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Almost Periodic Solutions of Abstract Impulsive Volterra Integro-Differential Inclusions

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Abstract: In this paper, we introduce and systematically analyze the classes of (pre-)$(\mathcal{B}, \rho, (t_k))$-piecewise continuous almost periodic functions and (pre-)$(\mathcal{B}, \rho, (t_k))$-piecewise continuous uniformly recurrent functions with values in complex Banach spaces. We weaken substantially, or remove completely, the assumption that the sequence (t_k) of possible first kind discontinuities of the function under consideration is a Wexler sequence (in order to achieve these aims, we use certain results about Stepanov almost periodic type functions). We provide many applications in the analysis of the existence and uniqueness of almost periodic type solutions for various classes of the abstract impulsive Volterra integro-differential inclusions.

Keywords: $(\mathcal{B}, \rho, (t_k))$-piecewise continuous almost periodic type functions; $(\mathcal{B}, \rho, (t_k))$-piecewise continuous uniformly recurrent type functions; Wexler sequences; abstract impulsive Volterra integro-differential inclusions; almost periodic type solutions

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1. Introduction and Preliminaries

Let $(X, \|\cdot\|)$ be a complex Banach space, let $I = \mathbb{R}$ or $I = [0, \infty)$, and let $c \in \mathbb{C}$ be such that $|c| = 1$. If a continuous function $f : I \to X$ and a number $\epsilon > 0$ are given, then we say that a number $\tau > 0$ an (ϵ, c)-period for $f(\cdot)$ if and only if $\|f(t+\tau) - cf(t)\| \leq \epsilon$ for all $t \in I$. By $\vartheta_c(f, \epsilon)$ we denote the set consisting of all (ϵ, c)-periods for $f(\cdot)$. It is said that $f(\cdot)$ is c-almost periodic if and only if for each $\epsilon > 0$ the set $\vartheta_c(f, \epsilon)$ is relatively dense in $[0, \infty)$. The usual class of (Bohr) almost periodic functions is obtained by plugging $c = 1$. For more details about almost periodic type functions and their applications, we refer the reader to the research monographs [1–23] and references cited therein.

In the recent research article [24] by M. Fečkan et al., we have extended the notion of c-almost periodicity by introducing and analyzing the notion of ρ-almost periodicity with ρ being a general binary relation on X. This class of functions will play an important role in our analysis (cf. Definition 1(i) below for the notion of Bohr $(\mathcal{B}, \rho)$-almost periodicity). On the other hand, in [25], we have recently provided some applications of (a, k)-regularized C-resolvent families to the abstract impulsive Volterra integro-differential inclusions in Banach spaces. The main aim of this paper is to reconsider the notion of a piecewise continuous almost periodic function. This notion has been thoroughly analyzed in the research monographs [26] by A. Halanay, D. Wexler and [27] by A. M. Samoilenko, N. A. Perestyuk, by introducing and systematically investigating the classes of (pre-)$(\mathcal{B}, \rho, (t_k))$-piecewise continuous almost periodic functions and (pre-)$(\mathcal{B}, \rho, (t_k))$-piecewise continuous uniformly recurrent functions. We also aim to continue the research study carried out in [25] by investigating the almost periodic type solutions for various classes of the abstract impulsive Volterra integro-differential inclusions. We consider here the functions of the form $F : I \times X \to E$, where $(X, \| \cdot \|)$ and $(E, \| \cdot \|_E)$ are complex Banach spaces and $I = \mathbb{R}$ or $I = [0, \infty)$. In the existing literature, it has been commonly assumed that the sequence (t_k) of possible first kind discontinuities of function $f(\cdot)$ under consideration is a Wexler sequence. Using certain results about Stepanov almost periodic type functions, we show that this condition is sometimes rather superfluous and almost completely irrelevant. Before proceeding any further, we would like to note that this is probably the first research article which investigates the existence and uniqueness of the uniformly recurrent type solutions, the Weyl almost periodic type solutions and the Besicovitch–Doss almost periodic type solutions to the abstract impulsive Volterra integro-differential equations. Furthermore, this is probably the first paper in the existing literature which investigates the almost periodic type solutions for certain classes of the abstract higher-order impulsive Cauchy problems. It should also be mentioned that we introduce here, for the first time in the existing literature, the class of Weyl-p-almost periodic sequences in the sense of the general approach of A. S. Kovanko [28], the class of Doss-p-almost periodic sequences ($1 \leq p < \infty$) and analyze their applications in the study of the existence and uniqueness of the Weyl-p-almost periodic solutions (Doss-p-almost periodic solutions) for certain kinds of the abstract impulsive Volterra integro-differential equations.

The organization and main ideas of this paper can be briefly summarized as follows. After explaining the notation used in the paper, we recall the basic definitions and results about ρ-almost periodic type functions in Section 1.1. The main aim of Section 1.2 is to recollect the basic facts about the class of piecewise continuous almost periodic functions, which has been commonly used in the existing literature. We extend the notion of piecewise continuous almost periodicity in Section 2, where we introduce and analyze various classes of $(\mathcal{B}, \rho)$-piecewise continuous almost periodic type functions. More precisely, in Definition 6, we introduce the classes of (pre-)$(\mathcal{B}, \rho, (t_k))$-piecewise continuous almost periodic functions and (pre-)$(\mathcal{B}, \rho, (t_k))$-piecewise continuous uniformly recurrent functions. The assumption that the corresponding sequence (t_k) of possible discontinuities is a Wexler sequence is almost completely irrelevant in the analysis. This is not the case with the quasi-uniformly continuity condition (QUC) from the formulation of Definition 6, which plays an important role in our study. In Examples 1 and 2, we present two illustrative examples of real-valued functions which are (t_k)-piecewise continuous almost periodic (cf. also Remarks 2–4 for some useful observations about the function spaces introduced in Definition 6). Several structural characterizations for introducing classes of piecewise continuous almost periodic type functions have been proved in Propositions 1–3; it should be specifically emphasized that the supremum formula holds for certain classes of pre-$(\mathcal{B}, T, (t_k))$-piecewise continuous uniformly recurrent functions (cf. Proposition 5).

In Section 3, we continue the analysis of L. Qi and R. Yuan from their remarkable paper [29] concerning the relations between the piecewise continuous almost periodic functions and the Stepanov almost periodic type functions. We improve some structural results obtained in [29] by removing the assumption that (t_k) is a Wexler sequence. The main results in this section are Theorems 1 and 2; some consequences of these results are presented in Theorems 3–5 and Propositions 6 and 7. Composition principles for $(\mathcal{B}, (t_k))$-piecewise continuous almost periodic type functions are investigated in Section 3.1. Section 4 examines the existence and uniqueness of almost periodic type solutions for certain classes of the abstract impulsive differential inclusions of integer order, while Section 5 examines the existence and uniqueness of almost periodic type solutions for certain classes of the abstract Volterra impulsive integro-differential inclusions. Section 4 is broken down into four subsections: Section 4.1 is devoted to the study of asymptotically almost periodic type solutions of the abstract impulsive differential Cauchy problem $(ACP)_{1;1}$, asymptotically Weyl almost periodic type solutions of $(ACP)_{1;1}$ are sought in Section 4.2, the Besicovitch almost periodic type solutions of $(ACP)_{1;1}$ are sought in Section 4.3, and the almost periodic type solutions of the abstract higher-order impulsive Cauchy problems are sought in Section 4.4 (let us only mention that the separation condition $\inf_{k \in \mathbb{N}}(t_{k+1} - t_k) > 0$ on the corresponding sequence (t_k) of possible discontinuities is not employed in some results). The final section of the paper is reserved for the conclusions and final remarks about the problems considered. In addition to the above, we present many illustrative examples and open problems.

We use the standard notation throughout the paper. By $(X, \|\cdot\|)$ and $(E, \|\cdot\|_E)$ we denote two complex Banach spaces; I denotes the identity operator on E. By $\mathcal{B}$, we denote the collection of non-empty subsets of X such that for every $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$. The abbreviation $C(K : X)$, where K is a non-empty compact subset of $\mathbb{R}$, stands for the space of all continuous functions from K into X; $C(K) \equiv C(K : \mathbb{C})$, $\mathbb{N}_n := \{1, \cdot \cdot \cdot, n\}$ and $\mathbb{N}_n^0 := \{0, 1, \cdot \cdot \cdot, n\}$, where $n \in \mathbb{N}$. Let $0 < \tau \leq \infty$ and $a \in L^1_{loc}([0, \tau))$. Then, we say that the function $a(t)$ is a kernel on $[0, \tau)$ if and only if for each $f \in C([0, \tau))$ the assumption $\int_0^t a(t-s) f(s)\, ds = 0$, $t \in [0, \tau)$ implies $f(t) = 0$, $t \in [0, \tau)$. Set $g_\alpha(t) := t^{\alpha-1}/\Gamma(\alpha)$, $t > 0$, where $\Gamma(\cdot)$ denotes the Euler Gamma function, and $g_0(t) := \delta(t)$, the Dirac delta distribution. We set $L(t, \delta) := (t - \delta, t + \delta)$, $B(t, \delta) := [t - \delta, t + \delta]$ $(t \in \mathbb{R}, \delta > 0)$, $S_1 := \{z \in \mathbb{C} : |z| = 1\}$, $\lfloor s \rfloor := \sup\{k \in \mathbb{Z} : k \leq s\}$ and $\lceil s \rceil := \inf\{k \in \mathbb{Z} : k \geq s\}$ $(s \in \mathbb{R})$. Unless stated otherwise, we will always assume that $I = \mathbb{R}$ or $I = [0, \infty)$ henceforth. If $\Omega \subseteq \mathbb{C}$, then Ω^c denotes its complement in $\mathbb{C}$; $\chi_A(\cdot)$ denotes the characteristic function of the set A. The notion of Caputo fractional derivative $\mathbf{D}_t^\alpha u(t)$, where $u : [0, \infty) \to X$ and $\alpha > 0$, is taken in the sense of Equation (3.1) [25]. By $P(X)$, we denote the power set of X.

Let $T > 0$. Then, the space of X-valued piecewise continuous functions on $[0, T]$ is defined by

$$PC([0,T] : X) \equiv \big\{u : [0,T] \to X : u \in C\big((t_i, t_{i+1}] : X\big),\ u(t_i-) = u(t_i) \text{ exist for any } i \in \mathbb{N}_l,\ u(t_i+) \text{ exist for any } i \in \mathbb{N}_l^0 \text{ and } u(0) = u(0+)\big\},$$

where $0 \equiv t_0 < t_1 < t_2 < \ldots < t_l < T \equiv t_{l+1}$, and the symbols $u(t_i-)$ and $u(t_i+)$ denote the left and the right limits of the function $u(t)$ at the point $t = t_i$, $i \in \mathbb{N}_{l-1}^0$, respectively. Let us recall that $PC([0, T] : X)$ is a Banach space endowed with the norm $\|u\| := \max\{\sup_{t \in [0,T)} \|u(t+)\|, \sup_{t \in (0,T]} \|u(t-)\|\}$. The space of X-valued piecewise continuous functions on $[0, \infty)$, denoted by $PC([0, \infty) : X)$, if defined as the union of those functions $f : [0, \infty) \to X$ such that the discontinuities of $f(\cdot)$ form a discrete set and that for each $T > 0$ we have $f_{|[0,T]}(\cdot) \in PC([0, T] : X)$. We similarly define the space $PC(\mathbb{R} : X)$.

If $\omega \in \mathbb{R}$, then $C_\omega([0, \infty) : X)$ stands for the space of all continuous functions $f : [0, \infty) \to X$ such that the function $t \mapsto e^{-\omega t}\|f(t)\|$, $t \geq 0$ is bounded; the space $PC_\omega([0, \infty) : X)$ stands for the space of all piecewise continuous functions $f : [0, \infty) \to X$ such that the function $t \mapsto e^{-\omega t}\|f(t)\|$, $t \geq 0$ is bounded.

Concerning the basic definitions and results about binary relations, see [24]. We refer the reader to [25] for more details concerning the multivalued linear operators and solution operator families subgenerated by them; unless stated otherwise, we will always assume henceforth that the operator $C \in L(X)$ is injective.

1.1. ρ-Almost Periodic Type Functions

In this subsection, we will recall the basic definitions and facts about (Stepanov) ρ-almost periodic type functions. We need the following notion [24]:

Definition 1. *Suppose that $\emptyset \neq I \subseteq \mathbb{R}$, $F : I \times X \to E$ is a continuous function, and ρ is a binary relation on E. Then, we say that:*

(i) *$F(\cdot;\cdot)$ is Bohr $(\mathcal{B}, \rho)$-almost periodic if and only if for every $B \in \mathcal{B}$ and $\epsilon > 0$ there exists $l > 0$ such that for each $t_0 \in I$ there exists $\tau \in B(t_0, l) \cap I$ such that, for every $t \in I$ and $x \in B$, there exists an element $y_{t;x} \in \rho(F(t;x))$ such that*

$$\big\|F(t+\tau;x) - y_{t;x}\big\| \leq \epsilon.$$

(ii) *$F(\cdot;\cdot)$ is $(\mathcal{B},\rho)$-uniformly recurrent if and only if for every $B \in \mathcal{B}$ there exists a sequence (τ_k) in I such that $\lim_{k\to+\infty}|\tau_k| = +\infty$ and that, for every $t \in I$ and $x \in B$, there exists an element $y_{t;x} \in \rho(F(t;x))$ such that*

$$\lim_{k\to+\infty} \sup_{t\in I; x\in B} \left\| F(t+\tau_k;x) - y_{t;x} \right\| = 0.$$

Any Bohr $(\mathcal{B},\rho)$-almost periodic function $F(\cdot;\cdot)$ is $(\mathcal{B},\rho)$-uniformly recurrent; the converse statement is not true in general [9]. If $X = \{0\}$, then we omit the term "$\mathcal{B}$" from the notation; furthermore, if $\rho = c\mathrm{I}$ for some complex number $c \in \mathbb{C}$, then we also say that the function $F(\cdot;\cdot)$ is Bohr $(\mathcal{B},c)$-almost periodic, respectively, $(\mathcal{B},c)$-uniformly recurrent.

We need the following notion (cf. [30] for the case in which ρ is single-valued):

Definition 2. *Suppose that $1 \le p < +\infty$, ρ is a binary relation on E and $F : I \times X \to E$. Then, we say that:*

(i) *$F(\cdot;\cdot)$ is Stepanov-p-$(\mathcal{B},\rho)$-almost periodic if and only if for every $B \in \mathcal{B}$ and $\epsilon > 0$ there exists $l > 0$ such that for each $t_0 \in I$ there exists $\tau \in B(t_0,l) \cap I$ such that, for every $t \in I$ and $x \in B$, there exists a mapping $F_{t,x} : [0,1] \to E$ such that $F_{t,x}(u) \in \rho(F(t+u;x))$ for a.e. $u \in [0,1]$, $F_{t,x} \in L^p([0,1]:E)$ and*

$$\left\| F(t+\tau+u;x) - F_{t,x}(u) \right\|_{L^p([0,1]:E)} \le \epsilon, \quad t \in I,\ x \in B.$$

(ii) *$F(\cdot;\cdot)$ is Stepanov-(p,ρ)-$\mathcal{B}$-uniformly recurrent if and only if for every $B \in \mathcal{B}$ there exists a sequence (τ_k) in I such that $\lim_{k\to+\infty}|\tau_k| = +\infty$ and that, for every $t \in I$ and $x \in B$, there exists a mapping $F_{t,x} : [0,1] \to E$ such that $F_{t,x}(u) \in \rho(F(t+u;x))$ for a.e. $u \in [0,1]$, $F_{t,x} \in L^p([0,1]:E)$ and*

$$\lim_{k\to+\infty} \sup_{t\in I; x\in B} \left\| F(t+\tau_k+u;x) - F_{t,x}(u) \right\|_{L^p([0,1]:E)} = 0.$$

If $X = \{0\}$, then it is also said that $F(\cdot)$ is Stepanov-(p,ρ)-almost periodic (Stepanov-(p,ρ)-uniformly recurrent). Finally, if $\rho = c\mathrm{I}$ for some $c \in \mathbb{C}$, then we also say that the function $F(\cdot)$ is Stepanov-(p,c)-almost periodic (Stepanov-(p,c)-uniformly recurrent); if $c = 1$, then it is also said that the function $f(\cdot)$ is Stepanov-p-almost periodic (Stepanov-p-uniformly recurrent).

1.2. Piecewise Continuous Almost Periodic Functions

The piecewise continuous almost periodic type solutions for various classes of impulsive integro-differential equations have been analyzed by numerous authors so far (see, e.g., the research monograph [21] by G. Tr. Stamov for a comprehensive survey of results). In this subsection, we analyze the piecewise continuous almost periodic type functions.

We say that an X-valued sequence $(x_n)_{n\in\mathbb{Z}}$ [$(x_n)_{n\in\mathbb{N}}$] is (Bohr) almost periodic if and only if, for every $\epsilon > 0$, there exists a natural number $N_0(\epsilon)$ such that among any $N_0(\epsilon)$ consecutive integers in $\mathbb{Z}$ [$\mathbb{N}$], there exists at least one integer $\tau \in \mathbb{Z}$ [$\tau \in \mathbb{N}$] satisfying that

$$\left\| x_{n+\tau} - x_n \right\| \le \epsilon, \quad n \in \mathbb{Z} \ \ [n \in \mathbb{N}].$$

Any almost periodic X-valued sequence is bounded. As in the case of functions, this number is said to be an ϵ-period of sequence (x_n). The equivalent concept of Bochner almost periodicity of X-valued sequences can be introduced, as well; see, e.g., [27] (Theorem 70, pp. 185–186 and its important consequences [27] (Theorems 71–73, pp. 186–188). It is well-known that a sequence $(x_k)_{k\in\mathbb{Z}}$ in X is almost periodic if and only if there exists an almost periodic function $f : \mathbb{R} \to X$ such that $x_k = f(k)$ for all $k \in \mathbb{Z}$; see, e.g., the proof of [31] (Theorem 2) given in the scalar-valued case. It is not difficult to prove that, for every almost periodic sequence $(x_k)_{k\in\mathbb{N}}$ in X, there exists a unique almost periodic sequence $(\tilde{x_k})_{k\in\mathbb{Z}}$ in X such that $\tilde{x_k} = x_k$ for all $k \in \mathbb{N}$, so that a sequence $(x_k)_{k\in\mathbb{N}}$ in X is almost periodic if and only if there exists an almost periodic function $f : [0, \infty) \to X$ such that $x_k = f(k)$ for all $k \in \mathbb{N}$.

Unless stated otherwise, we will always assume henceforth that $(t_k)_{k\in\mathbb{Z}}$ $[(t_k)_{k\in\mathbb{N}}]$ is a sequence in $\mathbb{R}$ [in $(0, \infty)$] such that $\delta_0 := \inf_{k\in\mathbb{Z}}(t_{k+1} - t_k) > 0$ $[\delta_0 := \inf_{k\in\mathbb{N}}(t_{k+1} - t_k) > 0]$. Set $t_k^j := t_{k+j} - t_k$, $j,\ k \in \mathbb{Z}$ $[j,\ k \in \mathbb{N}]$. We need the following definitions:

Definition 3. *The family of sequences $(t_k^j)_{k\in\mathbb{Z}}$ $[(t_k^j)_{k\in\mathbb{N}}]$, $j \in \mathbb{Z}$ $[j \in \mathbb{N}]$ is called equipotentially almost periodic if and only if, for every $\epsilon > 0$, there exists a relatively dense set Q_ϵ in $\mathbb{R}$ [in $[0, \infty)$] such that for each $\tau \in Q_\epsilon$ there exists an integer $q \in \mathbb{Z}$ $[q \in \mathbb{N}]$ such that $|t_{i+q} - t_i - \tau| < \epsilon$ for all $i \in \mathbb{Z}$ $[i \in \mathbb{N}]$.*

Definition 4. *The sequence $(t_k)_{k\in\mathbb{Z}}$ $[(t_k)_{k\in\mathbb{N}}]$ is said to be uniformly almost periodic if and only if, for every $\epsilon > 0$, there exists a relatively dense set Q_ϵ in $\mathbb{Z}$ [in $\mathbb{N}$] such that*

$$\left|t_{i+q}^j - t_i^j\right| < \epsilon, \quad i,\ j \in \mathbb{Z}\ \ [i,\ j \in \mathbb{N}],\ \ q \in Q_\epsilon.$$

We know that, if the sequence $(t_k)_{k\in\mathbb{Z}}$ $[(t_k)_{k\in\mathbb{N}}]$ is uniformly almost periodic, then the family of sequences $(t_k^j)_{k\in\mathbb{Z}}$ $[(t_k^j)_{k\in\mathbb{N}}]$, $j \in \mathbb{Z}$ $[j \in \mathbb{N}]$ is equipotentially almost periodic. See also [27] (p. 377) and [29] (Lemma 2.12; let us also note that the family of sequences $(t_k^j)_{k\in\mathbb{Z}}$, $j \in \mathbb{Z}$ is equipotentially almost periodic if and only if there exist a unique non-zero real number ζ and an almost periodic sequence $(a_k)_{k\in\mathbb{Z}}$ such that $t_k = \zeta k + a_k$ for all $k \in \mathbb{Z}$. It seems very plausible that a similar statement holds for the equipotentially almost periodic sequences $(a_k)_{k\in\mathbb{N}}$.

The usual definition of a piecewise continuous almost periodic function goes as follows (see [26,27] for more details about the subject):

Definition 5. *Suppose that the function $f : \mathbb{R} \to X$ $[f : [0, \infty) \to X]$ is piecewise continuous with the possible first kind discontinuities at the points of a fixed sequence $(t_k)_{k\in\mathbb{Z}}$ $[(t_k)_{k\in\mathbb{N}}]$. Then, we say that the function $f(\cdot)$ is (t_k)-piecewise continuous almost periodic if and only if the following conditions are fulfilled:*

- *(i) The family of sequences $(t_k^j)_{k\in\mathbb{Z}}$ $[(t_k^j)_{k\in\mathbb{N}}]$, $j \in \mathbb{Z}$ $[j \in \mathbb{N}]$ is equipotentially almost periodic, i.e., (t_k) is a Wexler sequence.*
- *(ii) For every $\epsilon > 0$, there exists $\delta > 0$ such that, if the points t_1 and t_2 belong to (t_i, t_{i+1}) for some $i \in \mathbb{Z}$ $[i \in \mathbb{N}_0;\ t_0 \equiv 0]$ and $|t_1 - t_2| < \delta$, then $\|f(t_1) - f(t_2)\| < \epsilon$.*
- *(iii) For every $\epsilon > 0$, there exists a relatively dense set S in $\mathbb{R}$ [in $[0, \infty)$] such that, if $\tau \in S$, then $\|f(t + \tau) - f(t)\| < \epsilon$ for all $t \in \mathbb{R}$ such that $|t - t_k| > \epsilon$, $k \in \mathbb{Z}$ $[k \in \mathbb{N}]$. Such a point τ is called an ϵ-almost period of $f(\cdot)$.*

For example, let the family of sequences $(t_k^j)_{k\in\mathbb{Z}}$, $j \in \mathbb{Z}$ be equipotentially almost periodic. Then, we know that the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(t) := \mu_i$ if $t \in (t_i, t_{i+1}]$ for some $i \in \mathbb{Z}$, is (t_k)-piecewise continuous almost periodic provided that the sequence $(\mu_i)_{i\in\mathbb{Z}}$ is almost periodic (cf. [27] (pp. 202–203) for the proof of the above fact).

For further information about piecewise continuous almost periodic functions and their applications, we refer the reader to the research articles [32] by H. R. Henríquez, B. de Andrade, M. Rabelo, [33] by L. Qi, R. Yuan, [20] by V. Tkachenko and references cited therein. Before proceeding with the original contributions about piecewise continuous almost periodic type functions, it would be worthwhile to mention that J. Xia has considered, in [34], the class of piecewise continuous almost periodic functions following a completely different approach (cf. also the research article [31] by L. Díaz and R. Naulin).

2. $(\mathcal{B}, \rho)$-Piecewise Continuous Almost Periodic Type Functions

We start this section by introducing the following notion:

Definition 6. *Suppose that ρ is a binary relation on E, the function $F : \mathbb{R} \times X \to E$ [$F : [0,\infty) \times X \to E$] satisfies that, for every $x \in X$, the function $t \mapsto F(t;x)$ is piecewise continuous with the possible first kind discontinuities at the points of a fixed sequence $(t_k)_{k\in\mathbb{Z}}$ [$(t_k)_{k\in\mathbb{N}}$]. Then, we say that the function $F(\cdot)$ is:*

- *(i) pre-$(\mathcal{B}, \rho, (t_k))$-piecewise continuous almost periodic if and only if, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exists a relatively dense set S in $\mathbb{R}$ [in $[0,\infty)$] such that, if $\tau \in S$, $x \in B$ and $t \in \mathbb{R}$ satisfies $|t - t_k| > \epsilon$ for all $k \in \mathbb{Z}$ [$k \in \mathbb{N}$], then there exists $y_{t,x} \in \rho(F(t;x))$ such that $\|F(t+\tau;x) - y_{t,x}\| < \epsilon$.*
- *(ii) $(\mathcal{B}, \rho, (t_k))$-piecewise continuous almost periodic if and only if the condition (i) from Definition 5 holds, $F(\cdot;\cdot)$ is pre-$(\mathcal{B}, \rho, (t_k))$-piecewise continuous almost periodic and (QUC) holds, where:*
  - *(QUC) For every $\epsilon > 0$ and $B \in \mathcal{B}$, there exists $\delta > 0$ such that, if $x \in B$ and the points t_1 and t_2 belong to (t_i, t_{i+1}) for some $i \in \mathbb{Z}$ [$i \in \mathbb{N}_0$; $t_0 \equiv 0$] and $|t_1 - t_2| < \delta$, then $\|F(t_1;x) - F(t_2;x)\| < \epsilon$.*
- *(iii) pre-$(\mathcal{B}, \rho, (t_k))$-piecewise continuous uniformly recurrent if and only if there exists a strictly increasing sequence (α_l) of positive real numbers tending to plus infinity and satisfying that, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exists an integer $l_0 \in \mathbb{N}$ such that, if $x \in B$, $l \geq l_0$ and $t \in \mathbb{R}$ satisfies $|t - t_k| > \epsilon$ for all $k \in \mathbb{Z}$ [$k \in \mathbb{N}$], then there exists $y_{t,x} \in \rho(F(t;x))$ such that $\|F(t+\alpha_l;x) - y_{t,x}\| < \epsilon$.*
- *(iv) $(\mathcal{B}, \rho, (t_k))$-piecewise continuous uniformly recurrent if and only if $F(\cdot;\cdot)$ is pre-$(\mathcal{B}, \rho, (t_k))$-piecewise continuous uniformly recurrent and the condition (QUC) holds.*

We say that the function $F(\cdot;\cdot)$ is (pre-)$(\mathcal{B}, \rho)$-piecewise continuous almost periodic [(pre-)$(\mathcal{B}, \rho)$-piecewise continuous uniformly recurrent] if and only if $F(\cdot;\cdot)$ is (pre-)$(\mathcal{B}, \rho, (t_k))$-piecewise continuous almost periodic [(pre-)$(\mathcal{B}, \rho, (t_k))$-piecewise continuous uniformly recurrent] for a certain sequence $(t_k)_{k\in\mathbb{Z}}$ [$(t_k)_{k\in\mathbb{N}}$] obeying the general requirements. If $\rho = c\mathrm{I}$ for some $c \in \mathbb{C}$, then we also say that $F(\cdot;\cdot)$ is (pre-)piecewise continuous c-almost periodic [(pre-)piecewise continuous c-uniformly recurrent]; furthermore, if $c = -1$, then we also say that $F(\cdot;\cdot)$ is (pre-)piecewise continuous almost anti-periodic [(pre-)piecewise continuous uniformly anti-recurrent]. We omit the term "$\mathcal{B}$" from the notation if $X = \{0\}$ and omit the term "c" from the notation if $c = 1$.

Remark 1. *In the notion introduced in Definition 6(i), we can also require that the inequality $\|F(t+\tau;x) - y_{t,x}\| < \epsilon$ holds provided that $|t - t_k| > M(\epsilon)$ for all $k \in \mathbb{Z}$ [$k \in \mathbb{N}$], where $M : (0,\infty) \to [0,\infty)$ satisfies $\liminf_{\epsilon\to 0+} M(\epsilon) = 0$. This notion is really not interesting because a very simple argument shows that a function $F : I \times X \to E$ obeys this condition if and only if $F(\cdot;\cdot)$ is pre-$(\mathcal{B}, \rho, (t_k))$-piecewise continuous almost periodic. The same holds in the case of consideration of parts (ii), (iii) and (iv) of Definition 6 so that we will always assume henceforth that $M(\epsilon) \equiv \epsilon$.*

Before proceeding any further, we would like to present the following illustrative examples:

Example 1. *Suppose that $c \in S_1$, $\omega > 0$, $t_1 \in (0, \omega]$ and $t_k = t_1 + (k-1)\omega$, $k \in \mathbb{Z}$ [$t_0 = 0$ and $t_k = t_1 + (k-1)\omega$, $k \geq 2$]. Suppose, further, that the function $F_1 : (t_1, t_1 + \omega] \times X \to E$ satisfies that, for every $x \in X$, $F_1(t_1 + \omega; x) \neq cF_1(t_1; x)$ as well as that $\lim_{t\to t_1+} F_1(t;x)$ exists in E. Then, we can extend the function $F_1(\cdot;\cdot)$ to a function $F : \mathbb{R} \times X \to E$ [$F : [0,\infty) \times X \to E$] such that, for every $x \in X$, the function $F(\cdot;x)$ is piecewise continuous, has the possible first kind discontinuities at the points of sequence $(t_k)_{k\in\mathbb{Z}}$ [$(t_k)_{k\in\mathbb{N}}$] and $F(t+\omega;x) = cF(t;x)$ for all $x \in X$ and $t \in \bigcup_{k\in\mathbb{Z}}(t_k, t_{k+1})$ [$t \in \bigcup_{k\in\mathbb{N}_0}(t_k, t_{k+1})$]. Since the set of all integers $k \in \mathbb{Z}$ [$k \in \mathbb{N}$] such that $c^k = c$ is relatively dense in $\mathbb{Z}$ [$\mathbb{N}$], with the meaning clear, a very simple argument shows that the function $F(\cdot;\cdot)$ is $(c, (t_k))$-piecewise continuous almost periodic. For example, if $P(t)$ is an (anti-)periodic non-zero trigonometric polynomial with real values, then the piecewise continuous function $f_0(\cdot)$ determined by the function $f(t) := sign(P(t))$, $t \in \mathbb{R}$ is (t_k)-piecewise continuous* *almost (anti-)periodic; here and hereafter,* $sign(0) = 0$, $sign(t) = 1$ *for* $t > 0$ *and* $sign(t) = -1$ *for* $t < 0$.

For the sequel, we will denote the collection of all functions $F : I \times X \to E$ *constructed in this way by* $PPC_{\omega,c;(t_k)}(I : X)$.

Example 2. *Suppose that* $m \in \mathbb{N}$, $y : \mathbb{R} \to \mathbb{R}$ *is a Bohr almost periodic function, and there exists an integer* $k_0 \in \mathbb{Z}$ *such that* $y(mk_0) \neq y(m(k_0 + 1))$. *Define*

$$f(t) := y\left(m\left\lfloor\frac{t+1}{m}\right\rfloor\right), \quad t \in \mathbb{R}.$$

In Figure 1, the plot of function $f(t)$ *is constructed for* $y(t) = \sin t$ *and* $m = 3$.

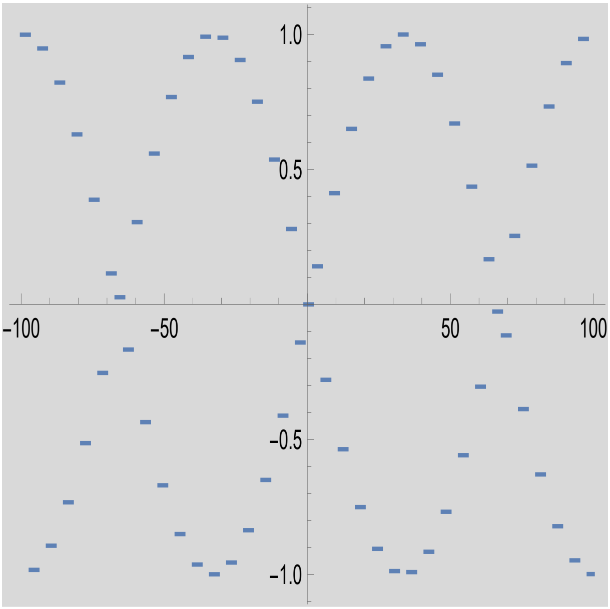

Figure 1. Graph for the case $y(t) = \sin t$ and $m = 3$.

Thus, we have $f(t) = y(mk)$ *if* $y \in [mk - 1, mk - 1 + m)$ *for some integer* $k \in \mathbb{Z}$ *so that the prescribed assumption implies that the function* $f(\cdot)$ *is not continuous on the real line. On the other hand, for every* $\epsilon > 0$ *there exists* $l > 0$ *such that any interval* $I \subseteq \mathbb{R}$ *of length* $\geq l$ *contains a point* τ *such that* $|f(t + \tau) - f(t)| \leq \epsilon$, $t \in \mathbb{R}$. *Towards this end, let us recall that, for a given* $\epsilon > 0$ *in advance, we can always find* $l > 0$ *such that any interval* $I \subseteq \mathbb{R}$ *of length* $\geq l$ *contains an integer* τ *such that* $|y(t + \tau) - y(t)| \leq \epsilon/m$, $t \in \mathbb{R}$; *the last estimate simply implies* $|y(t + m\tau) - y(t)| \leq \epsilon$, $t \in \mathbb{R}$ *so that, actually, we can always find a number* $l' = lm > 0$ *such that any interval* $I \subseteq \mathbb{R}$ *of length* $\geq lm$ *contains an integer* $m\tau$, *where* $\tau \in \mathbb{Z}$, *such that* $|y(t + m\tau) - y(t)| \leq \epsilon$, $t \in \mathbb{R}$. *Let it be the case; then we have*

$$|f(t + m\tau) - f(t)| = \left| y\left(m\left\lfloor\frac{t+1}{m}\right\rfloor + m\tau\right) - y\left(m\left\lfloor\frac{t+1}{m}\right\rfloor\right)\right| \leq \epsilon, \quad t \in \mathbb{R}.$$

Since the function $\check{f}(\cdot) \equiv f(-\cdot)$ *is continuous from the left side, the condition (i) from Definition 5 holds, and (QUC) holds. It readily follows that the function* $\check{f}(\cdot)$ *is* (t_k)*-piecewise continuous almost periodic.*

The proof of the following extension of [27] (Theorem 77) is simple and therefore omitted:

Proposition 1. *Suppose that* ρ *is a binary relation on* E, *and the function* $F : I \times X \to E$ *is (pre-)*$(\mathcal{B}, \rho, (t_k))$*-piecewise continuous almost periodic [(pre-)*$(\mathcal{B}, \rho, (t_k))$*-piecewise continuous* *uniformly recurrent]. If $(Z, \| \cdot \|_Z)$ is a complex Banach space, $\psi : E \to Z$ is uniformly continuous on the set $\rho(F(I \times B)) \cup F(I \times B)$ for each set $B \in \mathcal{B}$ and $\sigma = \{(\psi(y_1), \psi(y_2)) : y_1 \rho y_2\}$, then the function $\psi \circ F : I \times X \to Z$ is (pre-)$(\mathcal{B}, \sigma, (t_k))$-piecewise continuous almost periodic [(pre-)$(\mathcal{B}, \sigma, (t_k))$-piecewise continuous uniformly recurrent].*

We continue by providing several useful observations:

Remark 2. *(i) Condition (QUC) can be relaxed by assuming that, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exists $\delta > 0$ such that, if $x \in B$ and the points t_1 and t_2 belong to the set $\bigcup_{k \in \mathbb{Z}} [t_k + \epsilon, t_{k+1} - \epsilon]$ [$\bigcup_{k \in \mathbb{N}} [t_k + \epsilon, t_{k+1} - \epsilon]$] and $|t_1 - t_2| < \delta$, then $\|F(t_1; x) - F(t_2; x)\| < \epsilon$; cf. also [27] (Definition 7, p. 390) for this approach. We feel it is our duty to say that the condition (QUC) is primarily intended for the analysis of $(\mathcal{B}, \mathrm{I}, (t_k))$-piecewise continuous almost periodic type functions and that some problems naturally occur if $\rho \neq \mathrm{I}$.*

(ii) The introduction of class of (pre-)$(\mathcal{B}, \rho)$-piecewise continuous uniformly recurrent functions is strongly justified by the fact that the definition of a piecewise continuous almost periodic function is a bit restrictive due to condition (i). In actual fact, this condition does not allow one to consider the existence and uniqueness of the piecewise continuous solutions for a large class of the abstract impulsive Cauchy problems in which the corresponding sequence of the first kind discontinuities (t_k) is not of linear growth as $k \to +\infty$; for example, we cannot consider the case $t_{\pm k} = k^2$ for all $k \in \mathbb{Z}$, which is very legitimate from the point of view of the theory of the abstract impulsive Cauchy problems.

Remark 3. *It is clear that any Bohr $(\mathcal{B}, \rho)$-almost periodic [Bohr $(\mathcal{B}, \rho)$-uniformly recurrent] function $F(\cdot; \cdot)$ is pre-$(\mathcal{B}, \rho, (t_k))$-piecewise continuous almost periodic [pre-$(\mathcal{B}, \rho, (t_k))$-piecewise continuous uniformly recurrent] for any sequence (t_k) satisfying the general assumptions as well as that any Bohr $(\mathcal{B}, \rho)$-almost periodic [Bohr $(\mathcal{B}, \rho)$-uniformly recurrent] function $F(\cdot; \cdot)$ which is uniformly continuous on the set $I \times B$ for each $B \in \mathcal{B}$ is $(\mathcal{B}, \rho, (t_k))$-piecewise continuous almost periodic [$(\mathcal{B}, \rho, (t_k))$-piecewise continuous uniformly recurrent] for any sequence (t_k) satisfying the general assumptions. In the almost periodic case, the statements of [24] (Proposition 2.2, Proposition 2.7(ii)) show that this condition holds true if $\mathcal{B}$ is a family consisting of some compact subsets of X, $I = \mathbb{R}$, $R(F) \subseteq D(\rho)$, and ρ is single-valued on $R(F)$.*

The subsequent structural result is a generalization of [32] (Lemma 2.6):

Proposition 2. *Suppose that $F : I \times X \to E$ is pre-$(\mathcal{B}, T, (t_k))$-piecewise continuous almost periodic, where $\rho = T \in L(E)$ is a linear isomorphism, and the condition* (QUC) *holds. If $B \in \mathcal{B}$ is a compact subset of X, then the set $\{F(t; x) : t \in I,\ x \in B\}$ is relatively compact in E.*

Proof. We will basically consider the case in which $I = \mathbb{R}$ and explain the essential change in the case that $I = [0, \infty)$. Since $\rho = T \in L(E)$ is a linear isomorphism, it suffices to show that the set $T(\{F(t; x) : t \in \mathbb{R},\ x \in B\})$ is relatively compact in E. Let $\epsilon > 0$ be given. Then, there exists $\delta > 0$ such that, if $x \in B$ and the points t_1 and t_2 belong to (t_i, t_{i+1}) for some $i \in \mathbb{Z}$ and $|t_1 - t_2| < \delta$, then $\|F(t_1; x) - F(t_2; x)\| < \epsilon/2$. Let $\delta_1 \in (0, \min\{\delta, \epsilon/4\})$. After that, we find $l > 0$ such that, for every $t_0 \in \mathbb{R}$, the interval $[t_0, t_0 + l]$ contains a point $\tau \in I$ such that $\|F(t + \tau; x) - TF(t; x)\| \leq \delta_1$ for all $t \in \mathbb{R}$ such that $|t - t_k| \geq \delta_1$ for all $j \in \mathbb{Z}$. Fix now a point $t \in \mathbb{R}$ and consider the interval $I = [-t, l - t]$ (if $I = [0, \infty)$, then for each point $t \geq l$ we can consider the interval $[t - l, t]$ and a corresponding (δ_1, T)-almost period τ belonging to this set). The set $F([0, l] \times B)$ is compact in E; furthermore, if $\tau \in I$ and the above conditions are satisfied, then we easily obtain the existence of an integer $m \in \mathbb{N}$, the points $s_1, \ldots, s_m \in \mathbb{R}$ and the elements $x_1, \ldots, x_m \in B$ such that

$$
\begin{aligned}
TF(t; x) \in B\big(F(t + \tau; x), \delta_1\big) &\subseteq \bigcup_{y \in F([0,l] \times B)} B(y, \delta_1) \\
&\subseteq L\big(F(s_1; x_1), \epsilon/2\big) \cup \ldots \cup L\big(F(s_m; x_m), \epsilon/2\big), \quad x \in B,
\end{aligned}
$$

provided that $|t - t_k| \geq \delta_1$ for all $j \in \mathbb{Z}$. If $|t - t_k| < \delta_1$ for some $j \in \mathbb{Z}$, then there exists an element $t' \in [t_j + \delta_1, t_{j+1} - \delta_1] \cup [t_{j-1} + \delta_1, t_j - \delta_1]$ such that $\|F(t;x) - F(t';x)\| \leq \epsilon/2$, $x \in B$, which simply completes the proof of the theorem. $\square$

Remark 4. *(i) It is also worth noting that Proposition 2 provides a proper generalization of [29] (Lemma 3.3) as well as that this lemma holds even if the corresponding sequence $(\tau_j)_{j\in\mathbb{Z}}$ from its formulation is not a Wexler sequence.*

(ii) It is well-known that there exists a continuous Stepanov-1-almost periodic function $f : \mathbb{R} \to \mathbb{R}$ which is not bounded (see, e.g., [35]); therefore, $f(\cdot)$ cannot be piecewise continuous almost periodic due to Proposition 2.

We continue by stating the following results; the proofs are rather technical and therefore omitted (the statements of [24] (Theorem 2.11(ii)–(iv)) can also be simply reformulated in the new framework):

Proposition 3. *Suppose that ρ is a binary relation on E which satisfies that $D(\rho)$ is a closed subset of X and*

(C_ρ) : For every $\epsilon > 0$, there exists $\delta > 0$ such that, for every $y_1,\ y_2 \in E$ with $\|y_1 - y_2\| < \delta$, we have $\|z_1 - z_2\| < \epsilon$ for every $z_1 \in \rho(y_1)$ and $z_2 \in \rho(y_2)$.

Suppose, further, that for each $m \in \mathbb{N}$, the function $F_m : I \times X \to E$ satisfies that, for every $x \in X$, the function $t \mapsto F_m(t;x)$ is piecewise continuous with the possible first kind discontinuities at the points of a fixed sequence (t_k). Let $F : I \times X \to E$ and let $\lim_{m\to\infty} F_m(t;x) = F(t;x)$, uniformly on $I \times B$ for every fixed set $B \in \mathcal{B}$.

Then, for every $x \in X$, the function $t \mapsto F(t;x)$ is piecewise continuous with the possible first kind discontinuities at the points of sequence (t_k) and the following holds: If for each $m \in \mathbb{N}$ the function $F_m(\cdot;\cdot)$ is pre-$(\mathcal{B},\rho,(t_k))$-piecewise continuous almost periodic [$(\mathcal{B},\rho,(t_k))$-piecewise continuous almost periodic; pre-$(\mathcal{B},\rho,(t_k))$-piecewise continuous uniformly recurrent/$(\mathcal{B},\rho,(t_k))$-piecewise continuous uniformly recurrent], then the function $F(\cdot;\cdot)$ is pre-$(\mathcal{B},\rho,(t_k))$-piecewise continuous almost periodic [$(\mathcal{B},\rho,(t_k))$-piecewise continuous almost periodic; pre-$(\mathcal{B},\rho,(t_k))$-piecewise continuous uniformly recurrent/$(\mathcal{B},\rho,(t_k))$-piecewise continuous uniformly recurrent]. Furthermore, if the functions $F_m(\cdot;\cdot)$ satisfy condition (QUC)*, then the function $F(\cdot;\cdot)$ satisfies the same condition.*

Proposition 4. *Suppose that the function $F : I \times X \to E$ is pre-$(\mathcal{B},\rho,(t_k))$-piecewise continuous almost periodic [$(\mathcal{B},\rho,(t_k))$-piecewise continuous almost periodic; pre-$(\mathcal{B},\rho,(t_k))$-piecewise continuous uniformly recurrent/$(\mathcal{B},\rho,(t_k))$-piecewise continuous uniformly recurrent]. Then, the function $\|F(\cdot;\cdot)\|$ is pre-$(\mathcal{B},\sigma,(t_k))$-piecewise continuous almost periodic [$(\mathcal{B},\sigma,(t_k))$-piecewise continuous almost periodic; pre-$(\mathcal{B},\sigma,(t_k))$-piecewise continuous uniformly recurrent/$(\mathcal{B},\sigma,(t_k))$-piecewise continuous uniformly recurrent], where*

$$\sigma := \{(\|y_1\|, \|y_2\|) \mid \exists t \in I\ \exists x \in X\ :\ y_1 = F(t;x) \text{ and } y_2 \in \rho(y_1)\}.$$

It is worth noting that the supremum formula can be clarified for pre-$(\mathcal{B},T,(t_k))$-piecewise continuous uniformly recurrent functions.

Proposition 5. *Suppose that $\rho = T \in L(E)$ is a linear isomorphism and $F : I \times X \to E$ is a pre-$(\mathcal{B},T,(t_k))$-piecewise continuous uniformly recurrent function. Then, for each $a \in I$ and $B \in \mathcal{B}$, we have*

$$\sup_{t\in I;x\in B} \|F(t;x)\| \leq \left\|T^{-1}\right\| \sup_{t\in[a,\infty);x\in B} \|F(t;x)\|.$$

Proof. It suffices to show that for each fixed number $\epsilon > 0$ we have

$$\sup_{t \in I; x \in B} \|F(t;x)\| \le \|T^{-1}\| \sup_{t \in [a,\infty); x \in B} \|F(t;x)\| + \epsilon.$$

Clearly, there exists a sequence (δ_k) in $(0,\epsilon)$ such that $\lim_{k \to +\infty} \delta_k = 0$. After that, we find a strictly increasing sequence (α_k) of positive real numbers tending to plus infinity such that $\|F(t+\alpha_k;x) - TF(t;x)\| \le \delta_k$ provided that $|t - t_i| > \delta_k$ for all $i \in \mathbb{Z}$ [$i \in \mathbb{N}$]. If $t \notin \{t_i : i \in \mathbb{Z}\}$ [$t \notin \{t_i : i \in \mathbb{N}\}$], then there exists $k \in \mathbb{N}$ such that $|t - t_i| > \delta_k$ for some $i \in \mathbb{Z}$ [$i \in \mathbb{N}$]. Hence, we have $\|TF(t;x)\| \le \|F(t+\alpha_k;x)\| + \delta_k \le \|F(t+\alpha_k;x)\| + \epsilon$ and $\|F(t;x)\| \le \|T^{-1}\|(\|F(t+\alpha_k;x)\| + \epsilon)$. The final conclusion follows from the fact that, for every $x \in B$, the function $F(\cdot;x)$ is continuous from the left side. □

The statements of [36] (Propositions 2.17 and 2.18) continue to hold in our new framework, and we have the following:

(i) If $f : I \to \mathbb{R}$ is a pre-c-piecewise continuous uniformly recurrent function, then $c = \pm 1$; furthermore, if $f(t) \ge 0$ for all $t \in I$, then $c = 1$.
(ii) If $f : I \to E$ is a pre-c-piecewise continuous uniformly recurrent function, then $\lim_{t \to +\infty} f(t) \neq 0$.

It is well-known that, for every almost periodic function $f : [0,\infty) \to E$, there exists a unique almost periodic function $g : \mathbb{R} \to E$ such that $g(t) = f(t)$ for all $t \ge 0$; see H. Bart, S. Goldberg [37] as well as [7,9] for many similar results of this type. We close this section with the observation that is not clear whether we can state a satisfactory analogue of this result for certain subclasses of (pre-)$(\mathcal{B}, M, \rho)$-piecewise continuous almost periodic functions.

3. Relations with Stepanov Almost Periodic Type Functions

As observed by S. I. Trofimchuk in [27] (p. 389), a piecewise continuous almost periodic function $f : I \to X$ is Stepanov almost periodic under additional conditions that are not restrictive and that the interest in the spaces of piecewise continuous almost periodic functions comes from the fact that these spaces have stronger topologies than the spaces of Stepanov almost periodic functions. The main purpose of the following result is to indicate that any piecewise continuous almost periodic function $f : I \to X$ in the sense of Definition 5 is immediately Stepanov-p-almost periodic for any finite exponent $p \ge 1$, as well as that a much more general result holds true (cf. also [27] (Lemma 58, p. 400)):

Theorem 1. *Suppose that $\rho = T \in L(E)$, $1 \le p < +\infty$, $F : I \times X \to E$ is pre-$(\mathcal{B}, T, (t_k))$-piecewise continuous almost periodic [pre-$(T, (t_k))$-piecewise continuous uniformly recurrent] and, for every $B \in \mathcal{B}$, $\|f\|_{\infty,B} \equiv \sup_{t \in I, x \in B} \|F(t;x)\| < +\infty$. Then, the function $F(\cdot;\cdot)$ is Stepanov-$(\mathcal{B}, p, T)$-almost periodic [Stepanov-$(\mathcal{B}, p, T)$-uniformly recurrent] for any finite exponent $p \ge 1$.*

Proof. Without loss of generality, we may assume that $I = \mathbb{R}$ and $T = c\mathrm{I}$ for some $c \in \mathbb{C}$; we will consider only pre-$(\mathcal{B}, c)$-piecewise continuous almost periodic functions. Fix a number $\epsilon > 0$ and a set $B \in \mathcal{B}$. Let a point $x \in \mathbb{R}$ be also fixed, and let the interval $[x, x+1]$ contain the possible first kind discontinuities at the points $\{t_m, \ldots, t_{m+k}\} \subseteq [x, x+1]$. Then, $k \le \lceil 1/\delta_0 \rceil$ and we have the existence of a sufficiently small real number $\epsilon_0 > 0$ such that

$$\epsilon_0^p + 2\Big(\frac{1}{\delta_0} + 1\Big)\big((1+|c|)\|f\|_{\infty,B}\big)^p \epsilon_0 \le \epsilon^p. \tag{1}$$

Let S be a relatively dense set in $\mathbb{R}$ such that, if $\tau \in S$ and $b \in B$, then $\|F(t+\tau;x) - cF(t;x)\| < \epsilon_0$ for all $t \in \mathbb{R}$ such that $|t - t_k| > \epsilon_0$, $k \in \mathbb{Z}$. The function $t \mapsto F(t+\tau;b) - cF(t;b)$, $t \in [x, x+1]$ is less than or equal to ϵ_0 if $t \in [x, t_m - \epsilon_0] \cup (t_m + \epsilon_0, t_{m+1} - \epsilon_0] \cup \ldots \cup (t_{m+k}, x+1]$; otherwise, we have $\|F(t+\tau;b) - cF(t;b)\| \le (1+|c|)\|f\|_{\infty,B}$. This implies

$$\int_x^{x+1} \|F(t+\tau;b) - cF(t;b)\|^p \, dt \le \epsilon_0^p + 2\lceil 1/\delta_0 \rceil \big((1+|c|)\|f\|_{\infty,B}\big)^p \epsilon_0, \quad b \in B.$$

Taking into account (1) and a simple computation, we conclude that

$$\int_x^{x+1} \left\| F(t+\tau;b) - cF(t;b) \right\|^p dt \le \epsilon^p, \quad b \in B.$$

This simply completes the proof of the theorem. □

Remark 5. *(i) The condition $\delta_0 > 0$ is crucial for the proof of Theorem 1 to work. In [27] (Appendix A.5), S.I. Trofimchuk has also analyzed the class of piecewise continuous almost periodic functions which satisfies $\inf_{k\in\mathbb{Z}}(t_{k+1} - t_k) = 0$ [$\inf_{k\in\mathbb{N}}(t_{k+1} - t_k) = 0$]. If we allow the last condition, then a piecewise continuous almost periodic function need not be Besicovitch bounded (Besicovitch almost periodic); cf. [7] for the notion, and [27] (p. 400) for a counterexample of this type.*

(ii) Albeit sometimes inevitable, the condition $\delta_0 > 0$ is a little bit redundant. For example, if $P(t)$ is a non-periodic trigonometric polynomial with real values, then we know that the function $f(t) := sign(P(t))$, $t \in \mathbb{R}$ is Stepanov-p-almost periodic for any exponent $p \ge 1$; see [7] (Example 2.2.3). Clearly, the zeros of $P(\cdot)$ are the points of discontinuity of the piecewise continuous function $f_0(\cdot)$ determined by $f(\cdot)$. However, since $P(\cdot)$ is not periodic, its zeros cannot be separated; to illustrate this, let us consider the polynomial $P(t) := \sin t + \sin(\sqrt{2}t)$, $t \in \mathbb{R}$. Any zero of $P(\cdot)$ is of the form $t_k = 2k\pi/(1+\sqrt{2})$ for some $k \in \mathbb{Z}$ or $t'_m = (2m+1)\pi/(1-\sqrt{2})$ for some $m \in \mathbb{Z}$. It can be simply proved that $t_k \ne t'_m$ for all $k,\ m \in \mathbb{Z}$ as well as that for each $\epsilon > 0$ there exist two strictly increasing sequences $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ of positive integers such that $|t_{a_k} - t'_{b_k}| < \epsilon$ for all $k \in \mathbb{N}$; see, e.g., [19] (Definition 2, Theorem 2, Remark 1). This implies the required.

(iii) In the formulation of Theorem 1, we have assumed immediately that $c \in S_1$. The proof also works in the case that $c \notin S_1$, but then the obtained conclusion in combination with [36] (Proposition 2.6) shows that $f \equiv 0$.

(iv) Keeping in mind Theorem 1 and [17] (Theorem 2.3), we can extend the statements of [32] (Lemma 3.4, Theorem 3.7, Corollary 3.8) for any Stepanov-p-almost periodic inhomogeneity $f(\cdot)$ with the exponent $p > 1$; see also [7] (Theorem 2.14.6) for case $p = 1$.

In the case that $(t_k)_{k\in\mathbb{Z}}$ is a Wexler sequence, L. Qi and R. Yuan have proved that a piecewise continuous function $f : \mathbb{R} \to X$ which satisfies the condition (QUC) is (t_k)-piecewise continuous almost periodic if and only if the function $f(\cdot)$ is Stepanov-p-almost periodic for every (some) exponent $p \ge 1$; see [29] (Theorem 3.2). Taken together with the statements of Proposition 2 and Theorem 1, the subsequent result provides a proper generalization of [29] (Theorem 3.2). Here, we do not necessarily assume that $(t_k)_{k\in\mathbb{Z}}$ is a Wexler sequence and follow the idea from the proof of [27] (Lemma 59, pp. 401–402), which is slightly incorrect since it is not clear how we can directly deduce the estimate $|x(t+z'_n) - x(t+z_n)| \ge \epsilon/4$ for all $t \in [0,\kappa]$ or a similar estimate $|x(t+z'_n) - x(t+z_n)| \ge \epsilon/4$ for all $t \in [-\kappa, 0]$; see [27] (l. 1, p. 402) and observe that, in the above-described situation, we can have $z_n = t_j + \epsilon$ and $z'_n = t_p - \epsilon$ for some integers $j,\ n,\ p$ so that the quasi-uniform continuity argument cannot be directly applied here.

Theorem 2. *Suppose that $\rho = T \in L(E)$, $1 \le p < +\infty$ and $F : I \times X \to E$ is a Stepanov-$(\mathcal{B}, p, T)$-almost periodic function. If the condition* (QUC) *holds, then $F(\cdot;\cdot)$ is pre-$(\mathcal{B}, T, (t_k))$-piecewise continuous almost periodic.*

Proof. For the sake of convenience, we will assume that $I = \mathbb{R}$, $T = c\mathrm{I}$ for some $c \in S_1$ and $X = \{0\}$. Let $\epsilon > 0$ be given; then there exists $\delta \in (0, \min\{\epsilon/2, \delta_0/4\})$ such that, if the points t_1 and t_2 belong to the same interval (t_i, t_{i+1}) of the continuity of function $f(\cdot)$ and $|t_1 - t_2| < \delta$, then $\|f(t_1) - f(t_2)\| < \epsilon/4$. Let $\eta_k \in (0, \epsilon\delta^{1/p}/4)$ for all $k \in \mathbb{N}$ and let $\lim_{k\to+\infty} \eta_k = 0$. We claim that there exists $k_0 \in \mathbb{N}$ such that, for every $\tau \in \mathbb{R}$ with $\int_t^{t+1} \|f(s+\tau) - cf(s)\|^p\, ds \le \eta_{k_0}^p$, $t \in \mathbb{R}$, we have $\|f(t+\tau) - cf(t)\| \le \epsilon$ for all $t \notin \bigcup_{l\in\mathbb{Z}}(t_l - \epsilon, t_l + \epsilon)$. If we assume the contrary, then for each $k \in \mathbb{N}$ there exist points

$s_k \notin \bigcup_{l\in\mathbb{Z}}(t_l - \epsilon, t_l + \epsilon)$ and $\tau_k \in \mathbb{R}$ such that $\int_t^{t+1} \|f(s+\tau_k) - cf(s)\|^p \, ds \leq \eta_k^p$, $t \in \mathbb{R}$ and $\|f(s_k+\tau_k) - cf(s_k)\| > \epsilon$. Using the continuity of function $f(\cdot)$ from the left side, for each $k \in \mathbb{N}$ there exist points $s_k' \notin \bigcup_{l\in\mathbb{Z}}(t_l - (3\epsilon/4), t_l + (3\epsilon/4))$ and $\tau_k \in \mathbb{R}$ such that $\int_t^{t+1} \|f(s+\tau_k) - cf(s)\|^p \, ds \leq \eta_k^p$, $t \in \mathbb{R}$, $\|f(s_k'+\tau_k) - cf(s_k')\| > 3\epsilon/4$ and $s_k' + \tau_k \notin \{t_l : l \in \mathbb{Z}\}$. Since $\delta < \epsilon/2$, it follows that, for every $k \in \mathbb{N}$, the interval $(s_k' - \delta, s_k' + \delta)$ belongs to the same interval (t_j, t_{j+1}) of continuity of function $f(\cdot)$, for some $j \in \mathbb{Z}$. On the other hand, at least one of the intervals $(s_k' + \tau_k, s_k' + \tau_k + \delta)$ and $(s_k' + \tau_k - \delta, s_k' + \tau_k)$ belongs to the same interval (t_p, t_{p+1}) of continuity of function $f(\cdot)$, for some $p \in \mathbb{Z}$. If the integer $k \in \mathbb{N}$ is fixed, then we may assume without loss of generality that the above holds for the interval $(s_k' + \tau_k, s_k' + \tau_k + \delta)$; since $|c| = 1$, this readily implies:

$$\begin{aligned} &\Big\| \big[f(s+s_k'+\tau_k) - cf(s+s_k')\big] - \big[f(s_k'+\tau_k) - cf(s_k')\big] \Big\| \\ &\quad \leq \|f(s+s_k'+\tau_k) - f(s_k'+\tau_k)\| + \|f(s+s_k') - f(s_k')\| \leq \epsilon/2, \quad \text{a.e. } s \in [0,\delta]. \end{aligned}$$

Hence, for every $k \in \mathbb{N}$, we have:

$$\|f(s+s_k'+\tau_k) - cf(s+s_k')\| \geq \epsilon/4, \quad \text{a.e. } s \in [0,\delta],$$

and

$$\begin{aligned} \eta_k^p &\geq \int_{t_k}^{t_k+1} \|f(s+s_k'+\tau_k) - cf(s+s_k')\|^p \, ds \\ &\geq \int_{t_k}^{t_k+\delta} \|f(s+s_k'+\tau_k) - cf(s+s_k')\|^p \, ds \geq (\epsilon/4)^p \delta, \end{aligned}$$

which is a contradiction. This simply completes the proof of the theorem. □

The argument contained in the proof of [29] (Theorem 3.8) can be applied even if $(t_k)_{k\in\mathbb{Z}}$ is not a Wexler sequence. Keeping in mind this fact as well as Proposition 2, Theorem 1, Theorem 2 and [9] (Theorem 6.2.21), we can extend [29] (Theorem 3.8) in the following way:

Theorem 3. *Suppose that $F : I \times X \to E$ is pre-$(\mathcal{B}, (t_k))$-piecewise continuous almost periodic, the condition* (QUC) *holds, and any set B of the collection $\mathcal{B}$ is a compact subset of X. Then, $F(\cdot;\cdot)$ is Bohr $\mathcal{B}$-almost periodic if and only if $F(\cdot;\cdot)$ is continuous.*

We proceed with some applications of Theorems 1 and 2; the first result improves the statement of [27] (Lemma 31, pp. 204–206):

Theorem 4. *Suppose that $F_i : I \times X \to E$ is a pre-$(\mathcal{B}, (t_k))$-piecewise continuous almost periodic function ($i = 1, 2$), and every set B of collection $\mathcal{B}$ is compact in X. If the condition* (QUC) *holds for the functions $F_1(\cdot;\cdot)$ and $F_2(\cdot;\cdot)$, then the functions $(F_1, F_2)(\cdot;\cdot)$ and $\alpha F_1 \pm \beta F_2$ are pre-$(\mathcal{B}, (t_k))$-piecewise continuous almost periodic and satisfy the condition* (QUC)*.*

Proof. Due to Proposition 2 and Theorem 1, we have that the functions $F_1(\cdot;\cdot)$ and $F_2(\cdot;\cdot)$ are Stepanov-$(\mathcal{B}, p)$-almost periodic. An application of [9] (Proposition 6.2.17) shows that the function $(F_1, F_2)(\cdot;\cdot)$ is Stepanov-$(\mathcal{B}, p)$-almost periodic so that the function $(F_1, F_2)(\cdot;\cdot)$ is pre-$(\mathcal{B}, (t_k))$-piecewise continuous almost periodic by Theorem 2. This clearly implies that the function $\alpha F_1 \pm \beta F_2$ is pre-$(\mathcal{B}, (t_k))$-piecewise continuous almost periodic, as well. The condition *(QUC)* clearly holds for both functions. □

Observe that we have not assumed above that $(t_k)_{k\in\mathbb{Z}}$ is a Wexler sequence; in particular, if $f_1(\cdot)$ and $f_2(\cdot)$ are (t_k)-piecewise continuous almost periodic functions and the requirements of Theorem 4 hold, then for each number $\epsilon > 0$ there exists a relatively dense set of their common ϵ-almost periods, with the meaning clear. Keeping in mind Propositions 3 and 5 and Theorem 4, we can simply prove an analogue of [24] (Theorem 2.23) for (pre-)$(\mathcal{B},(t_k))$-piecewise continuous almost periodic functions.

Further on, as a simple application of Theorem 4, we have the following:

Proposition 6. *Suppose that $F_1 : I \times X \to \mathbb{C}$ and $F_2 : I \times X \to E$ are pre-$(\mathcal{B},(t_k))$-piecewise continuous almost periodic functions, and every set B of collection $\mathcal{B}$ is compact in X. If the condition* (QUC) *holds for the functions $F_1(\cdot;\cdot)$ and $F_2(\cdot;\cdot)$, then the function $(F_1 \cdot F_2)(\cdot;\cdot)$ is pre-$(\mathcal{B},(t_k))$-piecewise continuous almost periodic; moreover, the function $(F_1^{-1} \cdot F_2)(\cdot;\cdot)$ is pre-$(\mathcal{B},(t_k))$-piecewise continuous almost periodic, provided that for each set $B \in \mathcal{B}$ we have* $\inf_{t\in I;x\in B} |F_1(t;x)| > 0$.

The next result follows from the argument contained in the proofs of Theorems 1 and 2 and the corresponding result for the Stepanov-p-almost periodic functions:

Proposition 7. *Suppose that $F : I \to E$ is a pre-piecewise continuous almost periodic function, and the condition* (QUC) *holds. Let $\epsilon > 0$ be fixed. Then, for each number $\delta \in \mathbb{R} \setminus \{0\}$ there exists a relatively dense set S of integers such that the set $\delta \cdot S$ consists solely of the ϵ-almost periods of $F(\cdot)$.*

The Favard type theorems for piecewise continuous almost periodic functions have been considered in the research article [38] by L. Wang and M. Yu. Let us only mention that the authors have clarified, in [38] (Theorem 2.3), a sufficient condition for the primitive function of a scalar-valued piecewise continuous almost periodic function to be almost periodic; observe, however, that the established result is very unsatisfactory from the application point of view. On the other hand, using Proposition 2, Theorem 1 and the Bohl–Bohr–Amerio theorem (see, e.g., [13] (p. 80)), we can clarify the following simple result on the integration of piecewise continuous almost periodic type functions:

Theorem 5. *Suppose that $F : I \to E$ is a pre-piecewise continuous almost periodic function, and E is uniformly convex. If the function $t \mapsto F^{[1]}(t) \equiv \int_0^t F(s)\,ds$, $t \in I$ is bounded, then $F^{[1]}(\cdot)$ is almost periodic.*

The statement of [24] (Proposition 2.2) admits a satisfactory reformulation in the new framework provided that $\rho = T \in L(E)$ is a linear isomorphism; in order to see this, we can combine Proposition 2, Theorems 1 and 2 and [30] (Theorem 1(i)). Before proceeding to the next subsection, we observe that the statements of [36] (Proposition 2.9, Corollary 2.10, Proposition 2.11) admit satisfactory reformulations in the new context as well. For example, we can combine Proposition 2, Theorems 1 and 2 and [30] (Proposition 2) in order to see that the following generalization of [36] (Proposition 2.9) holds true:

Proposition 8. *Suppose that $\rho = T \in L(E)$ is a linear isomorphism and $F : I \times X \to E$ is (pre-)$(\mathcal{B},T,(t_k))$-piecewise continuous almost periodic [(pre-)$(\mathcal{B},T,(t_k))$-piecewise continuous uniformly recurrent]. Then, for each $l \in \mathbb{N}$ the function $F(\cdot;\cdot)$ is (pre-)$(\mathcal{B},T^l,(t_k))$-piecewise continuous almost periodic [(pre-)$(\mathcal{B},T^l,(t_k))$-piecewise continuous uniformly recurrent].*

3.1. Composition Principles for $(\mathcal{B},(t_k))$-Piecewise Continuous Almost Periodic Type Functions

In this subsection, we will prove two composition theorems for $(\mathcal{B},(t_k))$-piecewise continuous almost periodic type functions. In order to achieve this aim, we employ the relations between the $(\mathcal{B},(t_k))$-piecewise continuous almost periodic type functions and the Stepanov almost periodic type functions.

The first result reads as follows:

Theorem 6. *Suppose that $(Z,\|\cdot\|_Z)$ is a complex Banach space, $F : \mathbb{R} \times X \to E$ is a pre-$(\mathcal{B},(t_k))$-piecewise continuous almost periodic function, and $G : \mathbb{R} \times E \to Z$ is a pre-$(\mathcal{B}',(t_k))$-piecewise continuous almost periodic function, where $\mathcal{B}$ is a collection of all compact subsets of X, and $\mathcal{B}'$ is* *a collection of all compact subsets of E. If the condition* (QUC) *holds for the functions $F(\cdot;\cdot)$ and $G(\cdot;\cdot)$, and there exists $L > 0$ such that*

$$\|G(t;x) - G(t;y)\|_Z \leq L\|x - y\|, \quad t \in \mathbb{R},\ x,\ y \in E, \tag{2}$$

then the function $W : \mathbb{R} \times X \to Z$, given by $W(t;x) := G(t;F(t;x))$, $t \in \mathbb{R}$, $x \in X$, is a pre-$(\mathcal{B},(t_k))$-piecewise continuous almost periodic function, and the condition (QUC) *holds for $W(\cdot;\cdot)$.*

Proof. Let $\epsilon > 0$ and $B \in \mathcal{B}$ be fixed. Then, Proposition 2 implies that the set $B' := F(\mathbb{R} \times B)$ is relatively compact in E. Let $\delta_1 > 0$ be chosen from the condition (QUC) for the function $F(\cdot;\cdot)$, the number $\epsilon/2L$ and the set B; further on, let $\delta_2 > 0$ be chosen from the condition (QUC) for the function $G(\cdot;\cdot)$, the number $\epsilon/2$ and the set B'. Define $\delta := \min\{\delta_1, \delta_2\}$. Let $t',\ t'' \in (t_k, t_{k+1})$ for some $k \in \mathbb{Z}$, and let $|t' - t''| < \delta$. Then

$$\begin{aligned}
&\|G(t';F(t';x)) - G(t'';F(t'';x))\|_Z \\
&\leq \|G(t';F(t';x)) - G(t';F(t'';x))\|_Z + \|G(t';F(t'';x)) - G(t'';F(t'';x))\|_Z \\
&\leq L\|F(t';x) - F(t'';x)\| + \sup_{y \in B'}\|G(t';y) - G(t'';y)\|_Z \\
&\leq L(\epsilon/2L) + (\epsilon/2) = \epsilon.
\end{aligned}$$

Therefore, the condition (QUC) holds for $W(\cdot;\cdot)$; using a similar argument, we can show that for each $x \in X$ the mapping $t \mapsto W(t;x)$, $t \in \mathbb{R}$ is continuous from the left side, with the possible first kind of discontinuities at the points of the sequence $(t_k)_{k \in \mathbb{Z}}$. Consider now the functions $F_B : \mathbb{R} \to l_\infty(B : E)$ and $G_B : \mathbb{R} \to l_\infty(B' : Z)$ defined through $[F_B(t)](x) := F(t;x)$, $t \in \mathbb{R}$, $x \in B$ and $[G_{B'}(t)](y) := G(t;y)$, $t \in \mathbb{R}$, $y \in B'$, where $l_\infty(B : E)$ denotes the Banach space of all essentially bounded functions from B into E, equipped with the sup-norm. Due to Proposition 2, these mappings are well-defined. Using a simple argument involving the condition (QUC) for the functions $F(\cdot;\cdot)$ and $G(\cdot;\cdot)$, it follows that the functions $F_B(\cdot)$ and $G_B(\cdot)$ are pre-(t_k)-piecewise continuous almost periodic, and the condition (QUC) holds for them. Applying Theorems 1 and 2 and [9] (Proposition 6.2.17), we conclude that there exists a common set D of (t_k)-almost periods for these functions, with the meaning clear. If $\tau \in D$ and $|t - t_k| > \epsilon$ for some $k \in \mathbb{Z}$, then we have

$$\begin{aligned}
&\|G(t+\tau;F(t+\tau;x)) - G(t;F(t;x))\|_Z \\
&\leq \|G(t+\tau;F(t+\tau;x)) - G(t+\tau;F(t;x))\|_Z + \|G(t+\tau;F(t;x)) - G(t;F(t;x))\|_Z \\
&\leq L\|F(t+\tau;x) - F(t;x)\| + \sup_{y \in B'}\|G(t+\tau;y) - G(t;y)\|_Z.
\end{aligned}$$

This simply completes the proof of the theorem. □

The second structural result simply follows from Theorem 6 and the argument contained in the proof of [9] (Theorem 6.1.50) (cf. also [24] (Theorem 2.17) and [9] (Subsection 6.1.5) for similar results).

Theorem 7. *Suppose that $(Z, \|\cdot\|_Z)$ is a complex Banach space, $F_0 : \mathbb{R} \times X \to E$ is a pre-$(\mathcal{B},(t_k))$-piecewise continuous almost periodic function, $G_1 : \mathbb{R} \times E \to Z$ is a pre-$(\mathcal{B}',(t_k))$-piecewise continuous almost periodic function, where $\mathcal{B}$ is a collection of all compact subsets of X, and $\mathcal{B}'$ is a collection of all compact subsets of E. Suppose, further, that the condition* (QUC) *holds for the functions $F_0(\cdot;\cdot)$ and $G_1(\cdot;\cdot)$, there exists $L > 0$ such that* (2) *holds with the function $G(\cdot;\cdot)$ replaced therein with the function $G_1(\cdot;\cdot)$, the function $Q_0 : [0,\infty) \times X \to E$ [$Q_1 : [0,\infty) \times E \to Z$] satisfies that for each set $B \in \mathcal{B}$ [$B' \in \mathcal{B}'$] we have $\lim_{t \to +\infty} \sup_{x \in B} \|Q_0(t;x)\| = 0$ [$\lim_{t \to +\infty} \sup_{y \in B'} \|Q_1(t;y)\|_Z = 0$]. Then, the function $W : \mathbb{R} \times X \to Z$, given by $W(t;x) := [G_1 + Q_1](t;[F_0 + Q_0](t;x))$, $t \in \mathbb{R}$, $x \in X$, is strongly asymptotically pre-$(\mathcal{B},(t_k))$-piecewise continuous almost periodic in the sense that there exists a pre-$(\mathcal{B},(t_k))$-piecewise continuous almost periodic function $W_2 : \mathbb{R} \times X \to E$ obeying the condition* (QUC), *and a function Q_2 :*

$[0,\infty) \times X \to E$ *satisfying that for each set* $B \in \mathcal{B}$ *we have* $\lim_{t\to+\infty} \sup_{x\in B} \|Q_2(t;x)\| = 0$ *and* $W(t;x) = W_2(t;x) + Q_2(t;x)$ *for all* $t \geq 0$ *and* $x \in X$.

Concerning the composition principles for piecewise pseudo almost periodic type functions, we refer the reader to [39] (Section 3) for some results established by J. Liu and C. Zhang.

Before going any further, we would like to present the following simple application of Theorem 6 and some useful observations concerning this result:

Example 3. *Let* $\mathcal{X}$ *denote the set of all pre-*(t_k)*-piecewise continuous almost periodic functions* $f : \mathbb{R} \to X$ *satisfying the condition (QUC). Then, Proposition 3 and Theorem 4 together imply that* $(\mathcal{X}, \|\cdot\|_\infty)$ *is a complex Banach space. Consider now Theorem 6 with* $E = Z = X$ *and* $\mathcal{B} = \mathcal{B}'$ *being the collection of all compact subsets of* X. *Consider, further, the following simple equation*

$$u(t) = u_0 + G(t; u(t)), \quad t \in \mathbb{R}, \tag{3}$$

where $u_0 : \mathbb{R} \to X$ *is pre-*(t_k)*-piecewise continuous almost periodic and satisfies the condition (QUC) as well as* $G : \mathbb{R} \times X \to X$ *is pre-*$(\mathcal{B}, (t_k))$*-piecewise continuous almost periodic and satisfies the condition (QUC). Suppose that there exists* $L \in (0,1)$ *such that* (2) *holds. Then, the mapping* $u \ni \mathcal{X} \mapsto u_0 + G(\cdot; u(\cdot)) \in \mathcal{X}$ *is well-defined due to Proposition 3, Theorems 4 and 6. Moreover, this mapping is a contraction; therefore, there exists a unique function* $u \in \mathcal{X}$ *satisfying* (3). *For example, we can take* $X = \mathbb{C}$ *and*

$$G(t; x) = f_1(t)g_1(x) + \ldots . + f_k(t)g_k(x), \quad t \in \mathbb{R},\ x \in \mathbb{C} \ \ (k \in \mathbb{N}),$$

where the functions $F_j(\cdot)$ *are pre-*(t_k)*-piecewise continuous almost periodic and satisfy the condition (QUC), the functions* $g_j(\cdot)$ *are bounded, Lipschitz continuous with constants* $L_j > 0$ *and*

$$L_1\|f_1\|_\infty + \ldots + L_k\|f_k\|_\infty < 1.$$

On the other hand, it is very difficult to apply Theorem 6 to the abstract semilinear integro-differential equations of the form

$$u(t) = u_0 + \int_{-\infty}^{t} R(t-s)G(s; u(s))\, ds, \quad t \in \mathbb{R}, \tag{4}$$

if the operator family $(R(t))_{t>0} \subseteq L(X)$ *satisfies* $\int_0^{+\infty} \|R(t)\|\, dt < +\infty$ *and condition that the mapping* $t \mapsto R(t)x$, $t > 0$ *is (piecewise-)continuous for every element* $x \in X$. *Then, it is expected that the mapping* $t \mapsto \int_{-\infty}^{t} R(t-s)G(s; u(s))\, ds$, $t \in \mathbb{R}$ *is Bohr almost periodic in the usual sense, so that we can always use a more general assumption that* $G : \mathbb{R} \times X \to X$ *is Stepanov-p-*$\mathcal{B}$*-almost periodic for some* $p \geq 1$ *and apply the composition theorems for Stepanov almost periodic type functions [7,9] combined with some result of type [7] (Proposition 2.6.11); cf. also Remark 5(iv). We will not discuss here the well-posedness of problem* (4) *in the case that the mapping* $t \mapsto R(t)x$, $t > 0$ *is only Lebesgue measurable* ($x \in X$) *and* $\int_0^{+\infty} \|R(t)\|\, dt < +\infty$.

4. Almost Periodic Type Solutions of Abstract Impulsive Differential Inclusions of Integer Order

The main aim of this section is to analyze the almost periodic type solutions to the abstract impulsive differential inclusions of integer order. Of concern is the following abstract impulsive higher-order Cauchy inclusion

$$(ACP)_{n;1} : \begin{cases} u^{(n)}(t) \in \mathcal{A}u(t) + f(t), \quad t \in [0,T] \setminus \{t_1, \ldots, t_l\}, \\ (\Delta u^{(j)})(t_k) = u^{(j)}(t_k+) - u^{(j)}(t_k-) = Cy_j^k, \quad k \in \mathbb{N}_l,\ j \in \mathbb{N}_{n-1}^0, \\ u^{(j)}(0) = Cu_j, \quad j \in \mathbb{N}_{n-1}^0. \end{cases}$$

We refer the reader to [25] for the notion of a (pre-)solution of $(ACP)_{n;1}$ on $[0,T]$ and $[0,\infty)$. We will use the following result from [25]; it is worth noting that, in the second part, we do not need the separation condition $\delta_0 > 0$ on the sequence (t_k):

Lemma 1. *(i) Suppose that $\mathcal{A}$ is a closed subgenerator of a local (g_n, C)-regularized resolvent family $(R(t))_{t\in[0,\tau)}$, where $\tau > T$ and $n \in \mathbb{N}$. Suppose that the functions $C^{-1}f(\cdot)$ and $f_{\mathcal{A}}(\cdot)$ are continuous on the set $t \in [0,T] \setminus \{t_1, \ldots, t_l\}$, $f_{\mathcal{A}}(t) \in \mathcal{A}C^{-1}f(t)$ for all $t \in [0,T] \setminus \{t_1, \ldots, t_l\}$, as well as the right limits and the left limits of the functions $C^{-1}f(\cdot)$ and $f_{\mathcal{A}}(\cdot)$ exist at any point of the set $\{t_1, \ldots, t_l\}$. Define*

$$u(t) := R(t)u_0 + \sum_{j=1}^{n-1} \int_0^t g_j(t-s)R(s)u_j\,ds + \int_0^t \int_0^{t-s} g_{n-1}(t-s-r)R(r)\big(C^{-1}f\big)(s)\,dr\,ds + \omega(t), \quad t \in [0,T], \tag{5}$$

where

$$\omega(t) := \begin{cases} 0, \quad t \in [0,t_1], \\ \sum_{p=1}^{k} R\big(t-t_p\big)y_0^p + \sum_{p=1}^{k}\sum_{j=1}^{n-1}\int_0^{t-t_p} g_j(t-t_p-s)R(s)y_j^p\,ds, \\ \text{if } t \in \big(t_k, t_{k+1}\big] \text{ for some } k \in \mathbb{N}_{l-1}^0. \end{cases} \tag{6}$$

Then, the function $u(t)$ is a unique solution of the problem $(ACP)_{n;1}$, provided that $u_0, \ldots, u_l \in D(\mathcal{A})$ and $y_k^j \in D(\mathcal{A})$ for all $k \in \mathbb{N}_l$ and $j \in \mathbb{N}_{n-1}^0$.

(ii) Suppose that $\mathcal{A}$ is a closed subgenerator of a global (g_n, C)-regularized resolvent family $(R(t))_{t\geq 0}$, where $n \in \mathbb{N}$. Suppose, further, that $0 < t_1 < \ldots < t_l < \ldots < +\infty$, the sequence $(t_l)_l$ has no accumulation point, the functions $C^{-1}f(\cdot)$ and $f_{\mathcal{A}}(\cdot)$ are continuous on the set $[0,T] \setminus \{t_1, \ldots, t_l, \ldots\}$, $f_{\mathcal{A}}(t) \in \mathcal{A}C^{-1}f(t)$ for all $t \in [0,T] \setminus \{t_1, \ldots, t_l, \ldots\}$, as well as the right limits and the left limits of the functions $C^{-1}f(\cdot)$ and $f_{\mathcal{A}}(\cdot)$ exist at any point of the set $\{t_1, \ldots, t_l, \ldots\}$. Define the functions $u(t)$ and $\omega(t)$ for $t \in [0,T]$ by (5) and (6), respectively. Then, the function $u(t)$ is a unique solution of the problem $(ACP)_{n;1}$ for $t \in [0,T] \setminus \{t_1, \ldots, t_l, \ldots\}$, provided that $u_0, \ldots, u_l, \ldots \in D(\mathcal{A})$ and $y_j^k \in D(\mathcal{A})$ for all $k \in \mathbb{N}$ and $j \in \mathbb{N}_{n-1}^0$.

In this paper, we will mainly consider the case in which $n = 1$. We start with the observation that it is not so simple to analyze the existence and uniqueness of (ω, c)-periodic solutions of the abstract impulsive Volterra integro-differential inclusions on bounded domains unless some very restrictive assumptions are satisfied. Concerning this topic, which has recently been analyzed by some authors, we will only provide the following simple application of Lemma 1 with $n = 1$. Let $\omega = t_{k+1} - t_k > 0$ for all integers $k \in \mathbb{N}_n$, let $f(t) \equiv 0$, and let the (local) C-regularized semigroup with subgenerator $\mathcal{A}$ satisfy $R(t+\omega) = cR(t)$ for all $t \in [0, T-\omega]$. If $u_0 \in D(\mathcal{A})$, then the solution $u(t)$ of problem $(ACP)_{1;1}$ satisfies $u(T) = cu(0)$ if and only if

$$cu_0 = c^{l+1}u_0 + \Big[c^l y_0^1 + \ldots + c y_0^l\Big];$$

if $c = 1$, this simply means that $y_0^1 + \ldots + y_0^l = 0$. We divide the further investigations into four subsections:

4.1. *Asymptotically Almost Periodic Type Solutions of $(ACP)_{1;1}$*

Suppose that $\mathcal{A}$ is the integral generator of a global exponentially decaying C-regularized semigroup $(T(t))_{t\geq 0}$ on X; therefore, there exist finite real constants $\omega < 0$ and $M \geq 1$ such that $\|T(t)\| \leq Me^{\omega t}$, $t \geq 0$. Suppose, further, that the functions $C^{-1}f(\cdot)$ and $f_{\mathcal{A}}(\cdot)$ satisfy all requirements from Lemma 1(ii) with $n = 1$. For simplicity, we set $y_k \equiv y_0^k$.

1. In this part, we will only assume that the sequence (t_k) has no accumulation point; the separation condition $\delta_0 > 0$ is complete regardless. If $\sum_{k\geq 1} e^{-\omega t_k}\|y_k\| < +\infty$, then the function $\omega(\cdot)$ defined in the proof of Lemma 1(i) belongs to the space $PC_\omega([0,\infty) : X)$ since we have (cf. also [7] (Remark 2.6.14(i))):

$$\|\omega(t)\| \leq Me^{\omega t}\sum_{k=1}^{\infty} e^{-\omega t_k}\|y_k\| \to 0 \text{ as } t \to +\infty; \tag{7}$$

we will not further discuss here the sufficient conditions ensuring that the function $\omega(\cdot)$ belongs to some space of the weighted ergodic components in $\mathbb{R}$ (cf. [9] (Section 6.4) for more details about these spaces in the multi-dimensional setting). Concerning the function $C^{-1}f(\cdot)$, we can assume that there exists a bounded Stepanov-p-almost periodic function $g : \mathbb{R} \to X$ and a function $q \in PC_{\omega'}([0,\infty) : X)$, for some $\omega' \in \mathbb{R}$ and $p \in [1,\infty)$, such that $C^{-1}f(t) = g(t) + q(t)$ for all $t \geq 0$; see, e.g., the proofs of [7] (Propositions 2.6.11 and 2.6.13). A similar conclusion can be given in the case that there exist a bounded Stepanov-p-almost periodic function $g : \mathbb{R} \to X$ and a function $e^{-\omega'\cdot}q(\cdot) \in PAP_T^0([0,\infty) : X)$, for some $\omega' \in \mathbb{R}$ and $p \in [1,\infty)$, such that $C^{-1}f(t) = g(t) + q(t)$ for all $t \geq 0$; see [39] (p. 3 and Definition 2.7) for the notion, the argument contained in the proof of [7] (Lemma 2.12.3) and the decomposition used in the proof of [7] (Proposition 2.6.13). We can also use Stepanov-(p,c)-almost periodic functions here.
2. Suppose now, in place of condition $\sum_{k\geq 1} e^{-\omega t_k}\|y_k\| < +\infty$, that $(y_k)_{k\in\mathbb{N}}$ is an almost periodic sequence as well as that the family of sequences $(t_k^j)_{k\in\mathbb{N}}$, $j \in \mathbb{N}$ is equipotentially almost periodic. Then, the argument contained in the proofs of [32] (Lemmas 3.4 and 3.6, Theorem 3.7) shows that the function $\omega(\cdot)$ is piecewise continuous almost periodic.
3. In this issue, we are seeking for the uniformly recurrent analogues of the conclusions established in the previous issue. Suppose that (τ_m) is a strictly increasing sequence of positive real numbers such that $\lim_{m\to+\infty}\tau_m = +\infty$ and (q_l) is a strictly increasing sequence of positive integers. Let, for every $\epsilon > 0$ and $m \in \mathbb{N}$, there exist integers $s_1 \in \mathbb{N}$ and $s_2 \in \mathbb{N}$ such that, for every $l \geq s_1$ and $j \in \mathbb{N}$, we have $\|y_{j+q_l} - y_j\| + |t_{j+q_{s_2}} - t_j - \tau_m| \leq \epsilon$; see also [27] (Lemma 35). If the sequence $(y_k)_{k\in\mathbb{N}}$ is bounded [the sequences $(y_k)_{k\in\mathbb{N}}$ and $(Ay_k)_{k\in\mathbb{N}}$ are bounded], then Proposition 3 in combination with the argument contained in the proofs of [32] (Lemma 3.6, Theorem 3.7) shows that the function $\omega(\cdot)$ is (pre-)piecewise continuous uniformly recurrent; see also the statement (S) in the proof of Theorem 8 below.

For a concrete example, we need to recall that A. Haraux and P. Souplet have proved, in [18] (Theorem 1.1), that the function

$$f(t) := \sum_{m=1}^{\infty}\frac{1}{m}\sin^2\Big(\frac{t\pi}{2^m}\Big), \quad t \in \mathbb{R},$$

satisfies $\lim_{k\to+\infty} f(t+2^k) = f(t)$, uniformly in $t \in \mathbb{R}$. Take $y_j := f(j)$, $t_j := j$ for all $j \in \mathbb{N}$ and $\tau_m := 2^m$ for all $m \in \mathbb{N}$. Then, the above requirements are satisfied.

4.2. Asymptotically Weyl Almost Periodic Type Solutions of $(ACP)_{1;1}$

Suppose that $1 \leq p < \infty$ and $f \in L^p_{loc}(I : E)$. Let us recall that the function $f(\cdot)$ is called:

(i) equi-Weyl-p-almost periodic if and only if for each $\epsilon > 0$ we can find two real numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\sup_{x \in I} \left[\frac{1}{l} \int_x^{x+l} \left\| f(t+\tau) - f(t) \right\|^p dt \right]^{1/p} \leq \epsilon.$$

(ii) Weyl-p-almost periodic if and only if for each $\epsilon > 0$ we can find a real number $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\lim_{l \to \infty} \sup_{x \in I} \left[\frac{1}{l} \int_x^{x+l} \left\| f(t+\tau) - f(t) \right\|^p dt \right]^{1/p} \leq \epsilon.$$

In order to study the existence and uniqueness of asymptotically (equi-)Weyl-p-almost periodic solutions of the problem $(ACP)_{1;1}$, we will use the following conditions:

(ew-M1) For every $\epsilon > 0$, there exist $s \in \mathbb{N}$ and $L > 0$ such that every interval $I' \subseteq [0, \infty)$ of length L contains a point $\tau \in I'$ which satisfies that there exists an integer $q_\tau \in \mathbb{N}$ such that $|t_{i+q_\tau} - t_i - \tau| < \epsilon$ for all $i \in \mathbb{N}$ and

$$\sup_{|J|=s} \left[\frac{1}{s} \sum_{j \in J} \left\| y_{j+q_\tau} - y_j \right\|^p \right]^{1/p} < \epsilon, \tag{8}$$

where the supremum is taken over all segments $J \subseteq \mathbb{N}$ of length s.

(w-M1) For every $\epsilon > 0$, there exists $L > 0$ such that every interval $I' \subseteq [0, \infty)$ of length L contains a point $\tau \in I'$ which satisfies that there exist an integer $q_\tau \in \mathbb{N}$ and an integer $s_\tau \in \mathbb{N}$ such that $|t_{i+q_\tau} - t_i - \tau| < \epsilon$ for all integers $i \in \mathbb{N}$, and (8) holds for all integers $s \geq s_\tau$.

Condition (ew-M1), respectively, condition (w-M1), implies that the family of sequences $(t_k^j)_{k \in \mathbb{N}}$, $j \in \mathbb{N}$ is equipotentially almost periodic as well as that the sequence $(x_k)_{k \in \mathbb{N}}$ is equi-Weyl-p-almost periodic, respectively, Weyl-p-almost periodic, in the following sense:

(e-M1) For every $\epsilon > 0$, there exist $s \in \mathbb{N}$ and $L > 0$ such that every interval $I' \subseteq [0, \infty)$ of length L contains a point $\tau \in I' \cap \mathbb{N}$ which satisfies that (8) holds with the number q_τ replaced therein with the number τ.

(M1) For every $\epsilon > 0$, there exists $L > 0$ such that every interval $I' \subseteq [0, \infty)$ of length L contains a point $\tau \in I' \cap \mathbb{N}$ which satisfies that there exists an integer $s_\tau \in \mathbb{N}$ such that (8) holds for all integers $s \geq s_\tau$, with the number q_τ replaced therein with the number τ.

In the existing literature, the class of equi-Weyl-1-almost periodic sequences has been commonly used so far (see, e.g, the research articles [40] by V. Bergelson et al., [41] by T. Downarowicz, A. Iwanik and [42] by A. Iwanik). The class of Weyl-p-almost periodic sequences seems to be not considered elsewhere, even in the scalar-valued case. Before going further, let us mention that it is clear that condition (ew-M1) implies (w-M1) as well as that condition (e-M1) implies (M1).

Concerning the existence and uniqueness of asymptotically Weyl almost periodic solutions of problem $(ACP)_{1;1}$, we will state and prove the following result:

Theorem 8. *Suppose that* (ew-M1), *respectively,* (w-M1) *holds the functions* $(C^{-1}f)(\cdot)$ *and* $f_A(\cdot)$ *satisfy all requirements of* Lemma 1 *with* $n = 1$, $u_0 \in D(A)$ *and* $y_k \equiv y_k^0 \in D(A)$ *for all* $k \in \mathbb{N}$. *Suppose, further, that* (y_k) *and* (Ay_k) *are bounded sequences,* $q \in PC_0([0, \infty) : X)$, *the function* $g : \mathbb{R} \to X$ *is (equi-)Weyl-p-almost periodic and bounded as well as* $(C^{-1}f)(t) = g(t) + q(t)$ *for all* $t \geq 0$. *Then, there exist a bounded continuous (equi-)Weyl-p-almost periodic* *function $G_1 : \mathbb{R} \to X$, a bounded piecewise continuous (equi-)Weyl-p-almost periodic function $G_2 : [0, \infty) \to X$ and a function $Q_1 \in C_0([0, \infty) : X)$ such that the unique solution $u(t)$ of problem $(ACP)_{1;1}$ satisfies $u(t) = G_1(t) + G_2(t) + Q_1(t)$ for all $t \geq 0$.*

Proof. Keeping in mind [7] (Theorem 2.11.4) and the proof of [7] (Proposition 2.6.13), it readily follows that there exist a bounded (equi-)Weyl-p-almost periodic function $G_1 : \mathbb{R} \to X$ and a function $Q_1 \in C_0([0, \infty) : X)$ such that $T(t)u_0 + \int_0^t T(t-s)(C^{-1}f)(s)\,ds = G_1(t) + Q_1(t)$ for all $t \geq 0$; cf. the formulation of Lemma 1 with $R(t) \equiv T(t)$. It remains to be proved that the function $\omega(\cdot)$ from the formulation of Lemma 1(i) is bounded, piecewise continuous and (equi-)Weyl-p-almost periodic. Keeping in mind the argument contained in the proof of [32] (Theorem 3.7), the assumption that (y_k) is a bounded sequence and the fact that the statement of [7] (Proposition 2.3.5) continues to hold for the sequences of piecewise continuous bounded functions, it suffices to show that the function $\omega_1(\cdot)$, defined by $\omega_1(t) := 0$ if $0 \leq t \leq t_1$ and $\omega_1(t) := T(t - t_k)y_k$, if $t_k < t \leq t_{k+1}$ for some integer $k \in \mathbb{N}$, is (equi-)Weyl-p-almost periodic. The consideration is similar for both classes of functions, and we may assume, without loss of generality, that condition (ew-M1) holds. Since (Ay_k) is a bounded sequence, we have $T(t)y_k - T(s)y_k = [T(t)y_k - Cy_k] - [T(s)y_k - Cy_k] = \int_s^t T(r)Ay_k\,dr$ for all $t,\ s \geq 0$, and therefore, the following statement holds:

(S) For every $\epsilon > 0$, there exists $\delta \in (0, \epsilon)$ such that, if $t,\ s \geq 0$ and $|t - s| < \delta$, then $\|T(t)y_k - T(s)y_k\| \leq \epsilon/3$ for all $k \in \mathbb{N}$.

Let $\epsilon > 0$ be given. Then, we know that there exist $s \in \mathbb{N}$, as large as we want, and $L > 0$ such that every interval $I' \subseteq [0, \infty)$ of length L contains a point $\tau \in I'$ which satisfies that there exists an integer $q_\tau \in \mathbb{N}$ such that $|t_{i+q_\tau} - t_i - \tau| < \delta$ for all $i \in \mathbb{N}$ and (8) holds. Suppose now that $t > 0$, $|t - t_i| > \epsilon$, $|t - t_{i+1}| > \epsilon$ and $t_i < t < t_{i+1}$ for some integer $i \in \mathbb{N}$. Then, the argument contained in the proof of [32] (Lemma 3.6), with $\epsilon' = \beta = \epsilon$, shows that $t_{i+q_\tau} < t + \tau < t_{i+q_\tau+1}$. Therefore, since (S) holds and $|t_{i+q_\tau} - t_i - \tau| < \delta$, we have:

$$\begin{aligned}
\|\omega_1(t+\tau) - \omega_1(t)\| &= \|T(t+\tau - t_{i+q_\tau})y_{i+q_\tau} - T(t - t_i)y_i\| \\
&\leq \|T(t+\tau - t_{i+q_\tau})y_{i+q_\tau} - T(t - t_i)y_{i+q_\tau}\| + \|T(t - t_i)(y_{i+q_\tau} - y_i)\| \\
&\leq (\epsilon/3) + M\|y_{i+q_\tau} - y_i\|.
\end{aligned}$$

Suppose now that $x \geq 0$, $[x, x+l] \subseteq [t_r, t_{r+m}]$, $x \leq t_{r+1}$ and $x + l \geq t_{r+m-1}$ for some integers $r,\ m \in \mathbb{N}_0$. Since the separation condition $\delta_0 > 0$ holds, we have $l \geq (m-2)\delta_0$, and therefore $m \leq \lfloor l/\delta_0 \rfloor + 2$. Hence, there exist absolute real constants $M_1 > 0$ and $M_2 > 0$ such that

$$\begin{aligned}
&\left[\frac{1}{l}\int_x^{x+l} \|\omega_1(t+\tau) - \omega_1(t)\|^p\,dt\right]^{1/p} \\
&\leq \Bigg[\frac{1}{l}\Bigg(\int_{t_r}^{t_r+\epsilon} \|\omega_1(t+\tau) - \omega_1(t)\|^p\,dt + \int_{t_r+\epsilon}^{t_{r+1}-\epsilon} \|\omega_1(t+\tau) - \omega_1(t)\|^p\,dt \\
&+ \int_{t_{r+1}-\epsilon}^{t_{r+1}+\epsilon} \|\omega_1(t+\tau) - \omega_1(t)\|^p\,dt + \ldots\Bigg)\Bigg]^{1/p} \\
&\leq M_1\left[\frac{1}{l}\left(\epsilon + \left(\epsilon^p + \|y_{r+q_\tau} - y_r\|^p\right) + \epsilon + \ldots\right)\right]^{1/p} \\
&\leq M_1\left[\frac{1}{l}\left((m+1)\epsilon + (m+1)\epsilon^p + \sum_{w=r}^{r+m-1}\|y_{w+q_\tau} - y_w\|^p\right)\right]^{1/p} \\
&\leq M_1\left[\frac{1}{l}\left(\left(\lfloor l/\delta_0 \rfloor + 3\right)\epsilon + \left(\lfloor l/\delta_0 \rfloor + 3\right)\epsilon^p + \sum_{w=r}^{r+m-1}\|y_{w+q_\tau} - y_w\|^p\right)\right]^{1/p}
\end{aligned}$$

$$\leq M_2\left(\epsilon^{1/p}+\epsilon\right)+M_2\left[\frac{1}{l}\sum_{w=r}^{r+m-1}\left\|y_{w+q_\tau}-y_w\right\|^p\right]^{1/p}$$
$$\leq M_2\left(\epsilon^{1/p}+\epsilon\right)+M_2\left[\frac{1}{l}\sum_{w=r}^{r+\lfloor l/\delta_0\rfloor+1}\left\|y_{w+q_\tau}-y_w\right\|^p\right]^{1/p}.$$

Due to the assumption (8), the above calculation shows that we can take $l=\delta_0(s-2)$ in the corresponding definition of equi-Weyl-p-almost periodicity. The proof of the theorem is thereby complete. □

Remark 6. *If we replace the conjuction of condition (ew-M1), respectively, (w-M1), and the condition that (Ay_k) is a bounded sequence, by the condition that $\sum_{k\geq 1}e^{-\omega t_k}\|y_k\|<+\infty$, then the above argument and (7) together imply that there exist a bounded, continuous (equi-)Weyl-p-almost periodic function $G_1:\mathbb{R}\rightarrow X$ and a function $Q_1\in PC_0([0,\infty):X)$ such that the unique solution $u(t)$ of problem $(ACP)_{1;1}$ satisfies $u(t)=G_1(t)+Q_1(t)$ for all $t\geq 0$. Here, we can only assume that the sequence (t_k) has no accumulation point; the separation condition $\delta_0>0$ is complete regardless.*

Now, we would like to present the following simple example in which Theorem 8 can be applied ($X=\mathbb{C}$):

Example 4. *(i) Suppose that $t_i=i$ for all $i\in\mathbb{N}$, $m\in\mathbb{N}\setminus\{1\}$, $y_k=0$ for $1\leq k\leq m-1$ and $y_k=1$ for all $k\geq m$. Then, it is trivial to show that (ew-M1) holds with $L>k+1$ and $s\geq(k-1)\epsilon^{-p}$; on the other hand, it is clear that $(y_k)_{k\in\mathbb{N}}$ is not an almost periodic sequence.*

(ii) Suppose that $t_i=i$ for all $i\in\mathbb{N}$, $\sigma\in(0,1)$, $p\geq 1$, $(1-\sigma)p<1$ and $y_k=k^\sigma$ for all $k\in\mathbb{N}$. Then, the sequence $(y_k)_{k\in\mathbb{N}}$ is not equi-Weyl-p-almost periodic ($p\geq 1$); on the other hand, $(y_k)_{k\in\mathbb{N}}$ is Weyl-p-almost periodic for any exponent $p\geq 1$. Towards this end, it suffices to observe that there exists a finite constant $c_{\sigma,p}>0$ such that, for every $\tau, s_\tau,\ l\in\mathbb{N}$, we have

$$\sum_{j=l}^{l+s_\tau-1}\left[(j+\tau)^\sigma-j^\sigma\right]^p\leq c_{\sigma,p}\tau^p s_\tau^{1-(1-\sigma)p};$$

see the proof of [9] (Theorem 7.3.8, case 3, p. 566). The requirements of Theorem 8 hold with condition (w-M1) being satisfied.

We close this section with the observation that we can similarly analyze the existence and uniqueness of (equi-)Weyl-p-almost periodic solutions for a class of the abstract impulsive nonautonomous differential equation of the form [32] (1.1).

4.3. Besicovitch–Doss Almost Periodic Type Solutions of $(ACP)_{1;1}$

We start this subsection by recalling the following special case of the notion introduced in [43] (Definition 2.1):

Definition 7. *Suppose that $1\leq p<+\infty$, $F:I\rightarrow X$, $\phi:[0,\infty)\rightarrow[0,\infty)$ and $\mathrm{F}:(0,\infty)\rightarrow(0,\infty)$. Then, we say that the function $F(\cdot)$ belongs to the class $e-(\phi,\mathrm{F})-B^p(I:X)$ if and only if there exists a sequence $(P_k(\cdot))$ of trigonometric polynomials such that*

$$\lim_{k\rightarrow+\infty}\limsup_{t\rightarrow+\infty}\mathrm{F}(t)\left[\phi\left(\left\|F(\cdot)-P_k(\cdot)\right\|\right)\right]_{L^p([-t,t]\cap I)}=0,$$

where we assume that the term in brackets belongs to the space $L^p([-t,t]\cap I)$ for all $t>0$. If $\phi(x)\equiv x$, then we omit the term "ϕ" from the notation. The usual notion is obtained by plugging $\phi(x)\equiv x$ and $\mathrm{F}(t)\equiv t^{-1/p}$, when we say that the function $F(\cdot)$ is Besicovitch-p-almost periodic.

As an immediate consequence of [43] (Proposition 10) and the previous considerations, we have the following result (cf. also Remark 6; we only assume here that the sequence (t_k) has no accumulation point):

Proposition 9. *Suppose that the functions $(C^{-1}f)(\cdot)$ and $f_A(\cdot)$ satisfy all requirements of* Lemma 1 *with $n = 1$, $u_0 \in D(A)$ and $y_k \equiv y_k^0 \in D(A)$ for all $k \in \mathbb{N}$. Suppose, further, that $\sum_{k\geq 1} e^{-\omega t_k}\|y_k\| < +\infty$, $q \in PC_0([0,\infty) : X)$, $\alpha > 0$, $a > 0$, $\alpha p \geq 1$, $ap \geq 1$, the function $g : \mathbb{R} \to X$ is bounded and belongs to the class $e-(x^\alpha, t^{-a})-B^p(\mathbb{R} : X)$ as well as $(C^{-1}f)(t) = g(t) + q(t)$ for all $t \geq 0$. Then, there exist a bounded continuous function $G_1 : \mathbb{R} \to X$ belonging to the class $e-(x^\alpha, t^{-a})-B^p(\mathbb{R} : X)$ and a function $Q_1 \in PC_0([0,\infty) : X)$ such that the unique solution $u(t)$ of problem $(ACP)_{1;1}$ satisfies $u(t) = G_1(t) + Q_1(t)$ for all $t \geq 0$.*

Remark 7. *Is should be noted that the solution $u(\cdot) = G_1(\cdot) + Q_1(\cdot)$ belongs to the class $e-(x^\alpha, t^{-a})-B^p(\mathbb{R} : X)$, as well. In order to see this, it suffices to observe that $e-(x^\alpha, t^{-a})-B^p(\mathbb{R} : X)$ is a vector space (see the statement [43] ((i), p. 4221)) and $Q_1 \in e-(x^\alpha, t^{-a})-B^p(\mathbb{R} : X)$, which follows from a relatively simple computation with the sequence $(P_k \equiv 0)$ in Definition 7 and the assumption $ap \geq 1$.*

We also need the following notion (see, e.g., [7] (Definition 2.13.2)):

Definition 8. *Let $1 \leq p < \infty$ and let $f \in L^p_{loc}(I : X)$. Then, it is said that $f(\cdot)$ is Doss p-almost periodic if and only if, for every $\epsilon > 0$, the set of numbers $\tau \in I$ for which*

$$\limsup_{l\to+\infty}\left[\frac{1}{2l}\int_{-l}^{l}\|f(s+\tau)-f(s)\|^p\,ds\right]^{1/p} < \epsilon,$$

in the case that $I = \mathbb{R}$, respectively,

$$\limsup_{l\to+\infty}\left[\frac{1}{l}\int_{0}^{l}\|f(s+\tau)-f(s)\|^p\,ds\right]^{1/p} < \epsilon,$$

in the case that $I = [0,\infty)$, is relatively dense in I.

Now, we would like to re-examine the statement of Theorem 8 for Doss-p-almost periodic solutions. In order to do that, we need to introduce the following condition:

(ed-M1) For every $\epsilon > 0$, there exists $L > 0$ such that every interval $I' \subseteq [0,\infty)$ of length L contains a point $\tau \in I'$ which satisfies that there exists an integer $q_\tau \in \mathbb{N}$ such that $|t_{i+q_\tau} - t_i - \tau| < \epsilon$ for all $i \in \mathbb{N}$ and

$$\limsup_{s\to+\infty}\left[\frac{1}{s}\sum_{j=1}^{s}\left\|y_{j+q_\tau}-y_j\right\|^p\right]^{1/p} < \epsilon. \quad (9)$$

Condition (ed-M1) implies that the family of sequences $(t_k^j)_{k\in\mathbb{N}}$, $j \in \mathbb{N}$ is equipotentially almost periodic as well as that the sequence $(y_k)_{k\in\mathbb{N}}$ is Doss-p-almost periodic in the following sense:

(d-M1) For every $\epsilon > 0$, there exists $L > 0$ such that every interval $I' \subseteq [0,\infty)$ of length L contains a point $\tau \in I' \cap \mathbb{N}$ which satisfies that (9) holds with the number q_τ replaced therein with the number τ.

Before stating the next result, we would like to note that the class of Doss-p-almost periodic sequences has not been defined in the existing literature so far, even in the scalar-valued setting.

Theorem 9. *Suppose that* (ed-M1) *holds, the functions* $(C^{-1}f)(\cdot)$ *and* $f_A(\cdot)$ *satisfy all requirements of* Lemma 1 *with* $n = 1$, $u_0 \in D(A)$ *and* $y_k \equiv y_k^0 \in D(A)$ *for all* $k \in \mathbb{N}$. *Suppose, further, that* (y_k) *and* (Ay_k) *are bounded sequences,* $q \in PC_0([0,\infty) : X)$, *the function* $g : \mathbb{R} \to X$ *is Doss-p-almost periodic and bounded as well as* $(C^{-1}f)(t) = g(t) + q(t)$ *for all* $t \geq 0$. *Then, there exist a bounded continuous Doss-p-almost periodic function* $G_1 : \mathbb{R} \to X$, *a bounded piecewise continuous Doss-p-almost periodic function* $G_2 : [0,\infty) \to X$ *and a function* $Q_1 \in C_0([0,\infty) : X)$ *such that the unique solution* $u(t)$ *of problem* $(ACP)_{1;1}$ *satisfies* $u(t) = G_1(t) + G_2(t) + Q_1(t)$ *for all* $t \geq 0$.

Proof. We will only outline the main details of the proof (see also the proof of Theorem 8). In place of [7] (Theorem 2.11.4), we can use [7] (Theorem 2.13.10). If condition (ed-M1) holds in place of condition (ew-M1), then we can use the same arguments as in the proof of Theorem 8, with $x = 0$ and $r = 0$. The remainder of the proof is the same. □

Remark 8. *Due to* [7] *(Proposition 2.13.6), we have that the function* $Q_1(\cdot)$ *is also Doss-p-almost periodic.*

The pioneering results about Besicovitch-p-almost periodic sequences have been given in [3,44]. The first systematic study of scalar-valued Besicovitch-p-almost periodic sequences has been carried out by A. Bellow and V. Losert in [14] (Section 3); cf. also Bergelson et al. [40]. In the following definition, we introduce the vector-valued version of [14] (Definition 3.2):

Definition 9. *Suppose that* $1 \leq p < +\infty$ *and* $(y_k)_{k\in\mathbb{N}}$ *is a sequence in* X. *Then, we say that* $(y_k)_{k\in\mathbb{N}}$ *is Besicovitch-p-almost periodic if and only if for every* $\epsilon > 0$ *there exists a trigonometric polynomial* $P(\cdot)$ *such that*

$$\limsup_{s\to+\infty}\left[\frac{1}{s}\sum_{k=1}^{s}\left\|y_k - P(k)\right\|^p\right]^{1/p} < \epsilon.$$

It can be simply shown that any Besicovitch-p-almost periodic sequence $(y_k)_{k\in\mathbb{N}}$ is Besicovitch-p-bounded, i.e.,

$$\limsup_{k\to+\infty}\frac{1}{k}\sum_{l=1}^{k}\left\|y_l\right\|^p < +\infty;$$

see, e.g., the proof of [43] (Proposition 1(i)) for the continuous analogue of this statement. Therefore, the sequence $(k^\sigma)_{k\in\mathbb{N}}$ considered in Example 4(ii) is not Besicovitch-p-almost periodic since

$$\frac{1}{k}\sum_{l=1}^{k} l^{\sigma p} \sim (1+\sigma p)^{-1}k^{\sigma p} \text{ as } k \to +\infty.$$

It is completely out of the scope of this paper to reconsider the statements established in [14] for the vector-valued Besicovitch-p-almost periodic sequences. Before proceeding further, we would like to address the following questions:

(Q1) Is it possible to state a satisfactory analogue of Theorems 8 and 9 for Besicovitch-p-almost periodic solutions of problem $(ACP)_{1;1}$ $(1 \leq p < \infty)$?

(Q2) Suppose that the family of sequences $(t_k^j)_{k\in\mathbb{N}}$, $j \in \mathbb{N}$ is equipotentially almost periodic as well as that the sequence $(y_k)_{k\in\mathbb{N}}$ is equi-Weyl-p-almost periodic $(1 \leq p < \infty)$. Is it true that (ew-M1) holds true?

(Q3) Suppose that the sequence $(y_k)_{k\in\mathbb{N}}$ is (equi-)Weyl-p-almost periodic [Doss-p-almost periodic/Besicovitch-p-almost periodic] $(1 \leq p < \infty)$. Is it true that there exists a unique (equi-)Weyl-p-almost periodic [Doss-p-almost periodic/Besicovitch-p-almost periodic] sequence $(\tilde{y_k})_{k\in\mathbb{Z}}$ (defined in the obvious way) such that $\tilde{y_k} = y_k$ for all $k \in \mathbb{N}$?

(Q4) Is it true that the sequence $(y_k)_{k\in\mathbb{Z}}$ $[(y_k)_{k\in\mathbb{N}}]$ is (equi-)Weyl-p-almost periodic [Doss-p-almost periodic/Besicovitch-p-almost periodic] $(1 \leq p < \infty)$ if and only if there exists a continuous (equi-)Weyl-p-almost periodic [Doss-p-almost periodic/Besicovitch-p-almost periodic] function $f : \mathbb{R} \to X$ $[f : [0,\infty) \to X]$ such that $y_k = f(k)$ for all $k \in \mathbb{Z}$ $[k \in \mathbb{N}]$?

In connection with the problem (Q2), see also [27] (Lemma 35) and observe that we cannot expect the affirmative answer in the case of consideration of Weyl-p-almost periodic sequences and Doss-p-almost periodic sequences; cf. [9] for more details. It can be very simply shown that the class of Doss-p-almost periodic sequences is the most general since it contains all Weyl-p-almost periodic sequences and all Besicovitch-p-almost periodic sequences; it is also worth noting that the class of equi-Weyl-p-almost periodic sequences is contained in the class of Besicovitch-p-almost periodic sequences, which is no longer true for the class of Weyl-p-almost periodic sequences. A simple example of a Besicovitch-p-almost periodic sequence which is not Weyl-p-almost periodic is given as follows: If $1 \leq p < +\infty$ and $k \in [m^2, m^2 + \sqrt{m})$ for some integer $m \in \mathbb{N}$, then, we define $y_k := m^{1/2p}$; then the sequence $(y_k)_{k\in\mathbb{N}}$ enjoys the abovementioned properties (see also [45] (Example 6.24), [46] (p. 42) and [43] (Example 4)). We will examine in more detail the classes of (equi-)Weyl-p-almost periodic sequences, Doss-p-almost periodic sequences and Besicovitch-p-almost periodic sequences somewhere else $(1 \leq p < \infty)$.

4.4. Almost Periodic Type Solutions of the Abstract Higher-Order Impulsive Cauchy Problems

We will first explain how the results established in the previous three subsections can be used in the analysis of the existence and uniqueness of almost periodic type solutions for certain classes of the abstract higher-order impulsive (degenerate) Cauchy problems. Here, the idea is to convert these problems into the equivalent abstract impulsive (degenerate) Cauchy problems of first order on the product spaces.

Suppose, for example, that the operator A generates a strongly continuous semigroup on X as well as that B is a closed densely defined operator on X with $D(A) \subseteq D(B)$. Applying [47] (Theorem 3), we conclude that there exists a real number $\omega > 0$ such that, for every $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$, the matricial operator

$$\mathrm{D} := \begin{bmatrix} A-\lambda & I \\ B & -\lambda \end{bmatrix}$$

generates an exponentially decaying strongly continuous semigroup $(T(t))_{t\geq 0}$ on $X \times X$. Suppose, further, that $u_1 \in D(A)$, $u_2 \in X$, and $t \mapsto f_{1,2}(t)$, $t \geq 0$ are continuously differentiable and asymptotically almost periodic. Using Lemma 1(ii) and a simple computation with the variation of parameters formula, we can reformulate all established conclusions from the previous three subsections of this paper in the analysis of the abstract impulsive Cauchy problem

$$\begin{aligned} u''(t) - (A-2\lambda)u'(t) - [\lambda(A-\lambda)+B]u(t) = f_1'(t) + \lambda f_1(t) + f_2(t), \quad t \geq 0; \\ (\Delta u)(t_k) = y_0^k;\ (\Delta u')(t_k) = y_1^k + (A-\lambda)y_0^k; \\ u(0) = u_1,\ \ u'(0) = u_2 + (A-\lambda)u_1 + f_1(0). \end{aligned}$$

Without going into full details, we will only emphasize that the certain classes of the abstract degenerate second-order Cauchy problems with impulsive effects can also be analyzed in a similar manner by reduction to the abstract degenerate first-order Cauchy problems with impulsive effects on product spaces; see, e.g., [25] (Example 2.5(ii)), where we analyzed the well-posedness of the damped Poisson-wave type equations in L^p-spaces (it would be very tempting to apply the same method to the abstract higher-order Cauchy problems considered in [48]).

In the existing literature, the authors have found many interesting criteria ensuring that a strongly continuous operator family $(R(t))_{t\geq 0} \subseteq L(X)$ is (asymptotically) almost periodic, i.e., the mapping $t \mapsto R(t)x$, $t \geq 0$ is (asymptotically) almost periodic for every fixed element $x \in X$; cf. [7] for more details. Such operator families can be important in the analysis of the existence and uniqueness of the almost periodic type solutions for certain classes of the abstract impulsive Volterra integro-differenital inclusions, as the following illustrative example indicates.

Example 5. *The existence and uniqueness of almost periodic solutions for a class of the complete second-order Cauchy problems have been considered by T.-J. Xiao and J. Liang [22] (Section 7.1.2) under the assumption that the corresponding problem is strongly well-posed. Specifically, the authors analyzed the abstract Cauchy problem*

$$u''(t) + \big(aA_0 + bI\big)u'(t) + \big(cA_0 + dI\big)u(t) = 0, \quad t \geq 0,$$

where $a,\ b,\ c,\ d \in \mathbb{C}$ *and the operator* A_0 *is a closed linear operator with domain and range contained in a Banach space* X*. In the case that* $X := L^2[0,1]$ *and* A_0 *is the Dirichlet Laplacian, the authors have shown that both propagator families,* $(S_0(t))_{t\geq 0}$ *and* $(S_1(t))_{t\geq 0}$*, are almost periodic. If so, then we can consider the piecewise continuous almost periodic solutions of the following abstract impulsive Cauchy problem*

$$(ACP)_2 : \begin{cases} u''(t) + \big(aA_0 + bI\big)u'(t) + \big(cA_0 + dI\big)u(t) = 0, \\ t \in [0,\infty) \setminus \{t_1, \ldots, t_l, \ldots\}, \\ \big(\Delta u^{(j)}\big)(t_k) = u^{(j)}(t_k+) - u^{(j)}(t_k-) = y_j^k, \quad k \in \mathbb{N},\ j = 0,1; \\ u^{(j)}(0) = u_j, \quad j = 0,1, \end{cases}$$

where $0 < t_1 < \ldots < t_l < \ldots < +\infty$ *and the sequence* $(t_l)_l$ *has no accumulation point. Due to the consideration from [25], the function* $u(t) = S_0(t)u_0 + S_1(t)u_1 + \omega(t)$*,* $t \geq 0$*, where* $\omega(t) := \sum_{j=0}^{1}[S_j(t-t_1)y_j^1 + \ldots + S_j(t-t_k)y_j^k]$ *if* $t \in (t_k, t_{k+1}]$ *for some* $k \in \mathbb{N}$*, is a unique solution of* $(ACP)_2$*. Arguing as in Example 6 below, we may conclude that the assumptions* $\sum_{k\geq 1}\|y_0^k\| < +\infty$ *and* $\sum_{k\geq 1}\|y_1^k\| < +\infty$ *imply that there exist an almost periodic function* $f : [0,\infty) \to X$ *and a function* $q \in PC_0([0,\infty) : X)$ *such that* $u(t) = f(t) + q(t)$ *for all* $t \geq 0$.

5. Almost Periodic Type Solutions of the Abstract Volterra Integro-Differential Inclusions with Impulsive Effects

Let us consider the following abstract impulsive Volterra integro-differential inclusion:

$$\begin{aligned} \mathcal{B}u(t) \subseteq & \mathcal{A}\int_0^t a(t-s)u(s)ds + \mathcal{F}(t), \quad t \in [0,T] \setminus \{t_1, \ldots, t_l\}; \\ & (\Delta u)(t_m) = Cy_m, \quad m = 1, \ldots, l, \end{aligned} \tag{10}$$

where $0 \equiv t_0 < t_1 < \ldots < t_l < T \equiv t_{l+1}$, where $0 < T \leq \infty$, $a \in L^1_{loc}([0,\tau))$, $a \neq 0$, $\mathcal{F}\colon [0,\tau) \to P(E)$, and $\mathcal{A}\colon X \to P(E)$, $\mathcal{B}\colon X \to P(E)$ are two given mappings, as well as the well-posedness of the following abstract impulsive Volterra integro-differential inclusion:

$$\begin{aligned} \mathcal{B}u(t) \subseteq & \mathcal{A}\int_0^t a(t-s)u(s)ds + \mathcal{F}(t), \quad t \in [0,\infty) \setminus \{t_1, \ldots, t_l, \ldots\}; \\ & (\Delta u)(t_l) = Cy_l, \quad l \geq 1, \end{aligned} \tag{11}$$

where $0 \equiv t_0 < t_1 < \ldots < t_l < t_{l+1} < \ldots < +\infty$, the sequence $(t_l)_l$ has no accumulation point, $a \in L^1_{loc}([0,\infty))$, $a \neq 0$, $\mathcal{F}\colon [0,\infty) \to P(E)$, and $\mathcal{A}\colon X \to P(E)$, $\mathcal{B}\colon X \to P(E)$ are two given mappings.

The notion of a (pre-)solution of (10) [(11)] and the notion of a strong solution of (10) [(11)] have recently been introduced in [25]. We recall the following result from the same paper:

Lemma 2. *(i) Suppose that $a(t)$ and $k(t)$ are kernels, $k(0) = 1$, $C_2 \in L(X)$, and $\mathcal{A}$ is a closed subgenerator of a mild (a,k)-regularized C_2-uniqueness family $(R_2(t))_{t\in[0,\tau)} \subseteq L(X)$, where $\tau > T$. Define $\mathcal{F}(t) := 0$ for $t \in [0, t_1]$ and $\mathcal{F}(t) := \sum_{s=1}^{m} k(t - t_s)C_2 y_s$ if $t \in (t_m, t_{m+1}]$ for some integer $m \in \mathbb{N}_l$. Define also $u(t) := 0$ for $t \in [0, t_1]$ and $u(t) := \sum_{s=1}^{l} R_2(t - t_s)y_s$ if $t \in (t_m, t_{m+1}]$ for some integer $m \in \mathbb{N}_l$. If $y_1, \ldots, y_l \in D(\mathcal{A})$, then $u(t)$ is a unique strong solution of problem (10) on $[0, T]$.*

(ii) Suppose that $a(t)$ and $k(t)$ are kernels, $k(0) = 1$, $C_1 \in L(X, E)$, and $\mathcal{A}$ is a closed subgenerator of a mild (a,k)-regularized C_1-existence family $(R_1(t))_{t\in[0,\tau)} \subseteq L(X, E)$, where $\tau > T$. Define $\mathcal{F}(t)$ and $u(t)$ in the same way as above, with the operator C_2 replaced therein with the operator C_1 and the elements $y_1, \ldots, y_l \in X$. Then, $u(t)$ is a solution of problem (10) on $[0, T]$.

In a global version of Lemma 2, which can be very simply formulated, we do not need the separation condition $\delta_0 > 0$ on the sequence (t_k). For the sequel, we need to recall the following special consequences of [8] (Proposition 3.1.15(i)):

(i) Suppose that $\alpha \in (0,1)$, $u_0 \in D(\mathcal{A})$ as well as $C^{-1}f$, $f_{\mathcal{A}} \in C([0,\infty) : X)$, $f_{\mathcal{A}}(t) \in \mathcal{A}C^{-1}f(t)$, $t \geq 0$, and $\mathcal{A}$ is a closed subgenerator of a (g_α, C)-regularized resolvent family $(R(t))_{t\geq 0}$. Then, the function $u(t) := R(t)x + (R * C^{-1}f)(t)$, $t \geq 0$ is a unique solution of the following abstract fractional Cauchy inclusion:

$$\begin{cases} u \in C^1((0,\infty) : X) \cap C([0,\infty) : X), \\ \mathbf{D}_t^\alpha u(t) \in \mathcal{A}u(t) + \big(g_{1-\alpha} * f\big)(t),\ t \geq 0, \\ u(0) = Cu_0. \end{cases}$$

(ii) Suppose that $\alpha \in (1,2)$, $u_0 \in D(\mathcal{A})$ as well as $C^{-1}f$, $f_{\mathcal{A}} \in C([0,\infty) : X)$, $f_{\mathcal{A}}(t) \in \mathcal{A}C^{-1}f(t)$, $t \geq 0$, and $\mathcal{A}$ is a closed subgenerator of a (g_α, C)-regularized resolvent family $(R(t))_{t\geq 0}$. Set $v(t) := (g_{2-\alpha} * f)(t)$, $t \geq 0$. If $v \in C^1([0,\infty) : X)$, then the function $u(t) := R(t)x + (R * C^{-1}f)(t)$, $t \geq 0$ is a unique solution of the following abstract fractional Cauchy inclusion:

$$\begin{cases} u \in C^2((0,\infty) : X) \cap C^1([0,\infty) : X), \\ \mathbf{D}_t^\alpha u(t) \in \mathcal{A}u(t) + \frac{d}{dt}\big(g_{2-\alpha} * f\big)(t),\ t \geq 0, \\ u(0) = Cu_0,\ u'(0) = 0. \end{cases}$$

Keeping in mind this result, Lemma 2 and the second equality in [49] (1.21), we have the following:

Theorem 10. *Suppose that $u_0 \in D(\mathcal{A})$, $C^{-1}f$, $f_{\mathcal{A}} \in C([0,\infty) : X)$, $f_{\mathcal{A}}(t) \in \mathcal{A}C^{-1}f(t)$, $t \geq 0$, and $\mathcal{A}$ is a closed subgenerator of a global (g_α, C)-regularized resolvent family $(R(t))_{t\geq 0} \subseteq L(X)$. Define $\mathcal{F}_0(t) := 0$ for $t \in [0, t_1]$ and $\mathcal{F}_0(t) := \sum_{s=1}^{m} Cy_s$ if $t \in (t_m, t_{m+1}]$ for some integer $m \in \mathbb{N}$. Define also $\omega(t) := 0$ for $t \in [0, t_1]$ and $\omega(t) := \sum_{s=1}^{l} R(t - t_s)y_s$ if $t \in (t_m, t_{m+1}]$ for some integer $m \in \mathbb{N}$, and assume that $y_1, \ldots, y_l, \ldots . \in D(\mathcal{A})$.*

(i) Suppose that $\alpha \in (0,1)$. Then, the function $u(t) := R(t)u_0 + \int_0^t R(t-s)(C^{-1}f)(s)\,ds + \omega(t)$, $t \geq 0$ is piecewise continuous, and it is a unique strong solution of problem (10) on $[0,\infty)$ with $\mathcal{B} = \mathrm{I}$, $a(t) = g_\alpha(t)$ and $\mathcal{F}(t) = Cu_0 + \int_0^t f(s)\,ds + \mathcal{F}_0(t)$, $t \geq 0$.

(ii) Suppose that $\alpha \in (1,2)$ and the mapping $t \mapsto (g_{2-\alpha} * f)(t)$, $t \geq 0$ is continuously differentiable. Then, the function $u(t) := R(t)u_0 + \int_0^t R(t-s)(C^{-1}f)(s)\,ds + \omega(t)$, $t \geq 0$ is piecewise continuous, and it is a unique strong solution of problem (10) on $[0,\infty)$ with $\mathcal{B} = \mathrm{I}$, $a(t) = g_\alpha(t)$ and $\mathcal{F}(t) = Cu_0 + \int_0^t f(s)\,ds + \mathcal{F}_0(t)$, $t \geq 0$.

Concerning the existence and uniqueness of the asymptotically almost periodic type solutions of problem (10) on $[0,\infty)$ with $\mathcal{B} = \mathrm{I}$ and $\mathcal{F}(t) = Cu_0 + \int_0^t f(s)\,ds + \mathcal{F}_0(t)$, $t \geq 0$, we will present the following extremely important situation in which all conclusions established in the previous section continue to hold.

Suppose that A is a densely defined, closed linear operator, $1 < \alpha < 2$, $M > 0$, $\omega < 0$, $\Omega \equiv (\omega + \{\lambda \in \mathbb{C} \setminus \{0\} : \arg(-\lambda) < \theta\})^c \subseteq \rho_C(A)$ for some number $\theta \in [0, \pi(1-(\alpha/2)))$, $\|(\lambda - A)^{-1}C\| \leq M/|\lambda - \omega|$, $\lambda \in \Omega$, and the mapping $\lambda \mapsto (\lambda - A)^{-1}C$ is analytic in an open neighborhood of the set Ω. Then, we know that the operator A is a subgenerator of a global (g_α, C)-regularized resolvent family $(R(t))_{t\geq 0}$ satisfying that there exists $M' > 0$ such that $\|R(t)\| \leq M'/(1+|\omega|t^\alpha)$, $t \geq 0$; see, e.g., the proof of E. Cuesta's result [15] (Theorem 1) and [7] (Section 3.4) for many important generalizations of this result.

Suppose now that all requirements of Theorem 10 hold, the sequence (t_k) has no accumulation point (the separation condition $\delta_0 > 0$ is complete regardless here) and $\sum_{k\geq 1}(t_k^\alpha + 1)\|y_k\| < +\infty$. Then, the function $\omega(\cdot)$ defined in the proof of Theorem 10(ii) belongs to the space $PC_{\alpha;1}([0,\infty) : X) \equiv \{f \in PC([0,\infty) : X) ; \cdot^\alpha \|f(\cdot)\| \in L^\infty([0,\infty) : X)\}$ since for each $k \in \mathbb{N}$ and $t \in (t_k, t_{k+1}]$ we have

$$\begin{aligned} \|t^\alpha \omega(t)\| &\leq M' \sum_{l=1}^{k} \left[\frac{t^\alpha}{1+|\omega|(t-t_l)^\alpha} \|y_l\| \right] \leq M' 2^{\alpha-1} \sum_{l=1}^{k} \left[\frac{(t-t_l)^\alpha + t_l^\alpha}{1+|\omega|(t-t_l)^\alpha} \|y_l\| \right] \\ &\leq M' 2^{\alpha-1} \sum_{l=1}^{k} \left[\left((1/|\omega|) + t_l^\alpha \right) \|y_l\| \right] \leq M' 2^{\alpha-1} \sum_{l=1}^{\infty} \left[\left((1/|\omega|) + t_l^\alpha \right) \|y_l\| \right]. \end{aligned} \tag{12}$$

Concerning the function $C^{-1}f(\cdot)$, we can assume that there exists a bounded Stepanov-p-almost periodic function $g : \mathbb{R} \to X$ and a function $q \in PC_0([0,\infty) : X)$, for some $p \in [1,\infty)$, such that $C^{-1}f(t) = g(t) + q(t)$ for all $t \geq 0$. Then, the function $t \mapsto u(t) - \omega(t)$, $t \geq 0$ will be asymptotically almost periodic in the usual sense; see, e.g., [7] (Proposition 2.6.11, Remark 2.6.12, Proposition 2.6.13). Observe also that the obtained conclusion on the existence and uniqueness of asymptotically almost periodic solutions cannot be established in the case that $0 < \alpha < 1$ since, in this case, the resolvent $(R(t))_{t>0}$ is not uniformly integrable; for example, in Figure 2 we have constructed the graph of the $(g_{1/3}, \mathrm{I})$-regularized resolvent family $(R(t))_{t\geq 0} \equiv (E_{1/3}(-2t^{1/3}))_{t\geq 0}$ which satisfies an estimate of the type $\|R(t)\| \sim ct^{-(1/3)}$, $t \to +\infty$; of course, here we have $\mathcal{A} = A = -2\mathrm{I}$ and $X = \mathbb{C}$.

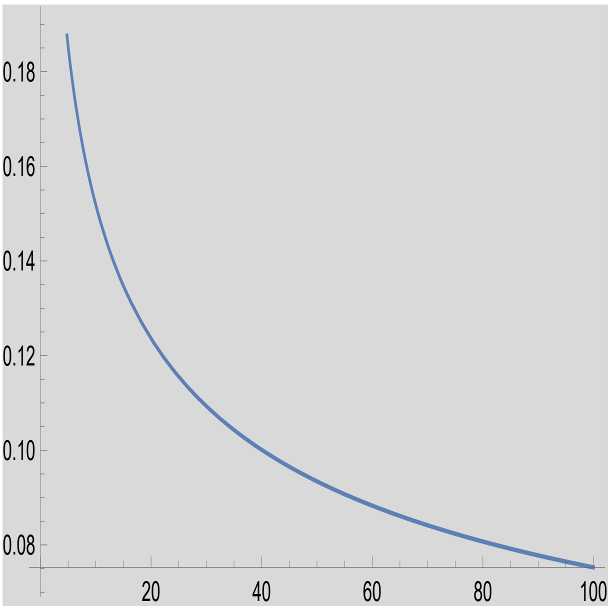

Figure 2. Graph of $E_{1/3}(-2t^{1/3})$, where $E_{1/3}(\cdot) \equiv E_{1/3,1}(\cdot)$ is the Mittag–Leffler function.

All other results established in the previous section continue to hold, as marked above. After a careful inspection of the proofs of [32] (Lemma 3.6, Theorem 3.7), it suffices to observe that the uniform convergence in the corresponding part of the proof of [32]

(Theorem 3.7; cf. (3.36), p. 14) is a consequence of the following simple computation, where we assume that the sequence (y_k) is bounded:

$$\begin{aligned}\left\|R(t_{i-k}-t)y_{i-k}\right\| &\leq N\sup_{i\in\mathbb{N}}\frac{1}{1+|\omega|(t-t_{i-k})^{\alpha}}\\ &\leq N\sup_{i\in\mathbb{N}}\frac{1}{1+|\omega|(t-t_i)^{\alpha}+|\omega|(t_i-t_{i-k})^{\alpha}}\leq N\frac{1}{1+|\omega|(k\delta_0)^{\alpha}},\end{aligned}$$

for some finite real constant $N>0$. In conclusion, we have the following: if $(y_k)_{k\in\mathbb{N}}$ is an almost periodic sequence, the separation condition $\delta_0>0$ holds, and the family of sequences $(t_k^j)_{k\in\mathbb{N}}$, $j\in\mathbb{N}$ is equipotentially almost periodic. Then, the function $\omega(\cdot)$ is piecewise continuous almost periodic.

It should be noted that the obtained results can be applied to the abstract (non-coercive) differential operators in L^p-spaces; cf. [50] (Section 2.5) for further information in this direction.

Remark 9. *Consider now the situation in which $\gamma\in(0,1)$, $u_0\in D(\mathcal{A})$, and $\mathcal{A}$ satisfies condition (P) analyzed in [51] and [7] (Subsection 2.9.1). If we consider the subordinated resolvent families $(S_\gamma(t))_{t>0}$ and $(R_\gamma(t))_{t>0}$ from [7], then the function $u_h(t):=S_\gamma(t)x+(R_\gamma*f)(t)$, $t\geq 0$ is a unique solution of the following abstract fractional Cauchy inclusion (under certain reasonable assumptions):*

$$\begin{cases}\mathbf{D}_t^{\alpha}u_h(t)\in\mathcal{A}u_h(t)+f(t),\ t>0,\\ u_h(0)=u_0.\end{cases}$$

Keeping in mind the second equality in [49] (1.21) and the initial condition $u(0)=u_0$, we simply conclude that the function $u_h(\cdot)$ is a unique strong solution of the associated Volterra inclusion

$$u_h(t)\in u_0+\big(g_\alpha*f\big)(t)+\mathcal{A}\big(g_\alpha*u_h\big)(t),\quad t\geq 0.$$

Suppose now that $\mathcal{F}_0(t):=0$ for $t\in[0,t_1]$ and $\mathcal{F}_0(t):=\sum_{s=1}^{m}y_s$ if $t\in(t_m,t_{m+1}]$ for some integer $m\in\mathbb{N}$, as well as that $y_1,\ldots,y_l,\ldots.\in D(\mathcal{A})$. Define $\omega(t):=0$ for $t\in[0,t_1]$ and $\omega(t):=\sum_{s=1}^{l}S_\gamma(t-t_s)y_s$ if $t\in(t_m,t_{m+1}]$ for some integer $m\in\mathbb{N}$, and assume that there exist vectors $z_1,\ldots,z_l,\ldots.$ from the continuity set of the resolvent operator family $(S_\gamma(t))_{t>0}$ such that $z_l\in\mathcal{A}y_l$ for all $l\in\mathbb{N}$; cf. also [25] (Example 2.5(i)). Then, the function $\omega(t)$ is a unique strong solution of the abstract impulsive Volterra inclusion

$$\omega(t)\in\mathcal{F}_0(t)+\mathcal{A}\big(g_\alpha*\omega\big)(t),\quad t\in[0,\infty)\setminus\{t_1,t_2,\ldots,t_l,\ldots\}.$$

Therefore, the function $u(t):=u_h(t)+\omega(t)$, $t\geq 0$ is a unique strong solution of the abstract impulsive Volterra inclusion

$$u(t)\in u_0+\big(g_\alpha*f\big)(t)+\mathcal{F}_0(t)+\mathcal{A}\big(g_\alpha*u\big)(t),\quad t\in[0,\infty)\setminus\{t_1,t_2,\ldots,t_l,\ldots\}.\tag{13}$$

Concerning the existence and uniqueness of asymptotically almost periodic solutions of (13), the situation is far from being simple because the operator family $(S_\gamma(t))_{t>0}$ has an integrable singularity at zero: we must impose certain extra assumptions in order for the proofs to work. This can be simply completed for the analogues of the equations [32] ((3.27)–(3.28)) but, unfortunately, this is almost impossible to be completed for the equation [32] (3.36) since the series $\sum_{k\geq 1}(k\delta_0)^{-\gamma}$ diverges. Even the computation carried out in (12) cannot be so simply reconsidered in a newly arisen situation.

We continue by providing the following instructive example:

Example 6. *Let $\alpha \in (0,2)$ and $\theta = \pi - \pi\alpha/2$, and let us consider the following fractional Cauchy problem*

$$\mathbf{D}_t^{\alpha} u(t,x) = e^{i\theta} u_{xx}(t,x), \quad 0 < x < 1,\ t \geq 0;$$

cf. also [49] (Example 2.20). Suppose that $X := L^2[0,1]$ and $A := e^{i\theta}\Delta$, where Δ denotes the Laplacian equipped with the Dirichlet boundary conditions. Then, we known that A is the integral generator of an asymptotically almost periodic (g_α, I)-resolvent family $(R(t))_{t\geq 0}$ as well as that $(R(t))_{t\geq 0}$ is not almost periodic if $\alpha \neq 1$; cf. [7] (Example 2.6.4).

Suppose now that $u_0, y_1, y_2, \ldots, y_l, \ldots \in D(A)$ and $\sum_{k\geq 1} \|y_k\| < +\infty$. Define the function $\omega(\cdot)$ as in the formulation of Lemma 2(i), with $k(t) \equiv 1$ and $C_2 = \mathrm{I}$. Then, it can be simply shown that the function $u(t) := R(t)u_0 + \omega(t)$, $t \geq 0$ is a unique strong solution of the abstract Volterra Equation (10) with $\mathcal{B} = C = \mathrm{I}$, $\mathcal{A} = A$, $a(t) \equiv g_\alpha(t)$ and $\mathcal{F}(t) \equiv u_0 + \mathcal{F}_0(t)$, where $\mathcal{F}_0(t) = y_1 + \ldots + y_k$ if $t_k < t \leq t_{k+1}$ for some $k \in \mathbb{N}_0$. By the assumption and the already mentioned result about the extension of almost periodic functions [37], we know that for each $k \in \mathbb{N}$ there exist an almost periodic function $R_{ap}^k : \mathbb{R} \to X$ and a function $Q \in C_0([0,\infty) : X)$ such that $R(t-t_k)y_k = R_{ap}^k(t-t_k) + Q(t-t_k)$ for all $t \geq t_k$. Define $F_k : [0,\infty) \to X$ and $Q_k : [0,\infty) \to X$ by $F_k(t) := R_{ap}^k(t-t_k)$, $t \geq 0$, $Q_k(t) := -R_{ap}^k(t-t_k)$ for $t \in [0,t_k]$ and $Q_k(t) := Q(t-t_k)$ for $t > t_k$. It can be simply shown that $F_k(\cdot)$ is almost periodic, $Q_k \in PC_0([0,\infty) : X)$, $\|F_k(\cdot)\|_\infty \leq \|R(\cdot)\|_\infty \cdot \|y_k\|$, $\|Q_k(\cdot)\|_\infty \leq 3\|R(\cdot)\|_\infty \cdot \|y_k\|$ as well as that $\chi_{[0,t_k]}(t)R(t-t_k)y_k = F_k(t) + Q_k(t)$ for all $t \geq 0$ and $k \in \mathbb{N}$; cf. also [4] (Lemma 4.28, Theorem 4.29). Since we have assumed that $\sum_{k\geq 1} \|y_k\| < +\infty$, the Weierstrass criterion implies that the series $\sum_{k\geq 1} F_k(t) =: F(t)$, $t \geq 0$ and $\sum_{k\geq 1} Q_k(t) =: Q(t)$, $t \geq 0$ are uniformly convergent. Since $PC_0([0,\infty) : X)$ is a Banach space, it readily follows that $Q \in PC_0([0,\infty) : X)$; on the other hand, it is clear that the function $F(\cdot)$ is almost periodic. Hence, the solution $u(t)$ is piecewise continuous asymptotically almost periodic since $u(t) = F(t) + Q(t)$ for all $t \geq 0$.

In [25], we have also considered the following Volterra integro-differential equation:

$$\begin{aligned} Bu(t) &= f(t) + \int_0^t a(t-s)Au(s)ds, \quad t \in [0,\infty) \setminus \{t_1, \ldots, t_l, \ldots\}; \\ B(\Delta u)(t_m) &= CBy_m, \ \ m = 1, \ldots, l, \ldots, \end{aligned} \tag{14}$$

where $t \mapsto f(t)$, $t \geq 0$ is a Lebesgue measurable mapping with values in X, $a \in L^1_{loc}([0,\infty))$, and A, B are closed linear operators with domain and range contained in X. The class of exponentially bounded (a,k)-regularized C-resolvent family for (14) has recently been introduced in [8] (Definition 2.2.2); cf. also [25] (Definition 4.4). We close this section with the observation that we can similarly analyze, with the help of [25] (Theorem 4.6) and the foregoing arguments, the existence and uniqueness of asymptotically almost periodic type solutions of problem (14). Details can be left to the interested readers.

6. Conclusions and Final Remarks

In this research article, we have introduced and systematically analyzed several new classes of piecewise continuous almost periodic type functions with values in complex Banach spaces. The existence and uniqueness of almost periodic type solutions for certain classes of the abstract impulsive Volterra integro-differential inclusions have been considered. In addition to the above, we have proposed many useful observations, illustrative examples and open problems.

We close the paper by emphasizing a few important topics not considered in the previous work:

1. As clearly marked in the final section of [35], the Levitan and Bebutov classes of almost periodic type functions can be further generalized using the approaches of Stepanov, Weyl and Besicovitch. The notion of a Stepanov 1-Levitan N-almost periodic

function $f : \mathbb{R} \to X$ has already been introduced in [27] (Definition 12, p. 402), and the analogues of [27] (Lemmas 58 and 59) for Stepanov 1-Levitan N-almost periodic functions have been clarified in [27] (Lemma 60). We will only note here that we can similarly define the notion of a Stepanov p-Levitan N-almost periodic function $f : I \to X$, where $1 \leq p < +\infty$. and prove an analogue of Theorem 1 for Stepanov p-Levitan N-almost periodic functions. Details can be left to the interested readers.

2. The notion of a Bochner spatially almost automorphic sequence $(t_k)_{k\in\mathbb{Z}}$ has recently been introduced by L. Qi and R. Yuan in [52] (Definition 3.1). Any Wekler sequence $(t_k)_{k\in\mathbb{Z}}$ is Bochner spatially almost automorphic, while the converse statement is not true in general. The authors have generalized the notion of piecewise continuous almost periodicity by introducing and examining the classes of Bohr, Bochner and Levitan piecewise continuous almost automorphic functions; in [52] (Theorem 4.8), the authors have proved that these classes coincide. We will consider piecewise continuous almost automorphic type functions and piecewise continuous almost automorphic solutions to the abstract impulsive Volterra integro-differenital equations somewhere else (cf. also the research article [16] by W. Dimbour and V. Valmorin for the notion of S-almost automorphy).
3. The following notion is also meaningful (cf. Example 1): A function $F : I \times X \to E$ is said to be semi-$(\mathcal{B}, \omega, c, (t_k))$-piecewise continuous periodic if and only if there exists a sequence $F_k : I \times X \to E$ in $PPC_{\omega,c;(t_k)}(I : X)$ which converges uniformly to the function $F(\cdot;\cdot)$ on $I \times B$ for each set $B \in \mathcal{B}$. The function $F : I \times X \to E$ is said to be semi-$(\mathcal{B}, \omega, c)$-piecewise continuous periodic if and only if $F(\cdot;\cdot)$ is semi-$(\mathcal{B}, \omega, c, (t_k))$-piecewise continuous periodic for some sequence (t_k) constructed in the same way as in Example 1; finally, we say that the function $F : I \times X \to E$ is semi-$(\mathcal{B}, c)$-piecewise continuous periodic if and only if $F(\cdot;\cdot)$ is semi-$(\mathcal{B}, \omega, c)$-piecewise continuous periodic for some number $\omega > 0$.

Besides the class of semi-$(\mathcal{B}, \omega, c)$-piecewise continuous periodic functions, we can also analyze many other classes of piecewise continuous almost periodic type functions such as $(S, \mathcal{B})$-asymptotically (ω, ρ)-periodic functions, quasi-asymptotically $(\mathcal{B}, \rho)$-almost periodic (uniformly recurrent) functions, $(\mathcal{B}, \rho)$-slowly oscillating functions and remotely $(\mathcal{B}, \rho)$-almost periodic (uniformly recurrent) functions; cf. [10] (Section 2.4) for more details about these classes of functions with $\mathbb{D} = \mathbb{R}$ or $\mathbb{D} = [0, \infty)$. Details and results will be given somewhere else.

Let us finally note that some numerical results about impulsive integro-differential equations are given in [53–55]; cf. also the references quoted therein.

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References

1. Bainov, D.; Simeonov, P. *Impulsive Differential Equations: Periodic Solutions and Applications*; Wiley: New York, NY, USA, 1993.
2. Bainov, D.; Simeonov, P. *Oscillation Theory of Impulsive Differential Equations*; International Publications: Orlando, FL, USA, 1998.
3. Besicovitch, A.S. *Almost Periodic Functions*; Dover Publications Inc.: New York, NY, USA, 1954.
4. Diagana, T. *Almost Automorphic Type and Almost Periodic Type Functions in Abstract Spaces*; Springer: New York, NY, USA, 2013.
5. Fink, A.M. *Almost Periodic Differential Equations*; Springer: Berlin, Germany, 1974.
6. N'Guérékata, G.M. *Almost Automorphic and Almost Periodic Functions in Abstract Spaces*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 2001.
7. Kostić, M. *Almost Periodic and Almost Automorphic Type Solutions of Abstract Volterra Integro-Differential Equations*; W. de Gruyter: Berlin, Germany, 2019.
8. Kostić, M. *Abstract Degenerate Volterra Integro-Differential Equations*; Mathematical Institute SANU: Belgrade, Serbia, 2020.
9. Kostić, M. *Selected Topics in Almost Periodicity*; W. de Gruyter: Berlin, Germany, 2022.
10. Kostić, M. *Advances in Almost Periodicity*; W. de Gruyter: Berlin, Germany, 2023.
11. Lakshmikantham, V.; Bainov, D.D.; Simeonov, P.S. *Theory of Impulsive Differential Equations*; World Scientific Publishing Co. Pte. Ltd.: Singapore, 1989.
12. Levitan, M. *Almost Periodic Functions*; G.I.T.T.L.: Moscow, Russia, 1953. (In Russian)
13. Levitan, B.M.; Zhikov, V.V. *Almost Periodic Functions and Differential Equations*; University Publishing House: Moscow, Russia, 1978.
14. Bellow, A.; Losert, V. The weighted pointwise ergodic theorem and the individual ergodic theorem along subsequences. *Trans. Am. Math. Soc.* **1985**, *288*, 307–345. [CrossRef]
15. Cuesta, E. Asymptotic behavior of the solutions of fractional integro-differential equations and some time discretizations. *Discret. Cont. Din. Syst. Suppl.* **2007**, *2007*, 277–285.
16. Dimbour, W.; Valmorin, V. *S*-Almost Automorphic Functions and Applications. HAL Preprints (2020), hal-03014691. Available online: https://hal.science/hal-03014691/ (accessed on 19 November 2020).
17. Ding, H.-S.; Long, W.; N'Guérékata, G.M. Almost periodic solutions to abstract semilinear evolution equations with Stepanov almost periodic coefficients. *J. Comput. Anal. Appl.* **2011**, *13*, 231–242.
18. Haraux, A.; Souplet, P. An example of uniformly recurrent function which is not almost periodic. *Fourier Anal. Appl.* **2004**, *10*, 217–220. [CrossRef]
19. Nawrocki, A. Diophantine approximations and almost periodic functions. *Demonstr. Math.* **2017**, *50*, 100–104. [CrossRef]
20. Tkachenko, V. Almost periodic solutions of evolution differential equations with impulsive action. In *Mathematical Modeling and Applications in Nonlinear Dynamics*; Nonlinear Systems and Complexity; Springer: Cham, Switzerland, 2016; Volume 14.
21. Stamov, G.T. *Almost Periodic Solutions of Impulsive Differential Equations*; Springer: Berlin/Heidelberg, Germany, 2012.
22. Xiao, T.-J.; Liang, J. *The Cauchy Problem for Higher–Order Abstract Differential Equations*; Springer: Berlin, Germany, 1998.
23. Zaidman, S. *Almost-Periodic Functions in Abstract Spaces*; Pitman Research Notes in Math; Pitman: Boston, MA, USA, 1985; Volume 126.
24. Fečkan, M.; Khalladi, M.T.; Kostić, M.; Rahmani, A. Multi-dimensional ρ-almost periodic type functions and applications. *Appl. Anal.* **2022**, *in press*. Available online: https://www.tandfonline.com/doi/abs/10.1080/00036811.2022.2103678?journalCode=gapa20 (accessed on 21 July 2022).
25. Du, W.-S.; Kostić, M.; Velinov, D. Abstract impulsive Volterra integro-differential inclusions. *Fractal Fract.* **2023**, *7*, 73. [CrossRef]
26. Halanay, A.; Wexler, D. *Qualitative Theory of Impulse Systems*; Mir: Moscow, Russia, 1971.
27. Samoilenko, A.M.; Perestyuk, N.A. *Impulsive Differential Equations*; World Scientific: Singapore, 1995.
28. Kovanko, A.S. Sur la compacié des sysémes de fonctions presque-périodiques généralisées de H. Weyl. *C.R. (Doklady) Ac. Sc. URSS* **1944**, *43*, 275–276.
29. Qi, L.; Yuan, R. A generalization of Bochner's theorem and its applications in the study of impulsive differential equations. *J. Dyn. Differ. Equ.* **2019**, *31*, 1955–1985. [CrossRef]
30. Kostić, M. Stepanov and Weyl classes of multi-dimensional ρ-almost periodic type functions. *Electron. Math. Anal. Appl.* **2022**, *10*, 11–35.
31. Díaz, L.; Naulin, R. A set of almost periodic discontinuous functions. *Pro. Math.* **2006**, *20*, 107–118.
32. Henríquez, H.R.; de Andrade, B.; Rabelo, M. Existence of almost periodic solutions for a class of abstract impulsive differential equations. *ISRN Math. Anal.* **2011**, *2011*, 632687. [CrossRef]
33. Qi, L.; Yuan, R. Factorization of scalar piecewise continuous almost periodic functions. *arXiv* **2018**, arXiv:1802.08854v1.
34. Xia, J. Piecewise continuous almost periodic functions and mean motions. *Trans. Am. Math. Soc.* **1985**, *288*, 801–811. [CrossRef]
35. Chaouchi, B.; Kostić, M.; Velinov, D. Metrical almost periodicity: Levitan and Bebutov concepts. *arXiv* **2022**, arXiv:2209.13576.
36. Khalladi, M.T.; Kostić, M.; Rahmani, A.; Pinto, M.; Velinov, D. *c*-Almost periodic type functions and applications. *Nonauton. Dyn. Syst.* **2020**, *7*, 176–193. [CrossRef]
37. Bart, H.; Goldberg, S. Characterizations of almost periodic strongly continuous groups and semigroups. *Math. Ann.* **1978**, *236*, 105–116. [CrossRef]
38. Wang, L.; Yu, M. Favard's theorem of piecewise continuous almost periodic functions and its application. *J. Math. Anal. Appl.* **2014**, *413*, 35–46. [CrossRef]

39. Liu, J.; Zhang, C. Composition of piecewise pseudo almost periodic functions and applications to abstract impulsive differential equations. *Adv. Differ. Equ.* **2013**, *2013*, 11. [CrossRef]
40. Bergelson, V.; Kułaga-Przymus, J.; Lemańczyk, M.; Richter, F.K. Rationally almost periodic sequences, polynomial multiple recurrence and symbolic dynamics. *Ergod. Theory Dyn. Syst.* **2018**, *39*, 2332–2383. [CrossRef]
41. Downarowicz, T.; Iwanik, A. Quasi-uniform convergence in compact dynamical systems. *Stud. Math.* **1988**, *89*, 11–25. [CrossRef]
42. Iwanik, A. Weyl almost periodic points in topological dynamics. *Colloq. Math.* **1988**, *56*, 107–119. [CrossRef]
43. Kostić, M. Multi-dimensional Besicovitch almost periodic type functions and applications. *Comm. Pure Appl. Anal.* **2022**, *21*, 4215–4250. [CrossRef]
44. Besicovitch, A.S. On the density of certain sequences of integers. *Math. Ann.* **1935**, *110*, 336–341. [CrossRef]
45. Andres, J.; Bersani, A.M.; Grande, R.F. Hierarchy of almost-periodic function spaces. *Rend. Math. Appl.* **2006**, *26*, 121–188.
46. Bertrandias, J.P. Espaces de fonctions bornees et continues en moyenne asymptotique d'ordre *p*. *Bull. Soc. Math. Fr. Mémoire* **1966**, *5*, 1–106. (In French) [CrossRef]
47. Neubrander, F. Well-posedness of higher order abstract Cauchy problems. *Trans. Am. Math. Soc.* **1986**, *295*, 257–290. [CrossRef]
48. Kostić, M. On the existence and uniqueness of solutions of certain classes of abstract multi-term fractional differential equations. *Funct. Anal. Appr. Comp.* **2014**, *6*, 13–33.
49. Bazhlekova, E. Fractional Evolution Equations in Banach Spaces. Ph.D. Thesis, Eindhoven University of Technology, Eindhoven, The Netherlands, 2001.
50. Kostić, M. *Abstract Volterra Integro-Differential Equations*; CRC Press: Boca Raton, FL, USA, 2015.
51. Favini, A.; Yagi, A. *Degenerate Differential Equations in Banach Spaces*; Pure and Applied Mathematics; Chapman and Hall/CRC: New York, NY, USA, 1998.
52. Qi, L.; Yuan, R. Piecewise continuous almost automorphic functions and Favard's theorems for impulsive differential equations in honor of Russell Johnson. *J. Dyn. Differ. Equ.* **2022**, *34*, 399–441. [CrossRef]
53. Akhmetov, M.U.; Zafer, A. Successive approximation method for quasilinear impulsive differential equations with control. *Appl. Math. Lett.* **2000**, *13*, 99–105. [CrossRef]
54. Bainov, D.D.; Kulev, G. Application of Lyapunov's functions to the investigation of global stability of solutions of system with impulses. *Appl. Anal.* **1988**, *26*, 255–270.
55. Ran, X.J.; Liu, M.Z.; Zhu, Q.Y. Numerical methods for impulsive differential equation. *Math. Comput. Model.* **2008**, *48*, 46–55. [CrossRef]

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