigma \in \{\xi, \varsigma\} \subset \{x_\jmath\}.$$

Then, $(\mathcal{X}, \perp, \tau)$ *is a* $\mathcal{O}$-*COMS. Define* $\mathcal{V} : \mathcal{X} \to \mathcal{X}$ *by*

$$\mathcal{V}(\xi) = \begin{cases} x_0, & \text{if } x_0 \le \xi \le x_1, \\ \frac{x_{\jmath-1}(x_{\jmath+1}-\xi)+x_\jmath(\xi-x_\jmath)}{x_{\jmath+1}-x_\jmath}, & \text{if } x_\jmath \le \xi \le x_{\jmath+1} \end{cases}$$

for each $\jmath \ge 1$. *Let* $\Theta : (0, \infty) \to (1, \infty)$ *given by*

$$\Theta(t) = e^{te^t}, \text{ fot } t > 0.$$

Then, $\Theta \in \Omega$. *Now, let* $\xi, \varsigma \in \mathcal{X}$ *with* $\xi \perp \varsigma$ *and* $\tau(\mathcal{V}(\xi), \mathcal{V}(\varsigma)) > 0$. *Without any loss of generality, we suppose that* $\xi < \varsigma$. *This signifies that* $\xi \in \{x_0, x_1\}$ *and* $\varsigma = x_i$ *for some* $i \in \mathbb{N}\backslash\{1\}$. *Subsequently, we obtain*

$$\frac{\tau(\mathcal{V}(\xi), \mathcal{V}(\varsigma))}{\tau(\xi, \varsigma)} e^{\tau(\mathcal{V}(\xi),\mathcal{V}(\varsigma))-\tau(\xi,\varsigma)} \le \left(\frac{x_{i-1}}{x_i - 1}\right) e^{[x_{i-1}-x_{i+1}]} < e^{-1}$$

for $\lambda = e^{-1} \in (0, 1)$.

Hence, all the conditions of Theorem 4 hold, and $\xi = x_0$ *is a unique fixed point of* $\mathcal{V}$.

The following theorem is a direct outcome of Theorem 4.

Theorem 5 ([12]). *Let* $(\mathcal{X}, \perp, \tau)$ *be an* $\mathcal{O}$-*COMS and let* $\mathcal{V} : (\mathcal{X}, \perp, \tau) \to (\mathcal{X}, \perp, \tau)$ *be a mapping such that*

(i) there exists $\lambda \in (0, 1)$ *such that*

$$\tau(\mathcal{V}\xi, \mathcal{V}\varsigma) \le \lambda\tau(\xi, \varsigma)$$

for all $\xi, \varsigma \in \mathcal{X}$, *with* $\xi \perp \varsigma$,

(ii) $\mathcal{V}$ *is* $\perp$-*preserving and* $\perp$-*continuous.*

Then, $\mathcal{V}$ *has a unique fixed point* $\xi^* \in \mathcal{X}$.

Theorem 6 ([2]). *Let* $(\mathcal{X}, \tau)$ *be a complete metric space and let* $\mathcal{V} : (\mathcal{X}, \tau) \to (\mathcal{X}, \tau)$ *be a* Θ-*contraction; then, there exists a unique* $\xi^* \in \mathcal{X}$ *such that* $\xi^* = \mathcal{V}\xi^*$.

Proof. Define a binary relation on $\mathcal{X}$ by

$$\xi \perp \varsigma \Leftrightarrow \Big[\tau(\mathcal{V}\xi, \mathcal{V}\varsigma) > 0 \Longrightarrow \Theta(\tau(\mathcal{V}\xi, \mathcal{V}\varsigma)) \le (\Theta(\tau(\xi, \varsigma)))^\lambda\Big].$$

Fix $\xi_0 \in \mathfrak{R}$. Since $\mathcal{V}$ is a Θ-contraction, we have $\xi_0 \perp \varsigma$ for all $\varsigma \in \mathcal{X}$. Hence, from Theorem 4, there exists a unique fixed point of $\mathcal{V}$. $\square$

4. Applications

In this section, we will investigate the solution for the nonlinear fractional differential equation

$$^C D^\eta(\xi(t)) = f(t, \xi(t)) \tag{19}$$

$(0 < t < 1,\ 1 < \eta \leq 2)$ via the integral boundary conditions

$$\xi(0) = 0\,,\ \xi(1) = \int_0^\beta \xi(s)ds, \quad (0 < \beta < 1)$$

where $\xi \in C([0,1],\mathbb{R})$ (family of all continuous functions). We symbolize and define the Caputo fractional derivative of order η as ${}^C D^\eta$ and

$${}^C D^\eta f(t) = \frac{1}{\Gamma(n-\eta)} \int_0^t (t-s)^{n-\eta-1} f^n(s)ds,$$

where $(n-1 < \eta < n,\ n = [\eta]+1)$ and $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$ is a continuous function. We take $\mathcal{X} = \{\xi : \xi \in C([0,1],\mathbb{R})\}$ along with $\|\xi\|_\infty = \sup_{t\in[0,1]}|\xi(t)|$. Then, $(\mathcal{X}, \|\cdot\|_\infty)$ is Banach space. Recall that the Riemann–Liouville fractional integral of order η is given as

$$I^\eta f(t) = \frac{1}{\Gamma(\eta)} \int_0^t (t-s)^{\eta-1} f(s)ds, \quad \text{with } \eta > 0.$$

Theorem 7. *Assume that $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$ is a continuous function satisfying the following condition:*

$$|f(t,\xi) - f(t,\varsigma)| \leq \aleph|\xi - \varsigma|$$

for all $t \in [0,1]$ and $\xi, \varsigma \in \mathcal{X}$ such that $\xi(t)\varsigma(t) \geq 0$ and a constant $\aleph$ with $\aleph\vartheta < 1$, where

$$\vartheta = \frac{t^\eta(2-\beta^2)(\eta+1) + 2t(\eta+\beta+1)}{(2-\beta^2)\eta(\eta+1)\Gamma(\eta)}$$

for $0 < \vartheta < 1$. Then, the differential equation (19) has a unique solution.

Proof. For all $t \in [0,1]$, suppose that the orthogonality relation on $\mathcal{X}$ is given as

$$\xi \perp \varsigma \quad \text{if} \quad \xi(t)\varsigma(t) \geq 0.$$

The set $\mathcal{X}$ is orthogonal with this orthogonality relation because, for all $\xi \in \mathcal{X}$, there exists $\varsigma(t) = 0$ such that

$$\xi(t)\varsigma(t) = 0.$$

Then, the metric d, defined by

$$d(\xi,\varsigma) = \sup_{t\in[0,1]} \|\xi(t) - \varsigma(t)\|$$

for all $t \in [0,1]$, is an orthogonal metric, and $(\mathcal{X}, \perp, d)$ is an $\mathcal{O}$-COMS (see ref. [30]). Define $\mathcal{V} : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ by

$$\begin{aligned} \mathcal{V}\xi(t) \quad = \quad & \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s,\xi(s))ds \\ & - \frac{2t}{(2-\beta^2)\Gamma(\gamma)} \int_0^1 (1-s)^{\gamma-1} f(s,\xi(s))ds \\ & + \frac{2t}{(2-\beta^2)\Gamma(\gamma)} \int_0^\beta \left(\int_0^s (s-m)^{\gamma-1} f(m,\xi(m))dm \right) ds \end{aligned}$$

for $t \in [0,1]$. Then, $\mathcal{V}$ is $\perp$-continuous. Now we show that $\mathcal{V}$ is $\perp$-preserving. Let $\xi(t) \perp \varsigma(t)$, for all $t \in [0,1]$. Now, we have

$$\begin{aligned}
\mathcal{V}\xi(t) &= \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s,\xi(s))ds \\
&\quad - \frac{2t}{(2-\beta^2)\Gamma(\gamma)} \int_0^1 (1-s)^{\gamma-1} f(s,\xi(s))ds \\
&\quad + \frac{2t}{(2-\beta^2)\Gamma(\gamma)} \int_0^\beta \left(\int_0^s (s-m)^{\gamma-1} f(m,\xi(m))dm \right) ds > 0
\end{aligned}$$

which implies that $\mathcal{V}\xi \perp \mathcal{V}\varsigma$, i.e., that $\mathcal{V}$ is $\perp$-preserving. Now, for $\xi(t) \perp \varsigma(t)$, we obtain

$$\begin{aligned}
|\mathcal{V}\xi(t) - \mathcal{V}\varsigma(t)| &= \left| \begin{array}{c}
\frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s,\xi(s))ds \\
- \frac{2t}{(2-\beta^2)\Gamma(\gamma)} \int_0^1 (1-s)^{\gamma-1} f(s,\xi(s))ds \\
+ \frac{2t}{(2-\beta^2)\Gamma(\gamma)} \int_0^\beta \left(\int_0^s (s-m)^{\gamma-1} f(m,\xi(m))dm \right) ds \\
- \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s,\varsigma(s))ds \\
+ \frac{2t}{(2-\beta^2)\Gamma(\gamma)} \int_0^1 (1-s)^{\gamma-1} f(s,\varsigma(s))ds \\
- \frac{2t}{(2-\beta^2)\Gamma(\gamma)} \int_0^\beta \left(\int_0^s (s-m)^{\gamma-1} f(m,\varsigma(m))dm \right) ds
\end{array} \right| \\
&\leq \frac{1}{\Gamma(\gamma)} \int_0^t (s-m)^{\gamma-1} |f(s,\xi(s)) - f(s,\varsigma(s))| ds \\
&\quad + \frac{2t}{(2-\beta^2)\Gamma(\gamma)} \int_0^1 (1-s)^{\gamma-1} |f(s,\xi(s)) - f(s,\varsigma(s))| ds \\
&\quad + \frac{2t}{(2-\beta^2)\Gamma(\gamma)} \int_0^\beta \left(\int_0^s (s-m)^{\gamma-1} |f(m,\xi(m)) - f(m,\varsigma(m))| dm \right) ds
\end{aligned}$$

which implies that

$$\begin{aligned}
&\leq \left(\begin{array}{c}
\frac{1}{\Gamma(\gamma)} \int_0^t (s-m)^{\gamma-1} ds \\
+ \frac{2t}{(2-\beta^2)\Gamma(\gamma)} \int_0^1 (1-s)^{\gamma-1} ds \\
+ \frac{2t}{(2-\beta^2)\Gamma(\gamma)} \int_0^\beta \left(\int_0^s (s-m)^{\gamma-1} dm \right) ds
\end{array} \right) \aleph \|\xi - \varsigma\| \\
&= \left(\frac{t^\gamma (2-\beta^2)(\gamma+1) + 2t(\gamma+\beta+1)}{(2-\beta^2)\gamma(\gamma+1)\Gamma(\gamma)} \right) \aleph \|\xi - \varsigma\| \\
&= \aleph\vartheta \|\xi - \varsigma\|
\end{aligned}$$

which implies that

$$\|\mathcal{V}\xi(t) - \mathcal{V}\varsigma(t)\| \leq \aleph\vartheta \|\xi - \varsigma\|.$$

Thus for each $\xi, \varsigma \in \mathcal{X}$, we have

$$d(\mathcal{V}\xi, \mathcal{V}\varsigma) \leq \aleph\vartheta d(\xi, \varsigma). \tag{20}$$

Now, taking $\Theta : (0,\infty) \to (1,\infty)$ defined by $\Theta(u) = e^{\sqrt{u}}$ for each $u > 0$, then $\Theta \in \Psi$, and we define $\alpha : \mathcal{X} \times \mathcal{X} \to [1, +\infty)$ by

$$\alpha(\xi, \varsigma) = 1$$

for all $\xi, \varsigma \in \mathcal{X}$. From inequality (20), we have

$$e^{\sqrt{d(\mathcal{V}\xi,\mathcal{V}\varsigma)}} \leq e^{\sqrt{\aleph\vartheta d(\xi,\varsigma)}} = \left(e^{\sqrt{d(\xi,\varsigma)}}\right)^{\lambda_1}$$

where $\lambda_1 = \sqrt{\aleph\vartheta}$. Since $\aleph\vartheta < 1$, $\lambda_1 \in (0,1)$. Thus for $\lambda_2 = \lambda_3 = \lambda_4 = 0$, we have

$$\begin{aligned}\alpha(\xi,\varsigma)\Theta(d(\mathcal{V}\xi,\mathcal{V}\varsigma)) &\leq \alpha(\xi,\varsigma)\Theta(\tau(\mathcal{V}\xi,\mathcal{V}\varsigma)) \\ &\leq [\Theta(\tau(\xi,\varsigma))]^{\lambda_1}\cdot[\Theta(\tau(\xi,\mathcal{V}\xi))]^{\lambda_2} \\ &\quad \cdot[\Theta(\tau(\varsigma,\mathcal{V}\varsigma))]^{\lambda_3}\cdot[\Theta(\tau(\xi,\mathcal{V}\varsigma)+\tau(\varsigma,\mathcal{V}\xi))]^{\lambda_4}\end{aligned}$$

for all $\xi, \varsigma \in \mathcal{X}$. Thus, all the hypotheses of Theorem 3 are satisfied, and ξ^* is a solution of differential Equation (19). □

5. Conclusions

In this manuscript, we have proven some fixed point theorems in $\mathcal{O}$-COMS for orthogonal Θ-contractions and orthogonal (α,Θ)-contraction. We have also explored the solution to a nonlinear fractional differential equation as the implementation of our foremost results. Furthermore, a significant example is also given to show the authenticity of the proved result.

In the context of $\mathcal{O}$-COMS, establishing fixed points and common fixed points of fuzzy mappings and set-valued mapping for orthogonal Θ-contractions and orthogonal (α,Θ)-contractions can be an interesting contribution in fixed point theory. Also, the solution to fractional differential inclusion can be investigated by applying these proposed outlines.

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