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Symmetry of Ancient Solution for Fractional Parabolic Equation Involving Logarithmic Laplacian

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Abstract: In this research, we focus on the symmetry of an ancient solution for a fractional parabolic equation involving logarithmic Laplacian in an entire space. In the process of studying the property of a fractional parabolic equation, we obtained some maximum principles, such as the maximum principle of anti-symmetric function, narrow region principle, and so on. We will demonstrate how to apply these tools to obtain radial symmetry of an ancient solution.

Keywords: ancient solution; logarithmic Laplacian; sub-solution

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1. Introduction

Reaction–diffusion equations have been widely studied in a number of domains such as general shadow [1], activator–inhibitor systems [2,3], nerve propagation [4], etc. In fact, a nonlocal operator can be better used to describe some intersecting phenomena such as population ecology and diffusion. In recent years, the number of studies on the fractional parabolic equation has increased, and numerous interesting results have also been published. Wu and Yuan [5] studied the well-posedness of a solution in a semilinear fractional dissipative equation [6], such as

$$\frac{\partial u(x,t)}{\partial t} + (-\Delta)^{\alpha} u(x,t) = F(u). \tag{1}$$

where $0 < \alpha < 1$ and

$$(-\Delta)^{\alpha} u(x,t) = CP.V \int_{R^n} \frac{u(x,t) - u(y,t)}{|x-y|^{n+2\alpha}} dy. \tag{2}$$

When $F(u) = u^p(x,t) - u(x,t)$, (1) becomes a fractional Schrödinger equation. Chen and Wu [7] proved a Liouville's type theorem of fractional equation

$$\frac{\partial u(x,t)}{\partial t} + (-\Delta)^{\alpha} u(x,t) = f(t,u), \quad (x,t) \in R^n \times R. \tag{3}$$

As we all know, Liouville's theorems play an important role in partial differential equations.

The ancient solution is the one where $t \in (-\infty, T]$. It was first introduced by Richard Hamilton in [8]. It was widely studied in geometric flows [9]. What is more, the ancient solution is associated with the solution in Ricci flow; for specific details, refer to [10,11]. In 2022, Wu and Chen [12] studied the ancient solution of

$$\frac{\partial u(x,t)}{\partial t} + (-\Delta)^{\alpha} u(x,t) = f(t,u), \quad (x,t) \in R^n \times (-\infty, T]. \tag{4}$$

They proved the radial symmetry and monotonicity of a solution for (4).

In 2019, Chen and Weth [13] gave a new concept of logarithmic Laplacian, and they proved a series of generalized functional analyses in the bounded region. Zhang and Nie proved the symmetry and monotonicity of

$$(-\Delta)^L u(x) + u(x) = u^p(x), \quad x \in R^n \tag{5}$$

in [14]. The Schrödinger equation [15,16] is crucial for the knowledge of quantum mechanics. For the system of equations, Liu [17] investigated the radial symmetry of a solution for the semi-linear Logarithmic Laplacian systems in a ball.

In this paper, we consider the following nonlocal parabolic equation such that

$$\frac{\partial u(x,t)}{\partial t} + (-\Delta)^L u(x,t) = f(t,u(x,t)), \quad (x,t) \in R^n \times (-\infty, T]. \tag{6}$$

Assume $u(x,t)$ and $f(t,u)$ satisfy following conditions:

- (a) $f(t,u)$ is of class C^1 in u uniformly with respect to t.
- (b) $f(t,0) = 0$ and there exists a constant $\sigma > 0$ such that

$$f_u(t,0) < -\sigma, \quad \forall t \leq T. \tag{7}$$

- (c)

$$\sup_{t \in (-\infty, T]} u(x,t) \to 0, \quad x \to \infty. \tag{8}$$

The expression for logarithmic Laplacian is as follows:

Proposition 1 ([13]). *If u is Dini continuous at some point $x \in R^n$, $\Omega \in R^n$ is an open subset and $x \in \Omega$, then we have the following integral representation*

$$(-\Delta)^L u(x) = C_n \int_\Omega \frac{u(x) - u(y)}{|x-y|^n} dy + C_n \int_{R^n \setminus \Omega} \frac{-u(y)}{|x-y|^n} dy + [h(x) + \mu] u(x) \tag{9}$$

where

$$h(x) = C_n \int_{B_1(x) \setminus \Omega} \frac{1}{|x-y|^n} dy - C_n \int_{\Omega \setminus B_1(x)} \frac{1}{|x-y|^n} dy, \tag{10}$$

$C_n = \pi^{-\frac{-\pi}{2}} \Gamma(\frac{n}{2})$ *and* $\mu = 2\log 2 + k(\frac{n}{2}) - \beta$, *where* Γ *is the Gamma function,* $k = \frac{\Gamma'}{\Gamma}$ *is Digamma function,* $\beta = -\Gamma'(1)$ *is the Euler–Mascheroni constant, and* $B_1(x)$ *is a sphere with* x *as its center and* 1 *as its radius.*

We say $u(x,t)$ is a classical ancient solution of (6) if

$$u(x,t) \in L_0^1(R^n) \times C^1(-\infty, T]. \tag{11}$$

It is easy to check that $(-\Delta)^L u(x,t)$ is well defined. Before the proof, we give some basic notations.

If we select an arbitrary direction in R^n as x_1, we have $x = (x_1, x') \in R^n$. This is denote as

$$T_\lambda = \{x \in R^n | x_1 = \lambda, \lambda \in R\}. \tag{12}$$

Let $x^\lambda = (2\lambda - x_1, x')$ be the reflection point of x with respect to T_λ.
Set

$$\Sigma_\lambda = \{x \in R^n | x_1 < \lambda\} \tag{13}$$

and

$$\Sigma_\lambda^c = \{x \in R^n | x_1 > \lambda\}. \tag{14}$$

Suppose that $u(x,t)$ is a solution of (6), we denote

$$u_\lambda(x,t) = u(x^\lambda, t) \tag{15}$$

and

$$w_\lambda(x,t) = u_\lambda(x,t) - u(x,t). \tag{16}$$

Next, we show our main conclusions and necessary maximum principles.

2. Basic Theorem

Theorem 1. *Assume $f(x,t)$ satisfies conditions (a) and (b). Suppose $u(x,t) \in L_0^1(R^n) \times C^1(-\infty,T]$ is a positive bounded solution and a Dini continuous function of (6) satisfies (c). Then, $u(x,t)$ is radial symmetry and decreases in some points in R^n.*

It is well known that the study of the symmetry and monotonicity of a solution is of great importance in partial differential equations. Many methods have been produced, such as the direct method of moving plane [18], the extension method [19], the sliding method [20,21], and so on. For a fractional elliptic equation, there are many results [22–25]. However, results in fractional parabolic equations are relatively scarce. Chen-Wang-Niu-Hu [26] proved the asymptotic symmetric of a solution for the parabolic Equation (4).

In 2023, Luo and Zhang [27] proved the asymptotic symmetry of solution of a nonlocal weighted fractional parobolic equation:

$$\frac{\partial u(x,t)}{\partial t} + A_{2s}u(x,t) = f(t,u(x,t),\nabla u(x,t)) + g(x,t), \quad (x,t) \in R^n \times R. \tag{17}$$

However, for a fractional parabolic equation involving logarithmic Laplacian, there are few articles. Therefore, it is important to study the properties of the solution of (6). In this paper, we study the radial symmetry of the bounded solution of a fractional parabolic equation involving logarithmic Laplacian.

Theorem 2. *We consider equation*

$$\frac{\partial u(x,t)}{\partial t} + (-\Delta)^L u(x,t) + \omega u(x,t) = u^p(x,t), \quad (x,t) \in R^n \times (-\infty,T]. \tag{18}$$

where $1 < p < \infty$ and $\omega > \sigma$ is a constant. Suppose $u(x,t) \in L_0^1(R^n) \times C^1(-\infty,T]$ is a positive Dini continuous solution of (18), which satisfies (c). Then, $u(x,t)$ is the radial symmetry of some point in R^n.

In order to complete the proof of Theorem 1, we need the following maximum principles.

Theorem 3. *This is the maximum principle for the anti-symmetric function. Let Ω be a bounded Lipschitz region in Σ_λ. Assume that $w_\lambda(x,t) \in L_0^1(R^n) \times C^1((-\infty,T])$ is a uniformly bounded continuous function on $\overline{\Omega}$, a Dini continuous function in Ω, and it satisfies*

$$\begin{cases} \dfrac{\partial w_\lambda}{\partial t} + (-\Delta)^L w_\lambda(x,t) = c(x,t)w_\lambda(x,t), & (x,t) \in \Sigma_\lambda \times (-\infty,T]. \\ w_\lambda(x^\lambda,t) = -w_\lambda(x,t), & (x,t) \in \Sigma_\lambda \times (-\infty,T] \end{cases} \tag{19}$$

where $c(x,t)$ is bounded from above, there exists $t_1 < T$ such that

$$w_\lambda(x,t) \geq 0, \quad (x,t) \in \Sigma_\lambda \times (t_1,T] \tag{20}$$

and $h(x) + \mu \geq 0$.

Then, there exists a constant m such that

$$w_\lambda(x,t) \geq e^{-m(t-t_1)} \min\{0, \inf_{x\in\Omega} w_\lambda(x,t_1)\}, \quad (x,t) \in \Sigma_\lambda \times [t_1,T]. \tag{21}$$

Furthermore, if

$$w_\lambda(x,t_1) \geq 0, \quad x \in \Omega. \tag{22}$$

we have

$$w_\lambda(x,t) \geq 0, \quad (x,t) \in \Sigma_\lambda \times [t_1, T]. \tag{23}$$

Proof. Set

$$v(x,t) = e^{mt} w_\lambda(x,t). \tag{24}$$

Placing it into (19), we have

$$\frac{\partial v(x,t)}{\partial t} + (-\Delta)^L v(x,t) = (c(x,t)+m)v(x,t), \quad (x,t) \in \Omega \times (-\infty, T]. \tag{25}$$

We want to prove

$$v(x,t) \geq \min\{0, \inf_{x\in\Omega} v(x,t_1)\}, \quad (x,t) \in \Omega \times [t_1, T]. \tag{26}$$

If (26) is invalid, there exists a point $(x^0, t_0) \in \Omega \times (t_1, T]$ such that

$$v(x^0,t_0) = \min_{(x,t)\in\Omega\times[t_1,T]} v(x,t) < \min\{0, \inf_{x\in\Omega} v(x,t_1)\} \leq 0, \tag{27}$$

Using a simple deduction, we have

$$\frac{\partial v(x^0,t_0)}{\partial t} \leq 0 \tag{28}$$

and

$$\begin{aligned} (-\Delta)^L v(x^0,t_0) &= C_n \int_{\Sigma_\lambda} \frac{v(x^0,t_0) - v(y,t_0)}{|x^0-y|^n} dy + \int_{\Sigma_\lambda^c} \frac{-v(y,t_0)}{|x^0-y|^n} dy + [h(x)+\mu]v(x^0,t_0) \\ &\leq C_n \int_{\Sigma_\lambda} \frac{v(x^0,t_0) - v(y,t_0)}{|x^0-y^\lambda|^n} dy + \int_{\Sigma_\lambda} \frac{v(y,t_0)}{|x^0-y^\lambda|^n} dy + [h(x)+\mu]v(x^0,t_0) < 0. \end{aligned} \tag{29}$$

We choose proper m such that

$$m + c(x,t) < 0. \tag{30}$$

Combining (28) and (29), we obtain a contradiction in (25). So, (26) is valid. Thus,

$$w_\lambda(x,t) \geq e^{-m(t-t_1)} \min\{0, \inf_{x\in\Omega} w_\lambda(x,t_1)\}. \tag{31}$$

With this, we have finished the proof of Theorem 3. □

Remark 1. *The maximum principle for an anti-symmetric function is an important tool in the process of constructing a sub-solution.*

Theorem 4. *This is the narrow region principle. Let Ω be a Lipschitz region in Σ_λ contained in $\{x|\lambda - l < x_1 < \lambda\}$ with small l and $h(x)+\mu \geq 0$. Suppose $w_\lambda(x,t) \in L_0^1(R^n) \times C^1(-\infty, T]$ is uniformly bounded continuous function on $\overline{\Omega}$, a Dini continuous function in Ω and satisfies*

$$\begin{cases} \dfrac{\partial w_\lambda(x,t)}{\partial t} + (-\Delta)^L w_\lambda(x,t) = c(x,t) w_\lambda(x,t), & (x,t) \in \Omega \times (-\infty, T]. \\ w_\lambda(x^\lambda, t) = -w_\lambda(x,t), & (x,t) \in \Sigma_\lambda \times (-\infty, T]. \end{cases} \tag{32}$$

where $c(x,t)$ is bounded from above, and there exists $t_1 < T$ such that

$$w_\lambda(x,t) \geq 0, \quad (x,t) \in (\Sigma_\lambda \setminus \Omega) \times (t_1, T]. \tag{33}$$

Then, for sufficiently small l, there exists a constant $m > 0$ such that

$$w_\lambda(x,t) \geq e^{-m(t-t_1)} \min\{0, \inf w_\lambda(x,t_1)\}. \tag{34}$$

Furthermore, if

$$w_\lambda(x,t) \geq 0, \quad (x,t) \in (\Sigma_\lambda \setminus \Omega) \times (-\infty, T]. \tag{35}$$

Then, we have

$$w_\lambda(x,t) \geq 0, \quad (x,t) \in \Sigma_\lambda \times (-\infty, T]. \tag{36}$$

Proof. Set

$$v(x,t) = e^{mt} w_\lambda(x,t) \tag{37}$$

Take it into (32), we have

$$\frac{\partial v(x,t)}{\partial t} + (-\Delta)^L v(x,t) = (m + c(x,t))v(x,t), \quad (x,t) \in \Omega \times (-\infty, T]. \tag{38}$$

We want to show

$$v(x,t) \geq \min\{0, \inf_{x\in\Omega} v(x,t_1)\}, \quad (x,t) \in \Omega \times [t_1, T]. \tag{39}$$

If (39) is invalid, there exists a point $(x^0, t_0) \in \Omega \times (t_1, T]$ such that

$$v(x^0, t_0) = \min_{(x,t)\in\Sigma_\lambda \times (t_1,T]} v(x,t) < \min\{0, \inf_{x\in\Omega} v(x,t_1)\}. \tag{40}$$

We can quickly obtain

$$\frac{\partial v(x^0,t_0)}{\partial t_0} \leq 0 \tag{41}$$

and

$$\begin{aligned}(-\Delta)^L v(x^0,t_0) &= C_n \int_{\Sigma_\lambda} \frac{v(x^0,t_0) - v(y,t_0)}{|x^0 - y|^n} dy + C_n \int_{\Sigma_\lambda^c} \frac{-v(y,t_0)}{|x^0 - y|^n} dy + [h(x^0) + \mu] v(x^0, t_0) \\ &\leq C_n \int_H \frac{v(x^0,t_0)}{|x^0 - y^\lambda|^n} dy\end{aligned} \tag{42}$$

where $H \subset \Sigma_\lambda$ is the region

$$H = \{y \in \Sigma_\lambda, l < |y_1 - x_1^0| < 1, |y' - x'| < 1\} \tag{43}$$

and

$$\int_{\Sigma_\lambda} \frac{1}{|x^0 - y|^n} dy \geq \int_H \frac{1}{|x^0 - y|^n} dy \to \infty, \quad as \quad l \to 0. \tag{44}$$

If we take (42) into the first equality in (32), we have

$$\frac{\partial v(x^0,t_0)}{\partial t} \geq (-\int_H \frac{1}{|x^0 - y|^n} + c(x^0,t_0) + m) v(x^0, t_0). \tag{45}$$

Combining (44) and (45), we obtain a contradiction in (38). So, (34) is valid. Because $w_\lambda(x,t)$ is uniformly bounded, we have

$$w_\lambda(x,t) \geq Ce^{-m(t-t_1)}, \quad (x,t) \in \Omega \times (t_1, T]. \tag{46}$$

We verified (36) by setting $t_1 \to -\infty$. $\square$

Theorem 5. *This is the maximum principle at infinity. Let Ω be an unbounded Lipschitz region in Σ_λ. Suppose that $w_\lambda(x,t) \in L_0^1(R^n) \times C^1(-\infty, T]$ is a uniformly bounded continuous function on $\overline{\Omega}$, a Dini continuous function in Σ_λ, and it satisfies*

$$\begin{cases} \dfrac{\partial w_\lambda(x,t)}{\partial t} + (-\Delta)^L w_\lambda(x,t) = c(x,t) w_\lambda(x,t), \quad (x,t) \in \Omega \times (-\infty, T]. \\ w_\lambda(x^\lambda, t) = -w_\lambda(x,t), \quad (x,t) \in \Sigma_\lambda \times (-\infty, T]. \\ \lim\limits_{|x| \to \infty} w_\lambda(x,t) = 0, \quad uniformly \quad t \in (-\infty, T]. \end{cases} \tag{47}$$

Assume that there exists a constant $\sigma > 0$ such that

$$c(x,t) \leq -\sigma \tag{48}$$

at points where $w_\lambda(x,t) < 0$ in $\Omega \times (-\infty, T]$, for some $t_1 < T$,

$$w_\lambda(x,t) \geq 0, \quad (x,t) \in (\Sigma_\lambda \setminus \Omega) \times (t_1, T] \tag{49}$$

and $h(x) + \mu \geq 0$ on Ω. Then, there exists a constant $m > 0$ such that

$$w_\lambda(x,t) \geq e^{-m(t-t_1)} \min\{0, \inf w_\lambda(x, t_1)\}, \quad (x,t) \in \Omega \times [t_1, T]. \tag{50}$$

Furthermore, if

$$w_\lambda(x,t) \geq 0, \quad (x,t) \in (\Sigma_\lambda \setminus \Omega) \times (-\infty, T]. \tag{51}$$

We have

$$w_\lambda(x,t) \geq 0, \quad (x,t) \in \Sigma_\lambda \times (-\infty, T]. \tag{52}$$

Proof. We need an auxiliary function

$$v(x,t) = e^{mt} w_\lambda(x,t) \tag{53}$$

where $m > 0$ is a fixed constant that will be chosen later. If we take it into the first equality of (47), we have

$$\frac{\partial v(x,t)}{\partial t} + (-\Delta)^L v(x,t) = (c(x,t) + m) v(x,t), \quad (x,t) \in \Omega \times (-\infty, T]. \tag{54}$$

Next, we will show

$$v(x,t) \geq \min\{0, \inf_{x \in \Omega} v(x, t_1)\}, \quad (x,t) \in \Omega \times [t_1, T]. \tag{55}$$

If (55) is invalid, there exists a point $(x^0, t_0) \in \Omega \times (t_1, T]$ such that

$$v(x^0, t_0) = \min_{(x,t) \in \Sigma_\lambda \times (t_1, T]} v(x,t) < \min\{0, \inf_{\Omega} v(x, t_1)\}. \tag{56}$$

We can quickly obtain

$$\frac{\partial v(x^0, t_0)}{\partial t} \leq 0 \tag{57}$$

and

$$\begin{aligned}(-\Delta)^L v(x^0,t_0) &= C_n \int_{\Sigma_\lambda} \frac{v(x^0,t_0)-v(y,t_0)}{|x^0-y|^n} dy + C_n \int_{\Sigma_\lambda^c} \frac{-v(y,t_0)}{|x^0-y|^n} dy + (h(x)+\mu)v(x^0,t_0) \\ &\leq C_n \int_{\Sigma_\lambda} \frac{v(x^0,t_0)}{|x^0-y^\lambda|^n} dy + (h(x)+\mu)v(x^0,t_0) < 0.\end{aligned} \tag{58}$$

We choose a proper m such that

$$m < \sigma. \tag{59}$$

Combining (58) and (59), we obtain a contradiction in (54). It has been verified in (55). So,

$$w_\lambda(x,t) \geq e^{-m(t-t_1)} \min\{0, \inf w_\lambda(x,t_1)\}. \tag{60}$$

Furthermore, by the uniform boundness of $w_\lambda(x,t)$ and setting $t_1 \to -\infty$, we have

$$w_\lambda(x,t) \geq Ce^{-m(t-t_1)} \geq 0, \quad (x,t) \in \Sigma_\lambda \times (-\infty, T]. \tag{61}$$

□

Theorem 6. *This is the maximum principle in a narrow region and the neighborhood of infinity. Let*

$$\Omega = \{x \in \Sigma_\lambda, d(x,T_\lambda) < l \quad or \quad |x| > R\}. \tag{62}$$

Here, $l > 0$ is small and $R > 0$ is sufficiently large. Assume that $w_\lambda(x,t) \in L_0^1(R^n) \times C^1((-\infty,T])$ is a uniformly bounded continuous function on $\overline{\Omega}$, a Dini continuous function in Ω and satisfies (8). Suppose $c(x,t)$ is bounded from above in $\{x \in \Sigma_\lambda | d(x,T_\lambda) < l\}$, and

$$c(x,t) < -\sigma, \quad x \in \Sigma_\lambda, |x| > R. \tag{63}$$

There exists $t_1 < T$ such that

$$w_\lambda(x,t) \geq 0, \quad (x,t) \in (\Sigma_\lambda \setminus \Omega) \times (t_1, T] \tag{64}$$

and $h(x)+\mu \geq 0$ on Ω.

Then, there exists $m > 0$ such that

$$w_\lambda(x,t) \geq e^{-m(t-t_1)} \min\{0, \inf w_\lambda(x,t_1)\}, (x,t) \in \Sigma_\lambda \times [t_1, T] \tag{65}$$

Furthermore, if

$$w_\lambda(x,t) \geq 0, (x,t) \in (\Sigma_\lambda \setminus \Omega) \times (-\infty, T], \tag{66}$$

we have

$$w_\lambda(x,t) \geq 0, \quad (x,t) \in \Sigma_\lambda \times (-\infty, T]. \tag{67}$$

Proof. The specific proof is omitted, and it is similar to Theorems 4 and 5.

If the minimum arrives at infinity, we obtain a contradiction using an example similar to Theorem 5. Similarly, if the minimum arrives in a narrow area, we are able to obtain a contradiction using a similar step as in Theorem 4. □

3. Symmetry of Solution in $R^n \times (-\infty, T]$

Using a simple derivation, we know that $w_\lambda(x,t)$ satisfies the following equation

$$\frac{\partial w_\lambda(x,t)}{\partial t} + (-\Delta)^L w_\lambda(x,t) = c_\lambda(x,t) w_\lambda(x,t) \tag{68}$$

where $c_\lambda(x,t) = f_u(\xi,t)$ and $\xi(x,t)$ between $u(x,t)$ and $u_\lambda(x,t)$. We first give a brief framework for the proof of Theorem 1.

1. We prove $\underset{t\in(-\infty,T]}{inf}\ w_\lambda(x,t) > 0, x \in \Sigma_\lambda$ when λ is sufficiently negative.

2. Holding $\underset{t\in(-\infty,T]}{inf}\ w_\lambda(x,t) > 0, x \in \Sigma_\lambda,$ move the plane T_λ to the limiting position .

We denote

$$\lambda_0^- = \sup\{\lambda | \inf_{t\in(-\infty,T]} w_\mu(x,t) > 0, \forall x \in \Sigma_\lambda, \mu \le \lambda\}. \tag{69}$$

Similarly, we have

$$\lambda_0^+ = \inf\{\lambda | \inf_{t\in(-\infty,T]} w_\mu(x,t) > 0, \forall x \in \Sigma_\mu^c, \mu \ge \lambda\}. \tag{70}$$

3. Finally, we will show that $\lambda_0^- = \lambda_0^+$.

Next, we begin the proof of Theorem 1.

Proof of Theorem 1. First, when λ sufficiently negative, we will show that

$$\inf_{t\in(-\infty,T]} w_\lambda(x,t) > 0, \quad x \in \Sigma_\lambda \tag{71}$$

using three steps.

In Step 1, we need to prove

$$w_\lambda(x,t) \ge 0, \quad (x,t) \in \Sigma_\lambda \times (-\infty,T]. \tag{72}$$

Combining $f_u(t,0) < -\sigma$ and the continuity of $f(t,u)$, there exists $\epsilon > 0$ such that

$$f_u(t,u) \le -\sigma, \quad (t,u) \in (-\infty,T) \times (0,\epsilon]. \tag{73}$$

Since $\lim_{|x|\to\infty} u(x,t) = 0$, there exists a sufficiently large R such that

$$c_\lambda(x,t) = f_u(t,\xi) < -\sigma, \quad (x,t) \in B_R^c \times (-\infty,T]. \tag{74}$$

Using Theorem 5, we obtain

$$w_\lambda(x,t) \ge 0, \quad (x,t) \in \Sigma_\lambda \times (-\infty,T], \lambda < -R. \tag{75}$$

In Step 2, we will show that

$$w_\lambda(x,t) > 0, \quad (x,t) \in \Sigma_\lambda \times (-\infty,T], \lambda < -R. \tag{76}$$

If (76) is not valid, there exists point $(x^0,t_0) \in \Sigma_\lambda \times (-\infty,T]$ such that

$$w_\lambda(x^0,t_0) = 0. \tag{77}$$

On the one hand, we obtain

$$\begin{aligned}(-\Delta)^L w_\lambda(x^0,t_0) &= -\frac{\partial w_\lambda}{\partial t}(x^0,t_0) + f(t_0,u_\lambda(x^0,t_0)) - f(t_0,u(x^0,t_0)) \\ &= -\frac{\partial w_\lambda}{\partial t}(x^0,t_0) \ge 0.\end{aligned} \tag{78}$$

On the other hand, we have

$$\begin{aligned}(-\Delta)^L w_\lambda(x^0,t_0) &= C_n \int_{\Sigma_\lambda} \frac{w_\lambda(x^0,t_0) - w_\lambda(y,t_0)}{|x^0-y|^n} dy - C_n \int_{\Sigma_\lambda^c} \frac{w_\lambda(y,t_0)}{|x-y|^n} dy + (h(x)+\mu) w(x^0,t_0) \\ &= C_n \int_{\Sigma_\lambda} w_\lambda(y,t_0)\left(\frac{1}{|x^0-y^\lambda|^n} - \frac{1}{|x^0-y|^n}\right) dy < 0.\end{aligned} \tag{79}$$

Combining (78) and (79), we have

$$w_\lambda(x, t_0) \equiv 0, \quad x \in \Sigma_\lambda \tag{80}$$

However, when $x = 0^\lambda$, we have

$$w_\lambda(0^\lambda, t_0) = u(0, t_0) - u(0^\lambda, t_0) > 0, \tag{81}$$

where λ is sufficiently negative. It contradicts (80).

Thus, we have verified that

$$w_\lambda(x, t) > 0. \tag{82}$$

In Step 3, we will show that

$$\inf_{t\in(-\infty,T]} w_\lambda(x, t) > 0, \quad x \in \Sigma_\lambda, \lambda \leq -R. \tag{83}$$

If (83) is invalid, there exists x^0 such that

$$\inf_{t\in(-\infty,T]} w_\lambda(x^0, t) = 0. \tag{84}$$

Due to the infinity of interval of t, the minimum of $w_\lambda(x, t)$ may not be attained. There exists sequence $\{t_k\}$ and $\{\epsilon_k\}$ such that

$$w_\lambda(x^0, t_k) = \epsilon_k \to 0, \quad as \quad k \to \infty. \tag{85}$$

We need to use the perturbation technique.

Consider an auxiliary function

$$v_\lambda^k(x, t) = w_\lambda(x, t) - \epsilon_k \psi_\delta(x)\eta_k(t), \quad (x, t) \in \Sigma_\lambda \times (-\infty, T] \tag{86}$$

where

$$\eta_k(t) = \begin{cases} 1, & |t - t_k| < \dfrac{1}{2}. \\ 0, & |t - t_k| \geq 1. \end{cases} \tag{87}$$

and

$$\psi_\delta(x) = \begin{cases} 1, & |x - x^0| < \dfrac{\delta}{2}. \\ 0, & |x - x^0| \geq \delta. \end{cases} \tag{88}$$

Therefore, we can obtain

$$v_\lambda^k(x, t) = w_\lambda(x, t) > 0, \quad (x, t) \in (\Sigma_\lambda \times (-\infty, T]) \setminus (B_\delta(x^0) \times [t_k - 1, \min\{t_k + 1, T\}]) \tag{89}$$

and

$$v_\lambda^k(x^0, t_k) = 0. \tag{90}$$

From (89) and (90), the minimum point can be obtained in $B_\delta(x^0) \times [t_k - 1, \min\{t_1 + 1, T\}]$. (denote it by (x_m, t_m)).

Then

$$v_\lambda^k(x_m, t_m) = \inf_{(x,t)\in\Sigma_\lambda\times(-\infty,T]} v_\lambda^k(x, t) \leq 0 \tag{91}$$

and

$$0 < w_\lambda(x_m, t_m) \leq \epsilon_k \psi_\delta(x_m)\eta_k(t_m) \leq \epsilon_k. \tag{92}$$

Furthermore, we have

$$\begin{aligned}\frac{w_\lambda(x_m,t_m)}{\partial t} &= \frac{\partial v_\lambda^k}{\partial t}(x_m,t_m) + \epsilon_k \psi_\delta(x_m)\frac{\partial \eta_k}{\partial t}(t_m) \\ &\le \epsilon_k \psi_\delta(x_m)\frac{\partial \eta_k}{\partial t}(t_m) \\ &\le C\epsilon_k\end{aligned} \tag{93}$$

and

$$\begin{aligned}(-\Delta)^L v_\lambda^k(x_m,t_m) &= (-\Delta)^L w_\lambda(x_m,t_m) - \epsilon_k \eta_k(t_m)(-\Delta)^L \psi_\delta(x_m) \\ &= -\frac{\partial w_\lambda(x_m,t_m)}{\partial t} + c_\lambda(x_m,t_m) w_\lambda(x_m,t_m) - \epsilon_k \eta_k(t_m)(-\Delta)^L \psi_\delta(x_m) \\ &\ge -C\epsilon_k + c_\lambda(x_m,t_m) w_\lambda(x_m,t_m) - \epsilon_k \eta_k(t_m)(-\Delta)^L \psi_\delta(x_m) \\ &\ge -C\epsilon_k - \epsilon_k \eta_k(t_m)(-\Delta)^L \psi_\delta(x_m) \to 0, k \to \infty.\end{aligned} \tag{94}$$

What is more,

$$\begin{aligned}(-\Delta)^L v_\lambda^k(x_m,t_m) &= C_n \int_{\Sigma_\lambda} \frac{v_\lambda^k(x_m,t_m) - v_\lambda^k(y,t_m)}{|x_m-y|^n} dy + C_n \int_{\Sigma_\lambda^c} \frac{-v_\lambda^k(y,t_m)}{|x_m-y|^n} dy + [\mu + h(x)] v_\lambda^k(x_m,t_m) \\ &= C_n \int_{\Sigma_\lambda} \frac{v_\lambda^k(x_m,t_m)}{|x_m-y^\lambda|^n} dy + C_n \int_{\Sigma_\lambda} (v_\lambda^k(x_m,t_m) - v_\lambda^k(y,t_m))(\frac{1}{|x_m-y|^n} - \frac{1}{|x_m-y^\lambda|^n}) dy \\ &+ [h(x) - \mu] v_\lambda^k(x_m,t_m).\end{aligned} \tag{95}$$

Since $v_\lambda^k(x_m,t_m) \le 0$, $v_\lambda^k(x_m,t_m) - v_\lambda(y,t_m) \le 0$, and $\frac{1}{|x_m-y|^n} - \frac{1}{|x_m-y^\lambda|^n} > 0$, we have

$$(-\Delta)^L v_\lambda^k(x_m,t_m) \le 0. \tag{96}$$

Because

$$0 \ge v_\lambda^k(x_m,t_m) = w_\lambda(x_m,t_m) - \epsilon_k \eta_k(t_m)\psi_\delta(x_m) \ge -\epsilon_k \to 0, as \quad k \to \infty, \tag{97}$$

if we combine (94) and (96), we have

$$w_\lambda(y,t_m) \to 0, \quad uniformly \quad for \quad y \in \Sigma_\lambda, \quad as \quad k \to \infty. \tag{98}$$

What is more, we have

$$w_\lambda(0^\lambda, t_m) > 0. \tag{99}$$

It contradicts (98). Thus, (83) is valid. Therefore, we have verified (71).

Next, we move the plane T_λ to a limiting position where (71) still holds. We can set

$$\lambda_0^- = \sup\{\lambda | \inf_{t\in(-\infty,T]} w_\mu(x,t) > 0, \forall x \in \Sigma_\lambda, \mu \le \lambda\} \tag{100}$$

Using a similar derivation of (71), we have

$$\inf_{t\in(\infty,T]} w_\lambda(x,t) > 0, \quad (x,t) \in \Sigma_\lambda^c \times (-\infty,T] \tag{101}$$

where $\lambda > R$.

We set

$$\lambda_0^+ = \inf\{\lambda | \inf_{t\in(-\infty,T]} w_\mu(x,t) > 0, \forall x \in \Sigma_\mu^c, \mu \ge \lambda\}. \tag{102}$$

We will show there exists a sequence $\{\underline{t_k}\}$ such that

$$w_{\lambda_0^-}(x,\underline{t_k}) \to 0, \quad as \quad k \to \infty. \tag{103}$$

uniformly for all $x \in \Sigma_{\lambda_0^-}$, and there exists a sequence $\{\overline{t_k}\}$ such that

$$w_{\lambda_0^+}(x, \overline{t_k}) \to 0, \quad as \quad k \to \infty. \tag{104}$$

uniformly for all $x \in \Sigma^c_{\lambda_0^+}$.

In order to obtain (103), from the definition of λ^-, there exists a sequence $\{\lambda_k\} \searrow \lambda_0^-$ such that

$$\inf_{(x,t)\in\Sigma_{\lambda_k}\times(-\infty,T]} w_{\lambda_k}(x,t) \le 0. \tag{105}$$

We consider two conditions:

1. There exists a subsequence of $\{\lambda_k\}$ (still denoted by $\{\lambda_k\}$) such that

$$\inf_{(x,t)\in\Sigma_{\lambda_k}\times(-\infty,T]} w_{\lambda_k}(x,t) < 0. \tag{106}$$

2. There exists a subsequence of $\{\lambda_k\}$ such that

$$\inf_{(x,t)\in\Sigma_{\lambda_k}\times(-\infty,T]} w_{\lambda_k}(x,t) = 0. \tag{107}$$

In the first case, we assume that

$$\inf_{(x,t)\in\Sigma_{\lambda_k}\times(-\infty,T]} w_{\lambda_k}(x,t) =: -m_k < 0. \tag{108}$$

There exists sequence $\{t_k\} \in (-\infty, T]$ and $\{\epsilon_k\} \searrow 0$ ($\epsilon_k = o(m_k)$) such that

$$w_{\lambda_k}(x(t_k), t_k) = \inf_{x\in\Sigma_{\lambda_k}} w_{\lambda_k}(x, t_k) = -m_k + \epsilon_k < 0. \tag{109}$$

We assume $x(t)$ to be the point where the given function can attain its minimum for every t, and we set

$$v_{\lambda_k}(x,t) = w_{\lambda_k}(x,t) - \epsilon_k \eta_k(t) \tag{110}$$

where

$$\eta_k(t) = \begin{cases} 1, & |t - t_k| < \dfrac{1}{2} \\ 0, & |t - t_k| \ge 1. \end{cases} \tag{111}$$

Furthermore,

$$v_{\lambda_k}(x(t_k), t_k) = -m_k + \epsilon_k - \epsilon_k = -m_k < 0 \tag{112}$$

and

$$v_{\lambda_k}(x,t) = w_\lambda(x,t) \ge -m_k, \quad t \in (-\infty, T] \setminus [t_k - 1, \min\{t_k + 1, T\}]. \tag{113}$$

It is easy to see that $v_{\lambda_k}(x,t)$ can attain a minimum in $\Sigma_{\lambda_k} \times [t_k - 1, \min\{t_k + 1, T\}]$, (denoted as $(\iota(\widehat{t_k}), \widehat{t_k})$)

$$v_{\lambda_k}(\iota(\widehat{t_k}), \widehat{t_k}) = \inf_{\Sigma_{\lambda_k}\times(-\infty,T]} v_{\lambda_k}(x,t) \le -m_k. \tag{114}$$

Using a simple derivation, we have

$$\frac{v_{\lambda_k}(\iota(\widehat{t_k}), \widehat{t_k})}{\partial t} \le 0 \tag{115}$$

and

$$-m_k \le w_{\lambda_k}(\iota(\widehat{t_k}), \widehat{t_k}) \le v_\lambda(\iota(\widehat{t_k}), \widehat{t_k}) + \epsilon_k \eta_k(\widehat{t_k}) \le -m_k + \epsilon_k < 0. \tag{116}$$

It follows that
$$\frac{\partial w_{\lambda_k}(\iota(\widehat{t}_k),\widehat{t}_k)}{\partial t} \le \epsilon_k \frac{\partial \eta_k(\widehat{t}_k)}{\partial t} \le C\epsilon_k. \tag{117}$$
On the one hand, we have
$$\begin{aligned}(-\Delta)^L v_{\lambda_k}(\iota(\widehat{t}_k),\widehat{t}_k) &= (-\Delta)^L w_{\lambda_k}(\iota(\widehat{t}_k),\widehat{t}_k)\\ &= -\frac{\partial w_{\lambda_k}(\iota(\widehat{t}_k),\widehat{t}_k)}{\partial t} + c_{\lambda_k}(\iota(\widehat{t}_k),\widehat{t}_k) w_{\lambda_k}(\iota(\widehat{t}_k),\widehat{t}_k)\\ &\ge -C\epsilon_k + c_{\lambda_k}(\iota(\widehat{t}_k),\widehat{t}_k) w_{\lambda_k}(\iota(\widehat{t}_k),\widehat{t}_k)\\ &\ge -C(\epsilon_k + m_k).\end{aligned} \tag{118}$$
On the other hand, we have
$$\begin{aligned}&(-\Delta)^L v_{\lambda_k}(\iota(\widehat{t}_k),\widehat{t}_k)\\ =&C_n \int_{\Sigma_{\lambda_k}} (v_{\lambda_k}(\iota(\widehat{t}_k),\widehat{t}_k) - v_{\lambda_k}(y,\widehat{t}_k))(\frac{1}{|\iota(\widehat{t}_k)-y|^n} - \frac{1}{|\iota(\widehat{t}_k)-y^\lambda|^n})dy +\\ &C_n \int_{\Sigma_{\lambda_k}} \frac{v_{\lambda_k}(\iota(\widehat{t}_k),\widehat{t}_k)}{|\iota(\widehat{t}_k)-y^\lambda|^n} dy + (h(\iota(\widehat{t}_k)) + \mu) v_{\lambda_k}(\iota(\widehat{t}_k),\widehat{t}_k).\end{aligned} \tag{119}$$
Combining
$$v_{\lambda_k}(\iota(\widehat{t}_k),\widehat{t}_k) - v_{\lambda_k}(y,\widehat{t}_k) \le 0, \quad y \in \Sigma_{\lambda_k}, \tag{120}$$
and
$$v_{\lambda_k}(\iota(\widehat{t}_k),\widehat{t}_k) < 0, \tag{121}$$
we obtain
$$-C(\epsilon_k + m_k) \le (-\Delta)^L v_{\lambda_k}(\iota(\widehat{t}_k),\widehat{t}_k) \le 0. \tag{122}$$
So, the integral of $\int_{\Sigma_\lambda} \frac{1}{|x(\widehat{t}_k)-y^{\lambda_k}|} dy$ is bounded, and it follows that $x(\widehat{t}_k)$ is bounded away from T_{λ_k}.

What is more, we have
$$(-\Delta)^L v_{\lambda_k}(\iota(\widehat{t}_k),\widehat{t}_k) \to 0, \quad as \quad k \to \infty \tag{123}$$
and
$$v_{\lambda_k}(\iota(\widehat{t}_k),\widehat{t}_k) - v_{\lambda_k}(y,\widehat{t}_k) \to 0. \tag{124}$$
So
$$w_{\lambda_k}(y,\widehat{t}_k) \to 0, \quad uniformly \quad y \in \Sigma_{\lambda_k} \quad as \quad k \to \infty. \tag{125}$$
We have therefore verified (103).

In the second condition, we have
$$\inf_{(x,t)\in\Sigma_{\lambda_k}\times(-\infty,T]} w_{\lambda_k}(x,t) = 0. \tag{126}$$
Using a similar derivation of (71), we can obtain
$$\inf_{\Sigma_{\lambda_k}\times(-\infty,T]} w_{\lambda_k}(x,t) > 0, \quad x \in \Sigma_{\lambda_k}. \tag{127}$$
This contradicts the definition of λ_0^-. Thus, the case will not happen. Using a similar derivation of (103), we can obtain (104).

Finally, we will show
$$\lambda_0^- = \lambda_0^+. \tag{128}$$

if (128) is invalid. Suppose $\lambda \in (\lambda_0^-, \lambda_0^+)$, and we investigate the property of $w_\lambda(x,t)$ for x in Σ_λ^c. For $x \in \Sigma_\lambda^c$, x^λ can be the reflection point of x with respect to T_λ, and $(x^\lambda)^{\lambda_0^+}$ can be the reflection point of x^λ with respect to $T_{\lambda_0^+}$.

For $x = (x_1, x') \in \Sigma_\lambda^c$, we have

$$\begin{aligned} w_\lambda(x, \overline{t_k}) &= u(x^\lambda, \overline{t_k}) - u(x, \overline{t_k}) \\ &= u(x^\lambda, \overline{t_k}) + u((x^\lambda)^{\lambda_0^+}, \overline{t_k}) - u((x^\lambda)^{\lambda_0^+}, \overline{t_k}) - u(x, \overline{t_k}) \\ &= w_{\lambda_0^+}((x^\lambda)^{\lambda_0^+}, \overline{t_k}) - w_{\lambda_0^+ + x_1 - \lambda}((x^\lambda)^{\lambda_0^+}, \overline{t_k}) \\ &\le \epsilon_k - w_{\lambda_0^+ + x_1 - \lambda}((x^\lambda)^{\lambda_0^+}, \overline{t_k}) \\ &< 0. \end{aligned} \tag{129}$$

where $\epsilon_k \to 0$ as $k \to \infty$. Using a similar derivation, for fixed $x \in \Sigma_\lambda^c$,

$$w_\lambda(x, \underline{t_k}) > 0. \tag{130}$$

where k is sufficiently large. Hence, for arbitrary subset $G \subset\subset \Sigma_\lambda^c$, there exists a constant $c_0 > 0$, for sufficiently large k,

$$w_\lambda(x, \overline{t_k}) < -c_0, \quad x \in \overline{G} \tag{131}$$

and

$$w_\lambda(x, \underline{t_k}) > c_0, \quad x \in \overline{G}. \tag{132}$$

We assume $\underline{t_k} < \overline{t_k}$. Based on the above conclusions, there exists $\sigma_k \in (\underline{t_k}, \overline{t_k})$ such that

$$w_\lambda(x,t) > 0, x \in \overline{G}, t \in [\underline{t_k}, \sigma_k) \tag{133}$$

and $w_\lambda(\cdot, \sigma_k)$ vanishes somewhere on $\overline{G}$.

Next, we give two estimates of $w_\lambda(x,t)$.

1. A global lower bound estimate in a parabolic cylinder $\Sigma_\lambda^c \times [\underline{t_k}, \sigma_k]$ such that
$$w_\lambda(x,t) \ge e^{-\theta(t-\underline{t_k})} \min\{0, \inf_{x \in \Sigma_\lambda^c} w_\lambda(x, \underline{t_k})\} \tag{134}$$
along with
$$\inf_{x \in \Sigma_\lambda^c} w_\lambda(x, \underline{t_k}) \ge -\epsilon_k \to 0, \quad as \quad k \to \infty \tag{135}$$
where θ is a constant and λ is close to λ_0^+.
2. A positive lower bound estimate on compact subset of G. There exists a subset $G_0 \subset\subset G$ and a constant $M > 0$ such that
$$w_\lambda(x,t) \ge M e^{-\theta(t-\underline{t_k})} > 0, \quad (x,t) \in G_0 \times [\underline{t_k}, \sigma_k]. \tag{136}$$

Now, we show estimate 1. Combining (130) and (8), we have (135). Because of (132), we just need to prove

$$w_\lambda(x, \underline{t_k}) \ge e^{-\theta(t-\underline{t_k})} \min\{0, \inf_{x \in \Sigma_\lambda^c} w_\lambda(x, \underline{t_k})\}, \quad (x,t) \in (\Sigma_\lambda^c \setminus G) \times [\underline{t_k}, \sigma_k]. \tag{137}$$

We consider the following problem:

$$\begin{cases} \dfrac{\partial w_\lambda(x,t)}{\partial t} + (-\Delta)^L w_\lambda(x,t) = c_\lambda(x,t) w_\lambda(x,t), & (x,t) \in (\Sigma_\lambda^c \setminus G) \times [\underline{t_k}, \sigma_k]. \\ w_\lambda(x,t) \ge 0, \quad (x,t) \in G \times [\underline{t_k}, \sigma_k]. \end{cases} \tag{138}$$

Using (130) and the continuity of $w_\lambda(x,t)$, we have verified the second equality of (138). We set R_1 be the radius such that

$$f(t,\xi(x,t)) := c_\lambda(x,t) < -\sigma, |x| > R_1, t \in (-\infty, T]. \tag{139}$$

where $R_1 > R$.

We choose $G = \Sigma^c_{\mu+\delta} \cap B_{R_2}(0)$, and denote $G_1 := \{x| \quad \lambda < x_1 < \mu+\delta, |x| \leq R_2\}$ and $G_2 := \{x \in \Sigma^c_\lambda \setminus G| \quad |x| > R_2\}$ where μ , δ are the undetermined constants and $R_2 > R_1$ because $c_\lambda(x,t)$ is bounded in G_1 and $c_\lambda(x,t) < -\sigma$ in G_2. We arrive at (137) by Theorem 6, which proves estimate 1.

Next, we will prove estimate 2 by constructing a sub-solution of $w_\lambda(x,t)$.

Denote

$$L_\lambda \xi(x,t) := \frac{\partial \xi(x,t)}{\partial t} + (-\Delta)^L \xi(x,t) - c_\lambda(x,t)\xi(x,t). \tag{140}$$

We have

$$L_\lambda w_\lambda(x,t) = 0. \tag{141}$$

In order to facilitate the derivation, we assume $\lambda = 0$ and $T_\lambda = T_0$. We set

$$p(x,t) = C_0 p_0(x,t) = C_0 e^{-\theta(t-t_k)}(v_\mu(x,t) - \tau g(x)) \tag{142}$$

where $0 < \theta < \sigma$, $C_0 = \frac{c_0}{\|p_0(x,\underline{t_k})\|_{L^\infty(G)}}$, $\mu > \lambda_0^+$, τ is an undetermined constant, and $g(x)$ and $v_\mu(x,t)$ are as follows:

$$g(x) = \begin{cases} -1, & x_1 < 0. \\ 1, & x_1 \geq 0. \end{cases} \tag{143}$$

and

$$v_\mu(x,t) = \begin{cases} w_\mu(x,t), & x_1 > \mu, t > \underline{t_k} \\ 0, & -\mu \leq x_1 \leq \mu, t > \underline{t_k} \\ w_\mu(x_1+2\mu, x', t), & x_1 < -\mu, t > \underline{t_k}. \end{cases} \tag{144}$$

Using the definition of v_μ(x,t), for $x \in \Sigma^c_\mu$, we have

$$\begin{aligned}
&(-\Delta)^L v_\mu(x,t) - (-\Delta)^L w_\mu(x,t) \\
&= C_n \int_{\Sigma_\mu} \frac{v_\mu(x,t) - v_\mu(y,t)}{|x-y|^n} dy - C_n \int_{\Sigma^c_\mu} \frac{v_\mu(y,t)}{|x-y|^n} dy - C_n \int_{\Sigma_\mu} \frac{w_\mu(x,t) - w_\mu(y,t)}{|x-y|^n} dy \\
&+ C_n \int_{\Sigma^c_\mu} \frac{w_\mu(y,t)}{|x-y|^n} dy + (h(x,t)+\mu)(v_\mu(x,t) - w_\mu(x,t)) \\
&= C_n \int_{R^n} \frac{w_\mu(y,t) - v_\mu(y,t)}{|x-y|^n} dy \\
&= C_n \int_{-\infty}^{\mu} \int_{R^{n-1}} \frac{w_\mu(y,t)}{|x-y|^n} dy + C_n \int_{-\infty}^{-\mu} \int_{R^{n-1}} \frac{v_\mu(y,t)}{(|x_1-y_1|^2 + |x'-y'|^2)^{\frac{n}{2}}} dy \\
&= C_n \int_{R^{n-1}} (\int_{-\infty}^{\mu} \frac{w_\mu(y,t)}{|x-y|^n} dy_1 - \int_{-\infty}^{-\mu} \frac{w_\lambda(y_1+2\mu, y', t)}{(|x_1-y_1|^2 + |x'-y'|^2)^{\frac{n}{2}}} dy_1) dy' \\
&= C_n \int_{-\infty}^{\mu} \int_{R^{n-1}} (\frac{w_\mu(y,t)}{|x-y|^n} dy_1 - \frac{w_\mu(y,t)}{(|x_1-y_1+2\mu|^2 + |x'-y'|^2)^{\frac{n}{2}}}) dy_1 dy' \\
&= C_n \int_{\Sigma_\mu} (\frac{1}{|x-y|^n} - \frac{1}{(|x'-y'|^2 + |x_1-y_1+2\mu|^2)^{\frac{n}{2}}}) w_\mu(y,t) dy < 0.
\end{aligned} \tag{145}$$

where

$$\frac{1}{|x-y|^n} - \frac{1}{(|x'-y'|^2 + |x_1-y_1+2\mu|^2)^{\frac{n}{2}}} dy > 0 \tag{146}$$

and

$$w_\mu(x,t) < 0, \quad (x,t) \in \Sigma_\mu \times (-\infty, T]. \tag{147}$$

We choose $\mu < d - \delta_0$ where δ_0 is sufficiently small, μ is close to λ, and $d > 0$ is a constant with $d > \lambda_0^+$. We denote $G_0 = (\Sigma_{\lambda+d}^c \cap B_{R_1}(0)) \subset\subset G$, and there exists a constant C_0 such that

$$w_\mu(x,t) > C_0, \quad (x,t) \in G_0 \times [\underline{t_k}, \sigma_k]. \tag{148}$$

Let τ be a sufficiently small constant such that

$$L_\lambda w_\mu(x,t) - L_\lambda p(x,t) \geq 0, \quad (x,t) \in G \times [\underline{t_k}, \sigma_k] \tag{149}$$

holds due to a similar derivation to Lemma 4.3 in [26].

For $t = \underline{t_k}$, using (132), we have

$$p(x,t) = \frac{c_0 p_0(x, \underline{t_k})}{\|p_0(x, \underline{t_k})\|_{L_\infty(G)}} \leq c_0, \quad x \in \overline{G}. \tag{150}$$

It has been verified that

$$w_\mu(x, \underline{t_k}) - v_\mu(x, \underline{t_k}) \geq 0. \tag{151}$$

Next, we consider $(x,t) \in \Sigma_\lambda^c \setminus G = G_1 \cup G_2$.

For $x \in G_1$, we have

$$\begin{cases} v_\mu(x,t) = 0, \quad -\mu < x_1 < \mu \\ v_\mu(x,t) = w_\mu(x,t), \mu < x_1 < \mu + \delta \end{cases} \tag{152}$$

where δ is a small constant. It follows that

$$v_\mu(x,t) \leq w_\mu(x,t) \tag{153}$$

Furthermore, if $x \in G_2$, there exists a sufficiently large R_2, such that

$$v_\mu(x,t) \leq \frac{\tau}{2} g(x). \tag{154}$$

Thus, we obtain

$$p(x,t) \leq -e^{\theta(t-\underline{t_k})} C \frac{\tau}{2}, \quad x \in \Sigma_\lambda^c \setminus G. \tag{155}$$

What is more, we have

$$w_\lambda(x,t) \geq -e^{\theta(t-\underline{t_k})} \epsilon_k. \tag{156}$$

As $k \to \infty$,

$$w_\lambda(x,t) - p(x,t) \geq 0, \quad (x,t) \in (\Sigma_\lambda^c \setminus G) \times [\underline{t_k}, \sigma_k]. \tag{157}$$

Thus, we have

$$w_\lambda(x,t) \geq p(x,t), \quad (x,t) \in G \times [\underline{t_k}, \sigma_k] \tag{158}$$

due to Theorem 3.

Because

$$p(x,t) \geq C e^{-\theta(t-\underline{t_k})}, \quad (x,t) \in G_0 \times [\underline{t_k}, \sigma_k]. \tag{159}$$

using (148).

So,

$$w_\lambda(x,t) \geq C e^{-\theta(t-\underline{t_k})}, \quad (x,t) \in G_0 \times [\underline{t_k}, \sigma_k]. \tag{160}$$

we have thus verified estimate 2.

Next, we will show (133). Because $w_\lambda(\cdot, \sigma_k)$ vanishes somewhere on $\overline{G}$, using the maximum principle, we assume $\overline{x} \in \partial G$ such that

$$w_\lambda(\overline{x}, \sigma_k) = 0. \tag{161}$$

where δ is a sufficiently small constant to satisfy

$$B_\delta(\overline{x}) \cap G_0 = \varnothing \tag{162}$$

and

$$w_\lambda(x, \underline{t_k}) > \frac{c_0}{2}, \quad x \in B_\delta(\overline{x}). \tag{163}$$

We set

$$S_\lambda(x,t) = \chi_{G_0 \cup G_0^\lambda} w_\lambda(x,t) + (x_1\psi(x) - b(x))\epsilon_k e^{-\theta(t-\underline{t_k})} \tag{164}$$

where

$$b(x) = \begin{cases} 1, & x_1 \geq \lambda. \\ -1, & x_1 < \lambda. \end{cases} \tag{165}$$

G_0^λ is the reflection region of G_0 (for simplicity of derivation, we assume $\lambda = 0$)

$$\chi_{G_0^\lambda \cup G_0} = \begin{cases} 1, & x \in G_0^\lambda \cup G_0 \\ 0, & x \notin G_0^\lambda \cup G_0. \end{cases} \tag{166}$$

and

$$\psi(x) = \begin{cases} 1, & x \in B_\delta(\overline{x}), \\ 0, & x \notin B_\delta(\overline{x}). \end{cases} \tag{167}$$

with $0 < |\psi(x)| \leq 1$. We can quickly obtain $|(-\Delta)^L(x_1\psi(x))| \leq C$. Obviously, $S_\lambda(x,t)$ is a symmetric function. We will show that

$$\begin{cases} L_\lambda w_\lambda \geq L_\lambda S_\lambda(x,t), & (x,t) \in B_\delta(\overline{x}) \times [\underline{t_k}, \sigma_k] \\ w_\lambda(x,t) \geq S_\lambda(x,t), & (x,t) \in (\Sigma_\lambda^c \setminus B_\delta(\overline{x})) \times [\underline{t_k}, \sigma_k] \\ w_\lambda(x,\underline{t_k}) \geq S_\lambda(x,\underline{t_k}) \end{cases} \tag{168}$$

When $t = t_k$, we have

$$S_\lambda(x,\underline{t_k}) = (x_1\phi(x) - b(x))\epsilon_k \leq \epsilon_k, x \in B_\delta(\overline{x}). \tag{169}$$

Thus, we have verified the third equality of (168).

Next, we consider $x \notin B_\delta(\overline{x})$. For $x \in G_0$, we have

$$S_\lambda(x,t) = w_\lambda(x,t) - \epsilon_k e^{-\theta(t-\underline{t_k})} \leq w_\lambda(x,t). \tag{170}$$

For $x \in \Sigma_\lambda^c \setminus G_0$ and $x \notin B_\delta(\overline{x})$, we have

$$S_\lambda(x,t) = -\epsilon_k e^{-\theta(t-\underline{t_k})} \leq w_\lambda(x,t), \quad t \in [\underline{t_k}, \sigma_k]. \tag{171}$$

Thus, we have verified the second equality of (168). Furthermore, for $x \in B_\delta(\overline{x})$, we have

$$\begin{aligned} (-\Delta)^L \chi_{G_0^\lambda \cup G_0}(x) w_\lambda(x,t) &= C_n \int_{\Sigma_\lambda} \frac{-\chi_{G_0^\lambda \cup G_0}(y) w_\lambda(y,t)}{|x-y|^n} dy - C_n \int_{\Sigma_\lambda^c} \frac{\chi_{G_0^\lambda \cup G_0}(y) w_\lambda(y,t)}{|x-y|^n} dy \\ &\quad + (h(x) + \mu)\chi_{G_0 \cup G_0^\lambda}(x) w_\lambda(x,t) \\ &\leq -Ce^{-\theta(t-\underline{t_k})} \int_{G_0} (\frac{1}{|x-y|^n} - \frac{1}{|x-y^\lambda|^n}) dy \\ &\leq -Ce^{-\theta(t-\underline{t_k})}. \end{aligned} \tag{172}$$

Thus, for $(x,t) \in B_\delta(\overline{x}) \times [\underline{t_k}, \sigma_k]$, we have

$$\begin{aligned} L_\lambda S_\lambda(x,t) \leq & (-\Delta)^L \chi_{G_0 \cup G_0^\lambda}(x) w_\lambda(x,t) - \theta(x_1\psi(x)-1)\epsilon_k e^{-\theta(t-\underline{t_k})} \\ & + \epsilon_k e^{-\theta(t-\underline{t_k})}(-\Delta)^L(x_1\psi(x)-1) + c_\lambda(x,t)(x_1\psi(x)-1)\epsilon_k e^{-\theta(t-\underline{t_k})} \\ & \leq (-\Delta)^L(\chi_{G_0\cup G_0^\lambda})(x) w_\lambda(x,t) + C\epsilon_k e^{-\theta(t-\underline{t_k})}. \end{aligned} \tag{173}$$

Combining (172) and (173), we have

$$L_\lambda w_\lambda(x,t) - L_\lambda S_\lambda(x,t) \geq 0, \quad (x,t) \in B_\delta(\overline{x}) \times [\underline{t_k}, \sigma_k]. \tag{174}$$

We obtain

$$w_\lambda(x,t) \geq S_\lambda(x,t) > 0, \quad (x,t) \in B_\delta(\overline{x}) \times [\underline{x}, \sigma_k]. \tag{175}$$

using Theorem 3. This contradicts $w_\lambda(\overline{x}, \sigma_k) = 0$. So, we have verified

$$\lambda_0^- = \lambda_0^+. \tag{176}$$

Therefore, we have completed Theorem 1. □

4. Application of Theorem 1

Proof. In this section, we will complete the proof of Theorem 2. Using a simple derivation, we have

$$\frac{\partial w_\lambda(x,t)}{\partial t} + (-\Delta)^L w_\lambda(x,t) = c(x,t) w_\lambda(x,t) \tag{177}$$

where $c(x,t) = \xi^{p-1} - \omega$ and ξ is between $u_\lambda(x,t)$ and $u(x,t)$. We define $u^p(x,t) - \omega u(x,t)$ to be $r(u,t)$. We can quickly check that

$$r(0,t) = 0, \quad x \in \Sigma_\lambda \tag{178}$$

What is more, there exists a constant ν such that

$$r'(0,t) < -\nu < -\sigma. \tag{179}$$

Thus, $r(u,t)$ satisfies (a) and (b). Through the application of Theorem 1, we can obtain that the solution of (18) is radial symmetry of some point in R^n. □

5. Conclusions

We prove the symmetry of an ancient solution of a fractional parabolic equation involving logarithmic Laplacian using moving plane methods. At same time, we give some maximum principles involving logarithmic Laplacian. We believe that these theorems are useful for some other equations. In future, we will investigate the blow-up theory of the logarithmic Laplacian by means of the symmetry of an ancient solution, the maximum principle, and the perturbation method.

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