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Faber Polynomial Coefficient Inequalities for a Subclass of Bi-Close-To-Convex Functions Associated with Fractional Differential Operator

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Abstract: In this study, we begin by examining the τ-fractional differintegral operator and proceed to establish a novel subclass in the open unit disk E. The determination of the nth coefficient bound for functions within this recently established class is accomplished by the use of the Faber polynomial expansion approach. Additionally, we examine the behavior of the initial coefficients of bi-close-to-convex functions defined by the τ-fractional differintegral operator, which may exhibit unexpected reactions. We established connections between our current research and prior studies in order to validate our significant findings.

Keywords: analytic functions; τ-fractional differintegral operator; Faber polynomial expansions; close-to-convex-functions

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1. Introduction and Definitions

Let $\mathcal{H}$ denote the set of all analytic functions in the open unit disc $E = \{z : |z| < 1\}$. Let $\mathcal{A}$ be the subset of $\mathcal{H}$ having the functions that are normalized by

$$f(0) = 0 = -1 + f'(0)$$

and each $f \in \mathcal{A}$ has the series of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

The members of $\mathcal{A}$ that are univalent in E form the subclass $\mathcal{S}$. Let the class $\mathcal{P} \subseteq \mathcal{H}$ be defined as:

$$\mathcal{P} = \{p \in \mathcal{H} : \ p(0) = 1 \text{ and } \mathrm{Re}\, p(z) > 0\}. \tag{2}$$

A function f having the form (1) is said to be starlike of order β $(0 \leq \beta < 1)$ if it satisfies the inequality

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta, \ \ z \in E.$$

The set of all functions that satisfies the above inequality is denoted by $\mathcal{S}^*(\beta)$. It is easy to prove that $\mathcal{S}^*(\beta) \subset \mathcal{S}^*(0) = \mathcal{S}^* \subset \mathcal{S}$.

In 1952, Kaplan introduced the class of close-to-convex functions, for every function having the form (1) and another function $g \in \mathcal{S}^*$ such that

$$\operatorname{Re}\left(\frac{zf'(z)}{g(z)}\right) > \alpha, \;\; 0 \leq \alpha < 1, \;\; z \in E.$$

The class of all such functions of order α is denoted by $\mathcal{C}(\alpha)$. Also, one can prove that $\mathcal{S}^*(\alpha) \subset \mathcal{C}(\alpha) \subset \mathcal{S}$ (see [1]).

For $f_1, f_2 \in \mathcal{A}$, and f_1 is subordinate to f_2 in E, denoted by

$$f_1(z) \prec f_2(z), \;\; z \in E,$$

if we have a function u_0, such that $u_0 \in \mathcal{A}$, $|u_0(z)| < 1$ and $u_0(0) = 0$,

$$f_1(z) = f_2(u_0(z)), \; z \in E.$$

Each univalent function f has an inverse $f^{-1} = F$, defined as:

$$F(f(z)) = z, \;\; z \in E$$

and

$$f(F(w)) = w, \; |w| < r_0(f), \; r_0(f) \geq \frac{1}{4}.$$

So each inverse function has the series form

$$\begin{aligned} F(w) &= w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \ldots \\ &= z + \sum_{n=2}^{\infty} A_n z^n. \end{aligned} \tag{3}$$

Also, every $g \in \mathcal{A}$ can be written as

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{4}$$

the series of the inverse function of (4) is given by

$$G(z) = z + \sum_{n=2}^{\infty} B_n z^n.$$

An analytic function f is said to be bi-univalent in E if both f and f^{-1} are univalent in E, and the set of all such type of bi-univalent functions is denoted by the symbol Σ. For $f \in \Sigma$, Levin [2] proved that $|a_2| < 1.51$, and after that Branan and Clunie [3] gave the improvement of $|a_2|$ and proved that $|a_2| \leq \sqrt{2}$. For $f \in \Sigma$, Netanyahu [4] proved that $\max|a_2| = \frac{4}{3}$ (see [5–7] for details), and all these bounds are non-sharp. A number of operators [8], subordination properties [9], generalization techniques [10], and the inverse of the square-root transform of the Koebe function [11] were used to derive recent coefficient estimates for general subclasses of m-fold symmetric analytic bi-univalent functions. Subclasses of m-fold symmetric bi-bazilevic functions related to modified Sigmoid functions were considered in [12], and those related to conic domains were considered in [13]. Aspects such as coefficient estimates were taken into consideration when extending, generalizing, and improving the starlikeness criteria for specific subclasses of analytic and bi-univalent functions [14]; additionally, specific coefficient estimates were provided for specific families of bi-Bazilevic functions of the Ma-Minda type that involve the Hohlov operator [15]. The study of complex functions began with the introduction of operators. Making use of them has made the verification of several prior results easier and allowed for the discovery of new results, particularly those relevant to the starlikeness and convexity of certain functions. The majority of studies using operators lead to introducing new classes of analytic functions. Using the fractional derivative of order τ [16], Owa and Srivastava [17] introduced the τ-fractional-differintegral operator $D^{\tau} : \mathcal{A} \to \mathcal{A}$, which is known as an extension of the fractional derivative and the fractional integral as follows:

Definition 1 ([17])**.** *The τ-fractional-differintegral operator $D^{\tau} : \mathcal{A} \to \mathcal{A}$ of the fractional derivative and the fractional integral is defined as follows:*

$$\begin{aligned} D^{\tau} f(z) &= \Gamma(2-\tau) z^{\tau} D^{\tau} f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{\Gamma(2-\tau)\Gamma(n+1)}{\Gamma(n+1-\tau)} a_n z^n, \ z \in E, \\ &= z + \sum_{n=2}^{\infty} \varphi_n a_n z^n, \ z \in E \end{aligned} \tag{5}$$

where,

$$\varphi_n = \frac{\Gamma(2-\tau)\Gamma(n+1)}{\Gamma(n+1-\tau)}$$

and

$$\tau < 2.$$

From the definition of $D^{\tau} f$, some properties can be given in the following form:

i.

$$\lim_{\tau \to 1} D^{\tau} f(z) = D f(z) = z f'(z).$$

ii.

$$D^{\tau}\left(D^{\delta} f(z)\right) = D^{\delta}(D^{\tau} f(z)) = z + \sum_{n=2}^{\infty} \frac{\Gamma(2-\tau)\Gamma(2-\delta)(\Gamma(n+1))^2}{\Gamma(n+1-\tau)\Gamma(n+1-\delta)} a_n z^n$$

and

$$\begin{aligned} \frac{D(D^{\tau} f(z))}{D^{\tau} f(z)} &= \frac{z f'(z)}{f(z)}, \ \textit{for} \ \ \tau = 0 \\ &= \left(1 + \frac{z f''(z)}{f'(z)}\right) \ \textit{for} \ \ \tau = 1. \end{aligned}$$

Recent advances in fractional calculus have shown its usefulness in a wide variety of fields, including engineering, turbulence, electric networks, computer graphics, biological systems with memory, and physics. One example is the Korteweg–de Vries equation, which is examined using a novel integral transform in which the fractional derivative is presented in the Caputo sense. The equation was designed to capture a wide range of physical characteristics of the development and connection of nonlinear waves [18]. The work on the Mittag–Leffler-confluent hypergeometric function [19] makes use of a novel fractional integral operator, while [20] presents applications of differential subordination theory to analytic and p-valent functions described by a generalized fractional differintegral operator. Hamidi and Jahangiri [21] have explored the bounds a_n for the coefficients of functions f that are in the class of bi-close-to-convex functions of order α $(0 \le \alpha < 1)$. They proved their main result by making use of the assertion that if $a_i = 0, 2 \le i \le n-1$, then $b_i = 0$, $2 \le i \le n-1$. However, Wang and Bulut [22] gave the counter example to contradict the above assertion. For example, by taking the functions f and g as:

$$f(z) = z \text{ and } g(z) = z - \frac{z^2}{2},$$

clearly, we see that $g \in \mathcal{S}^*$ and $f \in C$. It is worthy to note that for these functions $a_2 = 0$ but $b_2 = -\frac{1}{2} \neq 0$. Therefore, using the method of Wang and Bulut [22], we define the new subclass $C_\Sigma(\alpha, \tau)$ of the close-to-convex function of the order α associated with the τ-fractional-differintegral operator D^τ and investigate more general results by the implementation of the Faber polynomial technique.

First, we define a class of starlike functions of order β associated with τ-fractional-differintegral operator D^τ :

Definition 2. *Let g be the function of the form (4). Then, $g \in \mathcal{S}^*(\beta, \tau)$, if*

$$Re\left(\frac{D(D^\tau g(z))}{g(z)}\right) > \beta,$$

where $0 \leq \beta < 1$ and $\tau < 2$.

Now, we define a class $C_\Sigma(\alpha, \tau)$ of the close-to-convex function of order α associated with the τ-fractional-differintegral operator D^τ :

Definition 3. *Let f be the function of the form (1). Then, $f \in C_\Sigma(\alpha, \tau)$ if there is a function $g \in \mathcal{S}^*(\beta, \tau)$ satisfying*

$$Re\left(\frac{D(D^\tau f(z))}{g(z)}\right) > \alpha$$

and

$$Re\left(\frac{D(D^\tau F(w))}{G(w)}\right) > \alpha,$$

where $0 \leq \alpha < 1$, $\tau < 2$, $z, w \in E$.

Remark 1. *For $\tau = 0$, the above class will be reduced to the well-known class given in [21].*

2. Faber Polynomial Expansion Approach

Faber [23] introduced a new technique to investigate coefficient bounds $|a_n|$ for $n \geq 3$, known as the Faber polynomial expansion method. Gong [24] discussed the importance of Faber polynomials in mathematical sciences, particularly in Geometric Function Theory. New classes of bi-univalent functions were established by Hamidi and Jahangiri [21,25], who also employed the Faber polynomial expansion approach to establish coefficient bounds. Furthermore, many authors like [26–29] applied this technique and determined some useful results for bi-univalent functions (see for detail [21,30,31]). We also refer the reader to ([32–34]) for recent papers dealing with bi-close-to-convex functions.

It is possible to express the coefficients of the inverse map F of the analytic functions f using the Faber polynomial method, as shown below (see, [27,35]).

$$F(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{n}(a_2, a_3, \ldots, a_n) w^n,$$

where

$$\begin{aligned}
K_{n-1}^{-n} &= \frac{(-n)!}{(-2n+1)!(n-1)!}a_2^{n-1} + \frac{(-n)!}{[2(-n+1)]!(n-3)!}a_2^{n-3}a_3 \\
&+ \frac{(-n)!}{(-2n+3)!(n-4)!}a_2^{n-4}a_4 \\
&+ \frac{(-n)!}{[2(-n+2)]!(n-5)!}a_2^{n-5}\left[a_5 + (-n+2)a_3^2\right] \\
&+ \frac{(-n)!}{(-2n+5)!(n-6)!}a_2^{n-6}[a_6 + (-2n+5)a_3a_4] \\
&+ \sum_{i\geq 7} a_2^{n-i}\mathcal{Q}_i,
\end{aligned}$$

and $\mathcal{Q}_i$ is a homogeneous polynomial in the variables $a_2, a_3, \ldots, a_n$, for $7 \leq i \leq n$. Particularly, the first three terms of K_{n-1}^{-n} are

$$\begin{aligned}
\frac{1}{2}K_1^{-2} &= -a_2, \ \frac{1}{3}K_2^{-3} = 2a_2^2 - a_3, \\
\frac{1}{4}K_3^{-4} &= -(5a_2^3 - 5a_2a_3 + a_4).
\end{aligned}$$

In general, for any $r \in \mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}$ and $n \geq 2$, an expansion of K_{n-1}^r (see [27]) of the form:

$$K_{n-1}^r = ra_n + \frac{r(r-1)}{2}\mathcal{V}_{n-1}^2 + \frac{r!}{(r-3)!3!}\mathcal{V}_{n-1}^3 + \cdots + \frac{r!}{(r-n+1)!(n-1)!}\mathcal{V}_{n-1}^{n-1},$$

where,

$$\mathcal{V}_{n-1}^r = \mathcal{V}_{n-1}^r(a_2, a_3 \ldots),$$

and by [35], we have

$$\mathcal{V}_{n-1}^v(a_2, \ldots, a_n) = \sum_{n=1}^{\infty} \frac{v!}{\mu_1!, \ldots, \mu_n!}a_2^{\mu_1} \ldots a_n^{\mu_{n-1}}, \ \text{for } a_1 = 1 \text{ and } v \leq n.$$

The sum is taken over all non-negative integers $\mu_1, \ldots, \mu_n$ satisfying

$$\begin{aligned}
\mu_1 + \mu_2 + \cdots + \mu_{n-1} &= v, \\
\mu_1 + 2\mu_2 + \cdots + (n-1)\mu_{n-1} &= n-1.
\end{aligned}$$

Clearly,

$$\mathcal{V}_{n-1}^{n-1}(a_1, \ldots, a_n) = a_2^{n-1}$$

and

$$\mathcal{V}_{n-1}^1(a_1, \ldots, a_n) = a_{n-1}.$$

In this study, in Section 1, we briefly introduced the literature, which is very helpful to understanding the idea of this article. Also, in this section we defined a subclass of close-to-convex functions associated with τ-fractional-differintegral operators. In Section 2, we discussed the Faber polynomial method and its uses. In Section 3, we give some known and some new lemmas, which will be used to prove our main results. Using the Faber polynomial method, the nth coefficient bounds and unpredictable behavior of the initial coefficient estimates of bi-close-to-convex functions defined by the τ-fractional differintegral operator are investigated in Section 4. The consequences of the primary findings that are already known to us are also highlighted. In Section 5, we made concluding remarks.

3. Set of Lemmas

Lemma 1. *The Caratheodory Lemma (see [36]) if $p \in \mathcal{P}$ and*

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n,$$

then

$$|c_n| \leq 2.$$

Lemma 2 ([1])**.** *If $p \in \mathcal{P}$ and $\mu \in \mathbb{C}$, then*

$$\left|c_2 - \mu c_2^2\right| \leq 2 \max\{1, |2\mu - 1|\}.$$

Lemma 3. *Let a function $g \in \mathcal{A}$, be given by (4). If $g \in \mathcal{S}^*(\beta, \tau)$, then*

$$|b_n| \leq \frac{2(1-\beta)}{n\left(\frac{\Gamma(2-\tau)\Gamma(n+1)}{\Gamma(n+1-\tau)}\right) - 1} \prod_{j=1}^{n-2} \left(1 + \frac{2(1-\beta)}{\left(\frac{\Gamma(2-\tau)\Gamma(j+2)}{\Gamma(j+2-\tau)}(j+1) - 1\right)}\right), \quad n \geq 3 \tag{6}$$

and

$$|b_2| \leq \frac{2(1-\beta)}{2\left(\frac{\Gamma(2-\tau)\Gamma(3)}{\Gamma(3-\tau)}\right) - 1}, \tag{7}$$

where $0 \leq \beta < 1, \tau < 2$

Proof. Suppose $g \in \mathcal{S}^*(\beta, \tau)$; then,

$$\mathrm{Re}\left(\frac{D(D^\tau g(z))}{g(z)}\right) > \beta.$$

Setting

$$\begin{aligned} \frac{D(D^\tau g(z))}{g(z)} &= \beta + (1-\beta)p(z) \\ \frac{\frac{D(D^\tau g(z))}{g(z)} - \beta}{1-\beta} &= p(z), \end{aligned}$$

or, equivalently,

$$\begin{aligned} & D(D^\tau g(z)) \\ = \; & [(1-\beta)p(z) + \beta]g(z) \\ & \left(z + \sum_{n=2}^{\infty} \frac{\Gamma(2-\tau)\Gamma(n+1)}{\Gamma(2)\Gamma(n+1-\tau)} n b_n z^n\right) \\ = \; & \left(1 + (1-\beta)\sum_{n=1}^{\infty} c_n z^n\right)\left(z + \sum_{n=2}^{\infty} b_n z^n\right) \\ & \sum_{n=2}^{\infty}\left(\frac{\Gamma(2-\tau)\Gamma(n+1)}{\Gamma(2)\Gamma(n+1-\tau)} n - 1\right) b_n z^n \\ = \; & (1-\beta)\sum_{n=1}^{\infty} c_n z^{n+1} + (1-\beta)\sum_{n=2}^{\infty}\left(\sum_{j=1}^{n-1} c_j b_{n-j}\right) z^n \end{aligned}$$

Comparing the coefficient of z^n on both sides, we have

$$\left(n\left(\frac{\Gamma(2-\tau)\Gamma(n+1)}{\Gamma(n+1-\tau)}\right)-1\right)b_n=(1-\beta)\sum_{j=1}^{n-1}c_jb_{n-j}$$

and

$$\left|\left(n\left(\frac{\Gamma(2-\tau)\Gamma(n+1)}{\Gamma(n+1-\tau)}\right)-1\right)b_n\right|\leq(1-\beta)\sum_{j=1}^{n-1}|c_j||b_{n-j}|.$$

Using Lemma 1, we have

$$\left|\left(n\left(\frac{\Gamma(2-\tau)\Gamma(n+1)}{\Gamma(n+1-\tau)}\right)-1\right)b_n\right|\leq 2(1-\beta)\sum_{j=1}^{n-1}|b_j| \tag{8}$$

So for $n=2$, we have from (8)

$$|b_2|\leq\frac{2(1-\beta)}{2\left(\frac{\Gamma(2-\tau)\Gamma(3)}{\Gamma(3-\tau)}\right)-1} \tag{9}$$

from the above we conclude that (6) holds for $n=2$. Let us assume that (6) is true for $n\leq t$, i.e.,

$$|b_t|\leq\frac{2(1-\beta)}{\left(\frac{\Gamma(2-\tau)\Gamma(t+1)}{\Gamma(2)\Gamma(t+1-\tau)}[t]_q-1\right)}\prod_{j=1}^{t-2}\left(1+\frac{2(1-\beta)}{\left(\frac{\Gamma(2-\tau)\Gamma(j+2)}{\Gamma(2)\Gamma(j+2-\tau)}(j+1)-1\right)}\right).$$

Consider

$$\begin{aligned}
|b_{t+1}| &\leq \frac{2(1-\beta)}{\left((t+1)\left(\frac{\Gamma(2-\tau)\Gamma(t+2)}{\Gamma(t+2-\tau)}\right)-1\right)} \\
&\quad\times(1+|b_2|+|b_3|+\cdots+|b_t|) \\
&\leq \frac{2(1-\beta)}{\left((t+1)\left(\frac{\Gamma(2-\tau)\Gamma(t+2)}{\Gamma(t+2-\tau)}\right)-1\right)} \\
&\quad\left[\begin{array}{c}
1+\frac{2(1-\beta)}{2\left(\frac{\Gamma(2-\tau)\Gamma(3)}{\Gamma(3-\tau)}\right)-1}+\frac{2(1-\beta)}{3\left(\frac{\Gamma(2-\tau)\Gamma(4)}{\Gamma(4-\tau)}\right)-1}\left(1+\frac{2(1-\beta)}{2\left(\frac{\Gamma(2-\tau)\Gamma(3)}{\Gamma(3-\tau)}\right)-1}\right) \\
+\frac{2(1-\beta)}{4\left(\frac{\Gamma(2-\tau)\Gamma(5)}{\Gamma(5-\tau)}-1\right)}\left(1+\frac{2(1-\beta)}{2\left(\frac{\Gamma(2-\tau)\Gamma(3)}{\Gamma(3-\tau)}\right)-1}\right)\left(1+\frac{2(1-\beta)}{3\left(\frac{\Gamma(2-\tau)\Gamma(4)}{\Gamma(4-\tau)}\right)-1}\right) \\
+\frac{2(1-\beta)}{2\left(\frac{\Gamma(2-\tau)\Gamma(3)}{\Gamma(3-\tau)}\right)-1}\prod_{j=1}^{t-2}\left(1+\frac{2(1-\beta)}{(j+1)\left(\frac{\Gamma(2-\tau)\Gamma(j+2)}{\Gamma(2)\Gamma(j+2-\tau)}\right)-1}\right)
\end{array}\right] \\
&= \frac{2(1-\beta)}{\frac{\Gamma(2-\tau)\Gamma(t+2)}{\Gamma(2)\Gamma(t+2-\tau)}(t+1)-1}\prod_{j=1}^{t-1}\left(1+\frac{2(1-\beta)}{\frac{\Gamma(2-\tau)\Gamma(j+2)}{\Gamma(2)\Gamma(j+2-\tau)}(j+1)-1}\right).
\end{aligned}$$

This means that the result holds for $n=t+1$. Therefore, we have shown by mathematical induction that (6) is true for every n, $n\geq 2$. This completes the proof. □

Lemma 4. *Let a function $g\in\mathcal{A}$ be given by (4). If $g\in\mathcal{S}^*(\beta,\tau)$, then*

$$\left|b_3-\mu b_2^2\right|\leq\frac{2(1-\beta)}{3\varphi_3-1}\max\left[1,\left|1+\frac{2(1-\beta)}{2\varphi_2-1}\left(1-\mu\frac{(3\varphi_3-1)}{2\varphi_2-1}\right)\right|\right]$$

where

$$\begin{aligned} \varphi_3 &= \left(\frac{\Gamma(2-\tau)\Gamma(4)}{\Gamma(4-\tau)}\right)-1, \\ \varphi_2 &= \left(\frac{\Gamma(2-\tau)\Gamma(3)}{\Gamma(3-\tau)}\right)-1 \end{aligned} \tag{10}$$

and $0 \leq \beta < 1$, $\tau < 2$, $\mu \in \mathbb{C}$.

Proof. If $g \in \mathcal{S}^*(\beta, \tau)$, then we have

$$\mathrm{Re}\left(\frac{D(D^\tau g(z))}{g(z)}\right) > \beta$$

Then, $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{P}$ exist such that

$$\begin{aligned} \frac{D(D^\tau g(z))}{g(z)} &= \beta + (1-\beta)p(z) \\ &= 1 + (1-\beta)\sum_{n=1}^{\infty} c_n z^n \end{aligned} \tag{11}$$

From (11), we have

$$b_2 = \frac{(1-\beta)c_1}{2\varphi_2 - 1} \tag{12}$$

and

$$b_3 = \frac{(1-\beta)}{3\varphi_3 - 1}\left[c_2 + \frac{(1-\beta)c_1^2}{2\varphi_2 - 1}\right] \tag{13}$$

from (12) and (13)

$$\left|b_3 - \mu b_2^2\right| = \frac{(1-\beta)}{3\varphi_3 - 1}\left|c_2 - v c_1^2\right|,$$

where

$$v = -\frac{(1-\beta)}{2\varphi_2 - 1}\left(1 - \mu\frac{3\varphi_3 - 1}{2\varphi_2 - 1}\right).$$

By using Lemma 2, the result is now completed. □

4. Main Results

Theorem 1. *Let* $f \in C_\Sigma(\alpha, \tau)$ *be given by* (1) *with* $a_i = 0$, $2 \leq i \leq n-1$. *Then, for* $n \geq 3$

$$\begin{aligned} |a_n| \leq & \left(\frac{\Gamma(n+1-\tau)}{\Gamma(2-\tau)\Gamma(n+1)}\right)\frac{1}{n}\left\{\begin{array}{c} \frac{2(1-\beta)}{\left(\frac{\Gamma(2-\tau)\Gamma(n+1)}{\Gamma(n+1-\tau)}n-1\right)} \times \\ \prod_{j=1}^{n-2}\left(1 + \frac{2(1-\beta)}{\left(\frac{\Gamma(2-\tau)\Gamma(j+2)}{\Gamma(j+2-\tau)}(j+1)-1\right)}\right) \end{array}\right. \\ & \left. +2(1-\alpha) + \sum_{l=1}^{n-2}\left(\begin{array}{c} \frac{2(1-\beta)}{\left(\frac{\Gamma(2-\tau)\Gamma(n-l+1)}{\Gamma(n-l+1-\tau)}(n-l)-1\right)} \times \\ \prod_{j=1}^{n-l-2}\left(1 + \frac{2(1-\beta)}{\left(\frac{\Gamma(2-\tau)\Gamma(j+2)}{\Gamma(j+2-\tau)}(j+1)-1\right)}\right) \end{array}\right)\Upsilon(l)\right\}, \end{aligned}$$

where

$$\Upsilon(l) = \min\left(\left|K_l^{-1}(b_2, b_3, \ldots, b_{l+1})\right|, \left|K_l^{-1}(B_2, B_3, \ldots, B_{l+1})\right|\right).$$

Proof. For $f \in C_\Sigma(\alpha, \tau)$. Therefore, a function $g \in \mathcal{S}^*(\beta, \tau)$ exists satisfying

$$\mathrm{Re}\left(\frac{D(D^\tau f(z))}{g(z)}\right) > \alpha.$$

The Faber polynomial expansion of $\frac{D(D^\tau f(z))}{g(z)}$ is

$$\frac{D(D^\tau f(z))}{g(z)} = 1 + \sum_{n=2}^{\infty}\left[\begin{array}{c} \left(n\frac{\Gamma(2-\tau)\Gamma(n+1)}{\Gamma(n+1-\tau)}a_n - b_n\right) + \sum\limits_{l=1}^{n-2} K_l^{-1}(b_2, b_3, \ldots, b_{l+1}) \\ \left((n-l)\frac{\Gamma(2-\tau)\Gamma(n+1)}{\Gamma(n+1-\tau)}a_{n-l} - b_{n-l}\right) \end{array}\right] z^{n-1}. \quad (14)$$

For the inverse maps $F = f^{-1}$ and $G = g^{-1}$, we obtain

$$\frac{D(D^\tau F(w))}{G(w)} = 1 + \sum_{n=2}^{\infty}\left[\begin{array}{c} \left(n\frac{\Gamma(2-\tau)\Gamma(n+1)}{\Gamma(n+1-\tau)}A_n - B_n\right) + \sum\limits_{l=1}^{n-2} K_l^{-1}(B_2, B_3, \ldots, B_{l+1}) \\ \left((n-l)\frac{\Gamma(2-\tau)\Gamma(n+1)}{\Gamma(n+1-\tau)}A_{n-l} - B_{n-l}\right) \end{array}\right] w^{n-1}. \quad (15)$$

Since $\mathrm{Re}\frac{D(D^\tau f(z))}{g(z)} > \alpha$ in E, there is a real part function

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

so that

$$\begin{aligned} \frac{D(D^\tau f(z))}{g(z)} &= 1 + (1-\alpha)p(z) \\ &= 1 + (1-\alpha)\sum_{n=1}^{\infty} c_n z^n. \end{aligned} \quad (16)$$

Similarly, $\mathrm{Re}\frac{D(D^\tau F(w))}{G(w)} > \alpha$ in E, and there exists a positive real part function

$$q(w) = 1 + \sum_{n=1}^{\infty} d_n w^n$$

so that

$$\begin{aligned} \frac{D(D^\tau F(w))}{G(w)} &= 1 + (1-\alpha)q(w) \\ &= 1 + (1-\alpha)\sum_{n=1}^{\infty} d_n w^n. \end{aligned} \quad (17)$$

Comparing the coefficients of (14) and (16), for any $n \geq 2$, yields

$$\left\{\begin{array}{c} \left(n\left(\frac{\Gamma(2-\tau)\Gamma(n+1)}{\Gamma(n+1-\tau)}\right)a_n - b_n\right) + \sum\limits_{l=1}^{n-2} K_l^{-1}(b_2, b_3, \ldots, b_{l+1}) \\ \times\left((n-l)\left(\frac{\Gamma(2-\tau)\Gamma(n-l+1)}{\Gamma(n-l+1-\tau)}\right)a_{n-l} - b_{n-l}\right) \end{array}\right\} = (1-\alpha)c_{n-1}. \quad (18)$$

Evaluating the coefficients of the equations (15) and (17), for any $n \geq 2$, yields

$$\left\{ \begin{array}{c} \left(n\left(\frac{\Gamma(2-\tau)\Gamma(n+1)}{\Gamma(n+1-\tau)}\right)A_n - B_n\right) + \sum_{l=1}^{n-2} K_l^{-1}(B_2, B_3, \ldots, B_{l+1}) \\ \times\left((n-l)\left(\frac{\Gamma(2-\tau)\Gamma(n-l+1)}{\Gamma(n-l+1-\tau)}\right)A_{n-l} - B_{n-l}\right) \end{array} \right\} = (1-\alpha)d_{n-1}. \tag{19}$$

But under the assumption that $2 \leq i \leq n-1$, and $a_i = 0$, respectively, we find from (18) and (19) that

$$\begin{aligned} & \left(n\left(\frac{\Gamma(2-\tau)\Gamma(n+1)}{\Gamma(n+1-\tau)}\right)a_n - b_n\right) - \sum_{l=1}^{n-2} b_{n-l}K_l^{-1}(b_2, b_3, \ldots, b_{l+1}) \\ = \; & (1-\alpha)c_{n-1}, \end{aligned} \tag{20}$$

$$\begin{aligned} & -\left(n\left(\frac{\Gamma(2-\tau)\Gamma(n+1)}{\Gamma(n+1-\tau)}\right)A_n - B_n\right) - \sum_{l=1}^{n-2} B_{n-l}K_l^{-1}(B_2, B_3, \ldots, B_{l+1}) \\ = \; & (1-\alpha)d_{n-1}. \end{aligned} \tag{21}$$

Also, $a_i = 0$, $(2 \leq i \leq n-1)$, and we know that

$$A_n = -a_n.$$

Thus, (20) and (21) give

$$n\left(\frac{\Gamma(2-\tau)\Gamma(n+1)}{\Gamma(n+1-\tau)}\right)a_n = b_n + (1-\alpha)c_{n-1} + \sum_{l=1}^{n-2} b_{n-l}K_l^{-1}(b_2, b_3, \ldots, b_{l+1})$$

and

$$-n\left(\frac{\Gamma(2-\tau)\Gamma(n+1)}{\Gamma(n+1-\tau)}\right)a_n = B_n + (1-\alpha)d_{n-1} + \sum_{l=1}^{n-2} B_{n-l}K_l^{-1}(B_2, B_3, \ldots, B_{l+1}),$$

respectively. Now, taking the modulus on both sides we have

$$\left|n\left(\frac{\Gamma(2-\tau)\Gamma(n+1)}{\Gamma(n+1-\tau)}\right)\right||a_n| \leq |b_n| + (1-\alpha)|c_{n-1}| + \sum_{l=1}^{n-2}\left|b_{n-l}K_l^{-1}(b_2, b_3, \ldots, b_{l+1})\right|$$

and

$$\left|n\left(\frac{\Gamma(2-\tau)\Gamma(n+1)}{\Gamma(n+1-\tau)}\right)\right||a_n| \leq |B_n| + (1-\alpha)|d_{n-1}| + \sum_{l=1}^{n-2}\left|B_{n-l}K_l^{-1}(B_2, B_3, \ldots, B_{l+1})\right|.$$

On the other hand, since $g, G \in \mathcal{S}^*(\beta,\tau)$, we use the Lemma 3 to obtain

$$
\begin{aligned}
&\left| n\left(\frac{\Gamma(2-\tau)\Gamma(n+1)}{\Gamma(n+1-\tau)}\right)\right| |a_n| \\
\leq\ & \frac{2(1-\beta)}{\left(n\left(\frac{\Gamma(2-\tau)\Gamma(n+1)}{\Gamma(n+1-\tau)}\right)-1\right)} \prod_{j=1}^{n-2}\left(1+\frac{2(1-\beta)}{\left(\left(\frac{\Gamma(2-\tau)\Gamma(j+2)}{\Gamma(2)\Gamma(j+2-\tau)}\right)(j+1)-1\right)}\right) \\
&+(1-\alpha)|c_{n-1}| + \sum_{l=1}^{n-2}\left(\begin{array}{c} \frac{2(1-\beta)}{\left(\left(\frac{\Gamma(2-\tau)\Gamma(n-l+1)}{\Gamma(2)\Gamma(n-l+1-\tau)}\right)(n-l)-1\right)} \times \\ \prod_{j=1}^{n-l-2}\left(1+\frac{2(1-\beta)}{\left(\left(\frac{\Gamma(2-\tau)\Gamma(j+2)}{\Gamma(2)\Gamma(j+2-\tau)}\right)(j+1)-1\right)}\right) \end{array}\right) \\
&\times \left| K_l^{-1}(b_2, b_3, \ldots, b_{l+1})\right|
\end{aligned}
$$

and

$$
\begin{aligned}
&\left| n\left(\frac{\Gamma(2-\tau)\Gamma(n+1)}{\Gamma(n+1-\tau)}\right)\right| |a_n| \\
\leq\ & \frac{2(1-\beta)}{\left(\frac{\Gamma(2-\tau)\Gamma(n+1)}{\Gamma(n+1-\tau)}n-1\right)} \prod_{j=1}^{n-2}\left(1+\frac{2(1-\beta)}{\left(\left(\frac{\Gamma(2-\tau)\Gamma(j+2)}{\Gamma(j+2-\tau)}\right)(j+1)-1\right)}\right) \\
&+(1-\alpha)|d_{n-1}| + \sum_{l=1}^{n-2}\left(\begin{array}{c} \frac{2(1-\beta)}{\left(\frac{\Gamma(2-\tau)\Gamma(n-l+1)}{\Gamma(n-l+1-\tau)}n-1\right)} \times \\ \prod_{j=1}^{n-l-2}\left(1+\frac{2(1-\beta)}{\left(\frac{\Gamma(2-\tau)\Gamma(j+2)}{\Gamma(j+2-\tau)}(j+1)-1\right)}\right) \end{array}\right) \\
&\times \left| K_l^{-1}(B_2, B_3, \ldots, B_{l+1})\right|.
\end{aligned}
$$

Using the Lemma 1, we obtain

$$
\begin{aligned}
&\left| n\left(\frac{\Gamma(2-\tau)\Gamma(n+1)}{\Gamma(n+1-\tau)}\right)\right| |a_n| \\
\leq\ & \frac{2(1-\beta)}{\left(n\left(\frac{\Gamma(2-\tau)\Gamma(n+1)}{\Gamma(n+1-\tau)}\right)-1\right)} \prod_{j=1}^{n-2}\left(1+\frac{2(1-\beta)}{\left(\left(\frac{\Gamma(2-\tau)\Gamma(j+2)}{\Gamma(j+2-\tau)}\right)(j+1)-1\right)}\right) \\
&+2(1-\alpha) + \sum_{l=1}^{n-2}\left(\begin{array}{c} \frac{2(1-\beta)}{\left(\left(\frac{\Gamma(2-\tau)\Gamma(n-l+1)}{\Gamma(n-l+1-\tau)}\right)(n-l)-1\right)} \times \\ \prod_{j=1}^{n-l-2}\left(1+\frac{2(1-\beta)}{\left(\left(\frac{\Gamma(2-\tau)\Gamma(j+2)}{\Gamma(j+2-\tau)}\right)(j+1)-1\right)}\right) \end{array}\right) \\
&\times \left| K_l^{-1}(b_2, b_3, \ldots, b_{l+1})\right|.
\end{aligned}
\tag{22}
$$

and

$$
\begin{aligned}
&\left|n\left(\frac{\Gamma(2-\tau)\Gamma(n+1)}{\Gamma(n+1-\tau)}\right)\right||a_n| \\
\leq\ & \frac{2(1-\beta)}{\left(n\left(\frac{\Gamma(2-\tau)\Gamma(n+1)}{\Gamma(n+1-\tau)}\right)-1\right)}\prod_{j=1}^{n-2}\left(1+\frac{2(1-\beta)}{\left(\left(\frac{\Gamma(2-\tau)\Gamma(j+2)}{\Gamma(j+2-\tau)}\right)(j+1)-1\right)}\right) \\
&+2(1-\alpha)+\sum_{l=1}^{n-2}\left(\begin{array}{c}\frac{2(1-\beta)}{\left(\frac{\Gamma(2-\tau)\Gamma(n-l+1)}{\Gamma(n-l+1-\tau)}(n-l)-1\right)}\times \\ \prod_{j=1}^{n-l-2}\left(1+\frac{2(1-\beta)}{\left(\frac{\Gamma(2-\tau)\Gamma(j+2)}{\Gamma(j+2-\tau)}(j+1)-1\right)}\right)\end{array}\right) \\
&\times\left|K_l^{-1}(B_2,B_3,\ldots,B_{l+1})\right|.
\end{aligned} \tag{23}
$$

Consequently, comparing (22) and (23), we obtain the coefficient bounds $|a_n|$ as asserted in Theorem 1. □

For $\tau = 0$, we obtain a deduced result from Theorem 1 that was proven in [37].

Corollary 1. *[37]. For $0 \leq \alpha, \beta < 1$. Let $f \in C_\Sigma(\alpha,\beta)$, if $a_i = 0, 2 \leq i \leq n-1$. Then, for $n \geq 3$*

$$
\begin{aligned}
|a_n| \leq\ & \left\{\frac{1}{n!}\prod_{j=0}^{n-2}(j+2(1-\beta))\right. \\
&\left.+\frac{2(1-\alpha)}{n}+\frac{1}{n}\sum_{l=1}^{n-2}\left(\frac{1}{(n-l-1)!}\prod_{j=0}^{n-l-2}(j+2(1-\beta))\right)\Upsilon_1(l)\right\},
\end{aligned}
$$

where

$$\Upsilon_1(l) = \min\left(\left|K_l^{-1}(b_2,b_3,\ldots,b_{l+1})\right|,\left|K_l^{-1}(B_2,B_3,\ldots,B_{l+1})\right|\right).$$

For $\tau = 0$, and $\beta = 0$, we obtain a result that was proved in [22].

Corollary 2 ([22])**.** *For $0 \leq \alpha < 1$. Let $f \in C_\Sigma(\alpha)$ be given in (1), if $a_i = 0, 2 \leq i \leq n-1$. Then,*

$$
\begin{aligned}
&|a_n| \\
\leq\ & 1+\frac{2(1-\alpha)}{n}+\frac{1}{n}\sum_{l=1}^{n-2}(n-l)\min\left(\left|K_l^{-1}(b_2,b_3,\ldots,b_{l+1})\right|,\left|K_l^{-1}(B_2,B_3,\ldots,B_{l+1})\right|\right).
\end{aligned}
$$

Theorem 2. *Let $f \in C_\Sigma(\alpha,\tau)$ and f be given in (1). Then,*

$$
|a_2| \leq \min\left\{\begin{array}{c}\frac{(1-\alpha)}{\varphi_2}+\frac{(1-\beta)}{\varphi_2(2\varphi_2-1)}, \\ \sqrt{\frac{2(1-\alpha)}{3\varphi_3}+\frac{4q(1-\beta)}{3\varphi_3(2\varphi_2-1)}\left(\frac{(1-\alpha)}{1}+\frac{(1-\beta)}{(2\varphi_2-1)}\right)}\end{array}\right\} \tag{24}
$$

and

$$
|a_3| \leq \min\left\{\begin{array}{c}\frac{(1-\alpha)}{3\varphi_3}[1+\max(1,|V_1|)]+\frac{(1-\beta)}{3\varphi_3(3\varphi_3-1)}\max(1,|V_2|)+\frac{2(1-\alpha)(1-\beta)}{\varphi_2^2(2\varphi_2-1)}, \\ \frac{2(1-\alpha)}{3\varphi_3}+\frac{4q(1-\alpha)(1-\beta)}{3\varphi_3(2\varphi_2-1)}+\frac{4(q-1)(1-\beta)^2}{3\varphi_3(2\varphi_2-1)^2}+\frac{2(1-\beta)}{3\varphi_3(3\varphi_3-1)}\left(1+\frac{(1-\beta)}{\varphi_2-1}\right).\end{array}\right\} \tag{25}
$$

where φ_2 and φ_3 are given by (10).

Proof. If we set $n = 2$ and $n = 3$ in (18) and (19), respectively, we obtain

$$a_2 = \frac{(1-\alpha)}{2_q\varphi_2}c_1 + \frac{b_2}{2\varphi_2} \tag{26}$$

$$a_3 = \frac{(1-\alpha)}{3\varphi_3}c_2 + \frac{q(1-\alpha)}{3\varphi_3}b_2c_1 + \frac{q-1}{3\varphi_3}b_2^2 + \frac{b_3}{3\varphi_3} \tag{27}$$

$$-a_2 = \frac{(1-\alpha)}{2\varphi_2}d_1 - \frac{b_2}{2\varphi_2} \tag{28}$$

$$2a_2^2 - a_3 = \frac{(1-\alpha)}{3\varphi_3}d_2 - \frac{(1-\alpha)}{3\varphi_3}b_2d_1 + \frac{2}{3\varphi_3}b_2^2 - \frac{b_3}{3\varphi_3} \tag{29}$$

From (26) and (28), we find

$$c_1 = -d_1 \tag{30}$$

On the other hand, from (27) and (29), we obtain

$$a_2^2 = \frac{(1-\alpha)}{6\varphi_3}(c_2 + d_2) + \frac{q(1-\alpha)}{6\varphi_3}b_2(c_1 - d_1) + \frac{q}{3\varphi_3}b_2^2. \tag{31}$$

Taking the modulus of (26) and (31) together with Lemma 1, we obtain

$$|a_2| \le \frac{(1-\alpha)}{\varphi_2} + \frac{|b_2|}{2\varphi_2} \tag{32}$$

and

$$|a_2|^2 \le \frac{2(1-\alpha)}{3\varphi_3} + \frac{2(1-\alpha)}{3\varphi_3}|b_2| + \frac{1}{3\varphi_3}|b_2|^2. \tag{33}$$

Using inequality (7), we have

$$|a_2| \le \frac{(1-\alpha)}{\varphi_2} + \frac{(1-\beta)}{\varphi_2(2\varphi_2 - 1)} \tag{34}$$

and

$$|a_2| \le \sqrt{\frac{2(1-\alpha)}{3\varphi_3} + \frac{4q(1-\beta)}{3\varphi_3(2\varphi_2 - 1)}\left(\frac{(1-\alpha)}{1} + \frac{(1-\beta)}{(2\varphi_2 - 1)}\right)}. \tag{35}$$

Thus, we obtain the desired estimate as asserted in (24).

Next, in order to find the bound for $|a_3|$, we subtract (29) from (27)

$$2a_3 - 2a_2^2 = \frac{(1-\alpha)}{3\varphi_3}(c_2 - d_2) - \frac{2}{3\varphi_3}b_2^2 + \frac{2}{3\varphi_3}b_3 + \frac{(1-\alpha)}{3\varphi_3}(c_1 + d_1)b_2.$$

By (30), we obtain

$$a_3 = a_2^2 + \frac{(1-\alpha)}{6\varphi_3}(c_2 - d_2) + \frac{1}{3\varphi_3}\left(b_3 - b_2^2\right). \tag{36}$$

If we set the value of a_2^2 from (26) in (36), then we have

$$\begin{aligned} a_3 \quad = \quad & \frac{(1-\alpha)}{6\varphi_3}\left(c_2 - \left(\frac{-3\varphi_3(1-\alpha)}{2\varphi_2^2}\right)c_1^2\right) + \frac{1}{3\varphi_3}\left(b_3 - \left(1 - \frac{3\varphi_3}{4\varphi_2^2}\right)b_2^2\right) \\ & + \frac{(1-\alpha)}{2\varphi_2^2}c_1b_2 - \frac{1-\alpha}{6\varphi_3}d_2. \end{aligned}$$

So, using Lemma 1, inequality (7) of Lemma 3, Lemma 2, and Lemma 4, we obtain

$$|a_3| \leq \frac{(1-\alpha)}{3\varphi_3}[1+\max(1,|V_1|)] + \frac{(1-\beta)}{3\varphi_3(3\varphi_3-1)}\max(1,|V_2|) + \frac{2(1-\alpha)(1-\beta)}{\varphi_2^2(2\varphi_2-1)}, \tag{37}$$

where

$$\begin{aligned} V_1 &= 1+\frac{3\varphi_3(1-\alpha)}{\varphi_2^2}, \\ V_2 &= 1+\frac{2(1-\beta)}{(2\varphi_2-1)}\left[1-\left(1-\frac{3\varphi_3}{4\varphi_2^2}\right)\left(\frac{3\varphi_3-1}{2\varphi_2-1}\right)\right]. \end{aligned}$$

If we set the value of a_2^2 from (26) in (36), then we have

$$a_3 = \frac{(1-\alpha)}{3\varphi_3}c_2 + \frac{(1-\alpha)}{3\varphi_3}b_2c_1 + \frac{1}{3\varphi_3}b_3.$$

Using Lemma 1 and Lemma 3, we obtain

$$\begin{aligned} |a_3| \leq\ & \frac{2(1-\alpha)}{3\varphi_3} + \frac{4(1-\alpha)(1-\beta)}{3\varphi_3(2\varphi_2-1)} + \frac{4(q-1)(1-\beta)^2}{3\varphi_3(2\varphi_2-1)^2} \\ & + \frac{2(1-\beta)}{3\varphi_3(3\varphi_3-1)}\left(1+\frac{2(1-\beta)}{(2\varphi_2-1)}\right). \end{aligned} \tag{38}$$

Hence, (37) and (38) give the desired estimate on the coefficient $|a_3|$ as asserted in (25). □

In Theorem 2, we obtain the known corollary proved in [37] for $\tau = 0$.

Corollary 3 ([37]). *Let $f \in C_\Sigma(\alpha)$ be given by (1). Then,*

$$|a_2| \leq \min\left\{\begin{array}{c} 2-\alpha-\beta, \\ \\ \sqrt{\frac{1}{3}\{4(1-\beta)(2-\alpha-\beta)+2(1-\alpha)\}} \end{array}\right\}$$

and

$$|a_3| \leq \frac{1}{3}\left\{\begin{array}{cc} (3-2\beta)(3-2\alpha-\beta), & 0 \leq \alpha \leq \frac{2+\beta}{3} \\ \\ (1-\alpha)(5-3\alpha)+(1-\beta)(2-\beta)+6(1-\alpha)(1-\beta), & \frac{2+\beta}{3} \leq \alpha < 1. \end{array}\right\}$$

5. Conclusions

In this study, we show how the Faber polynomial method may be used to generalize previous work on classes of close-to-convex functions. The newly published note [22] provided the impetus for our research into a special category of bi-close-to-convex functions related to τ-fractional differintegral operators. Using the Faber polynomial expansion method, we found an upper bound for these functions at the nth coefficient. We also showed links between new and previous work and explored the unexpected behavior of the initial coefficients of bi-close-to-convex functions.

Since the beginning of the study of complex functions, operators have been in use. By applying them, many established findings have been shown to be easier, and new results may be produced. The most typical result of operator-based research is the introduction of new classes of analytic functions. We can make new subclasses of bi-close-to-convex functions using the same method as in this article. To do this, we need to consider the ordinary differential and integral operators, q-analogous differential and integral operators, and symmetrical q-calculus operators on known subclasses of bi-close-to-convex functions. The Faber polynomial expansion method can be used to talk about the general coefficient bound and how initial bounds can behave in unpredictable ways for functions in these classes. Use of the m-fold symmetric function-linked fractional differential operators allows for the generalization of the classes recently developed in this article, and extended results for functions belonging to m-fold symmetric functions may be studied.

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